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# On Spectrum Assignment in Elastic Optical Tree-Networks

Jean-Claude Bermond and Fatima Zahra Moataz\*

Univ. Nice Sophia Antipolis, CNRS, I3S, UMR 7271, 06900 Sophia Antipolis, France

Inria, France

## Abstract

To face the explosion of the Internet traffic, a new generation of optical networks is being developed; the Elastic Optical Networks (EONs). The aim with EONs is to use the optical spectrum efficiently and flexibly. The efficiency and flexibility are, however, accompanied by more difficulty in the resource allocation problems. In this report, we study the problem of Spectrum Allocation in Elastic Optical Tree-Networks. In trees, even though the routing is fixed, the spectrum allocation is NP-hard. We survey the complexity and approximability results that have been established for the SA in trees and prove new results for stars and binary trees.

## 1 Introduction

Elastic Optical Networks (EONs) [10] have been proposed recently as a potential candidate to replace the traditional Wavelength Division Multiplexing (WDM) networks. In EONs, new technologies such as optical OFDM, adaptive modulation techniques, bandwidth variable transponders, and flexible spectrum selective switches are used to ensure an efficient utilization of the optical resources and to enable a flexible grid as opposed to the WDM fixed-grid. In fact, the optical spectrum in EONs is subdivided into small channels, called slots, which are finer than the 50GHz wavelengths used under WDM. With these slots, small bitrates are not over-provisioned and big bitrates can be satisfied as single entities, under the constraint of contiguity. This constraint dictates that the slots used by a request should be consecutive. This results in an efficient use of the spectrum but it also makes the problems of resource allocation in EONs more difficult than their counterparts in WDM.

The key resource allocation problem in Elastic Optical Networks is referred to as Routing and Spectrum Assignment (RSA). For static RSA, the input is a set of traffic requests and the objective is to allocate to each request, a path in the optical network and an interval of spectrum slots along that path, minimizing the utilized spectrum. The spectrum allocated to a demand has to be contiguous (contiguity constraint), it has to be the same over all links of the routing path (continuity constraint) and demands sharing a link should be assigned disjoint spectrum intervals (non-overlapping constraint). If the routing is fixed, i.e., a path is predefined for each request, RSA reduces to the problem of **Spectrum Assignment (SA)**.

**Related work** Spectrum Assignment is a generalization of the well studied problem of Wavelength Assignment (WA). Since WA has been proved NP-complete in [4], SA is also NP-complete. In fact, SA remains NP-hard even in networks where WA is tractable, particularly in path networks. Indeed, SA has been proved to be equivalent to other problems studied in the literature which we describe in details in Section 2. Using

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the results obtained for the equivalent problems, SA is NP-complete in paths even if the requests' demands do not exceed 2 slots [2]. Furthermore, SA is NP-complete in paths with 4 links and unidirectional rings with 3 links [24]. On the positive side, SA can be approximated within a factor of  $2 + \epsilon$  in paths, a factor of  $4 + \epsilon$  in rings, and a factor of  $O(\log(k))$  in binary trees where  $k$  is the number of requests [23].

**Contribution** In this report, we study the spectrum assignment problem in trees. We focus on special cases where the tree is a star or where the demands of the requests are bounded by a constant and the tree is binary. By studying these special cases, we hope to gain more insight into the general problem in trees and design a constant-factor approximation algorithm or prove that such algorithm does not exist. We prove that SA is NP-hard in undirected stars of 3 links and in directed stars of 4 links, and show that it can be approximated within a factor of 4. Afterwards, we use the equivalence of SA with a graph coloring problem (interval coloring) and techniques used for the DSA problem to find constant-factor approximation algorithms for SA on binary trees when the demands are bounded.

This report is organized as follows. In Section 2, we formally define the problem of Spectrum Assignment (SA) and survey its relation to other problems and its complexity in path networks in particular. Afterwards, we present our results in stars and binary trees in Sections 3 and 4, respectively.

## 2 Problem statement and related problems

In this section, we first define the problem of Spectrum Allocation (SA) and then present some related problems and highlight their equivalence to SA. In the last subsection, we list implications of these equivalences on the complexity of SA in paths.

### 2.1 Spectrum Assignment

An instance  $(\mathcal{N}, \mathcal{R})$  of the problem consists in a graph  $\mathcal{N} = (N, L)$  and a set of requests  $\mathcal{R}$ . The graph  $\mathcal{N}$  models an optical network with  $N$  as set of nodes and  $L$  as set of links. A request  $r \in \mathcal{R}$  consists of a path  $P(r)$  in  $\mathcal{N}$  and a spectrum demand  $d(r) \in \mathbb{N}$  (number of slots). We say that two requests  $r, r' \in \mathcal{R}$  are *conflicting* if their paths  $P(r)$  and  $P(r')$  share a link. A spectrum assignment of  $(\mathcal{N}, \mathcal{R})$  is a mapping  $f$  from  $\mathcal{R}$  to  $\mathbb{N}^*$  such that for every pair of conflicting requests  $r, r' \in \mathcal{R}$ , we have  $\{f(r), \dots, f(r) + d(r) - 1\} \cap \{f(r'), \dots, f(r') + d(r') - 1\} = \emptyset$ . We say that all the slots in  $\{f(r), \dots, f(r) + d(r) - 1\}$  are *occupied* by  $r$ . In fact, the set of slots  $\{f(r), \dots, f(r) + d(r) - 1\}$  corresponds to the spectrum interval  $]f(r) - 1, f(r) + d(r) - 1]$  and we sometimes use them interchangeably.

The *span* of a spectrum assignment  $f$ , denoted  $s(f)$ , is the smallest integer  $s$  such that for each request  $r \in \mathcal{R}$ ,  $f(r) + d(r) - 1 \leq s$ . The span of an instance  $(\mathcal{N}, \mathcal{R})$ , denoted by  $s(\mathcal{N}, \mathcal{R})$  is the minimum of the spans over all possible spectrum assignments. We formulate the spectrum assignment problem as follows:

**Problem 1** (Spectrum Assignment (SA)). *Given an instance  $(\mathcal{N}, \mathcal{R})$ , compute  $s(\mathcal{N}, \mathcal{R})$ .*

For an instance  $(\mathcal{N}, \mathcal{R})$  of SA, the *load of a link*  $\ell$ , denoted by  $\pi(\ell)$ , is the sum of the demands of the requests using  $\ell$  and the *load of an instance*, denoted by  $\Pi(\mathcal{N}, \mathcal{R})$ , is the maximum load over all its links. It is straightforward that  $\Pi(\mathcal{N}, \mathcal{R}) \leq s(\mathcal{N}, \mathcal{R})$ . In the approximations we obtain for SA in this chapter, the span is usually upper bounded by a function of the maximum load.

The *greedy algorithm* for SA is an algorithm which assigns spectrum to requests ordered in a given order  $r_1, \dots, r_n$  such that a request  $r_i$  is assigned the smallest positive integer  $g(r_i)$  such that  $\{g(r_i), \dots, g(r_i) + d(i) - 1\} \cap \{g(r_j), \dots, g(r_j) + d_j - 1\} = \emptyset$  for each  $r_j$  in  $\{r_1, \dots, r_{i-1}\}$  conflicting with  $r_i$ . We will use this algorithm many times in the rest of this chapter.

## 2.2 Related problems

### 2.2.1 Scheduling Tasks on Multiprocessor Systems

It has been proved in [24] that SA in a network of  $k$  links can be reduced to the problem of Scheduling Tasks on Multiprocessor Systems (STMS) with  $k$  multiprocessors. In the STMS we are given a set of  $n$  tasks and a set of identical processors, a processing time  $d_j$  and a prespecified set  $P_j$  of processors for each task  $j$ ,  $j \in \{1, \dots, n\}$ . The objective is to schedule the tasks so as to minimize the makespan  $C_{max} = \max_j C_j$ , where  $C_j$  denotes the completion time of task  $j$ , under the following constraints: (1) preemptions (interruptions of a task) are not allowed, (2) each task must be processed simultaneously by all processors in  $P_j$ , and (3) each processor can work on at most one task at a time.

Given an instance  $(\mathcal{N}, \mathcal{R})$  of SA, an instance of STMS is constructed as follows. For each link  $\ell$  of  $\mathcal{N}$ , we associate a processor  $w_\ell$ , and for each request  $r$  in  $\mathcal{R}$  with path  $P(r)$  and demand  $d(r)$ , we associate a task  $t_r$  with processing time  $d(r)$  and a set of processors  $\{w_\ell \mid \ell \in P(r)\}$ .

**Complexity of STMS.** It has been shown in [15] that the problem of STMS is strongly NP-complete even if the number of used processors is at most 3. On the positive side, it has been proved in [11] and [16] that STMS can be approximated within  $\frac{7}{6}$  and 1.5 when the number of processors is 3 and 4, respectively. Theorem 1 follows from these approximations.

**Theorem 1.** *There are approximation algorithms with ratios  $\frac{7}{6}$  and 1.5 for SA in networks with 3 and 4 links, respectively.*

### 2.2.2 Dynamic Storage Allocation

When the network is a path, the SA problem is equivalent to the problem of Dynamic Storage Allocation (DSA). In the DSA problem, we are given a set  $A$  of items to be stored, each  $a \in A$  having size  $d(a)$ , an arrival time  $\alpha(a)$ , and a departure time  $\beta(a)$  (with  $\beta(a) > \alpha(a)$ ). A storage allocation for  $A$  is a function  $f : A \rightarrow \mathbb{N}^*$  which associates to each item  $a \in A$  a storage interval  $I(a) = \{f(a), \dots, f(a) + d(a) - 1\}$  such that for all  $a, a' \in A$  with  $a \neq a'$ , if  $I(a) \cap I(a')$  is non empty then  $]\alpha(a), \beta(a)] \cap ]\alpha(a'), \beta(a')]$  is empty. The storage size used by a storage allocation  $f$  denoted by  $s(f)$  is the smallest integer  $s$  such that for each item  $a \in A$ ,  $f(a) + d(a) \leq s$ . The objective in DSA is to find a storage allocation which minimizes the used storage size.

If we consider the time interval as a path network and each of the items to be stored as requests we can see the equivalence between the problem of SA in paths and the DSA problem. In more details, given an instance of SA on a path  $(v_1, \dots, v_k)$ , we associate to each request  $r$  with demand  $d(r)$ , an item  $a_r$  of size  $d(r)$ . We also associate to each vertex  $v_i$  of the path network time  $i$ . Let  $v_i$  and  $v_j$  be the endvertices of the path  $P(r)$  of the request  $r$  ( $i < j$ ), then we choose for the associated item  $a_r$  the arrival time  $\alpha(a_r) = i$  and the departure time  $\beta(a_r) = j$ . The fact that two requests  $r$  and  $r'$  are conflicting corresponds to the fact that the time intervals  $]\alpha(a_r), \beta(a_r)]$  and  $]\alpha(a_{r'}), \beta(a_{r'})]$  intersect. Then a spectrum assignment with value  $\gamma$  corresponds to a storage allocation using a storage size  $\gamma$ .

**Complexity of DSA.** The problem of DSA has been extensively studied. It has been proved that DSA is strongly NP-complete, even when restricted to instances where the storage size of all items is in  $\{1, 2\}$ . The proof of NP-completeness is by reduction from the 3-PARTITION problem and can be found in the appendix of [2]. On the positive side, many approximation algorithms have been proposed to solve DSA. The first proposed algorithms are based on a greedy algorithm called First Fit (FF) and its performance for online coloring of interval graphs. The relation between online coloring of interval graphs and dynamic storage allocation can be found in [5]. Using FF a linear approximation was proved in [17] and a ratio of 6 is given in [18]. Gergov has adopted another approach not using FF, yielding an approximation of 5 and 3 sequentially in [8] and [9]. In his approach, Gergov defines and uses a 2-allocation which is a storage allocation where two items but not three are allowed to overlap. A better approximation has been achieved in [3] where the

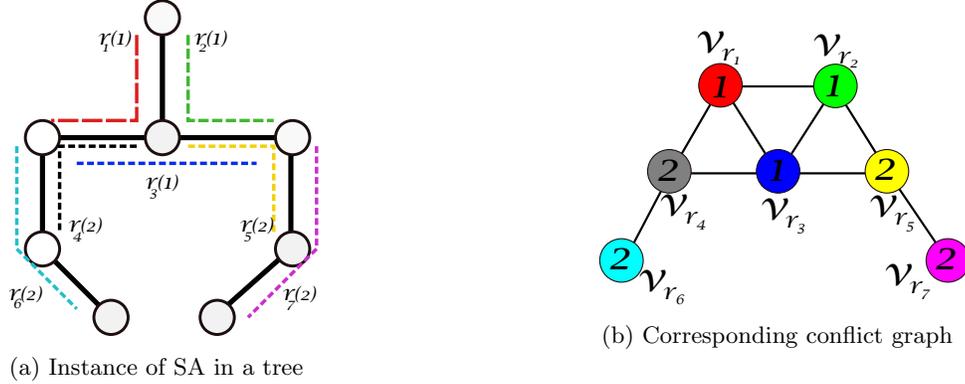


Figure 1: Example of the construction of the conflict graph

authors use the idea of boxing items to design a  $2 + \epsilon$ -approximation algorithm. Better approximations were achieved for DSA with restricted item size. In [19], the authors present a  $\frac{4}{3}$ -approximation algorithm when the maximum size is 2, and a 1, 7-approximation algorithm when the maximum size is 3. In [20], it is proved that for instances with sizes of 1 and  $X$ , an approximation of ratio  $2 - \frac{1}{X}$  can be guaranteed. All these results established for DSA apply, by equivalence, to SA in paths.

### 2.2.3 Interval Coloring

As pointed out in [23], the problem of SA is also equivalent to a graph coloring problem called Interval Coloring (IC). An interval coloring or a contiguous coloring [12] of a vertex-weighted graph  $(G = (V, E), w)$  is a mapping  $f : V \rightarrow \mathbb{N}^*$  such that for every  $v, v' \in V$ , if  $(v, v') \in E$  then  $\{f(v), \dots, f(v) + w(v) - 1\} \cap \{f(v'), \dots, f(v') + w(v') - 1\} = \emptyset$ . The number of colors used by an interval coloring  $f$ , denoted by  $\chi_f(G, w)$  is the smallest integer  $s$  such that for each vertex  $v \in V$ ,  $f(v) + w(v) - 1 \leq s$ . The interval chromatic number of a weighted graph  $(G, w)$ , denoted by  $\chi(G, w)$ , is the least number of colors needed to color the vertices with intervals, i.e. it is the minimum  $\chi_f(G, w)$  among all possible interval colorings  $f$  of  $(G, w)$ . The interval coloring problem is defined as follows.

**Problem 2** (Interval Coloring (IC)). *Given a vertex-weighted graph  $(G, w)$ , compute  $\chi(G, w)$ .*

To see the equivalence between SA and IC we do the following. For an instance  $(\mathcal{N}, \mathcal{R})$  of SA, we create a weighted graph  $(G = (V, E), w)$  modeling the dependency between the different requests and call it *the conflict graph*. We associate to every request  $r \in \mathcal{R}$  a vertex  $v_r$  in  $V$ . We add an edge between two vertices  $v_r$  and  $v_{r'}$  if the corresponding requests  $r$  and  $r'$  are conflicting. The weight  $w(v_r)$  of each vertex  $v_r$  is equal to the bandwidth demand of the corresponding request  $r$  (i.e.  $w(v_r) = d(r)$ ). Figure 1 illustrates an instance of SA on a tree ( $r_i(j)$  is request  $r_i$  with demand  $j$ ) and the corresponding conflict graph (weights of the vertices are inside the vertices).

If  $(\mathcal{N}, \mathcal{R})$  is an instance of SA and  $(G, w)$  is its conflict graph, then finding a spectrum assignment of  $(\mathcal{N}, \mathcal{R})$  is equivalent to finding an interval coloring of  $(G, w)$  and  $s(\mathcal{N}, \mathcal{R}) = \chi(G, w)$ .

**Complexity of IC.** The problem of IC has been introduced in [12] where its relation to other problem such as DSA has been highlighted. It has also been proven in [12] that IC is equivalent to the problem of finding, for a vertex-weighted graph, an acyclic orientation which minimizes the weight of the longest path, where the length of a path is the sum of the weights of its vertices. The complexity of DSA implies that IC is strongly NP-complete in interval graphs. IC is also strongly NP-complete in proper interval graphs [22]. On the positive side, IC is polynomial in comparability graphs [12] and can be approximated within a factor of  $2 + \epsilon$  in interval graphs [3], a factor of 2 for proper interval graphs [22] and claw-free chordal graphs [6], and a factor of  $O(\log(n))$  in chordal graphs where  $n$  is the number of vertices.

Since the conflict graph associated to an instance of SA in a path is an interval graph, all the results established for IC in interval graphs apply to SA in paths. In section 4, we will use the fact that the conflict graph associated to a binary tree is a chordal graph to obtain results for SA in binary trees using interval coloring of chordal graphs.

### 2.3 Spectrum Assignment in Paths

The results deduced from the equivalence to the problems defined above can be summarized as follows for path networks.

- With respect to the number of links, SA is NP-complete in path networks with 4 links and polynomial in paths with at most 3 links [24], and it can be approximated within a factor of  $\frac{7}{6}$  (resp.  $\frac{3}{2}$  in paths with 4 (resp. 5) links [24].
- With respect to the demands, SA is strongly NP-complete even if the requests have demands in the set  $\{1, 2\}$  [2]. It can be approximated within a factor of  $\frac{4}{3}$  and a factor of 1,7 when the maximum demand is 2 and 3, respectively [19]. It also can be approximated within a factor of  $2 - \frac{1}{X}$  when the demands are in the set  $\{1, X\}$  [20].
- In general, SA in paths can be approximated in paths within a factor of  $2 + \epsilon$  [3] and it can be approximated within a factor of 2 when the paths of the requests are such that no path is strictly included in another [22].

## 3 Spectrum assignment in stars: hardness and approximability results

A star is a tree-network with at most one node of degree at least 2. The problem of wavelength assignment (WA) is NP-complete in undirected stars but polynomial in directed stars [1]. The polynomiality of WA in directed stars was useful because optical networks are symmetrically directed and because it helped in the design of constant-factor approximation algorithms for WA in directed trees [1]. Such algorithm cannot be extended to SA since we prove in this section that SA is not only NP-complete in undirected stars but also in directed stars with 4 links. On the positive side, we prove the existence of a 4-approximation algorithm for the general case.

**Theorem 2.** *The problem of Spectrum Assignment is strongly NP-complete in undirected stars with 3 links.*

*Proof.* It was shown in [24] that the SA problem is strongly NP-complete in a 3-link unidirectional ring. Let us consider an instance of SA in a 3-link ring  $C = (l_1, l_2, l_3)$  with a request set  $\mathcal{R}$ . Let us build a star  $S$  with three links  $l'_1, l'_2$  and  $l'_3$ , and a set of requests  $\mathcal{R}'$  defined as follows. For each request  $r \in \mathcal{R}$  using at most 2 links, we create a request  $r'$  in  $\mathcal{R}'$  such that if the path of  $r$  in  $C$  is  $P(r) = l_i, i \in \{1, 2, 3\}$ , then the path of  $r'$  in  $S$  is  $P(r') = l'_i$ , and if  $P(r) = l_i l_j$ , then  $P(r') = l'_i l'_j$ . Solving SA in  $(C, \mathcal{R})$  is equivalent to solving SA in  $(S, \mathcal{R}')$ .  $\square$

**Theorem 3.** *The problem of Spectrum Assignment is weakly NP-complete in directed stars with 3 ingoing links and one outgoing link or 3 outgoing links and one ingoing link.*

*Proof.* The proof is by reduction from the 2-PARTITION problem. In the 2-PARTITION problem, we are given a set  $A$  of  $k$  integers  $a_1, a_2, \dots, a_k$  such that  $B = \sum_{j=1}^k a_j$  and the objective is to decide whether  $A$  can be partitioned into two disjoint sets  $A_1$  and  $A_2$  such that  $\sum_{a_j \in A_1} a_j = \sum_{a_j \in A_2} a_j$ .

Given an instance of the 2-PARTITION problem with a set of  $k$  integers  $A = \{a_1, a_2, \dots, a_k\}$  such that  $B = \sum_{j=1}^k a_j$ , we create an instance of spectrum assignment in a 4-link directed star network  $S$  (Figure 2a) and a set of requests  $\mathcal{R}$ . The star  $S$  has 3 ingoing links  $l_1, l_2$ , and  $l_3$  and one outgoing link  $l_4$ . The set of requests  $\mathcal{R}$  consists of the requests presented in Figure 2b: requests  $r_a, r_b, r_c, r_1$ , and  $r_2$  and for every

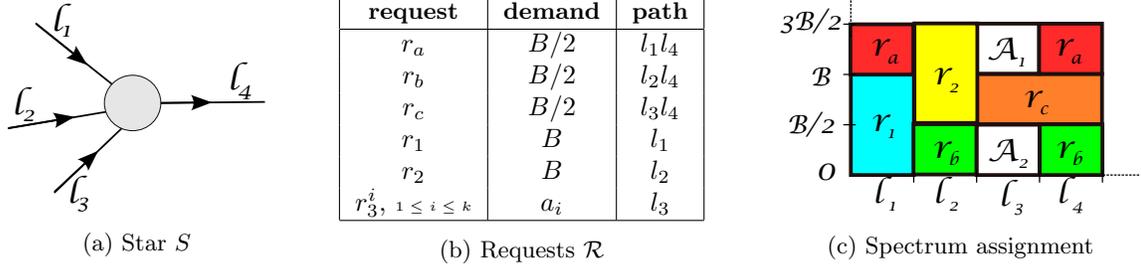


Figure 2: Reduction from 2-PARTITION to SA in a directed star

integer  $a_i$  in the set  $A$ , a request  $r_3^i$  with demand  $a_i$  and using link  $l_3$ . We prove that finding a spectrum assignment for  $(S, \mathcal{R})$  with span  $\frac{3}{2}B$  is equivalent to finding a partition of  $A$  into two sets  $A_1$  and  $A_2$  such that  $\sum_{a_j \in A_1} a_j = \sum_{a_j \in A_2} a_j = \frac{B}{2}$ . In fact, if there is a partition of  $A$  into  $A_1$  and  $A_2$  such that  $\sum_{a_j \in A_1} a_j = \sum_{a_j \in A_2} a_j = \frac{B}{2}$ , then we can assign spectrum as shown in Figure 2c. Now let us suppose there is a spectrum assignment for  $(S, \mathcal{R})$  with span  $\frac{3}{2}B$ . There are two possible symmetric assignments to the requests on links  $l_1$  and  $l_2$ . We suppose we assign to  $r_1, r_a, r_2$  and  $r_b$  spectrum intervals  $[1, B]$ ,  $[B + 1, \frac{3}{2}B]$ ,  $[\frac{B}{2} + 1, \frac{3}{2}B]$ , and  $[1, \frac{B}{2}]$ , respectively (the analysis is similar for the other assignment). This assignment forces request  $r_c$  to use the interval  $[\frac{B}{2} + 1, B]$  and the other requests on link  $l_3$  will have to be partitioned into two sets of the same  $\frac{B}{2}$ .  $\square$

**Theorem 4.** *The problem of Spectrum Assignment in directed stars with at most 3 links or exactly 2 ingoing links and 2 outgoing links can be solved in polynomial time.*

*Proof.* In all of these cases, the span is equal to the maximum load and the greedy algorithm with specific orders can achieve the optimal span.

- When the star has only ingoing or outgoing links, the problem is trivial since any conflicting requests use the same link and the greedy algorithm with any order can achieve the optimal span.
- For the case where the star is a directed path of length 2, an optimal spectrum assignment consists in using the greedy algorithm with an order where the requests using two links come first. This way, the spectrum span will be defined by the link with the maximum load.
- For the case where the star has two ingoing links  $l_1$  and  $l_2$  and one outgoing link  $l_3$  (or the opposite), an optimal spectrum assignment consists in using the greedy algorithm with an order where the requests using  $l_1$  and  $l_3$  come first and the requests using  $l_2$  and  $l_3$  come last.
- When the star has 2 ingoing links  $l_1$  and  $l_2$  and 2 outgoing links  $l_3$  and  $l_4$ , an optimal assignment consists in using the greedy algorithm with the following order: requests using  $l_1$  and  $l_3$ , then requests using  $l_2$  and  $l_4$ , then requests using only one link, then requests using  $l_1$  and  $l_4$ , and finally requests using  $l_2$  and  $l_3$ .

$\square$

**Theorem 5.** *Let  $(\mathcal{N}, \mathcal{R})$  be an instance of SA. If the length of the paths associated to the requests in  $\mathcal{R}$  is at most  $\alpha$ , then the greedy algorithm gives a  $2\alpha$ -approximation for the SA problem. In particular there is a  $4$ -approximation polynomial-time algorithm for the SA problem in stars.*

*Proof.* Let  $(\mathcal{N}, \mathcal{R})$  be an instance of SA. Let the requests of  $\mathcal{R}$  be ordered in the non-increasing order of demands  $r_1, r_2, \dots, r_q$  (i.e.,  $d(r_1) \geq d(r_2) \geq \dots \geq d(r_q)$ ). Let  $\Pi$  be the maximum load. We will use at most  $2\alpha\Pi$  slots to allocate spectrum to the requests of  $\mathcal{R}$ . Suppose that we have already assigned spectrum to

the first requests  $r_j$ ,  $j < i$  with the span  $2\alpha\Pi$  and consider the request  $r_i$  with demand  $d(r_i) = d$ . For each link  $l$  of the path  $P(r_i)$ , let  $R_i(l)$  be the set of requests conflicting with  $r_i$  on the link  $l$ . As the load of the link  $l$  is at most  $\Pi$ , the sum of the demands of the requests of  $R_i(l)$  is at most  $\Pi - d$ . Since each of these requests has demand at least  $d$ , we have at most  $\frac{\Pi-d}{d}$  requests in  $R_i(l)$ . This implies that the path  $P(r_i)$  has at most  $\frac{\alpha(\Pi-d)}{d}$  requests conflicting with  $r_i$  which have been already assigned spectrum. Consider the slots not occupied by these requests (available slots). If there exists an interval of  $d$  or more available slots below these requests or between two requests, we can assign to request  $r_i$  the first such interval.

Otherwise, between slot 1 and the first slot occupied by the conflicting requests and between the last slot occupied by a request and the first slot of the next request there are at most  $d - 1$  available slots. As there are at most  $\frac{\alpha(\Pi-d)}{d}$  requests conflicting with  $r_i$ , we have at most  $\frac{\alpha(\Pi-d)}{d}$  such intervals. As the requests in  $R_i(l)$  occupy at most  $(\Pi - d)$  slots, we have at most  $\alpha(\Pi - d)$  slots occupied by the conflicting requests and at most  $\frac{\alpha(\Pi-d)}{d}(d - 1)$  slots available where we cannot provision  $r_i$ . Altogether, we have a number of non usable slots equal to  $\alpha(\Pi - d) + \frac{\alpha(\Pi-d)}{d}(d - 1) = 2\alpha\Pi - 2\alpha d - \frac{\alpha(\Pi-d)}{d}$ . So, there is an interval of  $2\alpha d + \frac{\alpha(\Pi-d)}{d}$  available contiguous slots above all of the requests conflicting with  $r_i$ . We can allocate to  $r_i$   $d$  contiguous slots in this interval.  $\square$

This approximation algorithm for stars together with the  $2 + \epsilon$ -approximation algorithm for paths presented in [3], imply constant factor approximations for tree networks which are spiders or caterpillars. A *spider* is a tree with one vertex of degree at least 3 and all others with degree at most 2 and a *caterpillar* is a tree in which all the vertices are within distance 1 of a central path. The following theorem states the approximation result on SA in spiders and caterpillars.

**Theorem 6.** *There is a  $6 + \epsilon$ -approximation for SA in spiders and caterpillars.*

*Proof.* Let  $(S, \mathcal{R})$  be an instance of SA in a spider tree. Let  $v$  be the vertex of  $S$  of degree 3 and let  $S'$  be the star induced by  $v$  and all the vertices of  $S$  which are at distance 1 from  $v$ . The idea is to use the 4-approximation presented in Theorem 5, to allocate spectrum to the requests  $\mathcal{R}'$  using the star  $S'$  and then for each path  $P$  of the paths of  $S \setminus S'$ , use the  $2 + \epsilon$ -approximation algorithm to allocate spectrum to the requests using  $P$  and which are not in  $\mathcal{R}'$ . This would yield the  $6 + \epsilon$ -approximation algorithm.

Now, let  $(C, \mathcal{R})$  be an instance of SA in a caterpillar tree. Let  $P$  be the central path of  $C$ . The idea is to use the  $2 + \epsilon$ -approximation for paths to allocate spectrum to the requests  $\mathcal{R}'$  using the path  $P$  and then for each star  $S$  of the stars of  $C \setminus P$ , use the 4-approximation algorithm to allocate spectrum to the requests using  $S$  and which are not in  $\mathcal{R}'$ . This would yield the  $6 + \epsilon$ -approximation algorithm.  $\square$

## 4 Spectrum Assignment in Binary Trees

The problem of Spectrum Assignment in binary trees has been studied in [23]. It has been proved that SA can be approximated within a ratio of  $O(\log(k))$  where  $k$  is the number of requests. The proof is based on the equivalence between SA and the problem of Interval Coloring (IC). In fact, the conflict graph of an instance of SA in a binary tree corresponds to an edge intersection graph of paths in a binary tree. These graphs have been proved to be chordal graphs in [14, 13]. Using the problem of interval coloring in chordal graphs, we give in this section some constant-factor approximation results for the problem of spectrum assignment in binary trees with special demand profiles. Namely, we examine the cases where the demands are in a set  $\{k, kX\}$  ( $k, X \in \mathbb{N}^*$ ), in a set  $\{kX, k(X + 1)\}$  ( $k, X \in \mathbb{N}^*$ ), or bounded by  $D$ . For the latter case, we give a general approximation when the demands are bounded by  $D \in \mathbb{N}$  and then give better approximations for the cases where the demands are bounded by  $D \in \{3, 4, 5, 6\}$ . It is important to recall here that even if the network is a path and the demands are bounded by 2, SA is strongly NP-complete. We first start by giving some definitions and then we state our results.

## 4.1 Definitions

A chord of a cycle  $C$  in a graph is an edge of the graph connecting two vertices that are not adjacent in  $C$ . A graph  $G$  is chordal if every cycle of  $G$  with at least 4 vertices has a chord. One important property of chordal graphs is their perfect elimination order. The *perfect elimination order* (PEO) of a graph is an ordering  $x_1, x_2, \dots, x_n$  of the vertices of the graph such that for  $i = 1, \dots, n - 1$ , the neighbors of  $x_i$  in  $G[\{x_{i+1}, \dots, x_n\}]$ <sup>1</sup> form a clique. It is well known that a graph is chordal if and only if it has a perfect elimination order. Paper [25] describes a linear time algorithm called maximum cardinality search that can be used to determine if a given graph has a perfect elimination order and construct such an ordering if it exists. Throughout the remainder of this chapter, we use *the reverse perfect elimination order* (RPEO) in the design of some algorithms. Note that if  $v_1, v_2, \dots, v_n$  is a RPEO of the vertices of a chordal graph, then for  $i = 2, \dots, n$ , the neighbors of  $v_i$  in  $G[\{v_1, \dots, v_{i-1}\}]$  form a clique. Another tool we will be using is the *greedy algorithm* for IC. Defined similarly to the greedy algorithm for SA, the greedy algorithm for IC, also called the *First Fit algorithm* (FF) is an algorithm which assigns colors to vertices in a given order  $v_1, \dots, v_n$  such that a vertex  $v_i$  is assigned the smallest positive integer  $g(v_i)$  such that  $\{g(v_i), g(v_i) + w(v_i) - 1\} \cap \{g(v_j), g(v_j) + w(v_j) - 1\} = \emptyset$  for each  $v_j$  in  $\{v_1, \dots, v_{i-1}\}$  which is adjacent to  $v_i$ .

In a weighted graph  $(G = (V, E), w)$ , we define the weight of a subset  $S \subseteq V$  to be the quantity  $w(S) = \sum_{v \in S} w(v)$ . The maximum weighted clique is a clique with the biggest weight. The density of  $(G = (V, E), w)$  is the weight of the maximum weighted clique and is denoted by  $\Delta(G, w)$ . It is straightforward that  $\Delta(G, w) \leq \chi(G, w)$ , where  $\chi(G, w)$  is the interval chromatic number of  $(G, w)$ .

In the remainder of this section, we present our results for SA in binary trees with bounded demands as corollaries after proving theorems for IC in weighted chordal graphs with bounded weights. We note here that even though every conflict graph of an instance of SA in a binary tree is a chordal graph, the opposite is not true [13]. In what follows, we provide approximation algorithms for IC in weighted chordal graphs with bounded weights. These algorithms give an upper bound on the interval chromatic number in terms of the density. For some cases, we also provide examples giving lower bounds on the interval chromatic number in terms of the density. The results we prove for IC in weighted chordal graphs with bounded weights with respect to upper and lower bounds on  $\chi(G, w)$  in terms of  $\Delta(G, w)$  are summarized in Table 1<sup>2</sup>.

Weights	Upper bounds	Lower bounds
$\{k, kX\}$ $k, X \in \mathbb{N}^*$	$(2 - \frac{1}{X})\Delta(G, w) + k$	$(2 - \frac{1}{X})\Delta(G, w) - \epsilon$
$\{kX, k(X+1)\}$ $k, X \in \mathbb{N}^*$	$(1 + \frac{1}{X})\Delta(G, w)$	
$\leq 3$	$\frac{19}{10}\Delta(G, w) + \frac{8}{5}$	$\frac{3}{2}\Delta(G, w)$
$\leq 4$	$\frac{59}{27}\Delta(G, w) + \frac{67}{27}$	$\frac{3}{2}\Delta(G, w)$
$\leq 5$	$\frac{859}{336}\Delta(G, w) + \frac{229}{56}$	$\frac{5}{3}\Delta(G, w)$
$\leq 6$	$\frac{287}{100}\Delta(G, w) + \frac{885}{200}$	$\frac{5}{3}\Delta(G, w)$
$\leq W$	$2 \lceil \log_2(W) \rceil \Delta(G, w)$	

Table 1: Summary of the bounds obtained on  $\chi(G, w)$  for bounded weights

<sup>1</sup>For  $S \subseteq V$ , we define  $G[S]$  as the subgraph of  $G$  induced by the vertices of  $S$ , i.e. the subgraph of  $G$  containing the vertices of  $S$  and all the edges of  $G$  which have both endpoints in  $S$ .

<sup>2</sup>In Table 1, the presented upper bounds  $\beta$  are such that for each weighted chordal graph  $(G, w)$ ,  $\chi(G, w) \leq \beta$ . The lower bounds  $\gamma$ , on the other hand, are such that there exists a weighted chordal graph  $(G, w)$  such that  $\chi(G, w) \geq \gamma$ .

## 4.2 Demands $k$ and $kX$

In this section, we present an approximation algorithm for the SA problem when the demand of each request is either  $k$  or  $kX$ ,  $k, X \in \mathbb{N}^*$ . We start by proving the following theorem for interval coloring in chordal graphs.

**Theorem 7.** *Let  $(G, w)$  be a weighted chordal graph with weights in the set  $\{1, X\}$ . There exists a polynomial-time algorithm that finds an interval coloring of  $(G, w)$  with  $2\Delta(G, w) - \lfloor \frac{\Delta(G, w)}{X} \rfloor$  colors.*

*Proof.* It has been proved in [20] that there is an algorithm to find a  $(2 - \frac{1}{X})$ -approximation for the problem of interval coloring in interval graphs whenever we have only two weights 1 and  $X$ . We generalize this algorithm for chordal graphs as follows.

Let  $(G, w)$  be a weighted chordal graph with weights in  $\{1, X\}$  and let  $\Delta = \Delta(G, w)$  be its density. We will use  $2\Delta - \lfloor \frac{\Delta}{X} \rfloor$  colors to color  $(G, w)$  as follows. We partition the colors into two sets. The first set  $S_1$  contains colors from 1 to  $\Delta$  and the second set  $S_2$  contains colors from  $\Delta + 1$  to  $2\Delta - \lfloor \frac{\Delta}{X} \rfloor$ .

We order the vertices of  $G$  in the reverse perfect elimination order. Let  $v_1, \dots, v_n$  be the obtained ordering. Recall that the neighbors of  $v_i$  in  $\{v_1, \dots, v_{i-1}\}$  form a clique in the graph induced by  $\{v_1, \dots, v_{i-1}\}$ . We use the greedy algorithm to assign colors to the vertices in this order such that colors assigned to a vertex are either included in  $S_1$  or  $S_2$  (We cannot use colors from both sets). We prove that with this algorithm, all vertices will be assigned colors in  $S_1$  or  $S_2$ .

- All vertices of weight 1 will have a color in  $S_1$ . In fact, if a vertex  $v_i$  of weight 1 cannot be assigned a color in  $S_1$ , then its neighbors in  $\{v_1, \dots, v_{i-1}\}$  occupy all colors of  $S_1$ . This implies that the density of the graph is at least  $\Delta + 1$  which is not possible.
- For vertices of weight  $X$ , suppose that there is a vertex  $v_j$  of weight  $X$  to which we cannot assign colors in  $S_1$  nor in  $S_2$ . The minimum number of colors used in  $S_1$  that can make it not possible to color  $v_j$  with colors from  $S_1$  is  $\lfloor \frac{\Delta}{X} \rfloor$  ( $X - 1$  free colors then 1 occupied color, then  $X - 1$  free colors and 1 occupied color ...). The weight of the neighbors of  $v_j$  in  $\{v_1, \dots, v_{j-1}\}$  which use colors in  $S_1$  is at least  $\lfloor \frac{\Delta}{X} \rfloor$ . Since we cannot assign colors from  $S_2$  to  $v_j$  and knowing that only vertices of the same weight  $X$  use colors from  $S_2$  with the greedy algorithm, we deduce that the sum of the weights of the neighbors of  $v_j$  in  $\{v_1, \dots, v_{j-1}\}$  which use colors in  $S_2$  is at least  $|S_2| - (X - 1)$ . So  $v_j$  and its neighbors form a clique of size  $X + \lfloor \frac{\Delta}{X} \rfloor + |S_2| - (X - 1) \geq \Delta + 1$  as  $|S_2| = \Delta - \lfloor \frac{\Delta}{X} \rfloor$ . This implies that the density of  $G$  is at least  $\Delta + 1$ , which is not possible.

We have proved that when the weights are in the set  $\{1, X\}$ , we can color the graph with a number of colors smaller than  $(2 - \frac{1}{X})\Delta(G, w) + 1$ .  $\square$

**Corollary 1.** *Let  $(G, w)$  be a weighted chordal graph with weights in the set  $\{k, kX\}$ . There exists a polynomial time algorithm that finds an interval coloring of  $(G, w)$  with  $2\Delta(G, w) - k\lfloor \frac{\Delta(G, w)}{kX} \rfloor$  colors.*

*Proof.* Note that to color a graph  $(G, w)$  with weights in  $\{k, kX\}$ , we can transform it to a graph  $(G, w')$  with weights in  $\{1, X\}$ , color  $(G, w')$  and then transform the colors we found into intervals of colors of size  $k$ . The number of colors used for  $G$  will be then at most  $k$  times the number of colors used for  $G'$ . Note also that the density of  $(G, w)$  is  $k$  times the density of  $(G, w')$ . This means that if we can color  $(G, w')$  with  $a\Delta((G, w')) + b$  colors, then we can color  $(G, w)$  with  $a\Delta((G, w)) + kb$  colors.

This implies that when the weights are in the set  $\{k, kX\}$ , we can color the graph with less than  $(2 - \frac{1}{X})\Delta(G, w) + k$  colors.  $\square$

Thanks to Corollary 1, we can deduce the following corollary.

**Corollary 2.** *Let  $\mathcal{I}$  be an instance of SA in a binary tree such that the demands of requests are in the set  $\{k, kX\}$  and the span of  $\mathcal{I}$  is OPT. There is a polynomial-time algorithm that finds a spectrum assignment for  $\mathcal{I}$  with span less than  $(2 - \frac{1}{X})OPT + k$ .*

Now, in what follows of this subsection, we find a lower bound on the number of colors that can be used to find an interval coloring of a weighted chordal graph with weights in  $\{k, kX\}$ .

**Theorem 8.** *There exists a family of weighted graphs  $(G_m)_{m \in \mathbb{N}^*}$ , with weights in the set  $\{k, kX\}$ , for which the ratio between the interval chromatic number and the density tends to  $2 - \frac{1}{X}$  when  $m$  tends to infinity.*

*Proof.* For  $m > 0$ , we build the weighted graph  $G_m$  of density  $k(X^2m + 1)$  as follows.

- $X^2m + 1$  vertices of weight  $k$  each forming a "big" clique.
- For each subset  $S$  of  $Xm + 1$  vertices of the big clique, we add  $(X - 1)m$  new vertices of weight  $kX$  each. These vertices form a clique with the vertices of  $S$ .

In any contiguous coloring of  $G_m$ , there exists an integer  $\lambda$  in  $\{0, \dots, kX - 1\}$  such that the "big" clique uses  $X\alpha + 1$  colors congruent to  $\lambda$  modulo  $kX$ . Suppose that this is not true and that the big clique uses for each integer  $i$  in  $\{0, \dots, kX - 1\}$  at most  $mX$  colors which are congruent to  $i$  modulo  $kX$ . This means that the number of colors used is at most  $kmX^2$ . This is not possible since this maximum clique has weight  $k(mX^2 + 1)$ . Let  $S$  be the subset of vertices of the big clique using colors that are congruent to  $\lambda$  modulo  $kX$ . Vertices of  $S$  form a clique with  $m(X - 1)$  vertices of weight  $kX$ . Each of these vertices uses a color congruent to  $\lambda$  modulo  $kX$ . In total,  $m(2X - 1) + 1$  colors which are congruent to  $\lambda$  are used. This means that the number of colors used is at least  $kmX(2X - 1) + 1$ . The ratio between the chromatic number and the density is then at least  $\frac{kmX(2X-1)+1}{k(mX^2+1)} = 2 - \frac{1}{X} - \frac{(2k-1)X-k}{kX(mX^2+1)}$ . When  $m$  goes to infinity, this ratio goes to  $2 - \frac{1}{X}$ . □

### 4.3 Demands $kX$ and $k(X + 1)$

In this section, we present an approximation algorithm for the SA problem when the demand of each request is either  $kX$  or  $k(X + 1)$ . We start by proving the following theorem for interval coloring in chordal graphs.

**Theorem 9.** *Let  $(G, w)$  be a weighted chordal graph with weights in  $\{kX, k(X + 1)\}$ . There is a polynomial time algorithm to color  $G$  with at most  $\frac{X+1}{X}\Delta(G, w)$  colors.*

*Proof.* Let  $(G, w)$  be a weighted chordal graph with weights in  $\{kX, k(X + 1)\}$ . Let  $m = \lfloor \frac{\Delta(G, w)}{kX} \rfloor$ , we prove that we can color  $(G, w)$  with  $k(X + 1)m$  colors. We partition the set of colors  $\{1, \dots, k(X + 1)m\}$  into  $m$  contiguous intervals  $I_i$ ,  $1 \leq i \leq m$  of size  $k(X + 1)$  each. Let us order the vertices of  $(G, w)$  in the RPEO order. We use the greedy algorithm to color the vertices in this order using for each vertex colors from exactly one interval  $I_i$ ,  $1 \leq i \leq m$ . Suppose that we cannot color some vertex  $v_j$ , this means that each interval  $I_i$ ,  $1 \leq i \leq m$ , contains a neighbor of  $v_j$  with weight at least  $kX$  (recall that the weights are either  $kX$  or  $k(X + 1)$ ). Since the neighbors of  $v_j$  which appear first in the RPEO form a clique with  $v_j$ , we have a clique of weight at least  $mkX + kX > \Delta(G, w)$  which is not possible. Therefore we can color all the vertices. □

Theorem 9 implies the following corollary.

**Corollary 3.** *There is a  $\frac{X+1}{X}$ -approximation algorithm for SA in binary trees when the demands of the requests are in the set  $\{kX, k(X + 1)\}$ .*

### 4.4 Maximum demand $D$

In this section, we present an approximation algorithm for the SA problem when the maximum demand is  $D$ . The approximation algorithm proposed for Interval Coloring in chordal graphs in [21] and which gives  $2\log_2(n)$  approximation where  $n$  is the number of vertices, allows to prove the following theorem.

**Theorem 10.** *Let  $(G, w)$  be a weighted chordal graph with maximum weight  $W$ . There is a polynomial time algorithm that finds an interval coloring of  $(G, w)$  which uses at most  $2 \lceil \log_2(W) \rceil \Delta(G, w)$  colors.*

*Proof.* The proof is due to [21] which proposes a  $O(\log_2(n))$ -approximation where  $n$  is the number of vertices. We only replace  $n$  by the maximum weight. We include it for completeness.

We prove that there is a  $2 \lceil \log_2(W) \rceil$ -approximation for interval coloring of chordal graphs where  $W$  is the maximum weight. Let  $(G, w)$  be a vertex-weighted chordal graph with maximum weight  $W$ . We set  $k = \lceil \log_2(W) \rceil$ . Let us partition the set of vertices  $V$  into  $k$  subsets  $S_i$ ,  $i \in \{1, \dots, k\}$  such that for each vertex  $v \in S_i$ ,  $w(v) \in [\frac{W}{2^i}, \frac{W}{2^{i-1}}]$ . We first ignore the weights and optimally color each graph  $G_i$  induced by the subset  $S_i$ . As the graphs are chordal, we can color the vertices of  $G_i$  with  $\omega(G_i)$  colors where  $\omega(G_i)$  is the clique number of  $G_i$ . Afterwards, we replace the color of each vertex  $v \in G_i$  by an interval of  $w(v)$  colors. This way, we obtain an interval coloring of  $G_i$  with at most  $\frac{W}{2^{i-1}}\omega(G_i)$  colors. Therefore, the vertices of  $G$  can be colored with  $c$  colors where  $c = \sum_{i=1}^k \frac{W}{2^{i-1}}\omega(G_i)$ . Note that  $\frac{W}{2^i}\omega(G_i) \leq \Delta(G, w)$  which implies that  $c \leq \sum_{i=1}^k 2\Delta(G, w) = 2k\Delta(G, w)$ . The  $2 \lceil \log_2(W) \rceil$ -approximation follows.  $\square$

Theorem 10 implies the following corollary.

**Corollary 4.** *There is a  $2 \lceil \log_2(D) \rceil$ -approximation for SA in binary trees where  $D$  is the maximum demand.*

## 4.5 Maximum demand at most 6

In the previous subsection, a  $2 \lceil \log_2(D) \rceil$ -approximation algorithm for the SA problem in binary trees where the maximum demand is at most  $D$  has been presented. This approximation is achieved by partitioning the requests into subsets of close demands. This technique is used not only in binary trees but also in general graphs as a heuristic [26]. In what follows, we use different techniques to find better approximations for SA in binary trees for some given values of the maximum demand  $D$ . The techniques we use were introduced in [19] to approximate DSA. Results in [19] can extend directly to SA in path networks giving approximation algorithms with factors  $\frac{4}{3}$  and 1.7 when the spectrum demands are bounded by 2 and 3, respectively. In what follows we use the same techniques to design constant-factor approximations for SA in binary trees when the spectrum demand is bounded by 6.

We first prove the following theorem for interval coloring.

**Theorem 11.** *Let  $(G, w)$  be a weighted chordal graph. There are polynomial-time algorithms which find an interval coloring of  $(G, w)$  with at most  $\frac{19}{10}\Delta(G, w) + \frac{8}{5}$ ,  $\frac{59}{27}\Delta(G, w) + \frac{67}{27}$ ,  $\frac{859}{336}\Delta(G, w) + \frac{229}{56}$  and  $\frac{287}{100}\Delta(G, w) + \frac{885}{200}$  colors when the maximum weight is bounded by 3, 4, 5 and 6, respectively.*

*Proof.* As in the previous sections,  $\Delta(G, w)$  refers to the density of the weighted graph  $(G, w)$  and will be abbreviated in this proof to  $\Delta$ .

Let  $\mathcal{C}(d, S)$  denote the set of instances of IC in which the graph is chordal, the density is at most  $d$  and the weights are in the set  $S$ . Let  $c(d, S)$  denote the smallest integer  $\alpha$  such that for each instance of  $\mathcal{C}(d, S)$ , there is an interval coloring with at most  $\alpha$  colors (if such  $\alpha$  exists).

We present first the general approach to solve the problem for any maximum weight  $W$  before presenting the cases  $W \in \{3, 4, 5, 6\}$  in details.

**General Approach** Let  $(G, w)$  be a weighted chordal graph with maximum weight  $W$ . To color the graph  $G$ , we proceed in two phases as follows.

- *Partitioning the vertices into multi-level blocks:* in this phase, the vertices are partitioned into blocks. We will have for each  $i \in \{1, \dots, W\}$ , a set  $\mathcal{B}_i$  of  $n_i$  level- $i$  blocks  $B_i^1, \dots, B_i^{n_i}$  each of density  $d_i$ . We order the blocks in the lexicographic order: block  $B_j^i$  is before block  $B_{j'}^{i'}$  if  $i < i'$  or  $i = i'$  and  $j < j'$ .

Our algorithm consists in considering successively the vertices in the RPEO order and assigning a new vertex  $v$  to the first available block (in the block's order). In more details, we assign a vertex  $v$  to a block  $B$  if the weight of the clique induced by  $v$  and its neighbors in  $B$  does not exceed the density of

the block. The vertex  $v$  and its neighbors in  $B$  indeed form a clique since the graph is chordal and we consider the vertices in the RPEO order.

We will choose the parameters  $d_i$  and  $n_i$  (see details after) in such a way that the following property is satisfied:

**Property \***: Each vertex of weight  $i$  is assigned to some block in the set  $\mathcal{B}_l$  such that  $l \leq i$ .

In particular, this means that at the end of the algorithm each vertex is assigned to some block.

- *Solving the problem of interval coloring for each block*: in this second phase, the vertices of each block  $B_i^j$  are colored using an algorithm to solve instances with density  $d_i$  and weights in  $S_i = \{i, \dots, W\}$  (the possible weights of the vertices in  $B_i^j$ ). Note that the vertices of a block  $B_i^j$  induce a graph which belongs to  $\mathcal{C}(d_i, S_i)$ . The algorithm we use is designed to use no more than  $c(d_i, S_i)$  colors.

Therefore, the total number of colors used to color the whole graph is at most

$$\sum_{i=1}^W n_i c(d_i, \{i, \dots, W\})$$

The total number of colors depends on  $n_i$  and  $d_i$ . In fact, we will proceed as follows. For a chosen set of values of the densities  $d_i$ , we will choose the smallest possible  $n_i$  such that Property\* is satisfied. Afterwards, we will compute  $c(d_i, \{i, \dots, W\})$  and therefore the total number of colors for the chosen values of  $d_i$ . We will do this for many values of the densities  $d_i$  and keep the set of values which minimize the total number of colors.

**Choice of the  $n_i$**  Note that, if for some  $i$ ,  $d_i < i$ , then  $n_i = 0$  as a block of  $\mathcal{B}_i$  cannot be used to assign a vertex of weight  $\geq i$  (recall that the vertices of weight  $< i$  are by Property \* all assigned to blocks of  $\mathcal{B}_l$  with  $l < i$ ). So, in the following claims, we suppose  $d_i \geq i$  for all  $i$ .

**Claim 1.**  $n_1 = \left\lceil \frac{\Delta}{d_1} \right\rceil$

*Proof.* Suppose that a vertex  $v$  of weight 1 cannot be assigned to any block of  $\mathcal{B}_1$ . This means that, for each block  $B$  of  $\mathcal{B}_1$ , vertex  $v$  and its neighbors in  $B$  form a clique of size  $> d_1$  and so the weight of the neighbors of  $v$  in  $B$  is at least  $d_1$ . This implies that the weight of the neighbors of  $v$  in all of the blocks in  $\mathcal{B}_1$  is at least  $n_1 d_1$ . Since we are considering the vertices in the RPEO, this implies that the clique induced by  $v$  and its neighbors in  $\mathcal{B}_1$  is of weight  $n_1 d_1 + 1$  which exceeds  $\Delta$  for  $n_1 = \left\lceil \frac{\Delta}{d_1} \right\rceil$ . This is not possible.  $\square$

**Claim 2.**  $n_2 = \left\lceil \frac{\Delta - 1 - n_1(d_1 - 1)}{\Delta_2^2} \right\rceil$  where  $\Delta_2^2 = \max\{2, d_2 - 1\}$ .

*Proof.* Suppose that a vertex  $v$  of weight 2 cannot be assigned to any block of  $\mathcal{B}_1$  or  $\mathcal{B}_2$ . This means that, for each block  $B$  of  $\mathcal{B}_1$  (resp.  $\mathcal{B}_2$ ), vertex  $v$  and its neighbors in  $B$  form a clique of size  $> d_1$  (resp.  $> d_2$ ) and so the weight of the neighbors of  $v$  in  $B$  is at least  $d_1 - 1$  (resp.  $d_2 - 1$ ). However, if  $d_2 = 2$ , as all the vertices of weight 1 are assigned to blocks of  $\mathcal{B}_1$ ,  $v$  has necessarily one neighbor of weight 2 in each block of  $\mathcal{B}_2$ . Therefore, if we let  $\Delta_2^2 = \max\{2, d_2 - 1\}$ , the clique induced by  $v$  and its neighbors in the RPEO has a weight at least  $n_1(d_1 - 1) + n_2 \Delta_2^2 + 2$  which exceeds  $\Delta$  for  $n_2 = \left\lceil \frac{\Delta - 1 - n_1(d_1 - 1)}{\Delta_2^2} \right\rceil$ .  $\square$

**Example:** Consider the case  $W = 2$ , and let  $d_1 = 2$  and  $d_2 = 2$ . Applying the formula we get  $n_1 = \left\lceil \frac{\Delta}{2} \right\rceil$  and  $n_2 = \left\lceil \frac{\Delta - 1 - n_1}{2} \right\rceil$ . Using the fact that  $c(2, \{1, 2\}) = c(2, \{2\}) = 2$ , the number of colors used by the algorithm to color a chordal graph with weight bounded by 2 is at most  $2n_1 + 2n_2 \leq \Delta + \left\lceil \frac{\Delta}{2} \right\rceil$ .

**Claim 3.**  $n_i = \left\lceil \frac{\Delta + 1 - i - \sum_{l=1}^{i-1} n_l \Delta_l^l}{\Delta_i^i} \right\rceil$  where  $\Delta_i^l = \max\{l, d_l + 1 - i\}$ .

*Proof.* Suppose that a vertex  $v$  of weight  $i$  cannot be assigned to any block of  $\mathcal{B}_l$  with  $l \leq i$ . This means that, for each block  $B$  of  $\mathcal{B}_l$ , vertex  $v$  and its neighbors in  $B$  form a clique of size  $> d_l$  and so the weight of the neighbors of  $v$  in  $B$  is at least  $d_l + 1 - i$ . Furthermore, as all the vertices of weight  $< l$  are assigned to blocks of  $\mathcal{B}_j$  for  $j < l$ ,  $v$  has necessarily one neighbor of weight at least  $l$  in any block of  $\mathcal{B}_l$ . Therefore, if we let  $\Delta_i^l = \max\{l, d_l + 1 - i\}$ , the clique induced by  $v$  and its neighbors in the RPEO has a weight at least

$$\sum_{l=1}^i n_l \Delta_i^l + i \text{ which exceeds } \Delta \text{ for } n_i = \left\lceil \frac{\Delta + 1 - i - \sum_{l=1}^{i-1} n_l \Delta_i^l}{\Delta_i^i} \right\rceil. \quad \square$$

**Maximum weight 3** Let  $W = 3$ . We choose some values for  $d_i$  and using the claims above, we obtain the following values of  $n_i$ :

- $d_1 = d_2 = d_3 = 3$ .  $n_1 = \lceil \frac{\Delta}{3} \rceil$  and  $n_2 = \lceil \frac{\Delta - 1 - 2n_1}{2} \rceil$  and  $n_3 = \lceil \frac{\Delta - 2 - n_1 - 2n_2}{3} \rceil$ .
- $d_1 = 5, d_2 = d_3 = 3$ .  $n_1 = \lceil \frac{\Delta}{5} \rceil$  and  $n_2 = \lceil \frac{\Delta - 1 - 4n_1}{2} \rceil$  and  $n_3 = \lceil \frac{\Delta - 2 - 3n_1 - 2n_2}{3} \rceil$ .
- $d_1 = d_2 = 5, d_3 = 3$ .  $n_1 = \lceil \frac{\Delta}{5} \rceil$  and  $n_2 = \lceil \frac{\Delta - 1 - 4n_1}{4} \rceil$  and  $n_3 = \lceil \frac{\Delta - 2 - 3n_1 - 2n_2}{3} \rceil$ .

To compare the values of the total number of colors we need to compute  $c(3, S)$  for some basic sets  $S$ . We recall that  $c(d, S)$  is the minimum number of colors which can be used in an interval coloring of any chordal graph with density  $d$  and weights in  $S$ .

- $c(3, \{1, 2, 3\}) = 4$ .

We first prove that  $c(3, \{1, 2\}) \geq 4$ . Let us consider the example presented in Figure 3 in which the density is 3 and the maximum weight is 2. The graph in the example consists of a clique of 3 vertices of weight one, such that each vertex of weight one is joined to a vertex of weight 2. This graph cannot be colored using only 3 colors. If we suppose that it can be colored with 3 colors  $\{1, 2, 3\}$ , then one of the vertices of weight one will have to be assigned color 2. For this vertex, the neighbor of weight 2 cannot be colored since the only available colors are 1 and 3 which are not contiguous.

To prove that  $c(3, \{1, 2, 3\}) \leq 4$ , we use the First Fit algorithm in the RPEO which needs at most 4 colors .

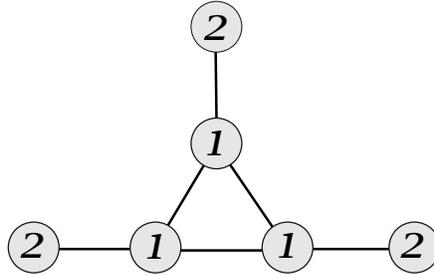


Figure 3: An example showing that  $c(3, \{1, 2\}) \neq 3$

- $c(3, \{2, 3\}) = c(3, \{3\}) = 3$ .

In fact in an instance of  $\mathcal{C}(3, \{2, 3\})$ , all vertices are isolated and we can hence easily color them with at most 3 colors.

- $c(4, \{1, 2, 3\}) = 6$ .

We first prove that  $c(4, \{1, 2, 3\}) \geq 6$ . Let us consider the example presented in Figure 4 which consists of a clique of four vertices of weight one. Each vertex of weight one is joined to a vertex of weight 3

and each pair of vertices of weight one is joined to a vertex of weight 2. Suppose that we only use 5 colors  $\{1, 2, 3, 4, 5\}$  to color this graph. If one of the vertices of weight 1 uses color 3, then its neighbor which has weight 3 cannot be colored. Otherwise, if the vertices of weight 1 use colors  $\{1, 2, 4, 5\}$ , then the vertex of weight 2 which is adjacent to the vertices of weight 1 which have colors 2 and 4 cannot be colored. This is the example which gives the lower bound  $\frac{3}{2}\Delta(G, w)$  in Table 1.

To color any instance in  $\mathcal{C}(4, \{1, 2, 3\})$  with at most 6 colors, we use the first fit algorithm in the RPEO.

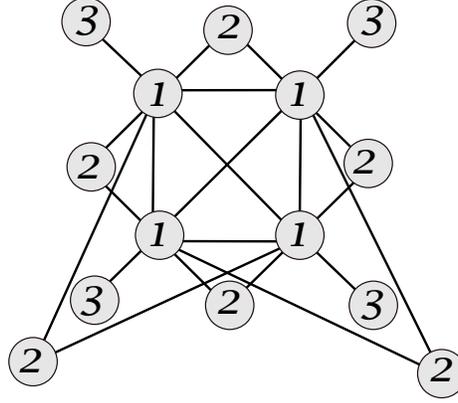


Figure 4: An example showing that  $c(4, \{1, 2, 3\}) \neq 5$

- $c(4, \{2, 3\}) = 4$ .

The First Fit algorithm in the RPEO, colors any instance in  $\mathcal{C}(4, \{2, 3\})$  with at most 4 colors.

- $c(5, \{1, 2, 3\}) = 7$ .

We first prove that  $c(5, \{1, 3\}) \geq 7$ . Let us consider the graph  $G$  consisting of a clique of 5 vertices each of weight 1 and such that each pair of vertices of the clique is connected to a vertex of weight 3. The graph  $G$  is chordal with density 5 and weights in  $\{1, 3\}$ . Let us suppose that we can color  $G$  with only six colors. There are either two vertices of weight one colored with colors 2 and 4 or two vertices of weight one colored with colors 3 and 5. In both cases the vertex of weight 3 adjacent to these two vertices cannot be colored.

Now let us describe an algorithm that takes an instance of  $\mathcal{C}(5, \{1, 2, 3\})$  and colors it with at most 7 colors. The algorithm is a First Fit algorithm in the RPEO of the vertices with the additional feature that colors 5 and 6 are forbidden for vertices of weight 1.

- If a vertex  $v$  of weight 3 is considered, then if  $v$  has a neighbor of weight 2 colored with  $\{\alpha, \alpha + 1\}$ , we color  $v$  with  $\{1, 2, 3\}$  if  $\alpha \geq 4$  or  $\{5, 6, 7\}$  if  $\alpha \leq 3$ . If  $v$  has two neighbors of weight 1; if color 7 is not used we color  $v$  with  $\{5, 6, 7\}$ . If color 7 is used, but not color 4 we color  $v$  with  $\{4, 5, 6\}$ . If both colors 4 and 7 are used, we color  $v$  with  $\{1, 2, 3\}$ .
- If a vertex  $v$  of weight 2 is considered, then if  $v$  has 3 neighbors of weight 1, we color  $v$  with  $\{5, 6\}$ . If it has one neighbor of weight 2 colored  $\{\alpha, \alpha + 1\}$  and one of weight 1 colored  $\beta$ , then we color  $v$  with  $\{5, 6\}$  if  $\alpha \leq 3$ ; with  $\{1, 2\}$  if  $\alpha \geq 4$  and  $\beta \geq 3$ ; with color  $\{6, 7\}$  if  $\alpha = 4$  and  $\beta \leq 2$  or  $\{3, 4\}$  if  $\alpha \geq 5$  and  $\beta \leq 2$ .

- $c(5, \{2, 3\}) = 5$ .

The First Fit algorithm in the RPEO in which we forbid color 3 to vertices of weight 2 uses at most 5 colors.

Now, we can compute the number of colors for the 3 cases considered above.

- If we set  $d_1 = d_2 = d_3 = 3$ , the number of colors used is  $n_1c(3, \{1, 2, 3\}) + n_2c(3, \{2, 3\}) + n_3c(3, \{3\}) = 4n_1 + 3n_2 + 3n_3$ . As  $n_3 \leq \frac{\Delta - n_1 - 2n_2}{3}$  the number of colors is at most  $\Delta + 3n_1 + n_2$  and as  $n_2 \leq \frac{\Delta - 2n_1}{2}$  it is at most  $\frac{3\Delta}{2} + 2n_1$ . Finally, as  $n_1 \leq \frac{\Delta}{3} + \frac{2}{3}$ , the number of colors used is at most  $\frac{13}{6}\Delta + \frac{4}{3}$ .
- If we set  $d_1 = 5, d_2 = d_3 = 3$ , the number of colors used is  $n_1c(5, \{1, 2, 3\}) + n_2c(3, \{2, 3\}) + n_3c(3, \{3\}) = 7n_1 + 3n_2 + 3n_3$ . As  $n_3 \leq \frac{\Delta - 3n_1 - 2n_2}{3}$  the number of colors is at most  $\Delta + 4n_1 + n_2$  and as  $n_2 \leq \frac{\Delta - 2n_1}{2}$  it is at most  $\frac{3\Delta}{2} + 2n_1$ . Finally, as  $n_1 \leq \frac{\Delta}{5} + \frac{4}{5}$ , the number of colors used is at most  $\frac{19}{10}\Delta + \frac{8}{5}$ .
- If we set  $d_1 = d_2 = 5$ , and  $d_3 = 3$ , the number of colors used is  $n_1c(5, \{1, 2, 3\}) + n_2c(5, \{2, 3\}) + n_3c(3, \{3\}) = 7n_1 + 5n_2 + 3n_3$ . As  $n_3 \leq \frac{\Delta - 3n_1 - 3n_2}{3}$  the number of colors is at most  $\Delta + 4n_1 + 2n_2$  and as  $n_2 \leq \frac{\Delta - 4n_1 + 2}{4}$  it is at most  $\frac{3\Delta}{2} + 2n_1 + 1$ . Finally, as  $n_1 \leq \frac{\Delta}{5} + \frac{4}{5}$ , the number of colors used is at most  $\frac{19}{10}\Delta + \frac{13}{5}$ .

We have tried other values of  $d_i$  and  $n_i$  but we obtained bigger numbers of colors. For example:

- If we set  $d_1 = 4, d_2 = d_3 = 3$ , using  $c(4, \{1, 2, 3\}) = 6$ , the number of colors is at most  $\frac{17}{8}\Delta + O(1)$ .
- If we set  $d_1 = d_2 = 4$ , and  $d_3 = 3$ , using  $c(4, \{2, 3\}) = 4$  the number of colors is at most  $\frac{13}{6}\Delta + O(1)$ .
- If we set  $d_1 = 6$ , and  $d_2 = d_3 = 3$ , then using  $c(6, \{1, 2, 3\}) = 9$  (to be computed after), the number of colors is at most  $\frac{23}{12}\Delta + O(1)$ .
- If we set  $d_1 = 6, d_2 = 5$  and  $d_3 = 3$ , then using  $c(5, \{2, 3\}) = 5$ , the number of colors is at most  $\frac{23}{12}\Delta + O(1)$ .

**In summary, the best approximation obtained for maximum weight 3 is with a multiplicative ratio of  $\frac{19}{10}$  and an additive constant of  $\frac{8}{5}$ .**

**Maximum weight 4** Let us first compute  $c(4, S)$  for some basic sets  $S$ .

- $c(4, \{2, 3, 4\}) = 4$ .

A First Fit algorithm in the RPEO uses at most 4 colors. In fact, in an instance of  $\mathcal{C}(4, \{2, 3, 4\})$ , vertices of weights 3 or 4 are isolated and it suffices to color vertices of weight 2 with  $\{1, 2\}$  or  $\{3, 4\}$ .

- $c(4, \{3, 4\}) = c(4, \{4\}) = 4$ .

In an instance of  $\mathcal{C}(4, \{3, 4\})$ , all vertices are isolated and can be colored independently.

- $c(6, \{1, 2, 3, 4\}) = c(6, \{1, 2, 3\}) = 9$ .

We first prove that  $c(6, \{1, 3\}) \geq 9$ . Let us consider the graph  $G$  consisting of a clique of 6 vertices of weight 1 each and such that the vertices of each triple of the clique are connected to a vertex of weight 3. The graph  $G$  is chordal with density 6 and weights in  $\{1, 3\}$ . Let us suppose that we can color  $G$  with only 8 colors. There are three vertices of weight 1 using colors  $\{1, 4, 7\}$  or three vertices of weight 1 using colors  $\{2, 5, 8\}$  or two vertices of weight 1 using colors  $\{3, 6\}$ . In any of these three cases, a vertex of weight 3 cannot be colored.

Now let us describe an algorithm that takes an instance of  $\mathcal{C}(6, \{1, 2, 3, 4\})$  and colors it with at most 9 colors. The algorithm is a First Fit algorithm in the RPEO of the vertices with two additional features: colors 6, 7 and 8 are forbidden to vertices of weight 1, and each vertex of weight 2 is assigned colors  $\{1, 2\}$ ,  $\{3, 4\}$ ,  $\{5, 6\}$ , or  $\{7, 8\}$  and not any other contiguous combination of two colors. This algorithm uses at most 9 colors.

- If a vertex  $v$  of weight 4 is considered, then if  $v$  has a neighbor of weight 2 which has been already colored, the possible sets of color used by this neighbor are  $\{1, 2\}$ ,  $\{3, 4\}$ ,  $\{5, 6\}$ , or  $\{7, 8\}$ . In any case,  $v$  can be colored with 4 contiguous colors. If  $v$  has two neighbors of weight 1 each that have been already colored, then either one of the colors 9 or 5 is not used by this neighbor, and in this case  $v$  can use it along with the colors  $\{6, 7, 8\}$  (which are forbidden for vertices of weight 1), or both colors 9 and 5 are used and  $v$  can use colors  $\{1, 2, 3, 4\}$ .
- If a vertex  $v$  of weight 3 is considered, then if  $v$  has a neighbor of weight 3 colored with  $\{\alpha, \alpha + 1, \alpha + 2\}$ , we color  $v$  with  $\{1, 2, 3\}$  if  $\alpha \geq 4$  or with  $\{7, 8, 9\}$  if  $\alpha \leq 3$ . If  $v$  has two neighbors one of weight 2 colored with  $\{\alpha, \alpha + 1\}$  and one of weight 1 colored with  $\beta$ , then we color  $v$  with  $\{6, 7, 8\}$  if  $\alpha = 1$  or 3;  $\{1, 2, 3\}$  if  $\alpha = 5$  or 7 and  $\beta \geq 4$ ;  $\{7, 8, 9\}$  if  $\alpha = 5$  and  $\beta \leq 3$ ;  $\{4, 5, 6\}$  if  $\alpha = 7$  and  $\beta \leq 3$ .
- If a vertex  $v$  of weight 2 is considered. If all its neighbors that have been already colored are of weight 1, then  $v$  can be assigned colors  $\{7, 8\}$ . If  $v$  has two colored neighbors of weight 2 each, or one colored neighbor of weight 2 and two other colored neighbors with weight 1, or one colored neighbor of weight 3 and another of weight 1, or one vertex of weight 4, then one of the channels  $\{1, 2\}$ ,  $\{3, 4\}$ ,  $\{5, 6\}$ , or  $\{7, 8\}$  is necessarily free to be used.

•  $c(6, \{2, 3, 4\}) = 8$ .

We first prove that  $c(6, \{2, 4\}) \geq 8$ . Let us consider the graph  $G$  which consists of a clique of 3 vertices of weight 2 each, and such that each vertex of weight 2 is connected to a vertex of weight 4. The graph  $G$  is chordal and has density 6 and weights in  $\{2, 4\}$ . Let us suppose that we can color  $G$  with only 7 colors. In any possible coloring of the vertices of weight 2, a vertex  $v$  of weight 2 has to use either colors  $\{3, 4\}$  or  $\{4, 5\}$ . In both cases, the neighbor of  $v$  which has weight 4 cannot be colored.

Now let us describe the algorithm that colors an instance of  $\mathcal{C}(6, \{2, 3, 4\})$ . The algorithm uses First Fit in the RPEO with the additional feature that the possible combinations of colors for vertices of weight 2 are the following:  $\{1, 2\}$ ,  $\{3, 4\}$ ,  $\{5, 6\}$ , and  $\{7, 8\}$ .

Now, we can compute the number of colors for  $d_1 = 6$  and  $d_2 = d_3 = d_4 = 4$ .

The number of colors used is  $n_1 c(6, \{1, 2, 3, 4\}) + n_2 c(4, \{2, 3, 4\}) + n_3 c(4, \{3, 4\}) + n_4 c(4, \{4\}) = 9n_1 + 4n_2 + 4n_3 + 4n_4$ .

We have  $n_1 = \lceil \frac{\Delta}{6} \rceil$ ;  $n_2 = \lceil \frac{\Delta - 1 - 5n_1}{3} \rceil$ ;  $n_3 = \lceil \frac{\Delta - 2 - 4n_1 - 2n_2}{3} \rceil$ , and  $n_4 = \lceil \frac{\Delta - 3 - 3n_1 - 2n_2 - 3n_3}{4} \rceil$ .

As  $n_4 \leq \frac{\Delta - 3n_1 - 2n_2 - 3n_3}{4}$  the number of colors is at most  $\Delta + 6n_1 + 2n_2 + n_3$ . As  $n_3 \leq \frac{\Delta - 4n_1 - 2n_2}{3}$  the number of colors is at most  $\frac{4\Delta}{3} + \frac{14}{3}n_1 + \frac{4}{3}n_2$ , and as  $n_2 \leq \frac{\Delta - 5n_1 + 1}{3}$  it is at most  $\frac{16\Delta}{9} + \frac{22}{9}n_1 + \frac{4}{9}$ . Finally, as  $n_1 \leq \frac{\Delta + 5}{6}$ , the number of colors is at most  $\frac{59}{27}\Delta + \frac{67}{27}$ .

We have computed the number of colors for other choices of the  $d_i$  but the values are bigger.

- For  $d_1 = d_2 = d_3 = d_4 = 4$ , we obtain  $\frac{91}{36}\Delta + O(1)$  colors.
- For  $d_1 = d_2 = 5$  and  $d_3 = d_4 = 4$ , we obtain  $\frac{73}{30}\Delta + O(1)$  colors.
- For  $d_1 = 5$  and  $d_2 = d_3 = d_4 = 4$ , we obtain  $\frac{109}{45}\Delta + O(1)$  colors.
- For  $d_1 = d_2 = d_3 = 6$ ,  $d_4 = 4$ , we obtain  $\frac{139}{60}\Delta + O(1)$  colors.
- For  $d_1 = d_2 = 6$  and  $d_3 = d_4 = 4$ , we obtain  $\frac{67}{30}\Delta + O(1)$  colors.

**The best approximation obtained for maximum weight 4 is with a multiplicative ratio of  $\frac{59}{27}$  and an additive constant of  $\frac{67}{27}$ .**

**Maximum weight 5** Let us first compute  $c(5, S)$  for some basic sets  $S$ .

- $c(5, \{1, 2, 3, 4, 5\}) = 8$ .

In fact, we know that  $c(5, \{1, \dots, 4\}) = 8$  and for any chordal graph with density 5 and maximum weight 5, vertices of weight 5 are isolated and can be colored independently from the others.

- $c(5, \{2, 3, 4, 5\}) = 5$ .

In a chordal graph of density 5 and weights in  $\{2, \dots, 5\}$ , vertices of weight 4 or 5 are isolated and can be easily colored. For other vertices, we know that  $c(5, \{2, 3\}) = 5$ .

- $c(5, \{3, 4, 5\}) = c(5, \{4, 5\}) = 5$ .

All vertices are isolated and can be easily colored.

- $c(6, \{1, 2, 3, 4, 5\}) = 10$ .

We first prove that  $c(6, \{1, \dots, 5\}) = 10$ . Let us consider the graph  $G$  which consists of a clique  $C$  of 6 vertices of weight 1 each such that each of the vertices of  $C$  is connected to a vertex of weight 5, and each pair of vertices of  $C$  is connected to a vertex of weight 4. Let us suppose that we can color  $G$  with 9 colors. No vertex of weight 1 can use color 5 because otherwise its neighbor of weight 5 cannot be colored. In any possible coloring of the vertices of weight 1 without using color 5, there are two vertices whose neighbor of weight 4 cannot be colored. This is the example which gives the lower bound  $\frac{5}{3}\Delta(G, w)$  in Table 1.

The First Fit in the RPEO colors any instance of  $\mathcal{C}(6, \{1, \dots, 5\})$  with at most 10 colors.

- $c(6, \{2, 3, 4, 5\}) = 8$ .

Vertices of weight 5 are isolated and can be easily colored. As for other vertices we have already proved that  $c(6, \{2, 3, 4\}) = 8$ .

- $c(6, \{3, 4, 5\}) = 6$ .

Vertices of weight 4 and 5 are isolated and vertices of weight 3 can be colored using First Fit in the RPEO with either the colors  $\{1, 2, 3\}$  or  $\{4, 5, 6\}$ .

- $c(7, \{1, 2, 3, 4, 5\}) \leq 12$ .

To obtain a coloring with 12 colors, we use the First Fit algorithm in the RPEO with the additional feature of forbidding colors  $\{8, 9, 10, 11, 12\}$  to vertices of weight 1 and colors  $\{8, 9\}$  and  $\{9, 10\}$  for vertices of weight 2. The proof that the algorithm works is done by considering the various possibilities when a new vertex is added.

Now we can compute the number of colors for  $d_1 = 7$  and  $d_2 = d_3 = d_4 = d_5 = 5$ .

The number of colors used is  $n_1c(7, \{1, 2, 3, 4, 5\}) + n_2c(5, \{2, 3, 4, 5\}) + n_3c(5, \{3, 4, 5\}) + n_4c(5, \{4, 5\}) + n_5c(5, \{5\}) = 12n_1 + 5n_2 + 5n_3 + 5n_4 + 5n_5$ .

We have  $n_1 = \lceil \frac{\Delta}{7} \rceil$ ,  $n_2 = \lceil \frac{\Delta - 1 - 6n_1}{4} \rceil$ ,  $n_3 = \lceil \frac{\Delta - 2 - 5n_1 - 3n_2}{3} \rceil$ ,  $n_4 = \lceil \frac{\Delta - 3 - 4n_1 - 2n_2 - 3n_3}{4} \rceil$ , and  $n_5 = \lceil \frac{\Delta - 4 - 3n_1 - 2n_2 - 3n_3 - 4n_4}{5} \rceil$ .

As in the preceding cases, we, successively, use upper bounds for the  $n_i$ . The number of colors is at most  $\Delta + 9n_1 + 3n_2 + 2n_3 + n_4$ , then  $\frac{5\Delta}{4} + 8n_1 + \frac{5}{2}n_2 + \frac{5}{4}n_3$ , then  $\frac{5\Delta}{3} + \frac{71}{12}n_1 + \frac{5}{4}n_2$ , then  $\frac{95\Delta}{48} + \frac{97}{24}n_1 + \frac{5}{8} \leq \frac{859}{336}\Delta + \frac{229}{56}$ .

We have computed the number of colors for other choices of the  $d_i$  but we obtained bigger values as we present in what follows.

- If we set  $d_1 = d_2 = d_3 = d_4 = d_5 = 5$ , then the number of colors used is  $8n_1 + 5n_2 + 5n_3 + 5n_4 + 5n_5$  which is at most  $\frac{679}{240}\Delta + O(1)$ .
- If we set  $d_1 = d_2 = d_3 = 6, d_4 = d_5 = 5$ , then the number of colors used is  $10n_1 + 8n_2 + 6n_3 + 5n_4 + 5n_5$  which is at most  $\frac{659}{240}\Delta + O(1)$ .

- If we set  $d_1 = 6, d_2 = d_3 = d_4 = d_5 = 5$ , then the number of colors used is  $10n_1 + 5n_2 + 5n_3 + 5n_4 + 5n_5$  which is at most  $\frac{763}{288}\Delta(G, w) + O(1)$ .

**The best approximation obtained for maximum weight 5 is with a multiplicative ratio of  $\frac{859}{336}$  and an additive constant of  $\frac{229}{56}$ .**

**Maximum weight 6** Let us first compute  $c(6, S)$  for some basic sets  $S$ . Note that with a density at most 6, any vertices of weight 6 are isolated. We can then deduce the following from what we have computed for a maximum weight of 5.

- $c(6, \{1, 2, 3, 4, 5, 6\}) = 10$ .
- $c(6, \{2, 3, 4, 5, 6\}) = 8$ .
- $c(6, \{3, 4, 5, 6\}) = c(6, \{4, 5, 6\}) = c(6, \{5, 6\}) = c(6, \{6\}) = 6$ .
- $c(7, \{1, 2, 3, 4, 5, 6\}) = 12$ .

We use the algorithm which gives  $c(7, \{1, 2, 3, 4, 5, \}) \leq 12$ . If a vertex  $v$  of weight 6 is added it is joined to at most one vertex of weight 1 of color  $\beta$ . If  $\beta \leq 6$ , we color  $v$  with colors  $\{7, 8, 9, 10, 11, 12\}$  and if  $\beta = 7$  with colors  $\{1, 2, 3, 4, 5, 6\}$  and so  $c(7, \{1, 2, 3, 4, 5, 6\}) \leq 12$ .

To show that  $c(7, \{1, 2, 3, 4, 5, 6\}) \geq 12$ , we consider the chordal graph consisting of a clique of 7 vertices of weight one such that each vertex of this clique is joined to a vertex of weight 6. Furthermore we join each pair of vertices of weight 1 to a vertex of weight 5. Suppose that we can color the graph with 11 colors. Color 6 cannot be used for any vertex of weight 1, otherwise its neighbor of weight 6 cannot be colored. Furthermore we can use for vertices of weight 1 at most one of the pair of colors  $\{2, 7\}$ ,  $\{3, 8\}$ ,  $\{4, 9\}$ ,  $\{5, 10\}$ . So we have altogether 5 colors forbidden for vertices of weight 1 and so only 6 available colors which is impossible.

Now we can compute the number of colors for  $d_1 = 7$  and  $d_2 = d_3 = d_4 = d_5 = d_6 = 6$ .

The number of colors used is  $n_1c(7, \{1, 2, 3, 4, 5, 6\}) + n_2c(6, \{2, 3, 4, 5, 6\}) + n_3c(6, \{3, 4, 5, 6\}) + n_4c(6, \{4, 5, 6\}) + n_5c(6, \{5, 6\}) + n_6c(6, \{6\}) = 12n_1 + 8n_2 + 6n_3 + 6n_4 + 6n_5 + 6n_6$ .

We have  $n_1 = \lceil \frac{\Delta}{7} \rceil$ ,  $n_2 = \lceil \frac{\Delta-1-6n_1}{5} \rceil$ ,  $n_3 = \lceil \frac{\Delta-2-5n_1-4n_2}{4} \rceil$ ,  $n_4 = \lceil \frac{\Delta-3-4n_1-3n_2-3n_3}{4} \rceil$ ,  $n_5 = \lceil \frac{\Delta-4-3n_1-2n_2-3n_3-4n_4}{5} \rceil$ , and  $n_6 = \lceil \frac{\Delta-5-2n_1-2n_2-3n_3-4n_4-5n_5}{6} \rceil$ .

Like in the preceding cases, we obtain upperbounds on  $n_i$ . The number of colors is at most  $\Delta + 10n_1 + 6n_2 + 3n_3 + 2n_4 + n_5$ , then  $\frac{6\Delta}{5} + \frac{47}{5}n_1 + \frac{28}{5}n_2 + \frac{12}{5}n_3 + \frac{6}{5}n_4$ , then  $\frac{3\Delta}{2} + \frac{41}{5}n_1 + \frac{47}{10}n_2 + \frac{3}{2}n_3$ , then  $\frac{15\Delta}{8} + \frac{253}{40}n_1 + \frac{16}{5}n_2 + \frac{3}{8}$ , then  $\frac{503\Delta}{200} + \frac{497}{200}n_1 + \frac{459}{200}n_2 \leq \frac{287}{100}\Delta + \frac{885}{200}$ .

If we set  $d_1 = d_2 = d_3 = 6 = d_4 = d_5 = d_6 = 6$ , the number of colors used is  $10n_1 + 8n_2 + 6n_3 + 6n_4 + 6n_5 + 6n_6$  which is at most  $\frac{603}{200}\Delta(G, w) + O(1)$ .

We could improve the value if we could prove that  $c(8, \{1, 2, 3, 4, 5, 6\}) \leq 14$  but that seems not possible. We can only prove  $c(8, \{1, 2, 3, 4, 5, 6\}) \leq 15$  which gives a bigger number of colors.

**For maximum weight 6, we obtain an approximation is with a multiplicative ratio of  $\frac{287}{100}$  and an additive constant of  $\frac{885}{200}$ .**  $\square$

Theorem 11 implies the following corollary.

**Corollary 5.** *Let  $\mathcal{I}$  be an instance of SA in a binary tree. Let  $OPT$  be the span of  $\mathcal{I}$ . There are polynomial-time algorithms which find a spectrum assignment for  $\mathcal{I}$  with a span less than  $\frac{19}{10}OPT + \frac{8}{5}$ ,  $\frac{59}{27}OPT + \frac{67}{27}$ ,  $\frac{859}{336}OPT + \frac{229}{56}$  and  $\frac{287}{100}OPT + \frac{885}{200}$  when the maximum request demand is bounded by 3, 4, 5 and 6, respectively.*

## 5 Conclusion

We have studied in this report the problem of spectrum assignment in tree networks. We have proved that SA is NP-complete in undirected stars with 3 links and directed stars with 4 links. We have also shown that there is a 4-approximation algorithm to solve the problem in general stars. Afterwards, we have focused on SA in binary trees with bounded demands and we have designed constant approximation algorithms for several cases. As future work, we would like to find approximation algorithms for interval coloring in chordal graphs in general and to SA in binary trees in particular. Towards this objective, we believe the following directions might be useful.

- It would be interesting to try to use the clique graph of the chordal graph [7] to find an acyclic orientation where the number of maximal cliques to which a path belongs is bounded. In fact, finding a  $k$ -approximation for interval coloring is equivalent to finding an acyclic orientation in which the longest directed path has vertices in at most  $k$  maximal cliques [12]. This approach has been used to find a 2-approximation for interval coloring in claw-free chordal graphs [6].
- It would be also helpful to try to use ideas from the approximation algorithms used for interval coloring in interval graphs. These algorithms were developed for the problem of Dynamic Storage Allocation (DSA) as we mentioned in Section 2.2.2 and they use mainly three techniques.
  - 2-coloring (2-allocation) [9]: in this technique, which yields a 3-approximation for IC in interval graphs, first, a 2-coloring is found where 2 adjacent vertices but not three might use the same color. This 2-coloring is transformed afterwards to a normal coloring. Is it possible to find a 2-coloring for chordal graphs in polynomial time? Knowing that if we directly apply the technique used by Gergov [9] to find a 2-coloring for an interval graph to chordal graphs, we will have instances where a color is shared by an unbounded number of vertices.
  - Boxing vertices [3]: in this technique, which yields a  $2 + \epsilon$  approximation for IC in interval graphs, vertices are modeled as rectangles (the dimensions of a rectangle corresponding to a vertex  $v$  are the weight of  $v$  and the interval corresponding to  $v$  in the interval representation of the graph). These rectangles are cleverly boxed or gathered in larger rectangles. Afterwards an exact algorithm to color these large rectangles. Is it possible to adapt such technique to chordal graphs and find a clever way to box the vertices?
  - Buddy-decreasing-size algorithm [5]: in this algorithm, which yields a 6-approximation for IC in interval graphs, vertices are colored in the decreasing order of their weights. Some of the challenges in this direction is that using it as it is for chordal graphs cannot give better than a  $\log(n)$ -approximation; there is a tight example in [21]. In the tight example however all the vertices have the same weight which means that there is an exponential number of possible orders. Maybe some clever order (something similar to lexicographic order?) can give a better bound.

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