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# On asymptotic normality of nonparametric estimate for stationary pairwise interaction point process

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## Abstract

Pairwise interacting point processes form a very popular and flexible class of models for spatial point patterns in a bounded region. In this class, the interaction between two points is described by pairwise interaction function.

We prove the asymptotic normality of nonparametric estimator of pairwise interaction function for stationary pairwise interaction point process characterized by the Papangelou conditional intensity, and observed in a bounded window of a sequence of cubes growing up to  $\mathbb{R}^d$ . Formula for the variance of the resulting estimator can be obtained using Papangelou conditional intensity of the point process. This is a random function satisfying the counterpart of the Georgii-Nguyen-Zessin formula. The proof of the asymptotic normality of the resulting estimator is based on the  $m_n$ -approximation method in the setting of dependent random fields indexed by  $\mathbb{Z}^d$  where  $d$  is a positive integer.

**Keywords:** Georgii-Nguyen-Zessin formula, kernel estimators, non parametric estimation, pairwise interaction point process, Papangelou conditional intensity, asymptotic normality.

# 1 Introduction

A special type of Gibbs processes that are noteworthy both for their abundant use in statistical physics are the pairwise interaction point processes. Several methods of estimation are available for parametric families of pairwise interaction point processes: approximate maximum likelihood (Ogata and Tanemura, [28], [29]), maximum pseudo-likelihood (Besag et al. [4]), Monte Carlo likelihood (Geyer [14]).

In this paper we are concerned with nonparametric statistics for stationary pairwise interaction point processes. They have been introduced by Ripley and Kelly [31], Daley and Vere-Jones [7] and Georgii [13]. These provide a large variety of complex patterns starting from simple potential functions (or pairwise interaction function) which are easily interpretable as attractive or repulsive forces acting among points and are of practical importance because of their ability to model a wide variety of spatial point patterns, especially those displaying some degree of spatial regularity. A great deal is understood about pairwise interaction models because they are very natural with respect to the derivation of Papangelou conditional intensity (Møller and Waagepetersen [23]).

Papangelou conditional intensity is a function  $\lambda : \mathbb{R}^d \times N_{lf} \rightarrow \mathbb{R}_+$  where  $N_{lf}$  is the space of locally finite configurations of points in  $\mathbb{R}^d$ . The Papangelou conditional intensity can be interpreted as follows: for any  $u \in \mathbb{R}^d$  and  $\mathbf{x} \in N_{lf}$ ,  $\lambda(u, \mathbf{x})du$  corresponds to the conditional probability of observing a point in a ball of volume  $du$  around  $u$  given the rest of the point process is  $\mathbf{x}$ .

The goal of the present paper is to establish the asymptotic normality of the new nonparametric estimator of the pairwise interaction function for stationary pairwise interaction point process characterized by the Papangelou conditional intensity. The asymptotic normality in statistics is based on central limit theorems for the sequences of random variables. These classical limit theorems have been extended to the setting of spatial processes. Some results on the central limit theorem and its functional versions are Bass [3], Basu and Dorea [1], Dedecker [9], [10] and El Machkouri [20]. Our proof is based on the  $m$ -approximation method ( $m_n \rightarrow \infty$  as  $n \rightarrow \infty$ ) in the setting of dependent random fields indexed by  $\mathbb{Z}^d$  where  $d$  is a positive integer. We apply a central limit theorem for triangular arrays of stationary  $m$ -dependent random fields with unbounded  $m$  established by Wang and Woodroffe [34]. This result improves a central limit theorem established by Heinrich [16]. The notion of  $m$ -dependence was first introduced by Hoeffding and Robbins [18]. Central limit theorems for  $m$ -dependent random fields has already been considered by many researchers. For example Cheng and Ho [6], Wang and Woodroffe [33] and El Machkouri et al. [22].

The paper is organized as follows. Section 2 sets up the basic tools of the point processes in  $\mathbb{R}^d$ . Section 3 contains the main asymptotic results on proposed estimators of stationary pairwise interaction point process. This concerns firstly the asymptotic behaviour of variance and covariance of the kernel-type estimator under certain smoothness conditions. The latter is based on the knowledge of Papangelou conditional intensity and the iterated Georgii-Nguyen-Zessin formula. Secondly, we are able to study asymptotic Gaussian of proposed estimators and proved in Section 4.

## 2 Basic tools

Let  $\mathcal{B}^d$  be the Borel  $\sigma$ -algebra (generated by open sets) in  $\mathbb{R}^d$  (the  $d$ -dimensional space) and  $\mathcal{B}_0^d \subseteq \mathcal{B}^d$  be the system of all bounded Borel sets. A point process  $\mathbf{X}$  in  $\mathbb{R}^d$  is a locally finite random subset of  $\mathbb{R}^d$ , i.e. the number of points  $N(\Lambda) = n(\mathbf{X}_\Lambda)$  of the restriction of  $\mathbf{X}$  to  $\Lambda$  is a finite random variable whenever  $\Lambda$  is a bounded Borel set of  $\mathbb{R}^d$  (see Daley and Vere-Jones [7]). We define the space of locally finite point configurations in  $\mathbb{R}^d$  as  $N_{lf} = \{\mathbf{x} \subseteq \mathbb{R}^d; n(\mathbf{x}_\Lambda) < \infty, \forall \Lambda \in \mathcal{B}_0^d\}$ , where  $\mathbf{x}_\Lambda = \mathbf{x} \cap \Lambda$ . The volume of a bounded Borel set  $\Lambda$  of  $\mathbb{R}^d$  is denoted by  $|\Lambda|$ . For all finite subset  $I$  of  $\mathbb{Z}^d$ , we denote  $|I|$  the number of elements in  $I$ .

In general it is not possible to deal with densities of infinite point processes. However, the Papangelou conditional intensity for a point process  $\mathbf{X}$  on  $\mathbb{R}^d$  can be indirectly defined. In fact two infinite point processes can share the same Papangelou conditional intensity. The Georgii-Nguyen-Zessin (GNZ) formula (see Papangelou [30], Zessin [35], Georgii [12], Nguyen and Zessin [27]) states that for any nonnegative measurable function  $h$  on  $\mathbb{R}^d \times N_{lf}$

$$\mathbb{E} \sum_{u \in \mathbf{X}} h(u, \mathbf{X} \setminus u) = \mathbb{E} \int_{\mathbb{R}^d} h(u, \mathbf{X}) \lambda(u, \mathbf{X}) du. \quad (1)$$

We will call the random integral on the right side of (1) the (Papangelou) compensator of the random sum on the left side. Using induction we obtain the iterated GNZ-formula: for nonnegative functions  $h : (\mathbb{R}^d)^n \times N_{lf} \rightarrow \mathbb{R}$

$$\mathbb{E} \sum_{\substack{\neq \\ u_1, \dots, u_n \in \mathbf{X}}} h(u_1, \dots, u_n, \mathbf{X} \setminus \{u_1, \dots, u_n\}) = \int \dots \int \mathbb{E} h(u_1, \dots, u_n, \mathbf{X}) \lambda(u_1, \dots, u_n, \mathbf{X}) du_1 \dots du_n \quad (2)$$

where  $\lambda(u_1, \dots, u_n, \mathbf{x})$  is Papangelou conditional intensity and is defined (not uniquely) by  $\lambda(u_1, \dots, u_n, \mathbf{x}) = \lambda(u_1, \mathbf{x}) \lambda(u_2, \mathbf{x} \cup \{u_1\}) \dots \lambda(u_n, \mathbf{x} \cup \{u_1, \dots, u_{n-1}\})$ .

Examples of Gibbs point process models and their conditional intensities are presented in Baddeley et al. [2], Møller and Waagepetersen ([23], [24]). The formula (2) will also be used extensively later in this document.

## 3 Assumptions and the main results

Throughout this paper, we assume the model is a stationary pairwise interaction point process and its Papangelou conditional intensity at a location  $u$  given by

$$\lambda_{\beta^*}(u, \mathbf{x}) = \beta^* \exp \left( - \sum_{v \in \mathbf{x} \setminus u} g(v - u) \right) \quad (3)$$

where  $\beta^*$  is the true value of the Poisson intensity parameter,  $g$  represents a pairwise interaction potential. Specifically,  $g$  is nonnegative measurable function; the practical significance of this assumption is that pairwise interaction processes are a

wide and useful class of models for spatial point processes with inhibition between points; for discussion see [11] and [32]. We denote  $G = \exp(-g)$  the pairwise interaction function. The basic assumption throughout this paper is the Papangelou conditional intensity (3) has a finite range  $R$ , i.e.

$$\lambda_{\beta^*}(u, \mathbf{x}) = \lambda_{\beta^*}(u, \mathbf{x}_{B(u,R)}), \quad (4)$$

for any  $u \in \mathbb{R}^d$ ,  $\mathbf{x} \in N_{lf}$ , where  $B(u, R)$  is the closed ball in  $\mathbb{R}^d$  with center  $u$  and radius  $R$ .

In order to account for edge-effects, we assume that a single realization of a point process  $\mathbf{X}$  is given in the erosion of  $\Lambda_n$  by distance  $2R$ , denoted by  $\Lambda_{n,R}$  and assume this has non-zero area, where  $(\Lambda_n)_{n \geq 1}$  is a sequence of cubes growing up to  $\mathbb{R}^d$ .

The nonparametric estimator  $\widehat{G}_n(t)$  of  $\beta^*G(t)$  is defined (for all positive integer  $n$  and any point  $t$  in the support  $T = \{t \in \mathbb{R}^d; g(t) > 0, \text{ for } \|t\| < R\}$  of the interaction function  $G = \exp(-g)$ ) by

$$\widehat{G}_n(t) = \frac{\widehat{H}_n(t)}{\widehat{F}_n(t)}, \quad (5)$$

where

$$\widehat{H}_n(t) = \frac{1}{b_n^d |\Lambda_{n,R}|} \sum_{\substack{\neq \\ u,v \in \mathbf{X} \\ v-u \in B(o,R)}} \mathbb{1}_{\Lambda_{n,R}}(u) h(u, \mathbf{X} \setminus \{u, v\}) h(v, \mathbf{X} \setminus \{u, v\}) K\left(\frac{v-u-t}{b_n}\right) \quad (6)$$

is a kernel-type estimator estimating  $\widetilde{H}(t) = \beta^{*2}G(t)\bar{F}(o, t)$ .

Here the  $\sum^{\neq}$  stretches over all pairs  $u, v$  of distinct points of the point process  $\mathbf{X}$ , and  $K : \mathbb{R}^d \rightarrow \mathbb{R}$  denotes a smoothing kernel function associated with a sequence  $(b_n)_{n \geq 1}$  is a parameter which converges slowly to zero.

$$\widehat{F}_n(t) = \frac{1}{|\Lambda_{n,R}|} \sum_{u \in \mathbf{X}_{\Lambda_{n,R}}} h(u, \mathbf{X} \setminus \{u\}) h(t-u, \mathbf{X} \setminus \{u\}) \quad (7)$$

meanwhile being an estimator of  $\beta^*\bar{F}(o, t)$ , where

$$\bar{F}(o, t) = \mathbb{P}(\mathbf{X} \cap B(o, R) = \emptyset, \mathbf{X} \cap B(t, R) = \emptyset) \quad (8)$$

where  $o \in \mathbb{R}^d$  denotes the origin and  $h$  is a nonnegative measurable function defined for all  $w \in \mathbb{R}^d$ ,  $\mathbf{x} \in N_{lf}$ , by

$$h(w, \mathbf{x}) = \mathbb{1}(\mathbf{x} \cap B(w, R) = \emptyset).$$

Conditions ensuring uniformly strong convergence of the estimator (5) were given in Morsli [25]. Moreover, we have to impose certain natural restrictions on the kernel function  $K$  and the sequence  $(b_n)_{n \geq 1}$ .

### Condition $K(d)$

The sequence of bandwidths  $(b_n)_{n \geq 1}$ , is a sequence of positive real numbers satisfying:

$$\lim_{n \rightarrow \infty} b_n = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} b_n^d |\Lambda_{n,R}| = \infty.$$

The kernel function  $K : \mathbb{R}^d \rightarrow \mathbb{R}$  is nonnegative and bounded with bounded support and satisfies:

$$\int_{\mathbb{R}^d} K(u) du = 1, \quad \int_{\mathbb{R}^d} K^2(z) dz < \infty.$$

### 3.1 Asymptotic behaviour of the variance and the covariance of the kernel-type estimator

In this section, We investigate the asymptotic behaviour of the variance and the covariance of the kernel-type estimator (6) estimating  $\beta^{*2}G(t)\bar{F}(o, t)$ . The latter is based on the knowledge of the iterated GNZ-formula (2). It is basically the known general formula for spatial point processes.

**Theorem 1.** *Consider a stationary pairwise interaction point process  $\mathbf{X}$  in  $\mathbb{R}^d$  with Papangelou conditional intensity (3) satisfying condition (4). Further the kernel function  $K$  satisfy Condition  $K(d)$ . We have*

$$\lim_{n \rightarrow \infty} b_n^d |\Lambda_{n,R}| \text{Var}(\hat{H}_n(t)) = \beta^{*2}G(t)\bar{F}(o, t) \int_{\mathbb{R}^d} K^2(z) dz$$

at any continuity point  $t \in T \setminus \{o\}$  of  $G$  and  $\bar{F}$ .

The following theorem presents an asymptotic representation of the covariance of the kernel-type estimator (6) in two points  $t_1, t_2$  in  $T$ .

**Theorem 2.** *Consider a stationary pairwise interaction point process  $\mathbf{X}$  in  $\mathbb{R}^d$  with Papangelou conditional intensity (3) satisfying condition (4). Further the kernel function  $K$  satisfy Condition  $K(d)$  and let  $t_1 \neq t_2 \in T$ . We have*

$$\lim_{n \rightarrow \infty} b_n^d |\Lambda_{n,R}| \text{Cov}(\hat{H}_n(t_1), \hat{H}_n(t_2)) = 0.$$

The proof Theorem 2 is similar to that of Theorem 1, and is omitted.

### Proof of Theorem 1

We determine the asymptotic behaviour of the variance of the estimator  $\hat{H}_n(t)$ . For this we use the following result.

**Lemma 1.** *Consider any Gibbs point process  $\mathbf{X}$  in  $\mathbb{R}^d$  with Papangelou conditional intensity  $\lambda$ . For any nonnegative measurable function  $f : \mathbb{R}^d \times \mathbb{R}^d \times N_{lf} \rightarrow \mathbb{R}$ ,*

we have

$$\begin{aligned}
& \text{Var} \left( \sum_{u,v \in \mathbf{X}}^{\neq} f(u, v, \mathbf{X} \setminus \{u, v\}) \right) \\
&= \mathbb{E} \int_{\mathbb{R}^{2d}} f(u, v, \mathbf{X}) [f(u, v, \mathbf{X}) + f(v, u, \mathbf{X})] \lambda(u, v, \mathbf{X}) \, du \, dv \\
&+ \mathbb{E} \int_{\mathbb{R}^{3d}} f(u, v, \mathbf{X}) [f(v, w, \mathbf{X}) + f(w, v, \mathbf{X})] \lambda(u, v, w, \mathbf{X}) \, du \, dv \, dw \\
&+ \mathbb{E} \int_{\mathbb{R}^{3d}} f(u, v, \mathbf{X}) [f(u, w, \mathbf{X}) + f(w, u, \mathbf{X})] \lambda(u, v, w, \mathbf{X}) \, du \, dv \, dw \\
&+ \mathbb{E} \int_{\mathbb{R}^{4d}} f(u, v, \mathbf{X}) f(w, y, \mathbf{X}) \lambda(u, v, w, y, \mathbf{X}) \, du \, dv \, dw \, dy \\
&- \int_{\mathbb{R}^{4d}} \mathbb{E} [f(u, v, \mathbf{X}) \lambda(u, v, \mathbf{X})] \mathbb{E} [f(w, y, \mathbf{X}) \lambda(w, y, \mathbf{X})] \, du \, dv \, dw \, dy.
\end{aligned}$$

*Proof.* After a simple calculation, but prolix (see Jolivet [19] and Heinrich [17]), we obtain the following decomposition:

$$\begin{aligned}
\left( \sum_{u,v \in \mathbf{X}}^{\neq} f(u, v, \mathbf{X} \setminus \{u, v\}) \right)^2 &= \sum_{u,v \in \mathbf{X}}^{\neq} f^2(u, v, \mathbf{X} \setminus \{u, v\}) \\
&+ \sum_{u,v \in \mathbf{X}}^{\neq} f(u, v, \mathbf{X} \setminus \{u, v\}) f(v, u, \mathbf{X} \setminus \{u, v\}) \\
&+ \sum_{u,v,w \in \mathbf{X}}^{\neq} f(u, v, \mathbf{X} \setminus \{u, v, w\}) f(v, w, \mathbf{X} \setminus \{u, v, w\}) \\
&+ \sum_{u,v,w \in \mathbf{X}}^{\neq} f(u, v, \mathbf{X} \setminus \{u, v, w\}) f(w, v, \mathbf{X} \setminus \{u, v, w\}) \\
&+ \sum_{u,v,w \in \mathbf{X}}^{\neq} f(u, v, \mathbf{X} \setminus \{u, v, w\}) f(u, w, \mathbf{X} \setminus \{u, v, w\}) \\
&+ \sum_{u,v,w \in \mathbf{X}}^{\neq} f(u, v, \mathbf{X} \setminus \{u, v, w\}) f(w, u, \mathbf{X} \setminus \{u, v, w\}) \\
&+ \sum_{u,v,w,y \in \mathbf{X}}^{\neq} f(u, v, \mathbf{X} \setminus \{u, v, w, y\}) f(w, y, \mathbf{X} \setminus \{u, v, w, y\}).
\end{aligned} \tag{9}$$

On the other hand using standard definition of variance, we have

$$\text{Var} \sum_{u,v \in \mathbf{X}}^{\neq} f(u, v, \mathbf{X} \setminus \{u, v\}) = \mathbb{E} \left( \sum_{u,v \in \mathbf{X}}^{\neq} f(u, v, \mathbf{X} \setminus \{u, v\}) \right)^2 - \left( \mathbb{E} \sum_{u,v \in \mathbf{X}}^{\neq} f(u, v, \mathbf{X} \setminus \{u, v\}) \right)^2. \tag{10}$$

After rearrangement with the formula of GNZ (2) and formulas (9) and (10), we obtain the desired result.  $\blacksquare$

In the sequel, we denote  $H(u, v, \mathbf{X}) = h(u, \mathbf{X})h(v, \mathbf{X})$ ,  $I_R(v - u) = \mathbb{1}(v - u \in B(o, R))$  and we keep in mind the function  $\bar{F}(o, t)$  defined by (8).

Substitute

$$f(u, v, \mathbf{x}) = \mathbb{1}_{\Lambda_{n,R}}(u)I_R(v - u)H(u, v, \mathbf{x})K\left(\frac{v - u - t}{b_n}\right)$$

the variance term of  $\widehat{H}_n(t)$  is now expanded using Lemma 1, therefore we find that

$$\text{Var } \widehat{H}_n(t) = T_1 + T_2 + T_3 + T_4 + T_5 + T_6 + T_7 + T_8,$$

with

$$\begin{aligned} T_1 &= \frac{1}{b_n^{2d}|\Lambda_{n,R}|^2} \mathbb{E} \int_{\mathbb{R}^{2d}} \mathbb{1}_{\Lambda_{n,R}}(u)I_R(v - u)H(u, v, \mathbf{X})K^2\left(\frac{v - u - t}{b_n}\right)\lambda_{\beta^*}(u, v, \mathbf{X})dudv, \\ T_2 &= \frac{1}{b_n^{2d}|\Lambda_{n,R}|^2} \mathbb{E} \int_{\Lambda_{n,R}^2} I_R(v - u, u - v)H(u, v, \mathbf{X})K\left(\frac{v - u - t}{b_n}\right)K\left(\frac{u - v - t}{b_n}\right)\lambda_{\beta^*}(u, v, \mathbf{X})dudv, \\ T_3 &= \frac{1}{b_n^{2d}|\Lambda_{n,R}|^2} \mathbb{E} \int_{\mathbb{R}^{2d}} \int_{\Lambda_{n,R}} I_R(v - u, w - v)H(u, v, w, \mathbf{X})K\left(\frac{v - u - t}{b_n}\right)K\left(\frac{w - v - t}{b_n}\right) \\ &\quad \times \lambda_{\beta^*}(u, v, w, \mathbf{X})dudvdw, \\ T_4 &= \frac{1}{b_n^{2d}|\Lambda_{n,R}|^2} \mathbb{E} \int_{\mathbb{R}^d} \int_{\Lambda_{n,R}^2} I_R(v - u, v - w)H(u, v, w, \mathbf{X})K\left(\frac{v - u - t}{b_n}\right)K\left(\frac{v - w - t}{b_n}\right) \\ &\quad \times \lambda_{\beta^*}(u, v, w, \mathbf{X})dudvdw, \\ T_5 &= \frac{1}{b_n^{2d}|\Lambda_{n,R}|^2} \mathbb{E} \int_{\mathbb{R}^{2d}} \int_{\Lambda_{n,R}} I_R(v - u, w - u)H(u, v, w, \mathbf{X})K\left(\frac{v - u - t}{b_n}\right)K\left(\frac{w - u - t}{b_n}\right) \\ &\quad \times \lambda_{\beta^*}(u, v, w, \mathbf{X})dudvdw, \\ T_6 &= \frac{1}{b_n^{2d}|\Lambda_{n,R}|^2} \mathbb{E} \int_{\mathbb{R}^d} \int_{\Lambda_{n,R}^2} I_R(v - u, u - w)H(u, v, w, \mathbf{X})K\left(\frac{v - u - t}{b_n}\right)K\left(\frac{u - w - t}{b_n}\right) \\ &\quad \times \lambda_{\beta^*}(u, v, w, \mathbf{X})dudvdw, \\ T_7 &= \frac{1}{b_n^{2d}|\Lambda_{n,R}|^2} \mathbb{E} \int_{\mathbb{R}^{4d}} \mathbb{1}_{\Lambda_{n,R}}(u)I_R(v - u, y - w)H(u, v, w, y, \mathbf{X})K\left(\frac{v - u - t}{b_n}\right)K\left(\frac{y - w - t}{b_n}\right) \\ &\quad \times \lambda_{\beta^*}(u, v, w, y, \mathbf{X})dudvdwdy \end{aligned}$$

and

$$\begin{aligned} T_8 &= -\frac{1}{b_n^{2d}|\Lambda_{n,R}|^2} \int_{\mathbb{R}^{4d}} \mathbb{1}_{\Lambda_{n,R}}(u)\mathbb{1}_{\Lambda_{n,R}}(w)I_R(v - u, y - w)K\left(\frac{v - u - t}{b_n}\right)K\left(\frac{y - w - t}{b_n}\right) \\ &\quad \times \mathbb{E} [H(u, v, \mathbf{X})\lambda_{\beta^*}(u, v, \mathbf{X})] \mathbb{E} [H(w, y, \mathbf{X})\lambda_{\beta^*}(w, y, \mathbf{X})]dudvdwdy. \end{aligned}$$

The asymptotic behaviour of the leading term  $T_1$  is obtained by applying the second order Papangelou conditional intensity given by:

$$\lambda_{\beta^*}(u, v, \mathbf{x}) = \lambda_{\beta^*}(u, \mathbf{x})\lambda_{\beta^*}(v, \mathbf{x} \cup \{u\}) \quad \text{for any } u, v \in \mathbb{R}^d \quad \text{and } \mathbf{x} \in N_{lf}.$$



And using the finite range property (4) for each function  $\lambda_{\beta^*}(u, \mathbf{x})$  and  $\lambda_{\beta^*}(v, \mathbf{x} \cup \{u\})$ . We recall when  $\mathbf{x} = \emptyset$ , this implies that

$$\lambda_{\beta^*}(u, \emptyset) = \beta^* \quad \text{and} \quad \lambda_{\beta^*}(v, \emptyset \cup \{u\}) = \beta^* G(v - u) \quad \text{for all } u, v \in \mathbb{R}^d.$$

By stationarity of  $\mathbf{X}$ , it results

$$\begin{aligned} T_1 &= \frac{1}{b_n^{2d} |\Lambda_{n,R}|^2} \mathbb{E} \int_{\mathbb{R}^{2d}} \mathbb{1}_{\Lambda_{n,R}}(u) I_R(v - u) H(u, v, \mathbf{X}) K^2\left(\frac{v - u - t}{b_n}\right) \lambda_{\beta^*}(u, v, \mathbf{X}) du dv \\ &= \frac{\beta^{*2}}{b_n^{2d} |\Lambda_{n,R}|^2} \int_{\mathbb{R}^{2d}} \mathbb{1}_{\Lambda_{n,R}}(u) I_R(v - u) \bar{F}(u, v) K^2\left(\frac{v - u - t}{b_n}\right) G(v - u) du dv \\ &= \frac{\beta^{*2}}{b_n^{2d} |\Lambda_{n,R}|} \int_{\mathbb{R}^d} I_R(s) \bar{F}(o, s) K^2\left(\frac{s - t}{b_n}\right) G(s) ds \\ &= \frac{\beta^{*2}}{b_n^d |\Lambda_{n,R}|} \int_{\mathbb{R}^d} I_R(b_n z + t) \bar{F}(o, b_n z + t) K^2(z) G(b_n z + t) dz. \end{aligned}$$

The continuity of  $G\bar{F}$  in  $t$  and the boundedness conditions on the kernel function yield the desired result by dominated convergence theorem, i.e.

$$\lim_{n \rightarrow \infty} b_n^d |\Lambda_{n,R}| T_1 = \beta^{*2} G(t) \bar{F}(o, t) \int_{\mathbb{R}^d} K^2(z) dz.$$

We apply in the following lemma of the sequence of observation windows  $(\Lambda_n)_{n \in \mathbb{N}}$ , established by (David [8], Lemma A.2).

**Lemma 2.** *Let  $(\Lambda_n)_{n \in \mathbb{N}}$  be a sequence of convex sets in  $\mathbb{R}^d$  growing up to  $\mathbb{R}^d$  and let  $y, z \in \mathbb{R}^d$ . Let  $(b_n)_{n \in \mathbb{N}}$  be a sequence of positive real numbers with  $b_n \xrightarrow[n \rightarrow \infty]{} 0$ . Then we have*

$$\lim_{n \rightarrow \infty} \frac{|\Lambda_n \cap (\Lambda_n - b_n y - z)|}{|\Lambda_n|} = 1.$$

We will now show that all the other terms of the variance of the estimator  $\widehat{H}_n(t)$  vanish. With similar arguments to those used to obtain the results above, we treat the term  $T_2$ .

$$\begin{aligned} T_2 &= \frac{1}{b_n^{2d} |\Lambda_{n,R}|^2} \\ &\mathbb{E} \int_{\mathbb{R}^{2d}} \mathbb{1}_{\Lambda_{n,R}}(u) \mathbb{1}_{\Lambda_{n,R}}(v) I_R(v - u, u - v) H(u, v, \mathbf{X}) K\left(\frac{v - u - t}{b_n}\right) K\left(\frac{u - v - t}{b_n}\right) \lambda_{\beta^*}(u, v, \mathbf{X}) du dv \\ &= \frac{\beta^{*2}}{b_n^{2d} |\Lambda_{n,R}|^2} \\ &\int_{\mathbb{R}^{2d}} \mathbb{1}_{\Lambda_{n,R}}(u) \mathbb{1}_{\Lambda_{n,R}}(v) I_R(v - u, u - v) \bar{F}(o, v - u) K\left(\frac{v - u - t}{b_n}\right) K\left(\frac{u - v - t}{b_n}\right) G(v - u) du dv \\ &= \frac{\beta^{*2}}{b_n^{2d} |\Lambda_{n,R}|} \int_{\mathbb{R}^d} \frac{|\Lambda_{n,R} \cap (\Lambda_{n,R} - s)|}{|\Lambda_{n,R}|} I_R(s, -s) \bar{F}(o, s) K\left(\frac{s - t}{b_n}\right) K\left(\frac{-s - t}{b_n}\right) G(s) ds \\ &= \frac{\beta^{*2}}{b_n^d |\Lambda_{n,R}|} \\ &\int_{\mathbb{R}^d} \frac{|\Lambda_{n,R} \cap (\Lambda_{n,R} - (b_n z + t))|}{|\Lambda_{n,R}|} I_R(b_n z + t, -b_n z - t) \bar{F}(o, b_n z + t) K(z) K(-z - \frac{2t}{b_n}) G(b_n z + t) dz. \end{aligned}$$

The continuity of  $\bar{F}G$  in  $t$  and the boundedness conditions on the kernel function by dominated convergence theorem and under condition  $t \neq 0$  and the fact that, in view of the geometric properties of the  $\Lambda_{n,R}$ 's (Lemma 2), we get  $b_n^d |\Lambda_{n,R}| T_2 \rightarrow 0$  as  $n \rightarrow \infty$ .

For the asymptotic behaviour of the third term  $T_3$ , we remember the third order Papangelou conditional intensity by,

$$\lambda_{\beta^*}(u, v, w, \mathbf{x}) = \lambda_{\beta^*}(u, \mathbf{x}) \lambda_{\beta^*}(v, \mathbf{x} \cup \{u\}) \lambda_{\beta^*}(w, \mathbf{x} \cup \{u, v\})$$

for any  $u, v, w \in \mathbb{R}^d$  and  $\mathbf{x} \in N_{lf}$ . Since  $\mathbf{X}$  is a point process to interact in pairs, the interaction terms due to triplets or higher order are equal to one, i.e. the potential  $g(\mathbf{y})$  is null when  $n(\mathbf{y}) > 2$ , for  $\emptyset \neq \mathbf{y} \subseteq \mathbf{x}$ . Using the finite range property (4) for each function  $\lambda_{\beta^*}(u, \mathbf{x})$ ,  $\lambda_{\beta^*}(v, \mathbf{x} \cup \{u\})$  and  $\lambda_{\beta^*}(w, \mathbf{x} \cup \{u, v\})$  and after a elementary calculation, we have

$$\lambda_{\beta^*}(u, v, w, \emptyset) = \begin{cases} \beta^{*3} G(v-u)G(w-v) & \text{if } u \notin B(w, R) \\ \beta^{*3} G(v-u) & \text{otherwise.} \end{cases} \quad (11)$$

Which ensures that  $\lambda_{\beta^*}(u, v, w, \emptyset)$  is a function that depends only variables  $v-u, w-v$ , denoted by  $G_3(v-u, w-v)$ . According to the stationarity of  $\mathbf{X}$ , it follows that

$$\begin{aligned} T_3 &= \frac{1}{b_n^{2d} |\Lambda_{n,R}|} \int_{\mathbb{R}^{2d}} \frac{|\Lambda_{n,R} \cap (\Lambda_{n,R} - v)|}{|\Lambda_{n,R}|} \bar{G}_3(v, w-v) K\left(\frac{v-t}{b_n}\right) K\left(\frac{w-v-t}{b_n}\right) dv dw \\ &= \frac{1}{|\Lambda_{n,R}|} \int_{\mathbb{R}^{2d}} \frac{|\Lambda_{n,R} \cap (\Lambda_{n,R} - (b_n z + t))|}{|\Lambda_{n,R}|} \bar{G}_3(b_n z + t, b_n z' + t) K(z) K(z') dz dz', \end{aligned}$$

where  $\bar{G}_3(v, w-v) = I_R(v, w-v) \bar{F}(-v, w-v) G_3(v, w-v)$ . By dominated convergence theorem and using the Lemma 2, we get  $b_n^d |\Lambda_{n,R}| T_3 \rightarrow 0$  as  $n \rightarrow \infty$ . Analogously, one may show the other terms  $T_4, T_5$  and  $T_6$ .

For the asymptotic behaviour of the leading term  $T_7$ , let  $u, v, w, y$  any points in  $\mathbb{R}^d$  and  $\mathbf{x} \in N_{lf}$ , we define Papangelou conditional intensity of the fourth order by

$$\lambda_{\beta^*}(u, v, w, y, \mathbf{x}) = \lambda_{\beta^*}(u, \mathbf{x}) \lambda_{\beta^*}(v, \mathbf{x} \cup \{u\}) \lambda_{\beta^*}(w, \mathbf{x} \cup \{u, v\}) \lambda_{\beta^*}(y, \mathbf{x} \cup \{u, v, w\}).$$

Next we introduce the finite range property (4) and reasoning analogous with the foregoing on  $\lambda_{\beta^*}(u, v, w, \emptyset)$  is defined by (11), which ensures that  $\lambda_{\beta^*}(u, v, w, y, \emptyset)$  is a function that depends only variables  $v-u, y-w, w-u, w-v$ , denoted by  $G_4(v-u, y-w, w-u, w-v)$ . Further by the stationarity of  $\mathbf{X}$ , this implies that

$$\begin{aligned} T_7 &= \frac{1}{b_n^{2d} |\Lambda_{n,R}|} \\ &\int_{\mathbb{R}^{3d}} \frac{|\Lambda_{n,R} \cap (\Lambda_{n,R} - w)|}{|\Lambda_{n,R}|} \bar{G}_4(v, y-w, w, w-v) K\left(\frac{v-t}{b_n}\right) K\left(\frac{y-w-t}{b_n}\right) dv dw dy \\ &= \frac{1}{|\Lambda_{n,R}|} \int_{\mathbb{R}^{3d}} \frac{|\Lambda_{n,R} \cap (\Lambda_{n,R} - w)|}{|\Lambda_{n,R}|} \bar{G}_4(b_n z + t, b_n z' + t, w, w - b_n z - t) K(z) K(z') dz dz' dw. \end{aligned}$$

$\bar{G}_4(v, y-w, w, w-v) = I_R(v, y-w) \bar{F}(v, w, y) G_4(v, y-w, w, w-v)$ . Similar arguments show that  $b_n^d |\Lambda_{n,R}| T_7 \rightarrow 0$  as  $n \rightarrow \infty$ .

We now finish the proof of the asymptotic behaviour of the term  $T_8$ . Under the conditions imposed in Theorem 1, we have

$$\begin{aligned}
T_8 &= -\frac{1}{b_n^{2d}|\Lambda_{n,R}|^2} \int_{\mathbb{R}^{4d}} \mathbb{1}_{\Lambda_{n,R}}(u) \mathbb{1}_{\Lambda_{n,R}}(w) I_R(v-u, y-w) K\left(\frac{v-u-t}{b_n}\right) K\left(\frac{y-w-t}{b_n}\right) \\
&\quad \times \mathbb{E} [H(u, v, \mathbf{X}) \lambda_{\beta^*}(u, v, \mathbf{X})] \mathbb{E} [H(w, y, \mathbf{X}) \lambda_{\beta^*}(w, y, \mathbf{X})] dudvdwdy \\
&= -\frac{\beta^{*4}}{b_n^{2d}|\Lambda_{n,R}|^2} \\
&\quad \int_{\mathbb{R}^{4d}} \mathbb{1}_{\Lambda_{n,R}}(u) \mathbb{1}_{\Lambda_{n,R}}(w) \bar{G}_5(u, v, w, y, v-u, y-w) K\left(\frac{v-u-t}{b_n}\right) K\left(\frac{y-w-t}{b_n}\right) dudvdwdy \\
&\quad \int_{\mathbb{R}^{3d}} \frac{|\Lambda_{n,R} \cap (\Lambda_{n,R} - w)|}{|\Lambda_{n,R}|} \bar{G}_5(o, v, w, y, v, y-w) K\left(\frac{v-t}{b_n}\right) K\left(\frac{y-w-t}{b_n}\right) dvdwdy \\
&= -\frac{\beta^{*4}}{|\Lambda_{n,R}|} \\
&\quad \int_{\mathbb{R}^{3d}} \frac{|\Lambda_{n,R} \cap (\Lambda_{n,R} - w)|}{|\Lambda_{n,R}|} \bar{G}_5(o, b_n z + t, w, b_n z' + t + w, b_n z + t, b_n z' + t) K(z) K(z') dw dz dz'
\end{aligned}$$

where

$$\bar{G}_5(u, v, w, y, v-u, y-w) = I_R(v-u, y-w) \bar{F}(u, v) \bar{F}(w, y) G(v-u) G(y-w).$$

By dominated convergence theorem, we get the desired result.  $\blacksquare$

### 3.2 Asymptotic normality of the kernel-type estimator

Asymptotic normality nonparametric estimate of the pairwise interaction function for stationary pairwise interaction point process is essentially based on multivariate central limit theorem for the estimator  $\hat{H}_n(t)$  of  $\tilde{H}(t) = \beta^{*2} G(t) \bar{F}(o, t)$ . The proving idea of the below Theorem 3 consists in approximating the sequence  $\hat{H}_n(t)$  by a triangular array of  $m_n$ -dependent random fields. The corresponding Lindeberg-type CLT for  $m_n$ -dependent random fields has been proved in Wang and Woodroffe [34].

A basic ingredient in the approach in this paper is that a point process can be viewed interchangeably as a lattice field, i.e. we split up the sampling window  $\Lambda_{n,R}$  into cubes such as  $\Lambda_{n,R} = \cup_{i \in \tilde{I}_n} \Lambda_i$ , where  $\Lambda_i$  are centered at  $i$  and assume that  $\tilde{I}_n$  increases towards  $\mathbb{Z}^d$  and setting  $X_i = X_{\Lambda_i}$ ,  $i \in \mathbb{Z}^d$ . We will consider estimation of  $\beta^{*2} G(t) \bar{F}(o, t)$  from  $\hat{H}_n(t)$ , where  $\Lambda_{n,R} = \cup_{i \in \tilde{I}_n} \Lambda_i$  and where the process is observed in  $\cup_{i \in \tilde{I}_n} \Lambda_i$ , i.e.

$$\hat{H}_n(t) = \frac{1}{b_n^d |\Lambda_{n,R}|} \sum_{i \in \tilde{I}_n} \sum_{\substack{u \in \mathbf{X}_i, v \in \mathbf{X} \\ v-u \in B(o, R)}}^{\neq} h(u, \mathbf{X} \setminus \{u, v\}) h(v, \mathbf{X} \setminus \{u, v\}) K\left(\frac{v-u-t}{b_n}\right).$$

We consider in this work a field  $\{X_i\}_{i \in \mathbb{Z}^d}$  ( $d \in \mathbb{N}$ ) in form of

$$X_i = \tilde{F}(\varepsilon_{i-k}, k \in \mathbb{Z}^d), i \in \mathbb{Z}^d \quad (12)$$

where  $\{\varepsilon_i\}_{i \in \mathbb{Z}^d}$  are i.i.d. random variables and  $\tilde{F}$  is measurable function. Let  $\{\varepsilon'_i\}_{i \in \mathbb{Z}^d}$  be an i.i.d. copy of  $\{\varepsilon_i\}_{i \in \mathbb{Z}^d}$  and consider for all positive integer  $n$  the coupled version

$X_i^*$  defined by  $X_i^* = \tilde{F}(\varepsilon_{i-k}^*, k \in \mathbb{Z}^d)$ , where

$$\varepsilon_j^* = \begin{cases} \varepsilon_j & \text{if } j \neq 0 \\ \varepsilon'_0 & \text{if } j = 0. \end{cases}$$

Let  $i \in \mathbb{Z}^d$  and  $p > 0$  be fixed. If  $X_i \in \mathbb{L}^p$ , we define  $\delta_{i,p} = \|X_i - X_i^*\|_p$ , where  $\|\cdot\|_p$  is the usual  $\mathbb{L}^p$ -norm. Thus, El Machkouri [21] obtained this sufficient condition:

$$\sum_{i \in \mathbb{Z}^d} |i|^{5d/2} \delta_{i,p} < \infty \quad (13)$$

for a random field of the form (12) for the kernel density estimator to be asymptotically normal. The desired sequence  $(m_n)_{n \geq 1}$  can be chosen as

$$m_n = \max \left\{ v_n, \left[ \left( \frac{1}{b_n^3} \sum_{|i| > v_n} |i|^{\frac{5d}{2}} \delta_{i,2} \right)^{\frac{1}{3d}} \right] + 1 \right\}$$

where  $v_n = \lfloor b_n^{-\frac{1}{2d}} \rfloor$  and  $\lfloor \cdot \rfloor$  denotes the integer part function.

Our results show essentially through the following lemma is a spatial version of a result by Bosq et al. [5].

**Lemma 3.** *If (13) holds, then*

$$m_n \rightarrow \infty, \quad m_n^d b_n^d \rightarrow 0 \quad \text{and} \quad \frac{1}{(m_n b_n)^{3d/2}} \sum_{|i| > m_n} |i|^{\frac{5d}{2}} \delta_{i,2} \rightarrow 0.$$

In order to establish the asymptotic normality, we need additional the assumption:

**Condition  $\mathcal{W}$ :** There exist a sequence of integers  $(m_n)_{n \geq 1}$  such that  $m_n \rightarrow \infty$  as  $n \rightarrow \infty$ , and the following limit holds,

$$\frac{m_n^d \log^d n}{b_n^d n^d} \rightarrow 0.$$

**Theorem 3.** *Consider a stationary pairwise interaction point process  $\mathbf{X}$  in  $\mathbb{R}^d$  with Papangelou conditional intensity (3) satisfying condition (4). Further the kernel function  $K$  satisfy Condition  $K(d)$  and the Lipschitz condition, and assume that Condition  $\mathcal{W}$  holds. Then, for any positive integer  $s$  and any distinct points  $t_1, \dots, t_s$  in  $T \setminus \{o\}$ , we have*

$$\sqrt{b_n^d |\Lambda_{n,R}|} \left( \hat{H}_n(t_i) - \mathbb{E} \hat{H}_n(t_i) \right)_{i=1}^s \xrightarrow{d} \mathcal{N}(0, \sigma_H)$$

where the covariance matrix  $\sigma_H = [r_{i,j}]_{i,j=1,\dots,s}$  is given by

$$r_{i,i} = \beta^{*2} G(t_i) \bar{F}(o, t_i) \int_{\mathbb{R}^d} K^2(z) dz, \quad \text{for } i = 1, \dots, s \quad \text{and} \quad r_{i,j} = 0 \quad \text{if } i \neq j.$$

**Remark 1.** By strengthening the continuity assumption on  $\tilde{H}$  in some neighborhood of the points  $t_1, \dots, t_s$  and the kernel  $K$  satisfies:

**Condition  $K(d, m)$ :** for  $m > 1$

$$\int_{\mathbb{R}^d} z_1^{a_1} \dots z_d^{a_d} K(z) dz = 0$$

with  $a_1, \dots, a_d \in \mathbb{N}^*$  such that  $0 < \sum_{i=1}^d a_i < m$ , the means  $E \hat{H}_n(t_i)$  can be replaced by the true values  $\tilde{H}(t_i)$ ,  $i = 1, \dots, s$ .

**Corollary 1.** Let, in addition to the assumptions of Theorem 3 and the kernel function  $K$  satisfy Condition  $K(d, m)$  and let  $b_n^{d+2m} |\Lambda_{n,R}| \rightarrow 0$  as  $n \rightarrow +\infty$ . Furthermore if  $\tilde{H}$  has bounded and continuous partial derivatives of order  $m$  in some neighborhood of the points  $t_1, \dots, t_s$ . Then

$$\sqrt{b_n^d |\Lambda_{n,R}|} \left( \hat{H}_n(t_i) - \tilde{H}(t_i) \right)_{i=1}^s \xrightarrow{d} \mathcal{N}(0, \sigma_H)$$

where the covariance matrix  $\sigma_H = [r_{i,j}]_{i,j=1,\dots,s}$  is given by

$$r_{i,i} = \beta^{*2} G(t_i) \bar{F}(o, t_i) \int_{\mathbb{R}^d} K^2(z) dz, \quad \text{for } i = 1, \dots, s \quad \text{and } r_{i,j} = 0 \quad \text{if } i \neq j.$$

### 3.3 Asymptotic normality of nonparametric estimator

The estimator (7) turn out to be unbiased estimator of  $\beta^* \bar{F}(o, t)$  and strongly consistent (the uniform strong consistency) as  $n$  tends infinity, since a classical ergodic theorem for spatial point processes obtained in [26]. This implies the following:

**Theorem 4.** Consider a stationary pairwise interaction point process  $\mathbf{X}$  in  $\mathbb{R}^d$  with Papangelou conditional intensity (3) satisfying condition (4). Further the kernel function  $K$  satisfy Condition  $K(d)$ ,  $K(d, m)$  and the Lipschitz condition, and assume that Condition  $\mathcal{W}$  and  $b_n^{d+2m} |\Lambda_{n,R}| \rightarrow 0$  as  $n \rightarrow \infty$  hold. Then for any positive integer  $s$  and any distinct points  $t_1, \dots, t_s$  in  $T \setminus \{o\}$ ,

$$\sqrt{b_n^d |\Lambda_{n,R}|} \left( \hat{G}_n(t_i) - \beta^* G(t_i) \right)_{i=1}^s \xrightarrow{d} \mathcal{N}(0, \sigma_G)$$

where the covariance matrix  $\sigma_G = [\sigma_{i,j}]_{i,j=1,\dots,s}$  is given by

$$\sigma_{i,i} = \frac{G(t_i)}{\bar{F}(o, t_i)} \int_{\mathbb{R}^d} K^2(z) dz \quad \text{for } i = 1, \dots, s \quad \text{and } \sigma_{i,j} = 0 \quad \text{if } i \neq j.$$

## 4 Proofs

### 4.1 Proof of Theorem 3

To prove the Theorem 3, we need (using the well-known Cramér-Wold (1936)). Without loss of generality, we consider only the case  $s = 2$ . Let  $\lambda_1$  and  $\lambda_2$  two

constants with  $\lambda_1 + \lambda_2 \neq 0$  and let  $t_1, t_2 \in T \setminus \{o\}$  the points be fixed,  $t_1 \neq t_2$  and  $|\Lambda_{n,R}| \sim cn^d$ . We split up the sampling window  $\Lambda_{n,R}$  into cubes such as  $\Lambda_{n,R} = \cup_{i \in \tilde{I}_n} \Lambda_i$ , where  $\Lambda_i$  are centered at  $i$  and assume that  $\tilde{I}_n$  increases towards  $\mathbb{Z}^d$ , and define the triangular array of random variables

$$\frac{1}{n^{d/2}} \sum_{i \in \tilde{I}_n} \bar{Y}_{n,i} = \lambda_1 (b_n^d n^d)^{1/2} \left( \widehat{H}_n(t_1) - \mathbb{E}[\widehat{H}_n(t_1)] \right) + \lambda_2 (b_n^d n^d)^{1/2} \left( \widehat{H}_n(t_2) - \mathbb{E}[\widehat{H}_n(t_2)] \right)$$

where

$$\bar{Y}_{n,i} = \lambda_1 \left( Y_i(t_1) - \mathbb{E}[Y_i(t_1)] \right) + \lambda_2 \left( Y_i(t_2) - \mathbb{E}[Y_i(t_2)] \right)$$

and for all  $j = 1, 2$ , we consider

$$Y_i(t_j) = (b_n^d)^{-1/2} \sum_{\substack{\neq \\ u \in X_{\Lambda_i}, v \in \mathbf{X} \\ v-u \in B(o,R)}} h(u, \mathbf{X} \setminus \{u, v\}) h(v, \mathbf{X} \setminus \{u, v\}) K \left( \frac{v-u-t_j}{b_n} \right).$$

The proving idea the below Theorem 3 consists in approximating the sequence  $\bar{Y}_{n,i}$  by a triangular array of  $m$ -dependent random fields. The corresponding Lindeberg-type CLT  $m$ -dependent random fields has been proved by Wang and Woodroffe [34]. Therefore, it proves convenient to switch notation from the text and to define

$$\tilde{Y}_{n,i} = \lambda_1 \left( \tilde{Y}_i(t_1) - \mathbb{E}[\tilde{Y}_i(t_1)] \right) + \lambda_2 \left( \tilde{Y}_i(t_2) - \mathbb{E}[\tilde{Y}_i(t_2)] \right)$$

and for all  $j = 1, 2$ , we consider

$$\tilde{Y}_i(t_j) = (b_n^d)^{-1/2} \sum_{\substack{\neq \\ u \in X_{\Lambda_{i-k}}, v \in \mathbf{X} \\ v-u \in B(o,R)}} h(u, \mathbf{X} \setminus \{u, v\}) h(v, \mathbf{X} \setminus \{u, v\}) K \left( \frac{v-u-t_j}{b_n} \right).$$

In this way, we assume that  $\{\tilde{Y}_{n,i}\}_{i \in \mathbb{Z}^d}$  is  $m_n$ -dependent, in the sense that  $Y_{n,i}$  and  $Y_{n,j}$  are independent if  $|i-j|_\infty \geq m$ , where  $|j|_\infty = \max\{|j_1|, \dots, |j_d|\}$ . We will use  $\{\tilde{Y}_{n,i}; i \in \mathbb{Z}^d\}_{n \in \mathbb{N}}$  to approximate  $\{\bar{Y}_{n,i}; i \in \mathbb{Z}^d\}_{n \in \mathbb{N}}$ . We decompose

$$\frac{1}{n^{d/2}} \sum_{i \in \tilde{I}_n} \bar{Y}_{n,i} = \frac{1}{n^{d/2}} \sum_{i \in \tilde{I}_n} \tilde{Y}_{n,i} + \frac{1}{n^{d/2}} \sum_{i \in \tilde{I}_n} (\bar{Y}_{n,i} - \tilde{Y}_{n,i})$$

and we assume that

$$\left\| \sum_{i \in \tilde{I}_n} \tilde{Y}_{n,i} \right\|_2 < \infty \quad \text{for all } n \in \mathbb{N}. \quad (14)$$

Then, it suffices to establish the following two results.

**Proposition 1.** *Under Conditions  $\mathcal{W}$  and (14). For  $n$  sufficiently large, we have*

$$\frac{1}{n^{d/2}} \sum_{i \in \tilde{I}_n} \tilde{Y}_{n,i} \xrightarrow{d} \mathcal{N}(0, \sigma^2),$$

where

$$\sigma^2 = \left( \beta^{*2} \lambda_1^2 \bar{F}(o, t_1) G(t_1) + \beta^{*2} \lambda_2^2 \bar{F}(o, t_2) G(t_2) \right) \int_{\mathbb{R}^d} K^2(z) dz. \quad (15)$$

**Proposition 2.** *Under condition of Lemma 3. We have for  $n$  sufficiently large,*

$$\frac{1}{n^{d/2}} \sum_{i \in \tilde{I}_n} (\bar{Y}_{n,i} - \tilde{Y}_{n,i}) \xrightarrow{\mathbb{P}} 0.$$

## Proof of Proposition 1

Taking into account of  $\{\tilde{Y}_{n,i}\}_{i \in \mathbb{Z}^d}$  are  $m_n$ -dependent random fields, we apply limit theorem for stationary triangular arrays of  $m$ -dependent random fields (Theorem 2 establish by Wang and Woodroffe [34]). We need to verify that the two fundamental conditions (16) and (17), i.e. it suffices to show, for  $l_n = m_n \log n$  such that  $m_n/l_n \rightarrow 0$  and  $l_n/n \rightarrow 0$  as  $n \rightarrow \infty$

$$\lim_{n \rightarrow \infty} \frac{1}{l_n^d} \mathbb{E} \left[ \left( \sum_{i \in \llbracket 1, l_n \rrbracket^d} \tilde{Y}_{n,i} \right)^2 \right] = \sigma^2 \quad (16)$$

and

$$\lim_{n \rightarrow \infty} \frac{1}{l_n^d} \mathbb{E} \left[ \left( \sum_{i \in \llbracket 1, l_n \rrbracket^d} \tilde{Y}_{n,i} \right)^2 \mathbb{1} \left\{ \left| \sum_{i \in \llbracket 1, l_n \rrbracket^d} \tilde{Y}_{n,i} \right| > n^{d/2} \epsilon \right\} \right] = 0 \quad \text{for all } \epsilon > 0 \quad (17)$$

where  $a, b \in \mathbb{Z}$ ,  $\llbracket a, b \rrbracket \equiv \{a, a+1, \dots, b\}$ .

Taking into account the definition of  $\sum_{i \in \llbracket 1, l_n \rrbracket^d} \tilde{Y}_{n,i}$  we get

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{1}{l_n^d} \text{Var} \left( \sum_{i \in \llbracket 1, l_n \rrbracket^d} \tilde{Y}_{n,i} \right) \\ &= \lim_{n \rightarrow \infty} b_n^d l_n^d \left( \lambda_1^2 \text{Var}(\hat{H}_{l_n}(t_1)) + \lambda_2^2 \text{Var}(\hat{H}_{l_n}(t_2)) + 2\lambda_1 \lambda_2 \text{Cov}(\hat{H}_{l_n}(t_1), \hat{H}_{l_n}(t_2)) \right). \end{aligned}$$

Using Theorem 1 entails replacing  $\hat{H}_n(t_j)$  by  $\hat{H}_{l_n}(t_j)$ , we have

$$\lim_{n \rightarrow \infty} b_n^d l_n^d \text{Var}(\hat{H}_{l_n}(t_j)) = \beta^{*2} G(t_j) \bar{F}(o, t_j) \int_{\mathbb{R}^d} K^2(u) du.$$

Using Theorem 2, we have

$$\lim_{n \rightarrow \infty} b_n^d l_n^d \text{Cov} \left( \hat{H}_{l_n}(t_1), \hat{H}_{l_n}(t_2) \right) = 0$$

then, it follows that

$$\lim_{n \rightarrow \infty} \frac{1}{l_n^d} \text{Var} \left( \sum_{i \in \llbracket 1, l_n \rrbracket^d} \tilde{Y}_{n,i} \right) = \sigma^2$$

where  $\sigma^2$  is defined by (15).

To prove (17), we write  $\xi_n = \sum_{i \in \llbracket 1, l_n \rrbracket^d} \tilde{Y}_{n,i}$ , therefore we get

$$\begin{aligned} \mathbb{E} [\xi_n^2 \mathbb{1}\{|\xi_n| > \varepsilon n^{d/2}\}] &\leq \|\xi_n\|_p^2 (\mathbb{P}(|\xi_n| > n^{d/2} \varepsilon))^{(p-2)/p} \\ &\leq \|\xi_n\|_p^2 \left( \frac{\|\xi_n\|_2^2}{n^d \varepsilon^2} \right)^{(p-2)/p}. \end{aligned}$$

This time, (14) yields

$$\|\xi_n\|_2 \leq C l_n^{d/2} \quad \text{for all } n \in \mathbb{N}.$$

For  $p > 2$ , observe that, since  $K$  is bounded,

$$\|\xi_n\|_p \leq C \left( \frac{1}{b_n^{(p-2)/2}} \|\xi_n\|_2^2 \right)^{1/p} \leq C l_n^d b_n^{-(p-2)/p}.$$

By the above inequalities, we have obtained

$$\frac{1}{l_n^d} \mathbb{E} [\xi_n^2 \mathbb{1}\{|\xi_n| > \varepsilon n^{d/2}\}] \leq C \left( \frac{l_n^d}{b_n^d n^d} \right)^{(p-2)/p}$$

now the Condition  $\mathcal{W}$  entails (17). ■

## Proof of Proposition 2

Let  $p > 1$  be fixed. For all  $i \in \mathbb{Z}^d$  and for all  $t \in T$ , we denote  $R_i = Y_i(t) - \tilde{Y}_i(t)$ . Since there exists a measurable function  $H$  such that  $R_i = H(\varepsilon_{i-s}; s \in \mathbb{Z}^d)$ . We recall that  $\delta_{i,p}^{(n)} = \|R_i - R_i^*\|_p$  where  $R_i^* = H(\varepsilon_{i-s}^*; s \in \mathbb{Z}^d)$ , with

$$\varepsilon_j^* = \begin{cases} \varepsilon_j & \text{if } j \neq 0 \\ \varepsilon_{t_0} & \text{if } j = 0. \end{cases}$$

Since  $(Y_i(t) - \tilde{Y}_i(t))^* = Y_i^*(t) - \tilde{Y}_i^*(t)$ , we derive  $\delta_{i,p}^{(n)} \leq 2\|Y_i(t) - Y_i^*(t)\|_p$ . Since  $K$  is Lipschitz, we get

$$\delta_{i,p}^{(n)} \leq \frac{2c\delta_{i,p}}{b_n^d} \tag{18}$$

where  $\delta_{i,p} = \|X_i - X_i^*\|_p$ . Moreover, we have also

$$\delta_{i,p}^{(n)} \leq 2\|Y_0(t) - \tilde{Y}_0(t)\|_p. \tag{19}$$

Combining (19) and the Burkholder inequality (c.f Hall Heyde [15]), we get

$$\delta_{i,p}^{(n)} \leq c \frac{\sqrt{8p}}{b_n^d} \sum_{|j| > m_n} \delta_{j,p}. \tag{20}$$



Finally, for every  $m_n$  positive we can write

$$\begin{aligned} \sum_{i \in \mathbb{Z}^d} \delta_{i,p}^{(n)} &\leq \frac{m_n^d \sqrt{8p}}{b_n^d} \sum_{|j| > m_n} \delta_{j,p} + \frac{2}{b_n^d} \sum_{|j| > m_n} \delta_{j,p} \\ &\leq \frac{2m_n^d \sqrt{8p}}{b_n^d} \sum_{|j| > m_n} \delta_{j,p}. \end{aligned} \quad (21)$$

We combine (18), (20), (21) and under conditions of Lemma 3, we get

$$\frac{1}{n^{d/2}} \left\| \sum_{i \in \tilde{I}_n} (\bar{Y}_{n,i} - \tilde{Y}_{n,i}) \right\|_2 \leq c \frac{|\lambda_1| + |\lambda_2|}{(m_n b_n)^{3d/2}} \sum_{|i| > m_n} |i|^{\frac{5d}{2}} \delta_{i,2} = o(1).$$

The proof of Proposition 2 is complete. ■

This completes the proof of Theorem 3. ■

## 4.2 Proof of Corollary 1

we get by Morsli [25]: if  $\tilde{H}(t_i)$  has bounded and continuous partial derivatives of order  $m$  in some neighborhood of the points  $t_1, \dots, t_s$ , and the kernel  $K$  satisfies

$$\int_{\mathbb{R}^d} z_1^{a_1} \dots z_d^{a_d} K(z) dz = 0$$

for  $m > 1$  with  $a_1, \dots, a_d \in \mathbb{N}^*$  such that  $0 < \sum_{i=1}^d a_i < m$ . Then

$$\mathbb{E} \hat{H}_n(t_i) = \beta^{*2} G(t_i) \bar{F}(o, t_i) + \mathcal{O}(b_n^m), i = 1, \dots, s \quad \text{as } n \rightarrow \infty.$$

Under the condition  $b_n^{d+2m} |\Lambda_{n,R}| \rightarrow 0$ , we get

$$\sqrt{b_n^d |\Lambda_{n,R}|} (\mathbb{E} \hat{H}_n(t_i) - \tilde{H}(t_i)) = \mathcal{O}\left(\sqrt{b_n^{d+2m} |\Lambda_{n,R}|}\right). \quad (22)$$

By with the application of Theorem 3 we conclude the result announced. ■

## 4.3 Proof of Theorem 4

Let  $t \in \{t_1, \dots, t_s\}$  be fixed. We may split the difference  $\hat{G}_n(t) - \beta^* G(t)$  as follows:

$$\begin{aligned} \hat{G}_n(t) - \beta^* G(t) &= \frac{L(t) \hat{H}_n(t) - \tilde{H}(t) \hat{F}_n(t)}{L(t) \hat{F}_n(t)} \\ &= \frac{\hat{H}_n(t) - \mathbb{E} \hat{H}_n(t)}{\hat{F}_n(t)} + \frac{\mathbb{E} \hat{H}_n(t) - \beta^* G(t) \hat{F}_n(t)}{\hat{F}_n(t)} \\ &= A_n^{(1)} + A_n^{(2)} \end{aligned}$$

The estimator  $\hat{F}_n(t)$  turn out to be unbiased estimator of  $\beta^* \bar{F}(o, t)$  and strongly consistent (the uniform strong consistency) (see Morsli [25]) as  $n$  tends infinity,

since a classical ergodic theorem for spatial point processes obtained in [26] and we apply Theorem 3, they imply that as  $n \rightarrow \infty$ ,  $\sqrt{b_n^d |\Lambda_{n,R}|} A_n^{(1)}$  tends to a Gaussian distribution with zero mean. Applying (22) under condition  $b_n^{d+2m} |\Lambda_{n,R}| \rightarrow 0$  as  $n \rightarrow \infty$  and from strong consistency of the estimator  $\widehat{F}_n(t)$ , they imply that as  $n \rightarrow \infty$ ,

$$\sqrt{b_n^d |\Lambda_{n,R}|} A_n^{(2)} \xrightarrow{\mathbb{P}} 0.$$

which completes the proof. ■

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