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To cite this version:
Gilles Puy, Mike E. Davies, Rémi Gribonval. Linear embeddings of low-dimensional subsets of a Hilbert space to $\mathbb{R}^m$. EUSIPCO - 23rd European Signal Processing Conference, Aug 2015, Nice, France. hal-01116153v2

HAL Id: hal-01116153
https://inria.hal.science/hal-01116153v2
Submitted on 8 Jun 2015

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Linear embeddings of low-dimensional subsets of a Hilbert space to \( \mathbb{R}^m \)

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ABSTRACT

We consider the problem of embedding a low-dimensional set, \( \mathcal{M} \), from an infinite-dimensional Hilbert space, \( \mathcal{H} \), to a finite-dimensional space. Defining appropriate random linear projections, we propose two constructions of linear maps that have the restricted isometry property (RIP) on the secant set of \( \mathcal{M} \) with high probability. The first one is optimal in the sense that it only needs a number of projections essentially proportional to the intrinsic dimension of \( \mathcal{M} \) to satisfy the RIP. The second one, which is based on a variable density sampling technique, is computationally more efficient, while potentially requiring more measurements.

Index Terms— Compressed sensing, restricted isometry property, box-counting dimension, variable density sampling.

1. INTRODUCTION

The compressed sensing (CS) theory shows that the information contained in objects with a small intrinsic dimension can be "captured" by few linear and non-adaptive measurements and recovered by non-linear decoders [1]. The restricted isometry property (RIP) is at the core of many of the theoretical developments in CS. A matrix \( A \in \mathbb{R}^{m \times n} \) satisfies the RIP on a general set \( S \subset \mathbb{R}^n \), if there exists a constant \( \delta \in (0, 1) \), such that for all \( x \in S \),

\[
(1 - \delta) \| x \|_2^2 \leq \| Ax \|_2^2 \leq (1 + \delta) \| x \|_2^2.
\]  

(1)

For example, if \( A \) satisfies the RIP on the set \( S \) of \( 2s \)-sparse with a sufficiently small constant \( \delta \) then every \( s \)-sparse vector \( x \) is accurately and stably recovered from its noisy measurements \( z = Ax + n \) by solving the Basis Pursuit problem [1]. For a more general low-dimensional model \( \Sigma \subset \mathbb{R}^n \), one needs to show that the matrix \( A \) satisfies the RIP on the secant set \( S = \Sigma - \Sigma \) to ensure stable recovery [2].

In this finite dimensional setting, random matrices with independent entries drawn from the centered Gaussian distribution with variance \( m^{-1} \) are examples of matrices that satisfy the RIP with high probability for many different low-dimensional models \( \Sigma \) in \( \mathbb{R}^n \); sparse signals [1], compact Riemannian manifold [3], etc. In these scenarios, the RIP holds for a number of measurements \( m \) essentially proportional to the dimension of \( \Sigma \).

In this work, we are interested in extending this theory to an infinite dimensional Hilbert space \( \mathcal{H} \). We consider a low-dimensional subset, \( \mathcal{M} \), of \( \mathcal{H} \) and our goal is to construct a linear map \( L \) from \( \mathcal{H} \) to \( \mathbb{R}^m \) that stably embeds \( \mathcal{M} \) in \( \mathbb{R}^m \) for a number of measurements \( m \) which is: 1) essentially proportional to the intrinsic dimension of \( \mathcal{M} \), and, 2) obviously independent of the (infinite) ambient dimension. These developments are important in CS to extend the theory to an analog setting [4], explore connections with the sampling of signals with finite rate of innovation [5], and also in machine learning to develop efficient methods to compute information-preserving sketches of probability distributions [6, 7].

In Section 2, we give the definition of dimension used and our assumption on the dimension of \( \mathcal{M} \). In Section 3, we detail a construction of a linear map that satisfies the RIP for a number of measurement \( m \) essentially proportional to the dimension of \( \mathcal{M} \). Even though this linear map has a number of measurements \( m \) reduced to its minimum, its interest can be computationally limited in certain settings. In Section 4, we propose a strategy more interesting in terms of computational cost. This strategy is based on a variable density sampling technique [4, 8] and its performance is driven by a generalised notion of coherence. Finally, in Section 5, we highlight the main key ingredients needed to construct a linear map \( L \) which satisfies the RIP. All proofs can be found in the appendices.

Notations: \( \mathcal{H} \) denotes a real Hilbert space with scalar product \( \langle \cdot, \cdot \rangle \) and associated norm denoted by \( \| \cdot \| \). The euclidean norm in \( \mathbb{R}^d \), for any \( d > 0 \), is denoted by \( \| \cdot \|_2 \).

2. THE DIMENSION OF \( \mathcal{M} \)

2.1. The normalised secant set

Our goal is to construct a linear map \( L : \mathcal{H} \rightarrow \mathbb{R}^m \) that satisfies

\[
(1 - \delta) \leq \|L(x_1 - x_2)\|_2/\|x_1 - x_2\| \leq (1 + \delta)
\]

(2)

Notice that the non-squared RIP (2) implies the squared RIP (1) with a RIP constant multiplied by 3.
for all pairs of distinct vectors $x_1, x_2 \in \mathcal{M}$. As $L$ is linear, this is equivalent to show that $\sup_{y \in S(\mathcal{M})} ||L(y)||_2 - 1| \leq \delta$, where $S(\mathcal{M})$ is the normalised secant set of $\mathcal{M}$:

$$S(\mathcal{M}) = \left\{ y = \frac{x_1 - x_2}{\|x_1 - x_2\|} \mid x_1, x_2 \in \mathcal{M}, x_1 \neq x_2 \right\}.$$ 

Therefore, $S(\mathcal{M})$ is our object of interest. It is the object that needs to have a small dimension. In the next section, we precise how we measure the dimension of $S(\mathcal{M})$. In this work, whenever we talk about the intrinsic dimension of $\mathcal{M}$, we implicitly refer to the dimension of $S(\mathcal{M})$.

We recall that the vectors $x_1 - x_2$ with $x_1, x_2 \in \mathcal{M}$ are called the chords of $\mathcal{M}$. From now on, we substitute $\mathcal{S}$ for $S(\mathcal{M})$ to simplify notations.

### 2.2. The upper box-counting dimension

We would like to highlight that several definitions of dimension exist. The reader can refer to, e.g., the monograph of Robinson [9], for an exhaustive list of definitions. In an infinite-dimensional space, one should be careful with the definition of dimension used. Indeed, as described in [9], there are examples of sets for which no stable linear embedding exists even though their dimension is finite (according to some definition). Therefore, there is no hope to construct a linear map that satisfies the RIP for these sets.

In this work, we use the upper box-counting dimension as our measure of dimension. This definition of the dimension is also at the centre of most of the developments in [9].

**Definition 1** (Upper box-counting dimension). Let $X$ be a metric space with metric denoted by $d$ and $A \subseteq X$. Let $N_A(\epsilon)$ be the minimum number of closed balls of radius $\epsilon > 0$ (with respect to the metric $d$) with centres in $A$ needed to cover $A$. The upper box-counting dimension of $A$ is

$$d(A) := \lim_{\epsilon \to 0} \sup_{\epsilon > 0} -\log(N_A(\epsilon))/\log(\epsilon).$$

From the definition, one can remark that if $d > d(A)$ then there exists $\epsilon_0 > 0$ such that $N_A(\epsilon) < e^{-d}$ for all $\epsilon < \epsilon_0$. In this paper, $A$ is always a subset of a (finite or infinite-dimensional) Hilbert space and the metric used for the definition of the upper box-counting dimension is always the norm induced by the scalar product in the ambient space.

We make the following assumption on the dimension of the normalised secant set $S$ of $\mathcal{M}$.

**Assumption 2.** The normalised secant set $S$ of $\mathcal{M}$ has a finite upper box-counting dimension which is strictly bounded by $s > 0$: $d(S) < s$. Therefore, there exist a constant $\epsilon_0 > 0$ such that $N_S(\epsilon) < e^{-s}$ for all $\epsilon < \epsilon_0$.

### 3. A TWO-STEP CONSTRUCTION OF $L$

We divide our first construction of the linear map $L$ into two simple steps. The first step is an orthogonal projection onto a well-chosen subspace $V$ of potentially large but finite dimension $d > 0$. This projection must preserve the norm of the vectors in $\mathcal{S}$ as well as possible. Here, we use the fact $\mathcal{S}$ has finite upper box-counting dimension to construct $V$. Once this projection is performed, we are coming back to a usual embedding problem in finite ambient dimension. In this setting, we show that the embedding dimension can be reduced by multiplication with a random matrix of size $m \times d$ with $m$ of the order of $s$. We highlight that the sufficient condition on $m$ to ensure a stable embedding is independent of the potentially very large dimension $d$ of $V$.

#### 3.1. Projection onto a finite-dimensional subspace

We fix a resolution $\epsilon \in (0, 1)$ and find a minimum cover of $\mathcal{S}$ with closed balls of radius $\epsilon$ and centres in $S$. Let $T(\epsilon)$ be the set of centres of these balls. The cardinality of $T(\epsilon)$ is at most $N_S(\epsilon)$. We denote $\mathcal{V}_\epsilon \subseteq \mathcal{H}$ the finite-dimensional linear subspace spanned by $T(\epsilon)$, and $P_{\mathcal{V}_\epsilon} : \mathcal{H} \to \mathcal{H}$ the orthogonal projection onto $\mathcal{V}_\epsilon$. By construction of $\mathcal{V}_\epsilon$, we have

$$\sup_{y \in S} ||y - P_{\mathcal{V}_\epsilon}(y)|| \leq \inf_{y \in S} \sup_{y \in T(\epsilon)} ||y - y_0|| \leq \epsilon. \quad (3)$$

The orthogonal projection onto $\mathcal{V}_\epsilon$ thus preserves the norm of all vectors in $\mathcal{S}$ with an error at most $\epsilon$. This projection allows us to transfer our problem from infinite ambient dimension to a problem in finite ambient dimension at the cost of an error at most $\epsilon$ on the norm of the normalised chords.

Let $(a_1, \ldots, a_d)$ be an orthonormal basis for $\mathcal{V}_\epsilon$, and $P_{\epsilon} : \mathcal{H} \to \mathbb{R}^d$ be the linear map $P_{\epsilon}(x) = (\langle a_i, x \rangle)_{1 \leq i \leq d}$. We remark that

$$||P_{\epsilon}(x)||_2 = ||P_{\mathcal{V}_\epsilon}(x)|| \leq ||x||, \quad (4)$$

for every $x \in \mathcal{H}$, and that

$$||P_{\epsilon}(y)||_2 = ||P_{\mathcal{V}_\epsilon}(y)|| \geq ||y|| - ||y - P_{\mathcal{V}_\epsilon}(y)|| \geq 1 - \epsilon, \quad (5)$$

for every $y \in \mathcal{S}$. To obtain the last inequality, we used inequality (3) and the fact that $||y|| = 1$ for any $y \in \mathcal{S}$. The inequalities (4) and (5) shows that $P_{\epsilon}$ is bi-Lipschitz on the set $\mathcal{M}$. However, the embedding dimension $d$ is still potentially very large. Indeed, we have $d \leq N_S(\epsilon) \sim \epsilon^{-s}$ while we would like an embedding dimension of the order of $s$. We thus need to reduce drastically the dimension of the embedding space. This reduction is achieved in the next section by a multiplication with a random flat matrix.

Let us highlight at this stage that we constructed the linear mapping $P_{\epsilon}$ using a covering of $\mathcal{S}$. However, the only properties that we require on $P_{\epsilon}$ hereafter is that its operator norm is lower and bounded on $\mathcal{S}$ (properties (4) and (5)). One may thus construct $P_{\epsilon}$ differently using other properties of $\mathcal{S}$.

#### 3.2. Dimensionality reduction by matrix multiplication

Let $P_{\epsilon}(S) \subset \mathbb{R}^d$ be the image of $\mathcal{S}$ under $P_{\epsilon}$. Our goal in this section is to map the vectors in $P_{\epsilon}(S)$ to a space of dimension
much smaller than \( \epsilon^{-s} \), ideally of the order of \( s \), while preserving their norm. To show that this reduction is possible, we first prove that \( P_\epsilon(S) \) has a small dimension.

**Lemma 3.** Let \( P_\epsilon(S) \) be the image of \( S \) under a linear mapping \( P_\epsilon : \mathcal{H} \to \mathbb{R}^d \) satisfying (4). If Assumption 2 holds, then the upper box-counting dimension of \( P_\epsilon(S) \) is strictly bounded by \( s \), and \( N_{P_\epsilon(S)}(\epsilon') < \epsilon'^{-s} \) for all \( \epsilon' < \epsilon_0 \).

**Proof.** From (4), we see that
\[
\|P_\epsilon(x_1) - P_\epsilon(x_2)\| \leq \|x_1 - x_2\| \quad \forall x_1, x_2 \in \mathcal{H}.
\]
(6)
We proceed as in the proof of Lemma 3.3.4 in [9]. Cover \( S \) with \( N_S(\epsilon') < \epsilon'^{-s} \) balls of radius \( \epsilon' < \epsilon_0 \). The image of this cover covers \( P_\epsilon(S) \). Consider one of these balls and let \( \epsilon \) be its centre. We denote this ball \( B(\epsilon, \epsilon') \). Using (6), we notice that \( \|P_\epsilon(c) - P_\epsilon(x)\| < \epsilon' \) for all \( x \in B(\epsilon, \epsilon') \). Therefore, the image of \( B(\epsilon, \epsilon') \) is contained in a closed ball of radius \( \epsilon' \) with centre \( P_\epsilon(c) \). We conclude that \( N_{P_\epsilon(S)}(\epsilon') < N_S(\epsilon') \) and that the upper box-counting dimension of \( P_\epsilon(S) \) is strictly bounded by \( s \).

As \( P_\epsilon(S) \) has a small dimension, it is tempting to reduce the dimension using a random matrix as done in CS. The next theorem shows that it is indeed possible with a random matrix of size \( m \times d \), where \( m \) is essentially proportional to \( s \).

**Theorem 4.** Let \( P_\epsilon : \mathcal{H} \to \mathbb{R}^d \) be a linear mapping satisfying (4) and (5). \( A \in \mathbb{R}^{m \times d} \) be a matrix whose entries are independent normal random variables with mean 0 and variance \( 1/m \), and \( \rho \in (0, 1) \).

There exist absolute constants \( D_1, D_2, D_3 > 0 \) with \( D_1 < 1 \) such that if Assumption 2 holds, then for any \( 0 < \delta < \min(D_1, \epsilon_0) \), we have, with probability at least \( 1 - \rho \),
\[
1 - \epsilon - \delta \leq \|AP_\epsilon(y)\| \leq 1 + \delta,
\]
uniformly for all \( y \in S \) provided that
\[
m \geq D\delta^{-2} \max\{s \log(D_3/\delta), \log(6/\rho)\}.
\]

The same result applies (with different absolute constants \( D_1, D_2, D_3 \)) if \( A \) is a matrix whose entries are independent \( \pm 1/\sqrt{m} \) Bernoulli random variables, or if its rows are independent random vectors drawn from the surface of the unit sphere using the uniform distribution.

**Proof.** The proof, available in Appendix C, is based on a technique used by Eftekhar et al. in [3] to show that random Gaussian matrices satisfy the RIP on compact Riemannian manifolds of \( \mathbb{R}^d \). The sufficient condition on \( m \) that they obtain is independent of the ambient dimension \( d \). It turns out that their result generalises to any set with finite upper box-counting dimension in \( \mathbb{R}^d \).

In view of Theorem 4, we remark that \( AP_\epsilon(\cdot) : \mathcal{H} \to \mathbb{R}^m \) has the RIP on \( \mathcal{M} \) if \( m \) and \( \epsilon \) are appropriately chosen.

**Corollary 5.** Let \( A \in \mathbb{R}^{m \times d} \) be as in Theorem 4, \( P_\epsilon : \mathcal{H} \to \mathbb{R}^d \) be a linear mapping satisfying (4) and (5), and \( \rho \in (0, 1) \).

There exist absolute constants \( D_1, D_2, D_3 > 0 \) with \( D_1 < 1 \) such that if Assumption 2 holds, then for any \( 0 < \delta < \min(D_1, \epsilon_0) \), if \( \epsilon \leq \delta/2 \) and
\[
m \geq D\delta^{-2} \max\{s \log(6/\delta), \log(6/\rho)\},
\]
then \( AP_\epsilon : \mathcal{H} \to \mathbb{R}^m \) satisfies the RIP (2) on \( \mathcal{M} \) with constant at most and probability at least \( 1 - \rho \).

### 3.3. Comparison with a known result in CS

Corollary 5 holds for low-dimensional signal models that satisfy Assumption 2 in an infinite-dimensional Hilbert space \( \mathcal{H} \), so it also holds for signal models in \( \mathbb{R}^n \) provided that their normalised secant set has finite upper box-counting dimension. Let us take one example in finite ambient space.

Consider the set \( \mathcal{M} \) of \( k \)-sparse signals in \( \mathbb{R}^n \). The normalised secant set \( \mathcal{S} \) is the set of \( 2k \)-sparse signals with \( \ell_2 \)-norm equal to 1. This set can be covered by at most \( \lceil 3en/(2k) \rceil \) balls of radius \( \epsilon < 1 \) [11]. Its box-counting dimension is thus \( 2k \). As we are in finite dimension, we can take \( V_\epsilon = \mathbb{R}^n \) (which implies \( \epsilon = 0 \)) and \( P_\epsilon \) is the identity map. The complete linear map \( L \) reduces to the matrix \( A \). Corollary 5 shows that \( A \) has the RIP if \( m \) satisfies (7) with \( s > 2k \).

In comparison, Theorem 9.2 in [1] shows that \( A \) has the RIP on \( \mathcal{S} \) with constant \( \delta \) and probability at least \( 1 - \rho \) provided that
\[
m \geq D\delta^{-2} (k \log(2n/k) + \log(2/\rho)),
\]
(8)
where \( D > 0 \) is an absolute constant. A fundamental difference seems to exist between (7) and (8). Our result seems to be independent of \( \log(n/k) \) while (8) is not. This dependence does actually exist in our result but is “hidden” in the variable \( \epsilon_0 \) in the statement of Corollary 5. We remark that \( \epsilon_0 \) is a constant such that \( 3en/(2k)e^{2k} < \epsilon^s \) for all \( \epsilon < \epsilon_0 \) with \( s > 2k \). Let us compute \( \epsilon_0 \). Writing \( s = 2k + \eta \) with \( \eta > 0 \), we should have \( (3en/(2k))^{2k} < \epsilon^{-\eta} \) for all \( \epsilon < \epsilon_0 \), or, equivalently, \( (3en/(2k))^{2k} \leq \epsilon_0^{-\eta} \). We take \( \epsilon_0 = 2k/(3en) \) and \( \eta = 2k \). For \( \delta < \epsilon_0 \), we have \( \log(1/\delta) > \log(3en/(2k)) \).

In this setting, we notice that we can slightly strengthen (8) to
\[
m \geq D\delta^{-2} (s \log(1/\delta) + \log(2/\rho)).
\]

This stronger condition on \( m \) is similar to ours. This shows that we recover results similar to known ones in CS.

### 4. AN EFFICIENT LINEAR EMBEDDING OF \( \mathcal{M} \)

In cases where the dimension \( d \) is very large (we recall that \( d \ll \epsilon^{-s} \)), the linear map \( AP_\epsilon \) proposed in the last section is a priori costly in terms of computations. Indeed, one needs to compute the scalar products between the signal of interest and all the vectors of the basis \( (a_1, \ldots, a_d) \) before multiplication.
with an unstructured matrix \( A \). In this section, we propose an computationally lighter solution where one computes the scalar products with only a small subset of the basis vectors. The method is related to the variable density sampling technique in CS [4, 8].

4.1. Construction of the linear map

To select a small subset of the basis vectors, we use a probability distribution on the discrete set \( \{1, \ldots, d\} \). We represent this probability distribution by a vector \( p = (p_i)_{1 \leq i \leq d} \in \mathbb{R}^d \), and the linear map \( L \) is created as follows.

1. We draw independently \( m \) indices \( \Omega = \{\omega_1, \ldots, \omega_m\} \) from \( \{1, \ldots, d\} \) according to the probability distribution \( p \): \( \mathbb{P}(\omega_k = j) = p_j \), for all \( k \in \{1, \ldots, m\} \) and \( j \in \{1, \ldots, d\} \).

2. We create the sparse normalized subsampling matrix \( R \in \mathbb{R}^{m \times d} \) with \( R_{\omega_k,j} = 1/\sqrt{m}\|p\|_\infty^{-1/2} \), for all \( k \in \{1, \ldots, m\} \), and 0 everywhere else.

3. We set \( L := RP \).

In order to compute \( L(x) \), one just needs to compute the \( m \) scalar products \( \langle a_{\omega_k}, x \rangle \), \( k = 1, \ldots, m \). We will next see that if \( p \) is properly designed then \( L \) satisfies the RIP with high probability.

Before continuing, we define two quantities that characterise the interaction between \( p, P, \) and \( M \). Let \( P \in \mathbb{R}^{d \times d} \) be the diagonal matrix with diagonal entries satisfying (4) and (5), and the linear map \( L \) is created as follows.

- The first quantity, \( \epsilon_p \), characterises how well the matrix \( P \) preserves the norm of the vectors in \( P_1(S) \):
  \[
  \epsilon_p := 1 - \inf_{y \in S} \left\{ \frac{\|P_1(y)\|_2}{\|P(y)\|_2} \right\}.
  \]

- The second quantity, \( \mu_p \), can be viewed as a generalised definition of coherence:
  \[
  \mu_p = \|p\|_\infty^{-1/2} \sup_{y \in S} \left\{ \frac{\|P_1(y)\|_\infty}{\|P(y)\|_2} \right\}.
  \]

We are now ready to state our main result.

**Theorem 6.** Let \( P : \mathcal{H} \to \mathbb{R}^d \) be a linear mapping satisfying (4) and (5), \( R \in \mathbb{R}^{m \times d} \) be the subsampling matrix created above and \( p \in (0, 1) \).

There exist absolute constants \( D_1, D_2 > 0 \) such that if Assumption 2 holds, then for any \( 0 < \delta < \min(1, \epsilon_0) \), we have, with probability at least \( 1 - \rho \),

\[
1 - \epsilon - \epsilon_p - \delta \leq \|RP_1(y)\|_2 \leq 1 + \delta,
\]

uniformly for all \( y \in S \) provided that

\[
m \geq D_1 \max \left\{ s \mu_p^2 \delta^{-2}, \frac{s}{\sqrt{2}} \right\} \cdot \log \left[ \frac{D_2/\|p\|_\infty}{\delta} \right].
\]

The proof, available in Appendix D, is similar to the proof of Theorem 4. We can now deduce sufficient conditions ensuring that \( RP_1(\cdot) : \mathcal{H} \to \mathbb{R}^m \) has the RIP on \( M \).

**Corollary 7.** Let \( R \in \mathbb{R}^{m \times d} \) be as in Theorem 4, \( P : \mathcal{H} \to \mathbb{R}^d \) be a linear mapping satisfying (4) and (5), and \( \rho \in (0, 1) \).

There exist absolute constants \( D_1, D_2 > 0 \) such that if Assumption 2 holds, then for any \( 0 < \delta < \min(1, \epsilon_0) \), if \( \epsilon \leq \delta^3, \epsilon_p \leq \delta/3 \) and

\[
m \geq D_2 \max \left\{ s \mu_p^2 \delta^{-2}, \frac{s}{\sqrt{2}} \right\} \cdot \log \left[ \frac{D_2/\|p\|_\infty}{\delta} \right],
\]

then \( RP_1 : \mathcal{H} \to \mathbb{R}^m \) satisfies the RIP (2) on \( M \) with constant at most \( \delta \) and probability at least \( 1 - \rho \).

Condition (10) shows that one should seek to reduce the coherence parameter \( \mu_p \) to optimise \( m \). To reduce the value of \( \mu_p \), the choice of \( p \) is obviously important. However, we want to highlight that the choice of the orthonormal basis \((a_1, \ldots, a_d)\) used to define \( P \) is also very important. Indeed, this choice will influence the value of \( \|P_1(y)\|_\infty \) appearing in the definition of \( \mu_p \). Therefore, if possible, one should choose an orthonormal basis that reduces \( \|P_1(y)\|_\infty \) to reduce \( \mu_p \).

4.2. Comparison with a known result in CS

As in Section 3.3, we consider the set of \( k \)-sparse signals in \( \mathcal{H} = \mathbb{R}^n \). We recall that \( \mathcal{S} \) is the set of \( 2k \)-sparse signals with \( \ell_2\)-norm equal to 1. We take \( \mathcal{V}_s = \mathbb{R}^n \) and \( P_s = H = \mathbb{H}^{s \times n} \) the Hadamard transform. The linear map \( L = RH \in \mathbb{R}^{n \times n} \), which is a random selection of \( m \) vectors of \( H \). This case is well-known in CS. The matrix \( H \) is optimally incoherent with the identity matrix, with coherence \( \|h_i\|_\infty = n^{-1/2} \) (\( h_i \) is the \( i \)th row-vector of \( H \)), and the RIP is satisfied for \( m \) essentially proportional to \( k \) when the measurements are selected according to the uniform probability measure (\( p_i = 1/n \) [1]).

In this setting, we have \( P = Id \) and thus \( \epsilon_p = 0 \). It is obvious from (10) that \( m \) should be at least proportional to \( s \).

Let us now compute \( \mu_p \). We have \( \|PHy\|_2 = \|y\|_2 = 1 \) for all \( y \in \mathcal{S} \).

Thus, \( \mu_p = \|p\|_\infty^{-1/2} \sup_{y \in \mathcal{S}} \|H_1 y\|_\infty = n^{-1/2} \sup_{y \in \mathcal{S}} \|h_i y\|_1 \leq n^{-1/2} \sup_{y \in \mathcal{S}} \|h_i\|_\infty \|y\|_1 \leq 1 \leq \sqrt{2s}, \)

and Corollary 7 requires a number of measurements essentially proportional to \( s^2 \). We do not recover an optimal result.

A careful inspection of the proof of the RIP for RH in [1] suggests that one needs to exploit additional properties on \( S \) than just its box-counting dimension to obtain an optimal result. Indeed, the proof is based on a clever evaluation of the covering number of \( S \) with an appropriate metric. The optimal result is obtained using two estimates of \( N_S \): a first one accurate for small radius of the balls (similar to the one used in this paper); a second one accurate for large radius. The combination of both bounds is essential to obtain \( m \) essentially proportional to \( s \) instead of \( s^2 \). It will be important in the future to determine what additional structure is needed in \( M \) or \( S \) to obtain optimal results with the variable density sampling technique.
5. KEY INGREDIENTS FOR THE DESIGN OF $L$

In this last section, we summarise the key steps that one should concentrate on for the design of $L$. For this discussion, we consider a particular case where the signal model $\mathcal{M}$ is a subset of $L_2([0, 1])$. One can think of $\mathcal{M}$ as, e.g., the set of $k$-sparse signals of the form $x = \sum_{i=1}^k x_i \psi_i$ where $||x||_0 \leq k$ and $\{\psi_i\}_{i \in \mathbb{N}}$ is the Haar wavelet basis in $L_2([0, 1])$.

The first step consists in identifying a finite orthonormal basis $\Phi = \{\phi_i\}_{1 \leq i \leq d}$ (with $d$ potentially large) that stably embeds $\mathcal{M}$. More precisely, we want (5) to hold. Intuitively, if one knows that the chords of $\mathcal{M}$ have a fast decaying spectrum, the Fourier basis $\{\sqrt{2}\cos(2\pi nt)\}_{n \in \mathbb{N}} \cup \{\sqrt{2}\sin(2\pi nt)\}_{n \in \mathbb{N}} \cup \{1\}$ seems a good candidate. The choice of $d$, i.e., the maximum frequency that can be probed, will be determined by the decay of the spectrum of the chords.

The second step consists in constructing a good probability distribution $p$ to select a subset of the basis vectors in $\Phi$. In CS for MRI, a common argument is that $p$ should favour the selection of low-frequency measurements because the energy of MR images is concentrated in this region. Let us confront this intuitive idea to our result.

According to Corollary 7, we see that $p$ must “capture” most of the energy of the chords, i.e., $\epsilon_p$ must be small. Obviously, taking the uniform discrete probability distribution, $p_i = 1/d$, ensures $\epsilon_p = 0$. However, $p_i = 1/d$ might not be the optimal choice to reduce $m$. The first reason is that $||p||_\infty$ appears in the log term of Condition (10). In the case where $d \approx e^{-s}$, Condition (10) imposes a number of measurements at least $s^2$ with $p_i = 1/d$. It could be wiser to have $\epsilon_p \neq 0$ and increase the value of $||p||_\infty$. The second reason is that it is also essential to minimise $\mu_p$. As the matrix $P$ should be such that $||P_1(y)||_2 \simeq ||P_1(y)||_2 \simeq 1$ for $y \in \mathcal{S}$, we have $\mu_p^2 \simeq \sup_{y \in \mathcal{S}} \max_i ||\langle \phi_i, y \rangle||_2^2 / ||p||_\infty$. Therefore, to minimise $\mu_p$, the maximum of $p$ should be as close as possible to $\max_i ||\langle \phi_i, y \rangle||^2$. To make sure that both $\epsilon_p$ and $\mu_p$ are small, we should have $p_i$ large whenever $||\langle \phi_i, y \rangle||$ is large and $p_i$ small whenever $||\langle \phi_i, y \rangle||$ is small. Let us highlight that the shape of $p$ is thus governed by the shape of the projection of the chords $x_1 - x_2$ on $\Phi$ and not the projection of the vectors in $\mathcal{M}$ themselves. The measurements need to be concentrated where the energy of the normalised difference between two signals in $\mathcal{M}$ is concentrated.

To conclude this paper, let us mention the related work of Dirksen [10] on a unified theory for dimensionality reduction with subgaussian matrices. His results also apply in infinite-dimensional Hilbert spaces. In particular, for separable Hilbert spaces, he gives an example of a linear map that satisfies the RIP with high probability but which requires the evaluation of an infinite number of scalar products. We also remark that Theorem 4 can be derived from the generic results presented in [10].

REFERENCES


A. CONCENTRATION INEQUALITIES

In this section, we use the notions of subgaussian and subexponential random vectors/variables. We let the reader refer to, e.g., [11] for more information about them. We recall here a few definitions and properties.

1. A subgaussian random variable $X$ is a random variable that satisfies $(E|X|^p)^{1/p} \leq K \sqrt{p}$ for all $p \geq 1$ with $K > 0$. The subgaussian norm of $X$, denoted by $\|X\|_{\psi_2}$, is the smallest constant $K$ for which the last property holds, i.e., $\|X\|_{\psi_2} = \sup_{p \geq 1} p^{-1/2} (E|X|^p)^{1/p}$ (Definition 5.7, [11]).

2. A subexponential random variable $X$ is a random variable that satisfies $(E|X|^p)^{1/p} \leq K_p$ for all $p \geq 1$ with $K_p > 0$. The subexponential norm of $X$, denoted by $\|X\|_{\psi_1}$, is the smallest constant $K$ for which the last property holds, i.e., $\|X\|_{\psi_1} = \sup_{p \geq 1} p^{-1} (E|X|^p)^{1/p}$ (Definition 5.13, [11]).

3. A random variable $X$ is subgaussian if and only if $X^2$ is subexponential and $\|X^2\|_{\psi_1} \leq 2 \|X\|_{\psi_2}$ (Lemma 5.14, [11]).

4. If $X$ is subexponential then so is $X - EX$, and we have $\|X - EX\|_{\psi_1} \leq 2 \|X\|_{\psi_1}$ (Remark 5.18, [11]).

5. A random vector $X$ in $\mathbb{R}^d$ is subgaussian if the one-dimensional marginals $x^T X$ are subgaussian random variables for all $x \in \mathbb{R}^d$. The subgaussian norm of $X$ is defined as $\|X\|_{\psi_2} = \sup_{x \in S^{d-1}} \|x^T X\|_{\psi_2}$, where $S^{d-1}$ is the unit sphere in $\mathbb{R}^d$ (Definition 5.22, [11]).

We start from generic concentration inequalities that we will use in specific cases after.

**Lemma 8** (Bernstein-type inequality). Let $X_1, \ldots, X_m$ be independent centered subexponential random variables with subexponential norm bounded by $K > 0$. Then, for every $t \in (0, K)$, we have

$$P \left\{ \sum_{i=1}^{m} X_i \geq tm \right\} \leq e^{-cmt^2/K^2}$$

$$P \left\{ \sum_{i=1}^{m} X_i \leq -tm \right\} \leq e^{-cmt^2/K^2},$$

and for every $t \geq K$,

$$P \left\{ \sum_{i=1}^{m} X_i \geq tm \right\} \leq e^{-cmt/K},$$

where $c > 0$ is an absolute constant.

The lemma above is a consequence of Proposition 5.16 in [11] and intermediate results in its proof.

**Lemma 9** ([12], Theorem 1.1). Let $X = X_1 + \ldots + X_m$ be the sum of $m$ random variables independently distributed in
[0, 1]. Then, for every \( t \in (0, 1) \), we have
\[
\begin{align*}
\mathbb{P} \{ X \geq (1 + t)\mathbb{E}(X) \} & \leq e^{-ct^2 \mathbb{E}(X)}, \\
\mathbb{P} \{ X \leq (1 - t)\mathbb{E}(X) \} & \leq e^{-ct^2 \mathbb{E}(X)},
\end{align*}
\]
and for every \( t > 0 \),
\[
\mathbb{P} \{ X \geq \mathbb{E}(X) + t \} \leq e^{-ct^2/m},
\]
where \( c > 0 \) is an absolute constant.

We now use Lemma 8 and Lemma 9 in respectively two scenarios of interest for dimensionality reduction. The first case below involves a multiplication with a dense random matrix as in Section 3.

**Lemma 10.** Let \( A \in \mathbb{R}^{m \times d} \) be a matrix whose entries are independent centered normal random variables with variance \( 1/m \), and \( x \in \mathbb{R}^d \) be a fixed vector.

For every \( \lambda \in (0, \lambda_0) \), we have
\[
\mathbb{P} \{ \| Ax \|_2 \geq (1 + \lambda) \| x \|_2 \} \leq e^{-c_0 m \lambda^2}, \tag{11}
\]
\[
\mathbb{P} \{ \| Ax \|_2 \leq (1 - \lambda) \| x \|_2 \} \leq e^{-c_0 m \lambda^2}, \tag{12}
\]
and for every \( \lambda \geq \lambda_0 \),
\[
\mathbb{P} \{ \| Ax \|_2 \geq (1 + \lambda) \| x \|_2 \} \leq e^{-c_1 m \lambda}, \tag{13}
\]
where \( c_0, c_1, \lambda_0 > 0 \) are absolute constants.

The same results apply (with different constants \( c_0, c_1, \lambda_0 \)) if, e.g., \( A \) is a matrix whose entries are independent \( \pm 1/\sqrt{m} \) Bernoulli random variables, or if its rows are independent random vectors drawn from the surface of the unit sphere using the uniform distribution.

**Proof.** Denote by \( a_1, \ldots, a_m \in \mathbb{R}^d \) the rows of \( A \). We have
\[
\| Ax \|_2^2 = \sum_{i=1}^m |a_i^\top x|^2,
\]
where \( a_1^\top x, \ldots, a_m^\top x \) are independent centered subgaussian random variables. Their subgaussian norm is bounded by \( C \| x \|_2 / \sqrt{m} \), where \( C > 0 \) is an absolute constant. We also have \( \mathbb{E}[a_i^\top x] = \| x \|_2^2/m \). Therefore, \( |a_i^\top x|^2 - \| x \|_2^2/m \) are independent centered subexponential random variables with subexponential norm bounded by \( C' \| x \|_2^2/m \), where \( C' > 0 \) is an absolute constant. Using Lemma 8 with \( X_i = |a_i^\top x|^2 - \| x \|_2^2/m \) and \( K = C' \| x \|_2^2/m \) shows that there exist absolute constants \( c_0, c_1, \lambda_0 > 0 \) such that, for every \( \lambda \in (0, \lambda_0) \),
\[
\mathbb{P} \{ \| Ax \|_2 \geq (1 + \lambda) \| x \|_2 \} \leq e^{-c_0 m \lambda^2},
\]
\[
\mathbb{P} \{ \| Ax \|_2 \leq (1 - \lambda) \| x \|_2 \} \leq e^{-c_0 m \lambda^2},
\]
and for every \( \lambda \geq \lambda_0 \),
\[
\mathbb{P} \{ \| Ax \|_2 \geq (1 + \lambda) \| x \|_2 \} \leq e^{-c_1 m \lambda}.
\]

To finish the proof, notice that
\[
\mathbb{P} \{ \| Ax \|_2 \geq (1 + \lambda) \| x \|_2 \} \leq \mathbb{P} \{ \| Ax \|_2 \geq (1 + \lambda) \| x \|_2 \}
\]
and
\[
\mathbb{P} \{ \| Ax \|_2 \leq (1 - \lambda) \| x \|_2 \} \leq \mathbb{P} \{ \| Ax \|_2 \leq (1 - \lambda) \| x \|_2 \}.
\]

We now transform the bounds above in a form directly usable in Lemma 14 for the proof of Theorem 4.

**Corollary 11.** Let \( A \in \mathbb{R}^{m \times d} \) be one of the matrices considered in Lemma 10.

1. For any fixed \( x \in \mathbb{R}^d \) with \( \| x \|_2 \leq 1 \) and every \( \lambda \in (0, \lambda_0) \), we have
\[
\mathbb{P} \{ \| Ax \|_2 \geq 1 + \lambda \} \leq e^{-c_0 m \lambda^2}.
\]

2. Let \( \epsilon \in (0, 1) \). For any fixed \( x \in \mathbb{R}^d \) with \( \| x \|_2 \geq 1 - \epsilon \) and every \( \lambda \in (0, \lambda_0) \), we have
\[
\mathbb{P} \{ \| Ax \|_2 \leq 1 - \lambda - \epsilon \} \leq e^{-c_0 m \lambda^2}. \tag{14}
\]

3. Let \( \epsilon' \in (0, 1) \) and \( j \in \mathbb{N} \). For any fixed \( x \in \mathbb{R}^d \) with \( \| x \|_2 \leq 2^{-j+1} \epsilon' \) and every \( \lambda \geq \lambda_1 \), we have
\[
\mathbb{P} \{ \| Ax \|_2 \geq j + 1/2 \epsilon' \} \leq e^{-c_1 m (8^{-j+1} (j+1) \lambda - 1)}. \tag{15}
\]

The constants \( \lambda_0, \lambda_1, c_0, c_1 \) are absolute and \( \lambda_1 > 8 \).

**Proof.** Inequality (11) in Lemma 10 yields
\[
\mathbb{P} \{ \| Ax \|_2 \geq 1 + \lambda \} \leq \mathbb{P} \{ \| Ax \|_2 \geq (1 + \lambda) \| x \|_2 \} \leq e^{-c_0 m \lambda^2},
\]
for any \( \lambda \in (0, \lambda_0) \). The first inequality is obtained by using the fact that \( 1 \geq \| x \|_2 \).

Inequality (12) yields
\[
\mathbb{P} \{ \| Ax \|_2 \leq 1 - \lambda - \epsilon \} \leq \mathbb{P} \{ \| Ax \|_2 \leq (1 - \lambda)(1 - \epsilon) \} \leq \mathbb{P} \{ \| Ax \|_2 \leq (1 - \lambda) \| x \|_2 \} \leq e^{-c_0 m \lambda^2},
\]
for any \( \lambda \in (0, \lambda_0) \). The first inequality is obtained by noticing that \( 1 - \lambda - \epsilon \leq (1 - \lambda)(1 - \epsilon) \), the second one by using the fact that \( 1 - \epsilon \leq \| x \|_2 \).

Inequality (13) yields
\[
\mathbb{P} \{ \| Ax \|_2 \geq j + 1/2 \epsilon' \} \leq \mathbb{P} \{ \| Ax \|_2 \geq j + 1/8 \lambda \| x \|_2 \} \leq e^{-c_1 m (8^{-j+1} (j+1) \lambda - 1)},
\]
for any \( \lambda \geq \lambda_1 := 8 \lambda_0 + 8 \). In the first line, we used the fact that \( 2^{-j+1} \epsilon' \geq \| x \|_2 \).
We consider now a second scenario for dimensionality reduction. It consists in randomly selecting few entries of a vector.

As in Section 4, the vector \( p \in \mathbb{R}^d \) represents a discrete probability distribution on \{1, \ldots, d\} and \( P \in \mathbb{R}^{d \times d} \) is the diagonal matrix with diagonal entries \( P_{ii} = (p_i / \| p \|_\infty)^{1/2} \). The set \( \Omega = \{\omega_1, \ldots, \omega_m\} \) is created by drawing independently \( m \) indices from \{1, \ldots, d\} according to \( p \). The sparse subsampling matrix \( R \in \mathbb{R}^{m \times d} \) has one non-zero entry on each line: \( R_{ik} = 1 / \sqrt{m \| p \|_\infty} \), for all \( k \in \{1, \ldots, m\} \).

**Lemma 12.** Let \( x \in \mathbb{R}^d \) be a fixed vector and define \( \mu_x = \| p \|^{-1/2} \| x \|_\infty / \|Px\|_2 \). For every \( \lambda \in (0, 1) \), we have

\[
\mathbb{P} \{ \|Rxi\|_2 \geq (1 + \lambda) \|Px\|_2 \} \leq e^{-c\lambda^2 \mu_x^2}, \tag{16}
\]

and, for every \( \lambda \geq 1 / \| p \|_\infty \),

\[
\mathbb{P} \{ \|Rxi\|_2 \geq (1 + \lambda) \|x\|_2 \} \leq e^{-c\mu_x \lambda \| p \|_\infty}, \tag{17}
\]

where \( c > 0 \) is an absolute constant.

**Proof.** We have \( x_j^2 / \|x\|_2^2 = x_j^2 / (\mu_x^2 \|x\|_\infty \|Px\|_2^2) \leq 1 \) for all \( j = 1, \ldots, d \). Therefore,

\[
S = \sum_{i=1}^m x_{\omega_i}^2 / \|p\|_\infty \|Px\|_2^2
\]

is a sum of \( m \) independent random variables bounded by \( 1 \). We have \( \mathbb{E}(S) = m / \mu_x^2 \) (recall that \( P_{ii}^2 = p_i / \| p \|_\infty \)). The first two inequalities in Lemma 9 yield

\[
\mathbb{P} \{ S \geq (1 + \lambda) \frac{m}{\mu_x^2} \} \leq e^{-c\lambda^2 m / \mu_x^2}, \text{ and}
\]

\[
\mathbb{P} \{ S \leq (1 - \lambda) \frac{m}{\mu_x^2} \} \leq e^{-c\lambda^2 m / \mu_x^2}.
\]

To obtain \(16\), we observe that

\[
\mathbb{P} \{ S \geq (1 + \lambda) \frac{m}{\mu_x^2} \}
\]

\[
= \mathbb{P} \{ \frac{\mu_x^2}{m} S \geq (1 + \lambda) \}
\]

\[
= \mathbb{P} \left\{ \sum_{i=1}^m \frac{x_{\omega_i}^2}{m \|p\|_\infty \|Px\|_2^2} \geq (1 + \lambda) \right\}
\]

\[
= \mathbb{P} \left\{ \|Rx\|_2^2 \geq (1 + \lambda) \|Px\|_2^2 \right\},
\]

and that

\[
\mathbb{P} \{ \|Rx\|_2 \geq (1 + \lambda) \|Px\|_2 \}
\]

\[
\leq \mathbb{P} \left\{ \|Rx\|_2^2 \geq (1 + \lambda) \|Px\|_2^2 \right\}.
\]

The same procedure with \( \mathbb{P} \{ S \leq (1 - \lambda)m / \mu_x^2 \} \) yields \(17\).

To obtain \(18\), we notice that \( x_j^2 \leq \|x\|_2^2 \) for all \( j = 1, \ldots, m \). Therefore,

\[
\tilde{S} = \sum_{i=1}^m \frac{x_{\omega_i}^2}{\|x\|_2^2}
\]

is a sum of \( m \) independent random variables bounded by \( 1 \). We have \( \mathbb{E}(S) = m / \| p \|_\infty \|Px\|_2^2 / \|x\|_2^2 \). The last inequality in Lemma 9 gives

\[
\mathbb{P} \left\{ \tilde{S} \geq \frac{m \| p \|_\infty \|Px\|_2^2 + t}{\|x\|_2^2} \right\} \leq e^{-ct^2 / m}.
\]

Then, we notice that

\[
\mathbb{P} \left\{ \tilde{S} \geq \frac{m \| p \|_\infty \|Px\|_2^2 + t}{\|x\|_2^2} \right\}
\]

\[
= \mathbb{P} \left\{ \|Rx\|_2^2 \geq \frac{\|Px\|_2^2 + \|x\|_2^2 t}{m \| p \|_\infty} \right\}
\]

\[
= \mathbb{P} \left\{ \sum_{i=1}^m \frac{x_{\omega_i}^2}{m \|p\|_\infty} \geq \|Px\|_2^2 + \|x\|_2^2 t \right\}
\]

\[
= \mathbb{P} \left\{ \|Rx\|_2^2 \geq \|Px\|_2^2 + \frac{\|x\|_2^2 t}{m \| p \|_\infty} \right\}.
\]

Therefore, for every \( t > 0 \),

\[
\mathbb{P} \left\{ \|Rx\|_2^2 \geq \|Px\|_2^2 + \frac{\|x\|_2^2 t}{m \| p \|_\infty} \right\} \leq e^{-ct^2 / m}
\]

or, equivalently,

\[
\mathbb{P} \left\{ \|Rx\|_2^2 \geq \|Px\|_2^2 + u \|x\|_2^2 \right\} \leq e^{-cm \| p \|_\infty^2 \|x\|_2^2} u^2,
\]

for every \( u > 0 \). Finally, we use the facts that \( e^{-cm \| p \|_\infty^2} \leq e^{-cm \| p \|_\infty^2 \|x\|_2^2} \) for every \( u \geq 1 / \| p \|_\infty \) and that

\[
\mathbb{P} \left\{ \|Rx\|_2^2 \geq (1 + u) \|x\|_2^2 \right\}
\]

\[
\leq \mathbb{P} \left\{ \|Rx\|_2^2 \geq \|x\|_2^2 + u \|x\|_2^2 \right\}
\]

\[
\leq \mathbb{P} \left\{ \|Rx\|_2^2 \geq \|Px\|_2^2 + u \|x\|_2^2 \right\},
\]

as \( \|x\|_2^2 \geq \|Px\|_2^2 \). \qed

We now transform the bounds above in a form directly usable in Lemma 14 for the proof of Theorem 6.

**Corollary 13.** For every \( x \in \mathbb{R}^d \), we associate the quantity

\( \mu_x = \|p\|^{-1/2} \|x\|_\infty / \|Px\|_2 \).

1. For any fixed \( x \in \mathbb{R}^d \) with \( \|x\|_2 \leq 1 \) and every \( \lambda \in (0, 1) \), we have

\[
\mathbb{P} \{ \|Rx\|_2 \geq 1 + \lambda \} \leq e^{-c\lambda^2 / \mu_x^2}. \tag{19}
\]
2. Let $\epsilon, \epsilon_p \in (0, 1)$ and assume that $\|Px\|_2 \geq (1 - \epsilon_p)\|x\|_2$. For any fixed $x \in \mathbb{R}^d$ with $\|x\|_2 \geq 1 - \epsilon$ and every $\lambda \in (0, 1)$, we have
\[
P\left\{ \|Rx\|_2 \leq 1 - \lambda - \epsilon - \epsilon_p \right\} \leq e^{-cm\lambda^2/\epsilon^2}.
\tag{20}
\]
3. Let $\epsilon' \in (0, 1)$. For any fixed $x \in \mathbb{R}^d$ with $\|x\|_2 \leq 2^{-j+1}\epsilon'$ and every $\lambda \geq 8/\|p\|_\infty + 8$, we have
\[
P\left\{ \|Rx\|_2 \geq \frac{j + 1}{2^{j+2}} \lambda \epsilon' \right\} \leq e^{-cm[\epsilon^{-1}(j+1)\lambda - 1]\|p\|_\infty}.
\tag{21}
\]

The constant $c > 0$ is an absolute constant.

Proof. We follow the same proof as the proof of Corollary 11.

For any $x \in \mathbb{R}^d$ such that $1 \geq \|x\|_2$, inequality (16) in Lemma 12 yields (19) for any $\lambda \in (0, 1)$ (using the fact that $\|x\|_2 \geq \|Px\|_2$).

As $1 - \epsilon - \epsilon_p \leq (1 - \epsilon_p)\|x\|_2 \leq \|Px\|_2$, inequality (17) yields (20) for any $\lambda \in (0, 1)$.

For any $x \in \mathbb{R}^d$ such that $2^{-j+1}\epsilon' \geq \|x\|_2$, inequality (18) yields (21).

\[\square\]

**B. CHAINING ARGUMENT**

This section contains the main result on which the proofs of Theorem 4 and Theorem 6 are based. In the lemma below, 1. $A$ is a subset of $\mathbb{R}^d$ with finite upper box-counting dimension strictly bounded by $s > 0$. Therefore, there exists $\epsilon_0' > 0$ such that its covering number satisfies $N_A(\alpha) < \alpha^{-s}$ for all $\alpha < \epsilon_0'$; 2. $\epsilon'$ is a fixed parameter in $(0, \epsilon_0')$ (its value will be chosen later on in Appendix C and Appendix D); 3. $B_j \subset A$, with $j \in \mathbb{N}$, is a set of centres of closed balls of radius $2^{-j}\epsilon'$ that covers $A$ with cardinality less than $2^{2s}\epsilon'^{-s}$; 4. $\pi_j$ is the mapping $\pi_j(y) \in \arg\min_{z \in B_j} \|y - z\|_2$; 5. $C_j$ is the finite set $\{\pi_{j+1}(y), \pi_j(y) \mid y \in A\}$.

Note that $\sup_{y \in A \setminus B_j} \|w - z\|_2 \geq 2^{-j+1}\epsilon'$.

The lemma below appears (in a slightly different form) in [3]. The proof is based on a chaining argument which is a powerful technique to obtain sharp bounds for the supremum of random processes [13–15].

**Lemma 14.** Let $B \in \mathbb{R}^{m \times d}$ be one of the random matrix considered in Theorem 4 or Theorem 6 and $t_1, t_2, u_1, u_2$ be any fixed values in $(0, \infty)$. Define $t := t_1 + t_2$ and $u := u_1 + u_2$. We have
\[
P\left\{ \inf_{y \in A} \|By\|_2 \leq 1 - t \right\} \leq e^{-s} \max_{y_0 \in B_0} P\{ \|By_0\|_2 \leq 1 - t_1 \}
+ \sum_{j \neq 0} 2^{2(j+1)s} \max_{(w, z) \in C_j} P\{ \|B(w - z)\|_2 \geq \frac{j + 1}{2^{j+2} t_2} \},
\tag{22}
\]
and
\[
P\left\{ \sup_{y \in A} \|By\|_2 \geq 1 + u \right\} \leq e^{-ss} \max_{y_0 \in B_0} P\{ \|By_0\|_2 \geq 1 + u_1 \}
+ \sum_{j \geq 0} 2^{2(j+1)s} \max_{(w, z) \in C_j} P\{ \|B(w - z)\|_2 \geq \frac{j + 1}{2^{j+2} t_2} \}.
\tag{23}
\]

Proof. We follow the same procedure as in [3].

We remark that any vector $y \in A$ can be written $y = \pi_0(y) + S(y)$ where $S(y) = \sum_{j \geq 0} (\pi_{j+1}(y) - \pi_j(y))$.

We start by proving (22). Noticing that
\[
\inf_{y \in A} \|B\pi_0(y)\|_2 - \sup_{y \in A} \|BS(y)\|_2 \leq \inf_{y \in A} \|By\|_2,
\]
we obtain
\[
P\left\{ \inf_{y \in A} \|By\|_2 \leq 1 - t \right\}
\leq P\left\{ \inf_{y \in A} \|B\pi_0(y)\|_2 - \sup_{y \in A} \|BS(y)\|_2 \leq 1 - t \right\}
= P\left\{ \inf_{y \in A} \|B\pi_0(y)\|_2 - \sup_{y \in A} \|BS(y)\|_2 \leq 1 - t_1 - t_2 \right\}.
\]
The union bound then yields
\[
P\left\{ \inf_{y \in A} \|B\pi_0(y)\|_2 - \sup_{y \in A} \|BS(y)\|_2 \leq 1 - t_1 - t_2 \right\}
\leq P\left\{ \inf_{y \in A} \|B\pi_0(y)\|_2 \leq 1 - t_1 \right\}
+ P\left\{ \sup_{y \in A} \|BS(y)\|_2 \geq t_2 \right\}
\leq P\left\{ \min_{y_0 \in B_0} \|By_0\|_2 \leq 1 - t_1 \right\}
+ P\left\{ \sup_{y \in A} \|BS(y)\|_2 \geq t_2 \right\}.
\tag{24}
\]
In the last step we used the fact that
\[
\inf_{y \in A} \|B\pi_0(y)\|_2 = \min_{y_0 \in B_0} \|By_0\|_2.
\]

Using again the union bound, we see that
\[
P\left\{ \min_{y_0 \in B_0} \|By_0\|_2 \leq 1 - t_1 \right\}
\leq e^{-ss} \max_{y_0 \in B_0} P\{ \|By_0\|_2 \leq 1 - t_1 \}.
\]

We continue by bounding the second term on the right-hand side of (24) to get a probability easier to evaluate. The trian-
gle inequality yields
\[ \mathbb{P} \left\{ \sup_{y \in A} \| BS(y) \|_2 \geq t_2 \right\} \leq \mathbb{P} \left\{ \sum_{j \geq 0} \sup_{y \in A} \| B (\pi_{j+1}(y) - \pi_j(y)) \|_2 \geq t_2 \right\}. \]

Then, noticing that \( \sum_{j \geq 0} 2^{-j-2}(j+1) = 1 \) (see, e.g., Section 4.2.3 in [16]), we obtain
\[
\mathbb{P} \left\{ \sum_{j \geq 0} \sup_{y \in A} \| B (\pi_{j+1}(y) - \pi_j(y)) \|_2 \geq t_2 \right\} = \mathbb{P} \left\{ \sum_{j \geq 0} \| B (\pi_{j+1}(y) - \pi_j(y)) \|_2 \geq \sum_{j \geq 0} \frac{j + 1}{2^{j+2} t_2} \right\} \leq \sum_{j \geq 0} \mathbb{P} \left\{ \sup_{y \in A} \| B (\pi_{j+1}(y) - \pi_j(y)) \|_2 \geq \frac{j + 1}{2^{j+2} t_2} \right\}.
\]

In the last step we used the union bound. We continue by observing that
\[
\sup_{y \in A} \| B (\pi_{j+1}(y) - \pi_j(y)) \|_2 = \max_{(w,z) \in C_j} \| B (w - z) \|_2.
\]

The cardinality of \( C_j \) is bounded by \( 2^{2(j+1) s} e^{-2 s} \). Using once more the union bound, we have
\[
\mathbb{P} \left\{ \max_{(w,z) \in C_j} \| B (w - z) \|_2 \geq \frac{j + 1}{2^{j+2} t_2} \right\} \leq \frac{2^{2(j+1) s}}{e^{2 s}} \max_{(w,z) \in C_j} \mathbb{P} \left\{ \| B (w - z) \|_2 \geq \frac{j + 1}{2^{j+2} t_2} \right\}.
\]

In total, we proved that
\[
\mathbb{P} \left\{ \inf_{y \in A} \| B g_y \|_2 \leq 1 - t \right\} \leq e^{-s} \max_{y_0 \in B_0} \mathbb{P} \left\{ \| B y_0 \|_2 \leq 1 - t_1 \right\} + \sum_{j \geq 0} \frac{2^{2(j+1) s}}{e^{2 s}} \max_{(w,z) \in C_j} \mathbb{P} \left\{ \| B (w - z) \|_2 \geq \frac{j + 1}{2^{j+2} t_2} \right\}.
\]

Repeating the same procedure for \( \sup_{y \in A} \| B g \|_2 \), one obtains (23).

\( \square \)

**C. PROOF OF THEOREM 4**

In this section, we prove Theorem 4.

Lemma 3 shows that \( N_{P, (S)} (\alpha) < \alpha^{-s} \) for all \( \alpha < \epsilon_0 \). As \( A \) is a random matrix, we are thus in the setting of Lemma 14 with \( A = P_2(S), B = A, \) and \( \epsilon_0 = \epsilon_0 \). Let \( t_1, B_0, C_j \) with \( j \in \mathbb{N} \) be as in Lemma 14. We will fix the value of \( \epsilon \) later on. We start by bounding the probabilities appearing in the right-hand side (rhs) of (22).

Let \( \delta \in (0, 1) \) and take \( t_1 = \epsilon + \delta/2 \) and \( t_2 = \delta/2 \) in Lemma 14. We recall that
\[
\sup_{y \in P_2(S)} \| y \|_2 \leq 1 \quad \text{and} \quad \inf_{y \in P_2(S)} \| y \|_2 \geq 1 - \epsilon,
\]
(see (4) and (5)) and that \( B_0 \subset P_2(S) \). If \( \delta < \min(2 \lambda_1, 1) \), then (14) in Lemma 11 shows that
\[
\max_{y_0 \in B_0} \mathbb{P} \left\{ \| A y_0 \|_2 \leq 1 - \epsilon - \delta/2 \right\} \leq e^{-c_0 \delta^2 / 4}.
\]
(25)

We have thus a bound on the first probability appearing in the rhs of (22). To bound the second one, we recall that
\[
\sup_{(w,z) \in C_j} \| w - z \|_2 \leq 2^{-j+1} e'.
\]
We can thus use the bound (15) of Lemma 11. In this lemma, we take \( \lambda = 8 \lambda_1 \) and \( e' = \delta/(16 \lambda_1) \) so that \( \lambda e' = \delta/2 = t_2 \). It is obvious that \( \lambda > \lambda_1 \). If \( \delta < \epsilon_0 \) then we have \( 2 \epsilon < \epsilon < \epsilon_0 \) as required in Lemma 14. Therefore,
\[
\max_{(w,z) \in C_j} \mathbb{P} \left\{ \| A (w - z) \|_2 \geq \frac{j + 1}{2^{j+2} t_2} \right\} \leq e^{-c_1 m (j+1) \lambda_1}.
\]
(26)

Combining (25) and (26), we arrive at
\[
\mathbb{P} \left\{ \inf_{y \in P_2(S)} \| A y \|_2 \leq 1 - t \right\} \leq \left( \frac{16 \lambda_1}{\delta} \right)^s e^{-c_0 \delta^2 / 4} + \left( \frac{32 \lambda_1}{\delta} \right)^{2 s} e^{c_1 m (\lambda - 1)} \sum_{j \geq 0} 2^{2 j s} e^{-c_1 m \lambda_1}.
\]

We observe that if
\[
m \geq \frac{\log(2)}{c_1 \lambda_1} (2s + 1),
\]
then \( \sum_{j \geq 0} 2^{2 j s} e^{-c_1 m \lambda_1} \leq 2 \). In addition, if
\[
m \geq \frac{4s}{c_1 (\lambda - 1)} \log \left( \frac{32 \lambda_1}{\delta} \right),
\]
then
\[
\left( \frac{32 \lambda_1}{\delta} \right)^{2 s} e^{-c_1 m (\lambda - 1)} \leq e^{-c_1 m (\lambda - 1)/2}.
\]

Finally, further imposing that
\[
m \geq \frac{8 s}{c_0 \delta^2} \frac{\log(2)}{c_1 \lambda_1}
\]
(29)
yields
\[
\mathbb{P} \left\{ \inf_{y \in P_2(S)} \| A y \|_2 \leq 1 - \epsilon - \delta \right\} \leq e^{-c_0 \delta^2 / 8} + 2 e^{-c_1 m (\lambda - 1)/2}.
\]

\(^{2}\text{Recall that } \lambda_1 > 8.\)
Define \( c_2 = \min(c_0/8, c_1 (\lambda_1 - 1)/2) \), then
\[
P \left\{ \inf_{y \in P_0(S)} \| Ay \|_2 \leq 1 - \epsilon - \delta \right\} \leq 3e^{-c_2 m \delta^2}.
\]
The right-hand side is less than \( \rho/2 \) provided that
\[
m \geq \frac{1}{c_2 \delta^2} \log \left( \frac{6}{\rho} \right).
\]
To conclude, we remark that there exists absolute constants \( D_1, D_2 > 0 \) such that if
\[
m \geq \frac{D_1}{\delta^2} \max \left\{ s \log \left( \frac{D_2}{\delta} \right), \log \left( \frac{6}{\rho} \right) \right\},
\]
then (27), (28), (29), and (30) hold. Notice that the final constraint on \( \delta \) is \( \delta < \min(1, 2\lambda_0, \epsilon_0) \).

Repeating the same procedure to bound
\[
P \left\{ \sup_{y \in P_1(S)} \| Ay \|_2 \geq 1 + u_1 + u_2 \right\},
\]
with \( u_1 = \delta/2 \) and \( u_2 = \delta/2 \), terminates the proof.

**D. PROOF OF THEOREM 6**

In this section, we prove Theorem 6.

We use Lemma 14 with \( \mathcal{A} = P_0(S) \), \( B = \mathbb{R} \), and \( \epsilon' = \epsilon_0 \).

Let \( \epsilon' \), \( B_0 \), \( C_j \) with \( j \in \mathbb{N} \) be as in Lemma 14. We start by bounding the probabilities appearing on the rhs of (22).

Let \( \delta \in (0, 1) \) and take \( t_1 = \epsilon + \epsilon_p + \delta/2 \) and \( t_2 = \delta/2 \) in Lemma 14. Inequality (20) yields
\[
\max_{y_0 \in B_0} P \left\{ \| R_{y_0} \| \leq 1 - \epsilon - \epsilon_p - \delta/2 \right\} \leq e^{-cm \delta^2/(4\mu_p^2)},
\]
(31)
where \( \mu_p \) is defined in (9). We have thus a bound on the first probability appearing on the rhs of (22). To bound the second one, we recall that \( \sup_{(w, z) \in C_j} \| w - z \|_2 \leq 2^{-j+1}\epsilon' \).

We can thus use the bound (21) of Lemma 11. In this lemma, we take \( \lambda = 16/\| p \|_\infty \) and \( \epsilon' = \delta \| p \|_\infty /32 \) so that \( \lambda \epsilon' = \delta/2 = t_2 \).

One can check that \( \lambda > 8/\| p \|_\infty + 8 \).

If \( \delta < \epsilon_0 \) then we have \( \epsilon' < \epsilon_0 \) as required in Lemma 14. Therefore,
\[
\max_{(w, z) \in C_j} P \left\{ \| R (w - z) \| \geq \frac{j + 1 \delta}{2^{j+2}} \right\} \leq e^{-cm [2(j+1)/\| p \|_\infty - 1] \| p \|_\infty},
\]
(32)
Gathering (31) and (32), we arrive at
\[
P \left\{ \inf_{y \in P_1(S)} \| R_y \|_2 \leq 1 - \epsilon \right\} \leq \left( \frac{32}{\delta \| p \|_\infty} \right)^s e^{-cm \delta^2/(4\mu_p^2)} + \left( \frac{64}{\delta \| p \|_\infty} \right)^{2s} e^{-c m (2-\| p \|_\infty)} \sum_{j \geq 0} 2^{2js} e^{-2cmj}.
\]
Then we observe that if
\[
m \geq \frac{\log(2)}{2c} (2s + 1),
\]
(33)
then \( \sum_{j \geq 0} 2^{2js} e^{-2cmj} \leq 2 \). In addition, if
\[
m \geq \frac{1}{c} \left( 2s \log \left( \frac{64}{\delta \| p \|_\infty} \right) + \log \left( \frac{1}{\rho} \right) \right),
\]
(34)
then
\[
\left( \frac{64}{\delta \| p \|_\infty} \right)^{2s} e^{-cm (2 - \| p \|_\infty)} \leq \rho.
\]

We used the fact that \( \| p \|_\infty \leq 1 \). Finally, also imposing that
\[
m \geq \frac{4\mu_p^2}{c^2} \left( s \log \left( \frac{32}{\delta \| p \|_\infty} \right) + \log \left( \frac{1}{\rho} \right) \right),
\]
(35)
yields
\[
P \left\{ \inf_{y \in P_1(S)} \| Ay \|_2 \leq 1 - \epsilon - \epsilon_p - \delta \right\} \leq 3\rho.
\]

To conclude, we remark that there exists absolute constants \( D_1, D_2 > 0 \) such that if
\[
m \geq D_1 \max \left\{ s \log \left( \frac{D_2}{\delta} \right), \log \left( \frac{6}{\rho} \right) \right\},
\]
then (33), (34), and (35) hold. Notice that the final constraint on \( \delta \) is \( \delta < \min(1, \epsilon_0) \).

Repeating the same procedure for
\[
P \left\{ \sup_{y \in P_1(S)} \| R_y \|_2 \geq 1 + u \right\},
\]
with \( u_1 = \delta/2 \) and \( u_2 = \delta/2 \), terminates the proof.