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On asymptotic normality of nonparametric estimate for stationary pairwise interaction point process

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Abstract

We prove the asymptotic normality of non parametric estimator of pairwise interaction function for stationary pairwise interaction point process characterized by the Papangelou conditional intensity, and observed on in a bounded window Λ_n where $(\Lambda_n)_{n \geq 1}$ is a sequence of cubes growing up to \mathbb{R}^d . The proof is based on the m-approximation method in the setting of dependent random fields indexed by \mathbb{Z}^d where d is a positive integer.

Keywords: Georgii-Nguyen-Zessin formula, kernel estimators, non parametric estimation, pairwise interaction point process, Papangelou conditional intensity, asymptotic normality, m-dependence.

1 Introduction

A special type of Gibbs processes that are noteworthy both for their abundant use in statistical physics are the pairwise interaction point processes. In this paper we are concerned with nonparametric statistics for stationary pairwise interaction point processes. They have been introduced by Ripley and Kelly [23], Daley and Vere-Jones [5] and Georgii [8]. Throughout this paper, we assume the model is a stationary pairwise interaction point process and its Papangelou conditional intensity at a location u given by

$$\lambda_{\beta^*}(u, \mathbf{x}) = \beta^* \exp \left(- \sum_{v \in \mathbf{x} \setminus u} g(v - u) \right) \quad (1)$$

where β^* is the true value of the Poisson intensity parameter, g represents a pairwise interaction potential. Specifically, g is nonnegative measurable function.

We observe \mathbf{X} in the erosion of Λ_n by distance $2R$, denoted by $\Lambda_{n,R}$ and assume this has non-zero area, where $(\Lambda_n)_{n \geq 1}$ is a sequence of cubes growing up to \mathbb{R}^d . The nonparametric estimator $\widehat{G}_n(t)$ of $\beta^*G(t)$ introduced by Morsli [19] is defined for all positive integer n and any point t in the support $T = \{t \in \mathbb{R}^d; g(t) > 0, \text{ for } \|t\| < R\}$ of the interaction function $G = \exp(-g)$ by

$$\widehat{G}_n(t) = \frac{\widehat{H}_n(t)}{\widehat{F}_n(t)}, \quad (2)$$

where

$$\widehat{H}_n(t) = \frac{1}{b_n^d |\Lambda_{n,R}|} \sum_{\substack{\neq \\ u, v \in \mathbf{X} \\ v-u \in B(o, R)}} \mathbb{1}_{\Lambda_{n,R}}(u) h(u, \mathbf{X} \setminus \{u, v\}) h(v, \mathbf{X} \setminus \{u, v\}) K \left(\frac{v - u - t}{b_n} \right) \quad (3)$$

is a kernel-type estimator estimating $\widetilde{H}(t) = \beta^{*2}G(t)\bar{F}(o, t)$.

Here the \sum^{\neq} stretches over all pairs u, v of distinct points of the point process \mathbf{X} , and $K : \mathbb{R}^d \rightarrow \mathbb{R}$ denotes a smoothing kernel function associated with a sequence $(b_n)_{n \geq 1}$ is a parameter which converges slowly to zero.

$$\widehat{F}_n(t) = \frac{1}{|\Lambda_{n,R}|} \sum_{u \in \mathbf{X}_{\Lambda_{n,R}}} h(u, \mathbf{X} \setminus \{u\}) h(t - u, \mathbf{X} \setminus \{u\}) \quad (4)$$

meanwhile being an estimator of $\beta^*\bar{F}(o, t)$, where

$$\bar{F}(o, t) = \mathbb{P}(\mathbf{X} \cap B(o, R) = \emptyset, \mathbf{X} \cap B(t, R) = \emptyset). \quad (5)$$

And h is a nonnegative measurable function defined for all $w \in \mathbb{R}^d$, $\mathbf{x} \in N_{lf}$, by

$$h(w, \mathbf{x}) = \mathbb{1}(\mathbf{x} \cap B(w, R) = \emptyset).$$

In a previous paper, Morsli [19] obtained strong convergence of the estimator (2). The goal of the present paper is to establish asymptotic normality nonparametric estimate of the pairwise interaction function for stationary pairwise interaction point process. Our proof is based on the m -approximation method in the setting of dependent random fields indexed by \mathbb{Z}^d where d is a positive integer. We apply a central limit theorem for triangular arrays of stationary m -dependent random fields with unbounded m established by Wang and Woodroffe [25]. This result improves a central limit theorem established by Heinrich [10]. The notion of m -dependence was first introduced by Hoeffding and Robbins [12]. In general, the m -approximation method has been successful in proving central limit theorems for random fields (see Cheng and Ho. [3], Wang and Woodroffe [24] and El Machkouri et al. [16]).

The paper is organized as follows. In Section 2 sets up the basic tools of the point processes in \mathbb{R}^d . Our assumptions and main results are formulated in Section 3, we first study the behaviour of variance and covariance of a kernel-type estimator. Proofs are given in Section 4

2 Basic tools

Gibbs point processes in \mathbb{R}^d can be defined and characterized through the Papangelou conditional intensity (Møller and Waagepetersen [17]) which is a function $\lambda : \mathbb{R}^d \times N_{lf} \rightarrow \mathbb{R}_+$ where N_{lf} is the space of locally finite configurations of points in \mathbb{R}^d . Heuristically, $\lambda(u, \mathbf{x})du$ can be interpreted as the conditional probability of \mathbf{X} having a point in an infinitesimal small region containing u and of size du given the rest of \mathbf{X} is \mathbf{x} . For Markov point processes (Ripley and Kelly [23]; van Lieshout [14]) it is convenient to model \mathbf{X} by λ rather than by density, since λ plays the same role as the local characteristics do for Markov random fields when specifying local Markov properties. The Georgii-Nguyen-Zessin (GNZ) formula (see Papangelou [22], Zessin [26], Georgii [7], Nguyen and Zessin [21]) states that for any nonnegative measurable function h on $\mathbb{R}^d \times N_{lf}$

$$\mathbb{E} \sum_{u \in \mathbf{X}} h(u, \mathbf{X} \setminus u) = \mathbb{E} \int_{\mathbb{R}^d} h(u, \mathbf{X}) \lambda(u, \mathbf{X}) du. \quad (6)$$

We will call the random integral on the right side of (6) the (Papangelou) compensator of the random sum on the left side. Using induction we obtain the iterated GNZ-formula: for nonnegative functions $h : (\mathbb{R}^d)^n \times N_{lf} \rightarrow \mathbb{R}$

$$\mathbb{E} \sum_{\substack{\neq \\ u_1, \dots, u_n \in \mathbf{X}}} h(u_1, \dots, u_n, \mathbf{X} \setminus \{u_1, \dots, u_n\}) = \int \dots \int \mathbb{E} h(u_1, \dots, u_n, \mathbf{X}) \lambda(u_1, \dots, u_n, \mathbf{X}) du_1 \dots du_n \quad (7)$$

where $\lambda(u_1, \dots, u_n, \mathbf{x})$ is Papangelou conditional intensity of order n . It is defined (not uniquely) by

$$\lambda(u_1, \dots, u_n, \mathbf{x}) = \lambda(u_1, \mathbf{x}) \lambda(u_2, \mathbf{x} \cup \{u_1\}) \dots \lambda(u_n, \mathbf{x} \cup \{u_1, \dots, u_{n-1}\}).$$

Examples of Markov and non-Markov Gibbs point process models and their conditional intensities are presented in Baddeley et al [1], Møller and Waagepetersen

([17], [18]), Coeurjolly and Morsli [4]. The formula (7) will also be used extensively later in this document.

3 Assumptions and the main result

The basic assumption throughout this paper is the Papangelou conditional intensity (1) has a finite range R , i.e.

$$\lambda_{\beta^*}(u, \mathbf{x}) = \lambda_{\beta^*}(u, \mathbf{x}_{B(u,R)}), \quad (8)$$

for any $u \in \mathbb{R}^d$, $\mathbf{x} \in N_{lf}$, where $B(u, R)$ is the closed ball in \mathbb{R}^d with centre u and radius R . Moreover, we have to impose certain natural restrictions on the kernel function K and the sequence $(b_n)_{n \geq 1}$:

Condition $K(d, m)$: The sequence of bandwidths $b_n > 0$ for $n \geq 1$, is chosen such that

$$\lim_{n \rightarrow \infty} b_n = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} b_n^d |\Lambda_{n,R}| = \infty.$$

The kernel function $K : \mathbb{R}^d \rightarrow \mathbb{R}$ is nonnegative and bounded with bounded support, and satisfies:

$$\int_{\mathbb{R}^d} K(u) du = 1, \quad \int_{\mathbb{R}^d} K^2(z) dz < \infty$$

and for $m > 1$

$$\int_{\mathbb{R}^d} z_1^{a_1} \dots z_d^{a_d} K(z) dz = 0$$

with $a_1, \dots, a_d \in \mathbb{N}^*$ such that $0 < \sum_{i=1}^d a_i < m$.

To establish asymptotic normality, we need a further smoothness condition on the kernel function.

Condition \mathcal{Z} : The kernel function K is Lipschitz.

The following theorem presents the asymptotic behavior of the variance of the kernel-type estimator (3).

Theorem 1. *Consider a stationary pairwise interaction point process \mathbf{X} in \mathbb{R}^d with Papangelou conditional intensity (1) satisfy condition (8) and under Condition $K(d, m)$. We have*

$$\lim_{n \rightarrow \infty} b_n^d |\Lambda_{n,R}| \text{Var}(\widehat{H}_n(t)) = \beta^{*2} \exp(-g(t)) \bar{F}(o, t) \int_{\mathbb{R}^d} K^2(z) dz$$

at any continuity point $t \in T \setminus \{o\}$ of g and \bar{F} .

Theorem 2. *Let the assumptions of Theorem 1 be satisfied and let $t_1 \neq t_2 \in T$. Then, we have*

$$\lim_{n \rightarrow \infty} b_n^d |\Lambda_{n,R}| \text{Cov}(\widehat{H}_n(t_1), \widehat{H}_n(t_2)) = 0.$$

The proof Theorem 2 is similar to that of Theorem 1, and is omitted.

Asymptotic normality nonparametric estimate of the pairwise interaction function for stationary pairwise interaction point process is essentially based on multivariate central limit theorem for the estimator $\widehat{H}_n(t)$ of $\widetilde{H}(t)$. Since the mixing property is often unverifiable and might be too restrictive, it is important to provide asymptotic normality for nonmixing and possibly nonlinear random fields. A basic ingredient in the approach in this paper is that a point process can be viewed interchangeably as a lattice field. We split up the sampling window $\Lambda_{n,R}$ into cubes such as $\Lambda_{n,R} = \cup_{i \in \widetilde{I}_n} \Lambda_i$, where Λ_i are centered at i and assume that \widetilde{I}_n increases towards \mathbb{Z}^d and setting $X_i = X_{\Lambda_i}$, $i \in \mathbb{Z}^d$. We will consider estimation of $\beta^{*2}G(t)\bar{F}(o, t)$ from $\widehat{H}_n(t)$, where $\Lambda_{n,R} = \cup_{i \in \widetilde{I}_n} \Lambda_i$ and where the process is observed in $\cup_{i \in \widetilde{I}_n} \Lambda_i$, i.e.

$$\widehat{H}_n(t) = \frac{1}{b_n^d |\Lambda_{n,R}|} \sum_{i \in \widetilde{I}_n} \sum_{\substack{\neq \\ u \in \mathbf{X}_i, v \in \mathbf{X} \\ v-u \in B(o,R)}} h(u, \mathbf{X} \setminus \{u, v\}) h(v, \mathbf{X} \setminus \{u, v\}) K\left(\frac{v-u-t}{b_n}\right).$$

We consider in this work a field $\{X_i\}_{i \in \mathbb{Z}^d}$ ($d \in \mathbb{N}$) in form of

$$X_i = \widetilde{F}(\varepsilon_{i-k}, k \in \mathbb{Z}^d), i \in \mathbb{Z}^d \quad (9)$$

where $\{\varepsilon_i\}_{i \in \mathbb{Z}^d}$ are i.i.d. random variables and \widetilde{F} is measurable function. Let $\{\varepsilon'_i\}_{i \in \mathbb{Z}^d}$ be an i.i.d. copy of $\{\varepsilon_i\}_{i \in \mathbb{Z}^d}$ and consider for all positive integer n the coupled version X_i^* defined by $X_i^* = \widetilde{F}(\varepsilon_{i-k}^*, k \in \mathbb{Z}^d)$, where

$$\varepsilon_j^* = \begin{cases} \varepsilon_j & \text{if } j \neq 0 \\ \varepsilon'_0 & \text{if } j = 0. \end{cases}$$

Let $i \in \mathbb{Z}^d$ and $p > 0$ be fixed. If $X_i \in \mathbb{L}^p$, we define $\delta_{i,p} = \|X_i - X_i^*\|_p$, where $\|\cdot\|_p$ is the usual \mathbb{L}^p -norm. Thus, El Machkouri [15] obtained this sufficient condition:

$$\sum_{i \in \mathbb{Z}^d} |i|^{5d/2} \delta_{i,p} < \infty \quad (10)$$

for a random field of the form (9) for the kernel density estimator to be asymptotically normal.

The desired sequence $(m_n)_{n \geq 1}$ can be chosen as

$$m_n = \max \left\{ v_n, \left[\left(\frac{1}{b_n^3} \sum_{|i| > v_n} |i|^{\frac{5d}{2}} \delta_{i,2} \right)^{\frac{1}{3d}} \right] + 1 \right\}$$

where $v_n = \lfloor b_n^{-\frac{1}{2d}} \rfloor$ and $[\cdot]$ denotes the integer part function.

Our results show essentially through the following technical lemma is a spatial version of a result by Bosq et al. [2].

Lemma 1. *If (10) holds, then*

$$m_n \rightarrow \infty, \quad m_n^d b_n^d \rightarrow 0 \quad \text{and} \quad \frac{1}{(m_n b_n)^{3d/2}} \sum_{|i| > m_n} |i|^{\frac{5d}{2}} \delta_{i,2} \rightarrow 0.$$

In order to establish the asymptotic normality, we need additional the assumption:

Condition \mathcal{W} : There exist a sequence of integers $(m_n)_{n \geq 1}$ such that $m_n \rightarrow \infty$ as $n \rightarrow \infty$, and the following limit holds,

$$\frac{m_n^d \log^d n}{b_n^d n^d} \rightarrow 0$$

Theorem 3. Consider a stationary pairwise interaction point process \mathbf{X} in \mathbb{R}^d with Papangelou conditional intensity (1) satisfy condition (8). Under Conditions \mathbf{K}^0 , \mathcal{Z} , \mathcal{W} and $b_n^{d+2m} |\Lambda_{n,R}| \rightarrow 0$ as $n \rightarrow \infty$. Then for any positive integer s and any distinct points t_1, \dots, t_s in $T \setminus \{o\}$,

$$\sqrt{b_n^d |\Lambda_{n,R}|} \left(\widehat{G}_n(t_i) - \beta^* G(t_i) \right)_{i=1}^s \xrightarrow{d} \mathcal{N}(0, \sigma_G)$$

where the covariance matrix $\sigma_G = [\sigma_{i,j}]_{i,j=1,\dots,s}$ is given by

$$\sigma_{i,i} = \frac{G(t_i)}{\bar{F}(o, t_i)} \int K^2(z) dz \quad \text{for } i = 1, \dots, s \quad \text{and} \quad \sigma_{i,j} = 0 \quad \text{if } i \neq j.$$

The proof of Theorem 3 is essentially based on multivariate central limit theorem for the estimator $\widehat{H}_n(t)$ of $\widetilde{H}(t)$ which is formulated and proved in this section.

Theorem 4. Consider a stationary pairwise interaction point process \mathbf{X} in \mathbb{R}^d with Papangelou conditional intensity (1) satisfy condition (8). Under Conditions \mathbf{K}^0 , \mathcal{Z} and \mathcal{W} . Then for any positive integer s and any distinct points t_1, \dots, t_s in $T \setminus \{o\}$,

$$\sqrt{b_n^d |\Lambda_{n,R}|} \left(\widehat{H}_n(t_i) - \mathbb{E} \widehat{H}_n(t_i) \right)_{i=1}^s \xrightarrow{d} \mathcal{N}(0, \sigma_H)$$

where the covariance matrix $\sigma_H = [r_{i,j}]_{i,j=1,\dots,s}$ is given by

$$r_{i,i} = \beta^{*2} \bar{F}(o, t_i) G(t_i) \int_{\mathbb{R}^d} K^2(z) dz, \quad \text{for } i = 1, \dots, s \quad \text{and} \quad r_{i,j} = 0 \quad \text{if } i \neq j.$$

Corollary 1. Under the conditions of Theorem 4 and let $b_n^{d+2m} |\Lambda_{n,R}| \rightarrow 0$ as $n \rightarrow +\infty$. Furthermore if \widetilde{H} has bounded and continuous partial derivatives of order m in some neighbourhood of the points t_1, \dots, t_s . Then

$$\sqrt{b_n^d |\Lambda_{n,R}|} \left(\widehat{H}_n(t_i) - \widetilde{H}(t_i) \right)_{i=1}^s \xrightarrow{d} \mathcal{N}(0, \sigma_H)$$

where the covariance matrix $\sigma_H = [r_{i,j}]_{i,j=1,\dots,s}$ is given by

$$r_{i,i} = \beta^{*2} \bar{F}(o, t_i) G(t_i) \int_{\mathbb{R}^d} K^2(z) dz, \quad \text{for } i = 1, \dots, s \quad \text{and} \quad r_{i,j} = 0 \quad \text{if } i \neq j.$$

4 Proofs

In the sequel, we denote $H(u, v, \mathbf{X}) = h(u, \mathbf{X})h(v, \mathbf{X})$, $I_R(v-u) = \mathbb{1}(v-u \in B(o, R))$ and keep in mind the function $\bar{F}(o, t)$ defined by (5).

4.1 Proof of Theorem 1

Proof. We determine the asymptotic behavior of the variance of the estimator $\widehat{H}_n(t)$. For this we use the following result.

Lemma 2. *Consider any Gibbs point process \mathbf{X} in \mathbb{R}^d with Papangelou conditional intensity λ . For any nonnegative measurable function $f : \mathbb{R}^d \times \mathbb{R}^d \times N_{lf} \rightarrow \mathbb{R}$, then*

$$\begin{aligned}
& \text{Var} \left(\sum_{\substack{\neq \\ u, v \in \mathbf{X}}} f(u, v, \mathbf{X} \setminus \{u, v\}) \right) \\
&= \mathbb{E} \int_{\mathbb{R}^{2d}} f(u, v, \mathbf{X}) [f(u, v, \mathbf{X}) + f(v, u, \mathbf{X})] \lambda(u, v, \mathbf{X}) \, du \, dv \\
&+ \mathbb{E} \int_{\mathbb{R}^{3d}} f(u, v, \mathbf{X}) [f(v, w, \mathbf{X}) + f(w, v, \mathbf{X})] \lambda(u, v, w, \mathbf{X}) \, du \, dv \, dw \\
&+ \mathbb{E} \int_{\mathbb{R}^{3d}} f(u, v, \mathbf{X}) [f(u, w, \mathbf{X}) + f(w, u, \mathbf{X})] \lambda(u, v, w, \mathbf{X}) \, du \, dv \, dw \\
&+ \mathbb{E} \int_{\mathbb{R}^{4d}} f(u, v, \mathbf{X}) f(w, y, \mathbf{X}) \lambda(u, v, w, y, \mathbf{X}) \, du \, dv \, dw \, dy \\
&- \int_{\mathbb{R}^{4d}} \mathbb{E} [f(u, v, \mathbf{X}) \lambda(u, v, \mathbf{X})] \mathbb{E} [f(w, y, \mathbf{X}) \lambda(w, y, \mathbf{X})] \, du \, dv \, dw \, dy.
\end{aligned}$$

Proof. After a simple calculation, but prolix (see Jolivet [13] and Heinrich [11]), we

obtain the following decomposition:

$$\begin{aligned}
\left(\sum_{u,v \in \mathbf{X}}^{\neq} f(u,v, \mathbf{X} \setminus \{u,v\}) \right)^2 &= \sum_{u,v \in \mathbf{X}}^{\neq} f^2(u,v, \mathbf{X} \setminus \{u,v\}) \\
&+ \sum_{u,v \in \mathbf{X}}^{\neq} f(u,v, \mathbf{X} \setminus \{u,v\}) f(v,u, \mathbf{X} \setminus \{u,v\}) \\
&+ \sum_{u,v,w \in \mathbf{X}}^{\neq} f(u,v, \mathbf{X} \setminus \{u,v,w\}) f(v,w, \mathbf{X} \setminus \{u,v,w\}) \\
&+ \sum_{u,v,w \in \mathbf{X}}^{\neq} f(u,v, \mathbf{X} \setminus \{u,v,w\}) f(w,v, \mathbf{X} \setminus \{u,v,w\}) \\
&+ \sum_{u,v,w \in \mathbf{X}}^{\neq} f(u,v, \mathbf{X} \setminus \{u,v,w\}) f(u,w, \mathbf{X} \setminus \{u,v,w\}) \\
&+ \sum_{u,v,w \in \mathbf{X}}^{\neq} f(u,v, \mathbf{X} \setminus \{u,v,w\}) f(w,u, \mathbf{X} \setminus \{u,v,w\}) \\
&+ \sum_{u,v,w,y \in \mathbf{X}}^{\neq} f(u,v, \mathbf{X} \setminus \{u,v,w,y\}) f(w,y, \mathbf{X} \setminus \{u,v,w,y\}).
\end{aligned} \tag{11}$$

On the other hand using standard definition of variance, we have

$$\text{Var} \sum_{u,v \in \mathbf{X}}^{\neq} f(u,v, \mathbf{X} \setminus \{u,v\}) = \mathbb{E} \left(\sum_{u,v \in \mathbf{X}}^{\neq} f(u,v, \mathbf{X} \setminus \{u,v\}) \right)^2 - \left(\mathbb{E} \sum_{u,v \in \mathbf{X}}^{\neq} f(u,v, \mathbf{X} \setminus \{u,v\}) \right)^2. \tag{12}$$

After rearrangement with the formula of GNZ (7) and formulas (11) and (12), we obtain the desired result. \blacksquare

substitute

$$f(u,v, \mathbf{x}) = \mathbb{1}_{\Lambda_{n,R}}(u) I_R(v-u) H(u,v, \mathbf{x}) K \left(\frac{v-u-t}{b_n} \right)$$

the variance term $H_n(t)$ is now expanded using Lemma 2, therefore we find that

$$\text{Var} \widehat{H}_n(t) = T_1 + T_2 + T_3 + T_4 + T_5 + T_6 + T_7 + T_8,$$

with

$$\begin{aligned}
T_1 &= \frac{1}{b_n^{2d} |\Lambda_{n,R}|^2} \mathbb{E} \int_{\mathbb{R}^{2d}} \mathbb{1}_{\Lambda_{n,R}}(u) I_R(v-u) H(u, v, \mathbf{X}) K^2\left(\frac{v-u-t}{b_n}\right) \lambda_{\beta^*}(u, v, \mathbf{X}) \mathrm{d}u \mathrm{d}v, \\
T_2 &= \frac{1}{b_n^{2d} |\Lambda_{n,R}|^2} \\
&\quad \mathbb{E} \int_{\Lambda_{n,R}^2} I_R(v-u, u-v) H(u, v, \mathbf{X}) K\left(\frac{v-u-t}{b_n}\right) K\left(\frac{u-v-t}{b_n}\right) \lambda_{\beta^*}(u, v, \mathbf{X}) \mathrm{d}u \mathrm{d}v, \\
T_3 &= \frac{1}{b_n^{2d} |\Lambda_{n,R}|^2} \mathbb{E} \int_{\mathbb{R}^{2d}} \int_{\Lambda_{n,R}} I_R(v-u, w-v) H(u, v, w, \mathbf{X}) K\left(\frac{v-u-t}{b_n}\right) K\left(\frac{w-v-t}{b_n}\right) \\
&\quad \times \lambda_{\beta^*}(u, v, w, \mathbf{X}) \mathrm{d}u \mathrm{d}v \mathrm{d}w, \\
T_4 &= \frac{1}{b_n^{2d} |\Lambda_{n,R}|^2} \mathbb{E} \int_{\mathbb{R}^d} \int_{\Lambda_{n,R}^2} I_R(v-u, v-w) H(u, v, w, \mathbf{X}) K\left(\frac{v-u-t}{b_n}\right) K\left(\frac{v-w-t}{b_n}\right) \\
&\quad \times \lambda_{\beta^*}(u, v, w, \mathbf{X}) \mathrm{d}u \mathrm{d}v \mathrm{d}w, \\
T_5 &= \frac{1}{b_n^{2d} |\Lambda_{n,R}|^2} \mathbb{E} \int_{\mathbb{R}^{2d}} \int_{\Lambda_{n,R}} I_R(v-u, w-u) H(u, v, w, \mathbf{X}) K\left(\frac{v-u-t}{b_n}\right) K\left(\frac{w-u-t}{b_n}\right) \\
&\quad \times \lambda_{\beta^*}(u, v, w, \mathbf{X}) \mathrm{d}u \mathrm{d}v \mathrm{d}w, \\
T_6 &= \frac{1}{b_n^{2d} |\Lambda_{n,R}|^2} \mathbb{E} \int_{\mathbb{R}^d} \int_{\Lambda_{n,R}^2} I_R(v-u, u-w) H(u, v, w, \mathbf{X}) K\left(\frac{v-u-t}{b_n}\right) K\left(\frac{u-w-t}{b_n}\right) \\
&\quad \times \lambda_{\beta^*}(u, v, w, \mathbf{X}) \mathrm{d}u \mathrm{d}v \mathrm{d}w,
\end{aligned}$$

$$\begin{aligned}
T_7 &= \frac{1}{b_n^{2d} |\Lambda_{n,R}|^2} \mathbb{E} \int_{\mathbb{R}^{4d}} \mathbb{1}_{\Lambda_{n,R}}(u) I_R(v-u, y-w) H(u, v, w, y, \mathbf{X}) K\left(\frac{v-u-t}{b_n}\right) K\left(\frac{y-w-t}{b_n}\right) \\
&\quad \times \lambda_{\beta^*}(u, v, w, y, \mathbf{X}) \mathrm{d}u \mathrm{d}v \mathrm{d}w \mathrm{d}y,
\end{aligned}$$

and

$$\begin{aligned}
T_8 &= - \frac{1}{b_n^{2d} |\Lambda_{n,R}|^2} \int_{\mathbb{R}^{4d}} \mathbb{1}_{\Lambda_{n,R}}(u) \mathbb{1}_{\Lambda_{n,R}}(w) I_R(v-u, y-w) K\left(\frac{v-u-t}{b_n}\right) K\left(\frac{y-w-t}{b_n}\right) \\
&\quad \times \mathbb{E} [H(u, v, \mathbf{X}) \lambda_{\beta^*}(u, v, \mathbf{X})] \mathbb{E} [H(w, y, \mathbf{X}) \lambda_{\beta^*}(w, y, \mathbf{X})] \mathrm{d}u \mathrm{d}v \mathrm{d}w \mathrm{d}y.
\end{aligned}$$

The asymptotic behaviour of the leading term T_1 is obtained by applying the second order Papangelou conditional intensity given by:

$$\lambda_{\beta^*}(u, v, \mathbf{x}) = \lambda_{\beta^*}(u, \mathbf{x}) \lambda_{\beta^*}(v, \mathbf{x} \cup \{u\}) \quad \text{for any } u, v \in \mathbb{R}^d \quad \text{and } \mathbf{x} \in N_{lf}.$$

And using the finite range property (8) for each function $\lambda_{\beta^*}(u, \mathbf{x})$ and $\lambda_{\beta^*}(v, \mathbf{x} \cup \{u\})$. Hence $\mathbf{x} = \emptyset$, this implies that

$$\lambda_{\beta^*}(u, \emptyset) = \beta^* \quad \text{and} \quad \lambda_{\beta^*}(v, \emptyset \cup \{u\}) = \beta^* G(v-u) \quad \text{for all } u, v \in \mathbb{R}^d.$$

By stationarity of \mathbf{X} , it results

$$\begin{aligned}
T_1 &= \frac{1}{b_n^{2d} |\Lambda_{n,R}|^2} \mathbb{E} \int_{\mathbb{R}^{2d}} \mathbb{1}_{\Lambda_{n,R}}(u) I_R(v-u) H(u, v, \mathbf{X}) K^2\left(\frac{v-u-t}{b_n}\right) \lambda_{\beta^*}(u, v, \mathbf{X}) \, dudv \\
&= \frac{\beta^{*2}}{b_n^{2d} |\Lambda_{n,R}|^2} \int_{\mathbb{R}^{2d}} \mathbb{1}_{\Lambda_{n,R}}(u) I_R(v-u) \bar{F}(u, v) K^2\left(\frac{v-u-t}{b_n}\right) G(v-u) \, dudv \\
&= \frac{\beta^{*2}}{b_n^{2d} |\Lambda_{n,R}|} \int_{\mathbb{R}^d} I_R(s) \bar{F}(o, s) K^2\left(\frac{s-t}{b_n}\right) G(s) \, ds \\
&= \frac{\beta^{*2}}{b_n^d |\Lambda_{n,R}|} \int_{\mathbb{R}^d} I_R(b_n z + t) \bar{F}(o, b_n z + t) K^2(z) G(b_n z + t) \, dz.
\end{aligned}$$

The continuity of $\bar{F}G$ in t and the boundedness conditions on the kernel function yield the desired result by dominated convergence theorem, i.e.

$$\lim_{n \rightarrow \infty} b_n^d |\Lambda_{n,R}| T_1 = \beta^{*2} \bar{F}(o, t) G(t) \int_{\mathbb{R}^d} K^2(z) \, dz.$$

We apply in the following lemma of the sequence of observation windows $(\Lambda_n)_{n \in \mathbb{N}}$, established by (David [6], Lemma A.2).

Lemma 3. *Let $(\Lambda_n)_{n \in \mathbb{N}}$ be a sequence of convex sets in \mathbb{R}^d growing up to \mathbb{R}^d and let $y, z \in \mathbb{R}^d$. Let $(b_n)_{n \in \mathbb{N}}$ be a sequence of positive real numbers with $b_n \xrightarrow[n \rightarrow \infty]{} 0$. Then we have*

$$\lim_{n \rightarrow \infty} \frac{|\Lambda_n \cap (\Lambda_n - b_n y - z)|}{|\Lambda_n|} = 1.$$

We will now show that all the other terms of the variance of the estimator $\widehat{H}_n(t)$ vanish. With similar arguments to those used to obtain the results above, we treat the term T_2 .

$$\begin{aligned}
T_2 &= \frac{1}{b_n^{2d} |\Lambda_{n,R}|^2} \\
&\mathbb{E} \int_{\mathbb{R}^{2d}} \mathbb{1}_{\Lambda_{n,R}}(u) \mathbb{1}_{\Lambda_{n,R}}(v) I_R(v-u, u-v) H(u, v, \mathbf{X}) K\left(\frac{v-u-t}{b_n}\right) K\left(\frac{u-v-t}{b_n}\right) \lambda_{\beta^*}(u, v, \mathbf{X}) \, dudv \\
&= \frac{\beta^{*2}}{b_n^{2d} |\Lambda_{n,R}|^2} \\
&\int_{\mathbb{R}^{2d}} \mathbb{1}_{\Lambda_{n,R}}(u) \mathbb{1}_{\Lambda_{n,R}}(v) I_R(v-u, u-v) \bar{F}(o, v-u) K\left(\frac{v-u-t}{b_n}\right) K\left(\frac{u-v-t}{b_n}\right) G(v-u) \, dudv \\
&= \frac{\beta^{*2}}{b_n^{2d} |\Lambda_{n,R}|} \int_{\mathbb{R}^d} \frac{|\Lambda_{n,R} \cap (\Lambda_{n,R} - s)|}{|\Lambda_{n,R}|} I_R(s, -s) \bar{F}(o, s) K\left(\frac{s-t}{b_n}\right) K\left(\frac{-s-t}{b_n}\right) G(s) \, ds \\
&= \frac{\beta^{*2}}{b_n^d |\Lambda_{n,R}|} \\
&\int_{\mathbb{R}^d} \frac{|\Lambda_{n,R} \cap (\Lambda_{n,R} - (b_n z + t))|}{|\Lambda_{n,R}|} I_R(b_n z + t, -b_n z - t) \bar{F}(o, b_n z + t) K(z) K(-z - \frac{2t}{b_n}) G(b_n z + t) \, dz.
\end{aligned}$$

The continuity of $\bar{F}G$ in t and the boundedness conditions on the kernel function by dominated convergence theorem and under condition $t \neq o$ and using the Lemma 3, we get $b_n^d |\Lambda_{n,R}| T_2 \rightarrow 0$ as $n \rightarrow \infty$.

For the asymptotic behavior of the third term T_3 , we remember the third order Papangelou conditional intensity by,

$$\lambda_{\beta^*}(u, v, w, \mathbf{x}) = \lambda_{\beta^*}(u, \mathbf{x})\lambda_{\beta^*}(v, \mathbf{x} \cup \{u\})\lambda_{\beta^*}(w, \mathbf{x} \cup \{u, v\})$$

for any $u, v, w \in \mathbb{R}^d$ and $\mathbf{x} \in N_{lf}$. Since \mathbf{X} is a point process to interact in pairs, the interaction terms due to triplets or higher order are equal to one, i.e. the potential $g(\mathbf{y})$ is null when $n(\mathbf{y}) > 2$, for $\emptyset \neq \mathbf{y} \subseteq \mathbf{x}$. Using the finite range property (8) for each function $\lambda_{\beta^*}(u, \mathbf{x})$, $\lambda_{\beta^*}(v, \mathbf{x} \cup \{u\})$ and $\lambda_{\beta^*}(w, \mathbf{x} \cup \{u, v\})$ and after a elementary calculation, we have

$$\lambda_{\beta^*}(u, v, w, \emptyset) = \begin{cases} \beta^{*3}G(v-u)G(w-v) & \text{if } u \notin B(w, R) \\ \beta^{*3}G(v-u) & \text{otherwise.} \end{cases} \quad (13)$$

Which ensures that $\lambda_{\beta^*}(u, v, w, \emptyset)$ is a function that depends only variables $v-u, w-v$, denoted by $G_3(v-u, w-v)$. According to the stationarity of \mathbf{X} , it follows that

$$\begin{aligned} T_3 &= \frac{1}{b_n^{2d}|\Lambda_{n,R}|} \int_{\mathbb{R}^{2d}} \frac{|\Lambda_{n,R} \cap (\Lambda_{n,R} - v)|}{|\Lambda_{n,R}|} \bar{G}_3(v, w-v) K\left(\frac{v-t}{b_n}\right) K\left(\frac{w-v-t}{b_n}\right) dv dw \\ &= \frac{1}{|\Lambda_{n,R}|} \int_{\mathbb{R}^{2d}} \frac{|\Lambda_{n,R} \cap (\Lambda_{n,R} - (b_n z + t))|}{|\Lambda_{n,R}|} \bar{G}_3(b_n z + t, b_n z' + t) K(z) K(z') dz dz', \end{aligned}$$

where $\bar{G}_3(v, w-v) = I_R(v, w-v)\bar{F}(-v, w-v)G_3(v, w-v)$. By dominated convergence theorem and using the Lemma 3, we get $b_n^d|\Lambda_{n,R}|T_3 \rightarrow 0$ as $n \rightarrow \infty$. Analogously, one may show the other terms T_4, T_5 and T_6 .

For the asymptotic behaviour of the leading term T_7 , let u, v, w, y any points in \mathbb{R}^d and $\mathbf{x} \in N_{lf}$, we define Papangelou conditional intensity of the fourth order by

$$\lambda_{\beta^*}(u, v, w, y, \mathbf{x}) = \lambda_{\beta^*}(u, \mathbf{x})\lambda_{\beta^*}(v, \mathbf{x} \cup \{u\})\lambda_{\beta^*}(w, \mathbf{x} \cup \{u, v\})\lambda_{\beta^*}(y, \mathbf{x} \cup \{u, v, w\}).$$

Next we introduce the finite range property (8) and reasoning analogous with the foregoing on $\lambda_{\beta^*}(u, v, w, \emptyset)$ is defined by (13), which ensures that $\lambda_{\beta^*}(u, v, w, y, \emptyset)$ is a function that depends only variables $v-u, y-w, w-u, w-v$, denoted by $G_4(v-u, y-w, w-u, w-v)$. Further by the stationarity of \mathbf{X} , this implies that

$$\begin{aligned} T_7 &= \frac{1}{b_n^{2d}|\Lambda_{n,R}|} \\ &\int_{\mathbb{R}^{3d}} \frac{|\Lambda_{n,R} \cap (\Lambda_{n,R} - w)|}{|\Lambda_{n,R}|} \bar{G}_4(v, y-w, w, w-v) K\left(\frac{v-t}{b_n}\right) K\left(\frac{y-w-t}{b_n}\right) dv dw dy \\ &= \frac{1}{|\Lambda_{n,R}|} \int_{\mathbb{R}^{3d}} \frac{|\Lambda_{n,R} \cap (\Lambda_{n,R} - w)|}{|\Lambda_{n,R}|} \bar{G}_4(b_n z + t, b_n z' + t, w, w - b_n z - t) K(z) K(z') dz dz' dw. \end{aligned}$$

$\bar{G}_4(v, y-w, w, w-v) = I_R(v, y-w)\bar{F}(v, w, y)G_4(v, y-w, w, w-v)$. Similar arguments show that $b_n^d|\Lambda_{n,R}|T_7 \rightarrow 0$ as $n \rightarrow \infty$.

We now finish the proof of the asymptotic behavior of the term T_8 . Under the conditions imposed in Theorem 1, we have

$$\begin{aligned}
T_8 &= -\frac{1}{b_n^{2d}|\Lambda_{n,R}|^2} \int_{\mathbb{R}^{4d}} \mathbb{1}_{\Lambda_{n,R}}(u) \mathbb{1}_{\Lambda_{n,R}}(w) I_R(v-u, y-w) K\left(\frac{v-u-t}{b_n}\right) K\left(\frac{y-w-t}{b_n}\right) \\
&\quad \times \mathbb{E} [H(u, v, \mathbf{X}) \lambda_{\beta^*}(u, v, \mathbf{X})] \mathbb{E} [H(w, y, \mathbf{X}) \lambda_{\beta^*}(w, y, \mathbf{X})] dudvdwdy \\
&= -\frac{\beta^{*4}}{b_n^{2d}|\Lambda_{n,R}|^2} \\
&\quad \int_{\mathbb{R}^{4d}} \mathbb{1}_{\Lambda_{n,R}}(u) \mathbb{1}_{\Lambda_{n,R}}(w) \bar{G}_5(u, v, w, y, v-u, y-w) K\left(\frac{v-u-t}{b_n}\right) K\left(\frac{y-w-t}{b_n}\right) dudvdwdy \\
&\quad \int_{\mathbb{R}^{3d}} \frac{|\Lambda_{n,R} \cap (\Lambda_{n,R} - w)|}{|\Lambda_{n,R}|} \bar{G}_5(o, v, w, y, v, y-w) K\left(\frac{v-t}{b_n}\right) K\left(\frac{y-w-t}{b_n}\right) dvdwdy \\
&= -\frac{\beta^{*4}}{|\Lambda_{n,R}|} \\
&\quad \int_{\mathbb{R}^{3d}} \frac{|\Lambda_{n,R} \cap (\Lambda_{n,R} - w)|}{|\Lambda_{n,R}|} \bar{G}_5(o, b_n z + t, w, b_n z' + t + w, b_n z + t, b_n z' + t) K(z) K(z') dw dz dz'
\end{aligned}$$

where

$$\begin{aligned}
\bar{G}_5(u, v, w, y, v-u, y-w) &= I_R(v-u, y-w) \bar{F}(u, v) \bar{F}(w, y) G(v-u) G(y-w). \\
\text{By dominated convergence theorem, we get the desired result.} &\quad \blacksquare
\end{aligned}$$

4.2 Proof of Theorem 4

Proof. To prove the Theorem 4, we need (using the well-known Cramér-Wold (1936)). Without loss of generality, we consider only the case $s = 2$. Let λ_1 and λ_2 two constantes with $\lambda_1 + \lambda_2 \neq 0$ and let $t_1, t_2 \in T \setminus \{o\}$ the points be fixed, $t_1 \neq t_2$ and $|\Lambda_{n,R}| \sim cn^d$. We split up the sampling window $\Lambda_{n,R}$ into cubes such as $\Lambda_{n,R} = \cup_{i \in \tilde{I}_n} \Lambda_i$, where \tilde{I}_n increases towards \mathbb{Z}^d and define the trianeular array of random variables

$$\frac{1}{n^{d/2}} \sum_{i \in \tilde{I}_n} \bar{Y}_{n,i} = \lambda_1 (b_n^d n^d)^{1/2} \left(\hat{H}_n(t_1) - \mathbb{E}[\hat{H}_n(t_1)] \right) + \lambda_2 (b_n^d n^d)^{1/2} \left(\hat{H}_n(t_2) - \mathbb{E}[\hat{H}_n(t_2)] \right)$$

where

$$\bar{Y}_{n,i} = \lambda_1 \left(Y_i(t_1) - \mathbb{E}[Y_i(t_1)] \right) + \lambda_2 \left(Y_i(t_2) - \mathbb{E}[Y_i(t_2)] \right), i \in \mathbb{Z}^d \quad (14)$$

for all $j = 1, 2$, we consider

$$Y_i(t_j) = (b_n^d)^{-1/2} \sum_{\substack{\neq \\ u \in X_{\Lambda_i}, v \in \mathbf{X} \\ v-u \in B(o, R)}} h(u, \mathbf{X} \setminus \{u, v\}) h(v, \mathbf{X} \setminus \{u, v\}) K\left(\frac{v-u-t_j}{b_n}\right).$$

The proving idea the below Theorem 4 consists in approximating the sequence $\bar{Y}_{n,i}$ by a triangular array of m-dependent random fields. The corresponding Lindeberg-type CLT m-dependent random fields has been proved by Wang and Woodroffe [25]. Therefore, it proves convenient to switch notation from the text and to define

$$\tilde{Y}_{n,i} = \lambda_1 \left(\tilde{Y}_i(t_1) - \mathbb{E}[\tilde{Y}_i(t_1)] \right) + \lambda_2 \left(\tilde{Y}_i(t_2) - \mathbb{E}[\tilde{Y}_i(t_2)] \right), \quad i \in \mathbb{Z}^d$$

for all $j = 1, 2$, we consider

$$\tilde{Y}_i(t_j) = (b_n^d)^{-1/2} \sum_{\substack{\neq \\ u \in X_{\Lambda_{i-k}}, v \in \mathbf{X} \\ v-u \in B(o,R)}} h(u, \mathbf{X} \setminus \{u, v\}) h(v, \mathbf{X} \setminus \{u, v\}) K \left(\frac{v - u - t_j}{b_n} \right).$$

In this way, we assume that $\{\tilde{Y}_{n,i}\}_{i \in \mathbb{Z}^d}$ is m_n -dependent, in the sense that $Y_{n,i}$ and $Y_{n,j}$ are independent if $|i - j|_\infty \geq m$, where $|j|_\infty = \max\{|j_1|, \dots, |j_d|\}$. We will use $\{\tilde{Y}_{n,i}; i \in \mathbb{Z}^d\}_{n \in \mathbb{N}}$ to approximate $\{\bar{Y}_{n,i}; i \in \mathbb{Z}^d\}_{n \in \mathbb{N}}$. We decompose

$$\frac{1}{n^{d/2}} \sum_{i \in \tilde{I}_n} \bar{Y}_{n,i} = \frac{1}{n^{d/2}} \sum_{i \in \tilde{I}_n} \tilde{Y}_{n,i} + \frac{1}{n^{d/2}} \sum_{i \in \tilde{I}_n} (\bar{Y}_{n,i} - \tilde{Y}_{n,i})$$

and we assume that

$$\left\| \sum_{i \in \tilde{I}_n} \tilde{Y}_{n,i} \right\|_2 < \infty \quad \text{for all } n \in \mathbb{N}. \quad (15)$$

Then, it suffices to establish the following two results.

Proposition 1. *Under Conditions \mathcal{W} and (15). For n sufficiently large, we have*

$$\frac{1}{n^{d/2}} \sum_{i \in \tilde{I}_n} \tilde{Y}_{n,i} \xrightarrow{d} \mathcal{N}(0, \sigma^2),$$

where

$$\sigma^2 = \left(\beta^{*2} \lambda_1^2 \bar{F}(o, t_1) G(t_1) + \beta^{*2} \lambda_2^2 \bar{F}(o, t_2) G(t_2) \right) \int_{\mathbb{R}^d} K^2(z) dz. \quad (16)$$

Proposition 2. *Under condition of Lemma 1. We have for n sufficiently large,*

$$\frac{1}{n^{d/2}} \sum_{i \in \tilde{I}_n} (\bar{Y}_{n,i} - \tilde{Y}_{n,i}) \xrightarrow{\mathbb{P}} 0.$$

Proof of Proposition 1

Proof. Taking into account of $\{\tilde{Y}_{n,i}\}_{i \in \mathbb{Z}^d}$ are m_n -dependent random fields, we apply limit theorem for stationary triangular arrays of m -dependent random fields (Theorem 2 establish by Wang and Woodroffe [25]). We need to verify that the two fundamental conditions (17) and (18).

It suffices to show, for $l_n = m_n \log n$ such that $m_n/l_n \rightarrow 0$ and $l_n/n \rightarrow 0$ as $n \rightarrow \infty$

$$\lim_{n \rightarrow \infty} \frac{1}{l_n^d} \mathbb{E} \left[\left(\sum_{i \in \llbracket 1, l_n \rrbracket^d} \tilde{Y}_{n,i} \right)^2 \right] = \sigma^2 \quad (17)$$

and

$$\lim_{n \rightarrow \infty} \frac{1}{l_n^d} \mathbb{E} \left[\left(\sum_{i \in \llbracket 1, l_n \rrbracket^d} \tilde{Y}_{n,i} \right)^2 \mathbb{1} \left\{ \left| \sum_{i \in \llbracket 1, l_n \rrbracket^d} \tilde{Y}_{n,i} \right| > n^{d/2} \epsilon \right\} \right] = 0 \quad \text{for all } \epsilon > 0 \quad (18)$$

where $a, b \in \mathbb{Z}$, $\llbracket a, b \rrbracket \equiv \{a, a+1, \dots, b\}$.

Taking into account the definition of $\sum_{i \in \llbracket 1, l_n \rrbracket^d} \tilde{Y}_{n,i}$ we get

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{1}{l_n^d} \text{Var} \left(\sum_{i \in \llbracket 1, l_n \rrbracket^d} \tilde{Y}_{n,i} \right) \\ &= \lim_{n \rightarrow \infty} \left(a_1^2 \text{Var} \left(\frac{1}{n^{d/2}} \sum_{i \in \tilde{I}_n} Z_{i, m_n}(t_1) \right) + a_2^2 \text{Var} \left(\frac{1}{n^{d/2}} \sum_{i \in \tilde{I}_n} Z_{i, m_n}(t_2) \right) \right) \\ &= \lim_{n \rightarrow \infty} b_n^d l_n^d \left(\lambda_1^2 \text{Var}(\hat{H}_{l_n}(t_1)) + \lambda_2^2 \text{Var}(\hat{H}_{l_n}(t_2)) + 2\lambda_1 \lambda_2 \text{Cov}(\hat{H}_{l_n}(t_1), \hat{H}_{l_n}(t_2)) \right). \end{aligned}$$

Using Theorem 1 entails replacing $\hat{H}_n(t_j)$ by $\hat{H}_{l_n}(t_j)$, we have

$$\lim_{n \rightarrow \infty} b_n^d l_n^d \text{Var}(\hat{H}_{l_n}(t_j)) = \beta^{*2} G(t_j) \bar{F}(o, t_j) \int_{\mathbb{R}^d} K^2(u) du.$$

Using Theorem 2, we have

$$\lim_{n \rightarrow \infty} b_n^d l_n^d \text{Cov} \left(\hat{H}_{l_n}(t_1), \hat{H}_{l_n}(t_2) \right) = 0$$

then, it follows that

$$\lim_{n \rightarrow \infty} \frac{1}{l_n^d} \text{Var} \left(\sum_{i \in \llbracket 1, l_n \rrbracket^d} \tilde{Y}_{n,i} \right) = \sigma^2$$

where σ^2 is defined by (16). To prove (18), we write $\xi_n = \sum_{i \in \llbracket 1, l_n \rrbracket^d} \tilde{Y}_{n,i}$, therefore we get

$$\begin{aligned} \mathbb{E} \left[\xi_n^2 \mathbb{1} \{ |\xi_n| > \epsilon n^{d/2} \} \right] &\leq \|\xi_n\|_p^2 \left(\mathbb{P}(|\xi_n| > \epsilon n^{d/2}) \right)^{(p-2)/p} \\ &\leq \|\xi_n\|_p^2 \left(\frac{\|\xi_n\|_2^2}{n^d \epsilon^2} \right)^{(p-2)/p}. \end{aligned}$$

so that assumption (15) implies that

$$\|\xi_n\|_2 \leq C l_n^{d/2} \quad \text{for all } n \in \mathbb{N}.$$

for $p > 2$, observe that, since K is bounded,

$$\|\xi_n\|_p \leq C \left(\frac{1}{b_n^{(p-2)/2}} \|\xi_n\|_2^2 \right)^{1/p} \leq C l_n^d b_n^{-(p-2)/p}.$$

By the above inequalities, we have obtained

$$\frac{1}{l_n^d} \mathbb{E} [\xi_n^2 \mathbb{1}\{|\xi_n| > \epsilon n^{d/2}\}] \leq C \left(\frac{l_n^d}{b_n^d n^d} \right)^{(p-2)/p}$$

now the Condition \mathcal{W} entails (18). ■

Proof of Proposition 2

Proof. Let $p > 1$ be fixed. For all $i \in \mathbb{Z}^d$ and for all $t \in T$, we denote $R_i = Y_i(t) - \tilde{Y}_i(t)$. Since there exists a measurable function \tilde{F} such that $R_i = \tilde{F}(\varepsilon_{i-s}; s \in \mathbb{Z}^d)$. We recall that $\delta_{i,p}^{(n)} = \|R_i - R_i^*\|_p$ where $R_i^* = \tilde{F}(\varepsilon_{i-s}^*; s \in \mathbb{Z}^d)$, with

$$\varepsilon_j^* = \begin{cases} \varepsilon_j & \text{if } j \neq 0 \\ \varepsilon_{t_0} & \text{if } j = 0. \end{cases}$$

applying the Burkholder inequality (c.f Hall Heyde [9] , page 23), we get

$$\left\| \sum_{i \in \tilde{I}_n} a_i R_i \right\|_p \leq \left(p \sum_{i \in \tilde{I}_n} a_i^2 \right)^{1/2} \sum_{i \in \mathbb{Z}^d} \delta_{i,p}^{(n)}.$$

Since $(Y_i(t) - \tilde{Y}_i(t))^* = Y_i^*(t) - \tilde{Y}_i^*(t)$, we derive $\delta_{i,p}^{(n)} \leq 2 \|Y_i(t) - Y_i^*(t)\|_p$. Since K is Lipschitz, we get

$$\delta_{i,p}^{(n)} \leq \frac{2c\delta_{i,p}}{b_n^d} \tag{19}$$

where $\delta_{i,p} = \|X_i - X_i^*\|_p$. Moreover, we have also

$$\delta_{i,p}^{(n)} \leq 2 \|Y_0(t) - \tilde{Y}_0(t)\|_p. \tag{20}$$

Combinons (20) and Lemma 3.3 of El Machkouri [15], we derive

$$\delta_{i,p}^{(n)} \leq \frac{\sqrt{8p}}{b_n^d} \sum_{|j| > m_n} \delta_{i,p}. \tag{21}$$

Finally, for every m_n positive we can write

$$\begin{aligned} \sum_{i \in \mathbb{Z}^d} \delta_{i,p}^{(n)} &\leq \frac{m_n^d \sqrt{8p}}{b_n^d} \sum_{|j| > m_n} \delta_{i,p} + \frac{2}{b_n^d} \sum_{|j| > m_n} \delta_{i,p} \\ &\leq \frac{2m_n^d \sqrt{8p}}{b_n^d} \sum_{|j| > m_n} \delta_{i,p}. \end{aligned} \tag{22}$$

We combine (19), (21), (22) and under conditions of Lemma 1, we get

$$\frac{1}{n^{d/2}} \left\| \sum_{i \in \tilde{I}_n} (\bar{Y}_{n,i} - \tilde{Y}_{n,i}) \right\|_2 \leq c \frac{|\lambda_1| + |\lambda_2|}{(m_n b_n)^{3d/2}} \sum_{|i| > m_n} |i|^{\frac{5d}{2}} \delta_{i,2} = o(1).$$

The proof of Proposition 2 is complete. ■

We have the desired result. ■

4.3 Proof of Corollary 1

Proof. Such as $\tilde{H}(t_i)$ has bounded and continuous partial derivatives of order m in some neighbourhood of the points t_1, \dots, t_s , therefore, we get by Morsli [19]

$$\mathbb{E} \hat{H}_n(t_i) = \beta^{*2} G(t_i) \bar{F}(o, t_i) + \mathcal{O}(b_n^m) \quad \text{as } n \rightarrow \infty.$$

Under the condition $b_n^{d+2m} |\Lambda_{n,R}| \rightarrow 0$, we get

$$\sqrt{b_n^d |\Lambda_{n,R}|} (\mathbb{E} \hat{H}_n(t_i) - \tilde{H}(t_i)) = \mathcal{O} \left(\sqrt{b_n^{d+2m} |\Lambda_{n,R}|} \right). \quad (23)$$

and by with the application of Theorem 4 we conclude the result announced. ■

4.4 Proof of Theorem 3

Proof. Let $t \in \{t_1, \dots, t_s\}$ be fixed. We may split the difference $\hat{G}_n(t) - \beta^* G(t)$ as follows:

$$\begin{aligned} \hat{G}_n(t) - \beta^* G(t) &= \frac{L(t) \hat{H}_n(t) - \tilde{H}(t) \hat{F}_n(t)}{L(t) \hat{F}_n(t)} \\ &= \frac{\hat{H}_n(t) - \mathbb{E} \hat{H}_n(t)}{\hat{F}_n(t)} + \frac{\mathbb{E} \hat{H}_n(t) - \beta^* G(t) \hat{F}_n(t)}{\hat{F}_n(t)} \\ &= A_n^{(1)} + A_n^{(2)} \end{aligned}$$

The estimator $\hat{F}_n(t)$ turn out to be unbiased estimator of $\beta^* \bar{F}(o, t)$ and strongly consistent (the uniform strong consistency) (see Morsli [19]) as n tends infinity, since a classical ergodic theorem for spatial point processes obtained in [20] and we apply Theorem 4, they imply that as $n \rightarrow \infty$, $\sqrt{b_n^d |\Lambda_{n,R}|} A_n^{(1)}$ tends to a Gaussian distribution with zero mean. Applying (23) under condition $b_n^{d+2m} |\Lambda_{n,R}| \rightarrow 0$ as $n \rightarrow \infty$ and from strong consistency of the estimator $\hat{F}_n(t)$, they imply that as $n \rightarrow \infty$,

$$\sqrt{b_n^d |\Lambda_{n,R}|} A_n^{(2)} \xrightarrow{\mathbb{P}} 0.$$

which completes the proof. ■

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