Homogeneous continuous finite-time observer for the triple integrator

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Abstract

In this paper we consider the continuous homogeneous observer defined in [1] in the case of the triple integrator. In [1], convergence of the algorithm was only proved when the degree of homogeneity was sufficiently *close to 0 without more tractable information. We show here that, in the case of the triple integrator, the observer presents global finite-time stability for* any *negative degree under constructive conditions on the gains. This is achieved with a homogeneous Lyapunov function design. Simulations of the proposed observer are also provided.*

1. Introduction

Even though mechanical or electrical systems often have a second order model, problems where third order systems appear are also common and the simplest and canonical form of these systems is a triple integrator. Most of the current techniques for linear or nonlinear feedback stabilization and observation provide an asymptotic or exponential stability: the obtained closed-loop dynamics is locally Lipschitz and the system trajectories settle at the origin when the time is approaching infinity. Such a rate of convergence is not admissible in many applications, this is why the Finite-Time Stability (FTS) notion has been quickly developing during the last decades: solutions of a FTS system reach the equilibrium point in a finite time. For example, for $x \in \mathbb{R}$ and $\alpha \in (0,1)$, the solutions of $\dot{x} = -\text{sign}(x)|x|^{\alpha}$ starting from $x_0 \in \mathbb{R}$ at $t_0 = 0$ are

$$\begin{cases} \text{sign}(x_0)[|x_0|^{1-\alpha} - (1-\alpha)t]^{\frac{1}{1-\alpha}} & \text{if} \quad 0 \le t \le \frac{|x_0|^{1-\alpha}}{1-\alpha} \\ 0 & \text{if} \quad t > \frac{|x_0|^{1-\alpha}}{1-\alpha} \end{cases}.$$

Let us note that the right hand side of the above differential equation is not Lipschitz. In fact, finite-time convergence implies non-uniqueness of solutions (in backward time) which is not possible in the presence of Lipschitz-continuous dynamics, where different maximal trajectories never cross.

The problem of finite-time stability has been developed for continuous systems giving sufficient and necessary condition (see [2, 3]). In addition, necessary and sufficient conditions appear for discontinuous systems (see [4]). It was observed in many papers that FTS can be achieved if the system is locally asymptotically stable and *homogeneous* with negative degree [5]. This is why the homogeneity plays a central role in the FTS system design. The reader may found additional properties and results on homogeneity in [6, 7, 8, 9, 10]. The homogeneity property was used many times to design FTS state controls [11, 12, 13, 14, 15, 16], FTS observers [1, 17, 18], consensus protocols [19] and FTS output feedback [20, 21].

In [1], a generic class of homogeneous continuous observers for an $n-th$ integrator was developed. Their simple shape, combined with FTS and robustness properties granted by homogeneity were making them very promising candidates for observation. However, the proof of stability relied on a continuity argument and the result only held when the homogeneity degree was sufficiently close to 0, i.e., when the observer was almost linear. No constructive procedure was known, given a degree of homogeneity, to check whether the observer was stable or not, for $n > 2$.

In this paper, we shall give explicit conditions under which the result of [1] holds in the case of the triple integrator. The paper is organized as follows. Section II is devoted to preliminaries. In section III we present the system of interest, the proposed observer and the corresponding error equation. In section IV, we introduce our

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candidate Lyapunov function and give conditions under which this function is positive definite. The derivative of this function is studied in section V, and the main results are compiled in theorems 1 and 2. Simulations are presented in section VI and finally a conclusion summarizes the paper.

2. Preliminaries

Through the paper the following notation will be used. For any real number $\alpha \geq 0$ and for all $x \in \mathbb{R}$ we define $\lfloor x \rceil^\alpha = \text{sign}(x)|x|^\alpha$.

Proposition 1 (Young's inequality)**.** *For any $x, y \in \mathbb{R}$, any $p, q > 0$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and any $\varepsilon > 0$, the following inequality holds*

$$|xy| \leq \varepsilon \frac{|x|^p}{p} + \frac{1}{\varepsilon} \frac{|y|^q}{q}.$$

This well known inequality will be used extensively throughout the paper.

2.1. Finite-time stabilization

Let us consider a continuous vector field f and the system

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n. \tag{1}$$

Definition 1. [22] The origin of the system (1) is *finite-time stable* (FTS) iff there exists a neighborhood of the origin $\mathscr{V}$ such that:

1. For any $x_0 \in \mathscr{V}$ there exists $t_0 \geq 0$ such that for any solution $x(t)$ of (1) such that $x(0) = x_0$ we have $x(t) = 0$ for all $t \geq t_0$. The infimum $T(x_0)$ of all such t_0 allows us to define the function $T : \mathscr{V} \to \mathbb{R}_+$, called the *settling-time function* of the system (1).

2. For any neighborhood of the origin $\mathscr{U}_1 \subset \mathscr{V}$, there exists a neighborhood of the origin $\mathscr{U}_2$ such that for any $x_0 \in \mathscr{U}_2$ and any solution $x(t)$ of (1) such that $x(0) = x_0$ we have $x(t) \in \mathscr{U}_1$ for all $t \geq 0$.

Moreover, if the neighborhood $\mathscr{V}$ can be chosen to be $\mathbb{R}^n$, then the origin of the system (1) is said to be *globally finite-time stable* (GFTS).

2.2. Homogeneity

Let $\mathbf{r} = (r_1, \ldots, r_n)$ be a $n-$uplet of positive real numbers, thereafter called a *generalized weight*. Then $\Lambda_r x = (\ldots, \lambda^{r_i} x_i, \ldots)$ for any positive number λ represents a mapping $x \mapsto \Lambda_\mathbf{r} x$ usually called a dilation (see [8]).

Definition 2. A function $h : \mathbb{R}^n \to \mathbb{R}$ is $\mathbf{r}$-homogeneous of degree $\kappa \in \mathbb{R}$ if for all $x \in \mathbb{R}^n$ and all $\lambda > 0$ we have $h(\Lambda_\mathbf{r} x) = \lambda^\kappa h(x)$.

Definition 3. A vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ is $\mathbf{r}$-homogeneous of degree κ if for all $x \in \mathbb{R}^n$ and all $\lambda > 0$ we have $f(\Lambda_r x) = \lambda^\kappa \Lambda_r f(x)$, or equivalently, if the coordinate functions f_i are $\mathbf{r}$-homogeneous of degree $\kappa + r_i$. When such a property holds, the corresponding system (1) is said to be $\mathbf{r}$-homogeneous of degree κ.

3. Problem formulation

Consider the following system defined for $z = (z_1, z_2, z_3)^T \in \mathbb{R}^3$

$$\begin{cases} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ \dot{z}_3 &= u \\ y &= z_1 \end{cases} \tag{2}$$

where z_1, z_2 and z_3 are the states of the system, u is the input and y is the output. We follow [1] and define, for $\beta \in (\frac{2}{3}, 1)$, the following estimator

$$\begin{cases} \dot{\hat{z}}_1 &= \hat{z}_2 + l_1 \lfloor y - \hat{z}_1 \rceil^\beta \\ \dot{\hat{z}}_2 &= \hat{z}_3 + l_2 \lfloor y - \hat{z}_1 \rceil^{2\beta - 1} \\ \dot{\hat{z}}_3 &= u + l_3 \lfloor y - \hat{z}_1 \rceil^{3\beta - 2} \end{cases}. \tag{3}$$

Consider now the estimation error $x = z - \hat{z}$. Its dynamics is given by

$$\begin{cases} \dot{x}_1 &= x_2 - l_1 \lfloor x_1 \rceil^\beta \\ \dot{x}_2 &= x_3 - l_2 \lfloor x_1 \rceil^{2\beta - 1} \\ \dot{x}_3 &= \quad -l_3 \lfloor x_1 \rceil^{3\beta - 2} \end{cases}. \tag{4}$$

We intend to find explicit conditions on the gains l_1, l_2 and l_3 and on the power β such that the origin is a GFTS equilibrium of the system (4).

Remark 1. The right-hand side of system (4) is not Lipschitz continuous. Even though we could select $\beta \geq 1$ to ensure Lipschitz continuity, we would lose the finite-time stability which is one of the desired properties of the algorithm. Therefore, we will stick to $\beta < 1$.

4. Lyapunov function design

Let us define, for $\rho, \delta > 0$, the following candidate Lyapunov's function

$$V(x) = (l_2 - \beta\rho l_3) \frac{|x_1|^{2\beta}}{2\beta} - \rho l_3 \lfloor x_1 \rceil^\beta x_2 + \frac{x_2^2}{2} - \frac{l_1}{l_2} x_2 \lfloor x_3 \rceil^{\frac{\beta}{2\beta - 1}} + \delta \frac{2\beta - 1}{2\beta} |x_3|^{\frac{2\beta}{2\beta - 1}} - x_1 x_3. \tag{5}$$

Using the Young's inequality, we find that V is definite positive if the following inequalities hold

$$l_2-\beta\rho l_3-\varepsilon_1\rho l_3\beta-\varepsilon_3 \quad >0 \tag{6}$$

$$1-\frac{\rho l_3}{\varepsilon_1}-\frac{l_1\varepsilon_2}{l_2} \quad >0 \tag{7}$$

$$\delta(2\beta-1)-\frac{l_1\beta}{l_2\varepsilon_2}-\frac{2\beta-1}{\varepsilon_3} \quad >0 \tag{8}$$

We claim that, under the two following conditions, we can find $\varepsilon_1,\varepsilon_2$ and ε_3 such that (6), (7) and (8) hold

$$l_2-\beta\rho l_3-\beta\rho^2 l_3^2>0 \tag{9}$$

$$\delta>\frac{\beta l_1^2 l_2-\beta^2\rho l_3 l_1^2+2\sqrt{2\beta-1}\beta\rho l_1 l_2 l_3+(2\beta-1)l_2^2}{(2\beta-1)l_2^2(l_2-\beta\rho l_3-\beta\rho^2 l_3^2)}. \tag{10}$$

Proposition 2. *Under conditions (9) and (10), the function V defined by (5) is positive definite.*

The proof is omitted due to space limitations.

Let us remark that the conditions (9) and (10) do not really constrain the gains. Indeed, ρ and δ are free parameters appearing only in the Lyapunov function V and taking ρ small enough ensures condition (9) while taking δ large enough ensures condition (10).

5. Study of the derivative of V

Our goal in this section is to prove that an adequate choice of the parameters δ and ρ together with conditions on the gains l_1,l_2 and l_3 can ensure that the derivative of V will be negative definite.

Let us start by mentioning that the function V is not differentiable on the plane $x_1=0$. It will not be a big deal because we will see that $\dot V$ is upper bounded by $-CV^{(\beta+1)/(2\beta)}$ with $C>0$ and because the only invariant subspace of the plane $x_1=0$ is $\{0\}$. However, to ensure the correctness of the following computations, we assume that $x_1\neq 0$. A direct computation gives

$$\begin{aligned}\dot V(x)&=(\rho l_3 l_2+l_3+l_1\beta\rho l_3-l_1 l_2)|x_1|^{3\beta-1}\\ &-\beta\rho l_3|x_1|^{\beta-1}x_2^2-\frac{l_1}{l_2}|x_3|^{\frac{3\beta-1}{2\beta-1}}\\ &\quad+\rho l_2\beta(l_1-1)\lfloor x_1\rceil^{2\beta-1}x_2+(l_1-\rho l_3)\lfloor x_1\rceil^{\beta}x_3\\ &\quad+l_1\lfloor x_1\rceil^{2\beta-1}\lfloor x_3\rceil^{\frac{\beta}{2\beta-1}}-\delta l_3\lfloor x_1\rceil^{3\beta-2}\lfloor x_3\rceil^{\frac{1}{2\beta-1}}\\ &\quad+\frac{l_1 l_3\beta}{l_2(2\beta-1)}\lfloor x_1\rceil^{3\beta-2}x_2\lfloor x_3\rceil^{\frac{1-\beta}{2\beta-1}}.\end{aligned}$$

From successive uses of the Young's inequality, we find that $\dot V(x)<0$ on $\{x_1\neq 0\}$ if there exist $\varepsilon_1,\varepsilon_2,\varepsilon_3>0$ such that the following conditions hold

$$l_1>\rho l_3 \tag{11}$$

$$\begin{aligned}&l_1[-(3\beta-1)l_2+\beta\varepsilon_2+\varepsilon_3(2\beta-1)]\\ &+l_3\left[(3\beta-1)\left(\rho l_2+1+l_1\beta\rho+\frac{\rho\beta|l_1-1|\varepsilon_1}{2}\right)\right.\\ &\left.-\rho\beta\varepsilon_2+\delta(3\beta-2)+\frac{l_1}{l_2}\frac{\beta(5\beta-3)}{2(2\beta-1)}\right]<0\end{aligned} \tag{12}$$

$$\rho\left[\frac{|l_1-1|}{\varepsilon_1}-2\right]+\frac{l_1}{l_2(2\beta-1)}<0 \tag{13}$$

$$\begin{aligned}&l_1\left[-\frac{3\beta-1}{l_2}+\frac{2\beta-1}{\varepsilon_2}+\frac{\beta}{\varepsilon_3}\right]\\ &+l_3\left[-\frac{\rho(2\beta-1)}{\varepsilon_2}+\delta+\frac{l_1\beta(1-\beta)}{l_2(2\beta-1)}\right]<0.\end{aligned} \tag{14}$$

There exists $\varepsilon_1>0$ such that inequation (13) holds if and only if

$$-2\rho+\frac{l_1}{l_2(2\beta-1)}<0 \tag{15}$$

and then ε_1 has to be chosen such that

$$\varepsilon_1>\frac{|l_1-1|l_2\rho(2\beta-1)}{2l_2\rho(2\beta-1)-l_1}. \tag{16}$$

Hence, there exist solutions $(\varepsilon_1,\varepsilon_2,\varepsilon_3)$ of inequalities (12) - (14) if and only if there exist solutions $(\varepsilon_2,\varepsilon_3)$ of

$$\begin{aligned}&l_1[-(3\beta-1)l_2+\beta\varepsilon_2+\varepsilon_3(2\beta-1)]\\ &+l_3\left[(3\beta-1)(\rho l_2+1+l_1\beta\rho+\frac{\rho^2\beta|l_1-1|^2 l_2(2\beta-1)}{4l_2\rho(2\beta-1)-2l_1})\right.\\ &\left.-\rho\beta\varepsilon_2+\delta(3\beta-2)+\frac{l_1}{l_2}\frac{\beta(5\beta-3)}{2(2\beta-1)}\right]<0,\end{aligned} \tag{17}$$

together with (11), (14) and (15). Now denoting

$$\begin{aligned}A_1&=-(3\beta-1)l_2+\beta\varepsilon_2+(2\beta-1)\varepsilon_3\\ A_3&=(3\beta-1)(\rho l_2+1+l_1\beta\rho+\frac{\rho^2\beta|l_1-1|^2 l_2(2\beta-1)}{4l_2\rho(2\beta-1)-2l_1})\\ &\quad-\rho\beta\varepsilon_2+\delta(3\beta-2)+\frac{l_1}{l_2}\frac{\beta(5\beta-3)}{2(2\beta-1)}\\ B_1&=-\frac{3\beta-1}{l_2}+\frac{2\beta-1}{\varepsilon_2}+\frac{\beta}{\varepsilon_3}\\ B_3&=-\frac{\rho(2\beta-1)}{\varepsilon_2}+\delta+\frac{l_1\beta(1-\beta)}{l_2(2\beta-1)}\end{aligned}$$

we see that (17) and (14) can be written $A_1l_1+A_3l_3<0$, $B_1l_1+B_3l_3<0$. Hence, the negativity of $\dot V$ would be ensured for small enough l_3 if we were able to prove that the terms A_1 and B_1 are negative.

Lemma 1. *For any $\beta\in(2/3,1)$ and any $l_2>0$ there exists a pair $(\varepsilon_2,\varepsilon_3)$ of positive reals such that*

$$\begin{cases}\beta\varepsilon_2+(2\beta-1)\varepsilon_3 < l_2(3\beta-1)\\ \frac{2\beta-1}{\varepsilon_2}+\frac{\beta}{\varepsilon_3} < \frac{3\beta-1}{l_2}\end{cases}. \tag{18}$$

The proof is omitted due to space limitations.

Finally we can state our main result.

Theorem 1. *For any $\beta \in (2/3,1)$, any $l_1, l_2 > 0$ and any ρ and δ such that*

$$\rho > \frac{l_1}{2(2\beta-1)l_2} \tag{19}$$

$$\delta > \frac{\beta l_1^2 + (2\beta-1)l_2}{(2\beta-1)l_2^2} \tag{20}$$

there exist $l_3 > 0$ such that V is positive definite and $\dot{V}(x) < 0$ for all $x \in \{x_1 \neq 0\}$.

Proof. From all the discussion beforehand, it suffices to find positive reals l_3, ε_2 and ε_3 such that the inequalities (9), (10), (11), (14), (15) and (17) hold.

Clearly, the condition (19) is just a rewritting of inequality (15). Inequality (9) is equivalent to

$$l_3 < \frac{1}{2\rho}\left[\sqrt{1+\frac{4l_2}{\beta}}-1\right]. \tag{21}$$

Condition (20) ensures that inequality (10) has solutions. Inequality (10) can then be rewritten under the form

$$l_3 < \kappa(l_1, l_2, \rho, \delta). \tag{22}$$

An explicit form of the function κ can be computed but is not presented here due to space limitations.

Inequality (11) can be rewritten

$$l_3 < \frac{l_1}{\rho}. \tag{23}$$

Now, granted that we choose ε_2 and ε_3 for which (18) holds, we see that $A_3 > 0$ and then inequalities (17) and (14) can be rewritten

$$l_3 \quad < \quad -\frac{A_1}{A_3} l_1 \tag{24}$$

$$B_3 \leq 0 \quad \text{or} \quad l_3 < -\frac{B_1}{B_3} l_1. \tag{25}$$

Finally, by lemma 1, we get $A_1 < 0$ and $B_1 < 0$ and all inequalities (21) - (25) hold if l_3 is chosen small enough, which concludes the proof. □

Let us remark that, although this theorem seems to only state the *existence* of a gain $l_3 > 0$ such that the Lyapunov function V is positive definite and its derivative $\dot{V}$ is negative for $x_1 \neq 0$, it is actually constructive. To compute admissible values of $l_3 > 0$, it suffices to

1. select $\beta \in (2/3,1)$, $l_1, l_2 > 0$, ρ and δ such that (19) and (20) hold;
2. compute a couple $(\varepsilon_2, \varepsilon_3)$ such that (18) holds;
3. select l_3 such that all inequalities (21) - (25) hold.

Example 1. *Let us select $\beta = 0.75$ and $l_1 = l_2 = 1$. Conditions (19) and (20) read $\rho > 1$ and $\delta > 2.5$ so we choose $\rho = 6$ and $\delta = 3$. Then we select $\varepsilon_2 = \frac{5}{6}$ and $\varepsilon_3 = \frac{6}{5}$ and easily check that (18) holds. Then inequalities (21) - (24) read (numerical values have been truncated)*

$$\begin{aligned} l_3 &\leq 0.126 \\ l_3 &\leq 0.145 \\ l_3 &\leq 0.166 \\ l_3 &\leq 0.002 \end{aligned}$$

while inequality (25) trivially holds because $B_3 < 0$. The choice $l_3 = 0.002$ then ensures that V is positive definite and $\dot{V} < 0$ for $x_1 \neq 0$.

We can now express the result in a more theoretical form.

Theorem 2. *For any $\beta \in (2/3,1)$, for any $l_1 > 0$ and $l_2 > 0$, there exists $l_3 > 0$ such that the observer (3) recovers in finite time the state z of the system (2) for any initial state $z_0 \in \mathbb{R}^3$.*

Proof. We need to prove that the origin is a GFTS equilibrium for the error equation (4). Let us select $\rho, \delta, \varepsilon_2, \varepsilon_3$ and l_3 such that inequalities (19), (20), (18) and (21) - (25) hold.

Let us denote $-a = \sup\{\dot{V}(x) : V(x) = 1, x_1 \neq 0\}$. Given that $\dot{V} < 0$ on $\{V(x) = 1, x_1 \neq 0\}$, we have $a \geq 0$. Moreover, $\{V(x) = 1\}$ is compact: indeed, any continuous homogeneous function is proper [11] and a straightforward computation shows that V and $\dot{V}$ are $(1, \beta, 2\beta-1)$-homogeneous functions of degree 2β and $3\beta - 1$. Since $0 \notin \{V(x) = 1\}$, we get $a > 0$. Classical manipulations on homogeneous functions then show that for any $\{x \in \mathbb{R}^3 : x_1 \neq 0\}$ we have

$$\dot{V}(x) \leq -aV(x)^{(3\beta-1)/(2\beta)}.$$

Now, given that no solution of (4), except $x(t) = 0$, stays on $\{x_1 = 0\}$, the set $\{V \leq 1\}$ is strictly positively invariant and therefore the origin is globally asymptotically stable for system (4) [11]. But system (4) is $(1, \beta, 2\beta-1)$-homogeneous of degree $\beta - 1 < 0$, and therefore the origin is GFTS [11].

□

Sometimes the conditions (21) - (25) may lead to very small values of the gain l_3. It is possible to rescale it with the following procedure.

Proposition 3. *Set $\beta \in (2/3, 1)$. If the origin is GFTS for system (4) with a set of gains (l_1, l_2, l_3), then for any $\lambda > 0$ the origin is also GFTS for system (4) with the set of gains $(\lambda l_1, \lambda^2 l_2, \lambda^3 l_3)$.*

The proof is omitted due to space limitations.

Remark that, for this new set of gains, corresponding values δ, ρ, ε_2 and ε_3 such that all the conditions (20) - (25) hold may not exist or be different, but the finite-time stability is still ensured.

Example 2 (Example 1 continued)**.** *We take again $\beta = 0.75$, $\rho = 6$, $\delta = 3$, $\varepsilon_2 = \frac{5}{6}$ and $\varepsilon_3 = \frac{6}{5}$. The set of gains $(1, 1, 0.002)$ is valid, thus by proposition 3, taking $\lambda = 10^3$, the set of gains $(10^3, 10^6, 2.10^6)$ is also valid. However, inequalities (19) and (15) read $\rho > 10^{-3}$ and $\rho < 5.10^{-4}$ and thus this set of gains could not have been found by the direct method exposed after theorem 1.*

A local approach can also be used to extend our result to any linear system with an output of relative degree 3.

Corollary 1. *Consider a linear system of dimension 3 such that the output y is of relative degree 3. The system reads $\dot{x} = Ax + Bu$ with $x = [y, \dot{y}, \ddot{y}]^T$, $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ c_1 & c_2 & c_3 \end{pmatrix}$ and $B = [0, 0, \lambda]^T$ where c_1, c_2, c_3 and λ are real coefficients. Then, if the gains l_i are selected according to the conditions presented before, the following observer leads the origin of the error equation to being locally finite-time stable:*

$$\dot{\hat{x}} = A\hat{x} + Bu + \begin{pmatrix} l_1 \lfloor x_1 - \hat{x}_1 \rceil^{\beta} \\ l_2 \lfloor x_1 - \hat{x}_1 \rceil^{2\beta-1} \\ l_3 \lfloor x_1 - \hat{x}_1 \rceil^{3\beta-2} \end{pmatrix}.$$

Proof. Computing a local homogeneous approximation of the error equation leads to system (4) which is finite-time stable. Therefore the original system is locally finite-time stable for gains selected according to the conditions presented before. For details about local homogeneous approximations, see [23]. □

6. Examples and Simulations

In this section, we would like to illustrate the applicability of the proposed algorithm in showing how to use it on concrete examples and demonstrate the efficiency of the proposed observer compared to a Luenberger observer.

Consider an electric motor acting on a rigid arm turning around a central point in the horizontal plane. Let denote θ the angle of the arm and $\dot{\theta}$ the angular velocity of the arm. The force generated by the motor is Ki where i is the intensity of the current and $K > 0$ a constant. We find

$$\begin{aligned} m\ddot{\theta} &= F - b\theta \\ u &= L\frac{di}{dt} + Ri + k\dot{\theta} \end{aligned}$$

where u is the input tension, $b > 0$ is a coefficient of friction, m is the mass of the arm and L, R and k are positive coefficients of the motor. The output is supposed to be θ and is of relative degree 3. Following Corollary 1, we get a locally finite-time stable observer.

Now, we would like to compare the proposed observer with a classical Luenberger observer. As in Example 1, we take $\beta = 0.75$, $\rho = 6$, $\delta = 3$, $\varepsilon_2 = \frac{5}{6}$ and $\varepsilon_3 = \frac{6}{5}$. The set of gains $(1, 1, 0.002)$ is valid, thus by proposition 3, taking $\lambda = 20$, the set of gains $(20, 400, 16)$ is also valid. We will compare the estimation error given by (4) with the estimation error

$$\left\{ \begin{array}{lcl} \dot{x}_1 &=& x_2 - 20x_1 \\ \dot{x}_2 &=& x_3 - 400x_1 \\ \dot{x}_3 &=& -16x_1 \end{array} \right. . \qquad (26)$$

coming from the following observer of system (2)

$$\left\{ \begin{array}{lcl} \dot{\hat{z}}_1 &=& \hat{z}_2 + 20(y - \hat{z}_1) \\ \dot{\hat{z}}_2 &=& \hat{z}_3 + 400(y - \hat{z}_1) \\ \dot{\hat{z}}_3 &=& u(y) + 16(y - \hat{z}_1) \end{array} \right. . \qquad (27)$$

The results of simulations are presented in Fig. 1. The eigenvalues of the matrix corresponding to the linear system (26) are $-9.980 + 17.309i, -9.980 - 17.309i$ and -0.040, ensuring the asymptotic stability. However, the small value of the third eigenvalue does not allow the linear observer to recover quickly the signal. We can see that in about $1.2s$ the homogeneous observer has converged, while x_3 is still around -0.2 after $1.6s$.

7. Conclusion

In this paper, we gave explicit conditions under which the observer (3) converges to the state of the system (2) so that the origin is a GFTS equilibrium of the corresponding error equation (4). We introduced a homogeneous Lyapunov function and gave explicit conditions under which this function is positive definite and its derivative is negative. Examples of use of these conditions were provided, as well as a method to derive other valid choices for the gains. Computer simulations demonstrated the efficiency of the proposed algorithm.

Future works include an optimization of the parameters ρ and δ to let the gain l_3 take values as large as

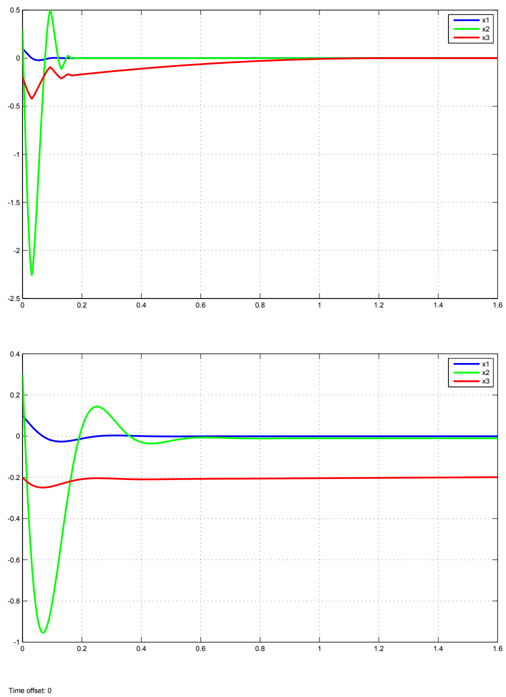

Figure 1. Comparison between homogeneous (top) and linear (bottom) estimation errors.

possible. Also, we would like to go further and try to extend this method to n^{th}-integrators.

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