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# A transfer principle and applications to eigenvalue estimates for graphs

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## A transfer principle and applications to eigenvalue estimates for graphs

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**Abstract:** In this paper, we prove a variant of the Burger-Brooks transfer principle which, combined with recent eigenvalue bounds for surfaces, allows to obtain upper bounds on the eigenvalues of graphs as a function of their genus. More precisely, we show the existence of a universal constants  $C$  such that the  $k$ -th eigenvalue  $\lambda_k^{nr}$  of the normalized Laplacian of a graph  $G$  of (geometric) genus  $g$  on  $n$  vertices satisfies

$$\lambda_k^{nr} \leq C \frac{d_{\max}(g+k)}{n},$$

where  $d_{\max}$  denotes the maximum valence of vertices of the graph. This result is tight up to a change in the value of the constant  $C$ .

We also use our transfer theorem to relate eigenvalues of the Laplacian on a metric graph to the eigenvalues of its simple graph models, and discuss an application to the mesh partitioning problem.

**Key-words:** Laplacian, graph partitioning, Riemannian geometry

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## Un principe de transfert et ses applications à l'estimation des spectres de graphes

**Résumé :** Dans cet article, nous démontrons une variante du principe de transfert de Burger-Brooks qui, combinée avec des estimés récents sur les valeurs propres des surfaces, permet d'obtenir des bornes supérieures sur les valeurs propres des graphes en fonction de leur genre. Plus précisément, nous montrons l'existence d'une constante universelle  $C$  telle que la  $k$ -ième valeur propre  $\lambda_k^{nr}$  du laplacien normalisé d'un graphe  $G$  de genre géométrique  $g$  et à  $n$  sommets vérifie

$$\lambda_k^{nr} \leq C \frac{d_{\max}(g+k)}{n},$$

où  $d_{\max}$  est le degré maximum de  $G$ . Ce résultat est optimal à une constante multiplicative près.

Nous utilisons également notre résultat pour relier les valeurs propres d'un graphe métrique aux valeurs propres de ses modèles simples. Une autre application au problème du partitionnement de maillage est présentée.

**Mots-clés :** Laplacien, partitionnement de graphes, géométrie Riemannienne

## Contents

### 1 Introduction

The spectrum of the Laplacian of a finite graph reflects a lot of information about the structural properties of the graph and has been successfully used in a large variety of applications to other domains, *e.g.*, recently this was used in order to prove finiteness results on the number of points of bounded degree on towers of curves over number fields [?], to control the gonality of Drinfeld modular curves [?], and to provide bounds on the gonality of metric graphs [?, ?], to name some recent applications in arithmetic and algebraic geometry.

The study of the spectrum of a finite graph is in many ways related to the spectral theory of Riemannian manifolds. Starting from the pioneering works of Burger and Brooks, which connect the eigenvalues of a covering space to the ones of their Cayley-Schreier graphs [?, ?], and that of Colin de Verdière and Colbois on degenerations of Riemannian manifold to (metric) graphs [?, ?], finite graphs have been used as a way to get control on the eigenvalues of Riemannian manifolds (see *e.g.* [?] and also the earlier work of Buser [?]). While bounds on graph eigenvalues are usually used to prove bounds for Riemannian manifolds, it is somehow believed that the spectra of graphs should be more subtle, and it is intriguing to see that the rich spectral theory of Riemannian manifolds has not been much used so far to provide information on the spectral properties of general finite graphs. Results in geometric analysis have been, in many cases, a source of inspiration to state and prove corresponding results concerning finite graphs, essentially from scratch. These proofs, interestingly, usually bear some high level similarity to the ones used in Riemannian geometry. In this paper, we take a different route, and show how eigenvalue bounds for surfaces combined with basic spectral theory of (singular) surfaces, and a suitable transfer principle, allows to obtain eigenvalue estimates for graphs in terms of their geometric genus, providing tight bounds and improving recent results of [?]. As we will show in Section ??, our transfer principle can be used as well to provide uniform upper and lower bounds on the eigenvalues of metric graphs in terms of the eigenvalues of their simple graph models. In addition, it allows to generalize the mesh partitioning results of [?, ?] to arbitrary meshes, a result we discuss at the end of this paper.

#### 1.1 Eigenvalue estimates in graphs and surfaces: an overview

The eigenvalues of a bounded degree graph provide information on the existence of good clusterings of that graph. For clusterings in two classes, this is the statement of the celebrated discrete Cheeger inequality by Alon and Milman [?, ?]: a sparse balanced cut may be found if and only the second eigenvalue, or Fiedler value, is small. More recent results [?, ?] show that similar statements hold for  $k$ -way clusterings, whose optimal quality is shown to relate to the  $k$ -th eigenvalue. While this connection does in general get looser as  $k$  grows, it is about as tight as in the two-way case when the density of states near  $k$  is large enough. The proofs of these results are constructive and show that specific spectral clustering algorithms find a clustering whose quality is controlled by the spectrum of the graph.

In particular, upper bounds on the eigenvalues of a class of graphs directly translate into efficient clustering algorithms with quality guarantees. This motivated a series of work, starting with Spielman and Teng [?], who gave an  $O(1/n)$  bound for the Fiedler value of a bounded degree planar graph on  $n$  vertices, using a suitably centered circle packing representation of the graph. As a consequence, spectral clustering may be used to find a balanced separator of size  $O(\sqrt{n})$ . Kelner extended this result to an  $O((g + 1)/n)$  bound for (geometric) genus  $g$  graphs [?]. The argument is more involved and uses Riemann-Roch theorem to find a circle packing representation of the graph. In addition, this approach cannot work directly and a subdivision process has to be first performed so that a conclusion may be reached. Recently, Kelner, Lee, Price and Teng proved an  $O((g + 1) \log(g + 1)^2 k/n)$  upper bound for the  $k$ -th

eigenvalue [?]. Their bound comes from a metric graph partitioning result by Klein, Plotkin and Rao [?], which is applied to a suitably uniformized metric on the graph, found by solving a multicommodity flow problem.

Eigenvalue bounds for manifolds have a somewhat parallel history. Hersch [?] first proved an  $O(1/\text{vol}(M))$  bound for the Neumann value of the sphere  $\mathbb{S}^2$  equipped with a Riemannian metric. Yang and Yau [?] then showed that for genus  $g$  surfaces an  $O((g+1)/\text{vol}(M))$  bound holds, and Li and Yau improved the latter result by replacing the genus with the finer conformal invariant they defined [?]. It is interesting to notice that these proofs are quite similar at a high level to the ones later used in the graph setting. Conformal uniformization was used in place of circle packing representations, but the very same topological argument for centering the packing in the discrete case was used in the manifold case as well. For higher eigenvalues, Korevaar [?] established an  $O((g+1)k/\text{vol}(M))$  for genus  $g$  surfaces, and Hassannezhad [?] improved this to  $O((g+k)/\text{vol}(M))$  by combining the two methods of constructing disjoint capacitors of Grigor'yan, Netrusov and Yau [?], and Colbois and Maerten [?].

We extend the result of [?] to the graph setting using a suitable variant of the Burger-Brooks transfer method, c.f. Theorem ??, providing improvements and a uniform proof of the results of [?, ?, ?] on bounded genus graphs.

## 1.2 Statement of the main theorem on eigenvalues of bounded genus graphs

Let  $G = (V, E)$  be a finite simple graph, that we assume connected all through the paper. For two vertices  $u, v \in V$ , we write  $u \sim v$  if the two vertices  $u$  and  $v$  are connected by an edge in  $G$ . The valence of a vertex  $v$  of  $G$  is denoted by  $d_v$ . We denote by  $d_{\max}$  the maximum degree of vertices of the graph, and by  $n$  the number of vertices. The geometric genus of  $G$  is by definition the minimum integer  $g$  such that  $G$  can be embedded with no crossing on the compact orientable surface of genus  $g$ .

Denote by  $C(G)$  the vector space of all real valued functions  $f$  defined on the set of vertices of  $G$ . The (discrete) Laplacian  $\Delta$  and the normalized Laplacian  $\mathcal{L}$  of  $G$  are defined as follows: the Laplacian  $\Delta : C(G) \rightarrow C(G)$  is the linear operator which sends a function  $f \in C(G)$  to  $\Delta(f) \in C(G)$  defined by

$$\Delta(f)(v) = \sum_{u:u\sim v} f(v) - f(u).$$

The normalized Laplacian  $\mathcal{L} : C(G) \rightarrow C(G)$  compensates for uneven valence distribution. Namely, let  $S$  be the diagonal matrix whose entries are the valences of the vertices of  $G$ . The random walk Laplacian is defined as the operator  $S^{-1}\Delta$ , while the normalized Laplacian is the operator  $S^{-1/2}\Delta S^{-1/2}$ . It is not hard to see that both operators have the same eigenvalues. However, the latter is usually preferred because it is symmetric.

We denote by

$$\lambda_0 = 0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_{n-1}$$

the set of eigenvalues of  $\Delta$ , which we call the standard spectrum of  $G$ , and by

$$\lambda_0^{\text{nr}} = 0 < \lambda_1^{\text{nr}} \leq \dots \leq \lambda_{n-1}^{\text{nr}}$$

the set of all eigenvalues of the normalized Laplacian  $\mathcal{L}$ , which we call the normalized spectrum. Normalized eigenvalues are sometimes more natural than standard ones. For example, the normalized spectrum is, up to a reflection, the same as the spectrum of the natural random walk on the graph, while no such relation exists for the standard spectrum. The standard and normalized spectrum of  $G$  are easily seen to satisfy the inequalities  $d_{\min}\lambda_k^{\text{nr}} \leq \lambda_k \leq d_{\max}\lambda_k^{\text{nr}}$  for any  $k$ . In this paper we prove the following theorem.

**Theorem 1.1** *There exists a universal constant  $C$  such that the eigenvalues of the normalized Laplacian of any graph  $G$  on  $n$  vertices satisfy:*

$$\forall k \quad \lambda_k^{\text{nr}}(G) \leq C \frac{d_{\max}(g+k)}{n},$$

where  $d_{\max}$  and  $g$  are the maximum valence and the geometric genus of  $G$ , respectively.

The linear dependance in the maximum degree is clearly optimal, as can be seen by considering star graphs, which have lower bounded Fiedler value.

The above result also implies a similar bound for the eigenvalues of the standard Laplacian, at the expense of an extra  $d_{\max}$  factor. We note that Kelner, Lee, Price and Teng [?] give a similar bound for the standard spectrum with a linear rather than quadratic dependence in  $d_{\max}$ . However, their bound has a  $gk \log(g+1)^2$  dependence instead of our  $(g+k)$  dependence. In addition to simplifying and improving the result of [?] for bounded genus graphs, we note that the dependence in  $g$  and  $k$  of our estimate is tight, at least when  $g$  is sufficiently high.

Indeed, it is shown in [?] that for large  $g$ , there are area 1 and genus  $g$  Riemannian surfaces  $S$  with

$$\lambda_k(S) \geq \frac{4\pi}{5}(g-1) + 8\pi(k-1) - \epsilon$$

for any  $\epsilon > 0$ . Now, the subdivision argument behind the proof of theorem 5.2 in [?], implies the existence of a bounded degree genus  $g$  graph  $G$  with  $n$  vertices such that  $\lambda_k(G) \geq C\lambda_k(S)/n$ . Hence, at least for large enough  $n$  and  $g$ , there are graphs whose eigenvalues match the behaviour of our estimate.

We note that informally, the improvement over [?] means that the asymptotic behavior of graphs' eigenvalues do not depend on the (geometric) genus of the graph. This fact, which may be seen as a one-sided discrete form of Weyl's law for surfaces, is consistent with the intuition that at a small scale, bounded genus graphs behave like planar graphs.

### 1.3 Two-fold covers and discretized Laplacians

The proof of our theorem is based on a variant of the Burger-Brooks transfer principle which relates the Laplacian eigenvalues of a metric measured space to the eigenvalues of a certain discretized Laplacian.

Let  $M$  be a metric measured space regular enough to carry a Laplacian or a theory of Laplacian eigenvalues. This includes in particular the cases where either  $M$  is a smooth manifold with a smooth Riemannian metric  $g$ , or  $M$  is a compact smooth surface equipped with a conformal class of smooth Riemannian metrics  $g$  and a Radon measure  $\mu$ , c.f. Section ??, or  $M$  is a metric graph with the Lebesgue measure and the continuous Laplacian, c.f. Section ?. Let  $\mu$  be the measure on  $M$ , which coincides with the measure  $\mu_g$  in the case of a smooth Riemannian manifold, is equal to the Radon measure in the case of a measured metric surface, and is the Lebesgue measure in the case of a metric graph. A *2-fold cover* of  $M$  is a finite collection  $\mathcal{U} = (U_v)_{v \in V}$  of open subsets  $U_v$  such that almost every point in  $M$  is covered by exactly two subsets. Given a 2-fold cover  $\mathcal{U} = (U_v)_{v \in V}$ , we can form a graph  $G = (V, E)$  on the set of vertices  $V$  and with edges  $\{u, v\} \in E$  for two vertices  $u, v$  such that  $U_v \cap U_u \neq \emptyset$ . We define a weight function  $\omega : E \rightarrow \mathbb{R}$  which to any edge  $e = \{u, v\}$  of  $G$ , associates the weight  $\omega(e) = \mu(U_u \cap U_v)$ . The weighted valence  $d_v^\omega$  of a vertex  $v$  of  $G$  is defined by

$$d_v^\omega = \sum_{u: u \sim v} \mu(U_u \cap U_v).$$

We denote by  $\mathcal{L}_{\mathcal{U}}$  the normalized graph Laplacian associated to the weighted graph  $(G, \omega)$ , which we obtain from the standard weighted Laplacian by normalizing using the weighted valence, as in the previous section. Denote by  $\lambda_k(\mathcal{L}_{\mathcal{U}})$  the  $k$ -th smallest eigenvalue of  $\mathcal{L}_{\mathcal{U}}$ .

Our version of the Burger-Brooks transfer principle is stated as follows:

**Theorem 1.2** *Assume all the elements in a 2-fold cover  $\mathcal{U}$  of  $M$  have Neumann value at least  $\eta$ . Then for all  $k$  we have:*

$$\lambda_k(\mathcal{L}_{\mathcal{U}}) \leq 2 \frac{\lambda_k(M)}{\eta}.$$

The main difference with the classical versions of the transfer principle [?, ?, ?] is that we discretize the continuous Laplacian as a weighted normalized graph Laplacian instead of a combinatorial one, which allows for a closer connection between the two. The classical form of the transfer principle [?, ?] was used to connect the eigenvalues of a covering space to the ones of its Cayley-Schreier graph. Other instances applying to Vietoris-Rips graphs of dense point sample were found in [?]. Our variant here uses a different notion of graph approximation that involves particular weights. In addition, the above mentioned results take as input a partition of  $M$ , while our theorem is expressed in terms of two-fold covers, which add more flexibility. This version seems to be required to get our Theorem ?? on eigenvalues of bounded (geometric) genus graphs.

## 1.4 Applications to eigenvalue estimates in metric graphs and mesh partitioning

In addition to the previously mentioned application of Theorem ?? to eigenvalue estimates for graphs of bounded geometric genus, we provide another application of this result in Section ?? to the eigenvalues of metric graphs, leading to a uniform quantitative complement to Faber's theorem [?] on the spectral convergence of finite graphs to metric graphs, and give in Section ?? an algorithmic application to mesh partitioning in numerical analysis, generalizing the results of [?, ?] to anisotropic meshes.

## 2 Transfer principle for 2-fold covers: proof of Theorem ??

Let  $M$  be a measured metric space carrying a Laplacian or a theory of Laplacian eigenvalues. More specifically, consider one of the following situations:

- (i)  $M$  is a smooth Riemannian manifold;
- (ii)  $M$  is a smooth compact Riemannian surface equipped with a Radon measure  $\mu$ , c.f. Section ??.
- (iii)  $M$  is a metric graph (with its intrinsic metric and the Lebesgue measure), c.f. Section ??.

In any of the above cases, denote by  $\mathfrak{g}$  the corresponding Riemannian metric and by  $\mu_{\mathfrak{g}}$  the volume form induced by  $\mathfrak{g}$ . In addition, in case (ii), where  $M$  is a measured metric surface, we suppose that the Radon measure  $\mu$  is absolutely continuous with respect to  $\mu_{\mathfrak{g}}$ , and in case (i) and (iii), we simply let  $\mu = \mu_{\mathfrak{g}}$  in what follows.

Let  $U$  be an open subset of  $M$ , and denote by  $\bar{U}$  the topological closure of  $U$  in  $M$ . The Neumann value  $\eta(U)$  of  $U$  is defined as the infimum of the Rayleigh ratio  $\int_U |\nabla_{\mathfrak{g}} f|^2 d\mu_{\mathfrak{g}} / \int_U f^2 d\mu$  over all smooth functions  $f$  on  $U$  which extend continuously to  $\bar{U}$  and which satisfy  $\int_U f d\mu = 0$ :

$$\eta(U) := \inf_{f: \int_U f d\mu = 0} \frac{\int_U \|\nabla_{\mathfrak{g}} f\|^2 d\mu_{\mathfrak{g}}}{\int_U f^2 d\mu}.$$

Let now  $M$  be as above and consider a 2-fold cover  $\mathcal{U} = (U_v)_{v \in V}$  of  $M$ . Denote by  $\eta$  the minimum of  $\eta(U_v)$  for  $v \in V$ . Let  $G$  be the associated weighted graph with vertex set  $V$  and weight matrix  $[W_{\mathcal{U}}]_{u,v} = \mu(U_u \cap U_v)$  for  $u \neq v$ . Let  $\mathcal{L}_{\mathcal{U}}$  be the matrix of the associated normalized graph Laplacian,  $\mathcal{L}_{\mathcal{U}} = I - D_{\mathcal{U}}^{-1/2} W_{\mathcal{U}} D_{\mathcal{U}}^{-1/2}$ , where the matrix  $D_{\mathcal{U}}$  is diagonal with entries given by the vector  $W_{\mathcal{U}} \mathbf{1}$ .

**Proof:** [Proof of Theorem ??] Let  $v \in V$  and  $f$  any smooth function on  $M$ . By restricting  $f$  to  $U_v$  and subtracting the mean over  $U_v$ , we get:

$$\begin{aligned} \int_{U_v} \|\nabla_{\mathfrak{g}} f\|^2 d\mu_{\mathfrak{g}} &\geq \eta(U_v) \int_{U_v} \left( f - \frac{1}{\mu(U_v)} \int_{U_v} f d\mu \right)^2 d\mu \\ &\geq \eta \left( \int_{U_v} f^2 d\mu - \frac{1}{\mu(U_v)} \left( \int_{U_v} f d\mu \right)^2 \right), \end{aligned}$$

Summing the last inequalities over  $v \in V$  yields:

$$\frac{2}{\eta} \int_M \|\nabla_{\mathfrak{g}} f\|^2 d\mu_{\mathfrak{g}} \geq 2\|f\|_2^2 - \sum_v \frac{1}{\mu(U_v)} \left( \int_{U_v} f d\mu \right)^2, \quad (1)$$

where the  $L^2$  norm  $\|\cdot\|_2$  is with respect to the measure  $\mu$ . Denote by  $\mathbf{1}_{U_v}$  the characteristic function of the open set  $U_v$ , and let  $\phi_v = \mu(U_v)^{-1/2} \mathbf{1}_{U_v}$ . Define  $\Phi : L^2(M) \rightarrow C(G)$ , by

$$\Phi(f)(v) := \int_M f \phi_v,$$

on any vertex  $v$  of  $G$ . We see that the quadratic form in  $f$  in the right hand side of Equation (??) is given by  $2\|f\|_2^2 - \|\Phi f\|_2^2$ .

Let  $\epsilon > 0$ , and denote by  $\Lambda_{k+1}^\epsilon$  a  $(k+1)$ -dimensional space of smooth functions on  $M$  such that for any  $f \in \Lambda_{k+1} \setminus \{0\}$ , we have

$$\frac{\int_M \|\nabla_{\mathfrak{g}} f\|^2 d\mu_{\mathfrak{g}}}{\int_M f^2 d\mu} \leq (1 + \epsilon) \lambda_k(M).$$

Note that by the variational characterization of the eigenvalues (see e.g. (??) and (??)), such a space exists. For any  $f \in \Lambda_{k+1}^\epsilon$ , by inequality (??), we have:

$$\frac{2(1 + \epsilon) \lambda_k(M)}{\eta} \|f\|_2^2 \geq \frac{2}{\eta} \int_M \|\nabla_{\mathfrak{g}} f\|^2 d\mu_{\mathfrak{g}} \geq 2\|f\|_2^2 - \|\Phi f\|_2^2.$$

That is:

$$\|\Phi f\|_2^2 \geq 2 \left( 1 - \frac{(1 + \epsilon) \lambda_k(M)}{\eta} \right) \|f\|_2^2.$$

Let  $\Phi^*$  denote the adjoint of the operator  $\Phi : L^2(M) \rightarrow C(G)$ . From the variational characterization of the eigenvalues, this implies that the compact self-adjoint operator  $\Phi^* \Phi$  on  $L^2(M)$  has at least  $k+1$  eigenvalues greater than or equal to  $2(1 - (1 + \epsilon) \lambda_k(M) / \eta)$ . We can assume that this latter quantity is positive, otherwise there is nothing to prove since all the eigenvalues of the normalized Laplacian are at most 2. Since the non zero eigenvalues of  $\Phi^* \Phi$  are the same as the non zero eigenvalues of  $\Phi \Phi^*$ , we thus deduce that

$$\lambda_k(2I - \Phi \Phi^*) \leq \frac{2(1 + \epsilon) \lambda_k(M)}{\eta}. \quad (2)$$

To conclude the proof, it suffices to notice that

$$[\Phi \Phi^*]_{u,v} = \int_M \phi_u \phi_v = \frac{\mu(U_u \cap U_v)}{(\mu(U_u) \mu(U_v))^{1/2}}.$$

Because  $\mathcal{U}$  is a 2-fold cover, the  $v$ -th entry of the diagonal matrix  $D_{\mathcal{U}}$  is equal to  $\mu(U_v)$ . We thus easily check that  $2I - \Phi \Phi^* = \mathcal{L}_{\mathcal{U}}$ . Therefore, inequality (??) gives

$$\lambda_k(\mathcal{L}_{\mathcal{U}}) \leq \frac{2(1 + \epsilon) \lambda_k(M)}{\eta}.$$

Since this holds for any  $\epsilon > 0$ , the theorem follows.  $\square$

### 3 Eigenvalues of bounded genus graphs

#### 3.1 Eigenvalues on measured surfaces

We recall the variational approach to study eigenvalue problems for surfaces with measures [?], which provides a setting to study eigenvalue problems for singular surfaces. This was needed to state the application of Theorem ?? to measured metric surfaces, and will also be used in the proof of Theorem ??.

Let  $M$  be a smooth compact surface, possibly with boundary, which we suppose equipped with a smooth Riemannian metric  $g$ . Denote by  $\mu_g$  the induced volume form on  $M$ . Let  $\mu$  be a Radon measure on  $M$  which we suppose absolutely continuous with respect to the measure  $\mu_g$ . For a  $C^\infty$ -smooth function  $f \in L^2(M, \mu)$ , the Rayleigh quotient  $R_{M_g}(f, \mu)$  is defined by

$$R_{M_g}(f, \mu) := \frac{\int_M |\nabla_g f|^2 d\mu_g}{\int_M f^2 d\mu}.$$

The eigenvalues of the measured metric surface  $(M_h, \mu)$  are defined by the variational formula:

$$\lambda_k(M_g, \mu) := \inf_{\Lambda_{k+1}} \sup_{f \in \Lambda_{k+1}^*} R_{M_g}(f, \mu), \quad (3)$$

where  $\Lambda_{k+1} \subset L^2(M, \mu)$  varies over subspaces of dimension  $k+1$  which consist only of smooth functions on  $M$ , and  $\Lambda_{k+1}^* = \Lambda_{k+1} \setminus \{0\}$ . Note that in the case  $\mu = \mu_g$ , we recover the usual variational characterization of the eigenvalues of the Laplacian  $\Delta_g$  associated to the Riemannian surface  $M_g$ .

To see the point of introducing this formalism, assume that the two metrics  $g$  and  $h$  on  $M$  are conformally equivalent. From the conformal invariance of the Dirichlet integral, we see that  $R_{M_g}(f, \mu) = R_{M_h}(f, \mu)$ . In particular, letting  $\mu = \mu_h$ , we see that the spectra of the metric  $h$  within the conformal class of  $M_g$  coincides with the spectra of measured surface  $(M_g, \mu)$  for an appropriate Radon measure  $\mu$ . Now, if  $h$  is a metric with conical singularities, it is a classical fact that  $M$  is conformally equivalent to a constant curvature metric  $g$ , the conformal factor being square integrable with respect to the corresponding area form  $\mu_g$ . Thus the framework of measured metric surfaces allows in particular to define spectra of surfaces with conical singularities as the spectra of the measured metric surface  $(M_g, \mu_h)$  [?, ?, ?].

#### 3.2 Proof of Theorem ??

We apply Theorem ?? to get the claimed bounds on the eigenvalues of bounded genus graphs.

**Proof:** [Proof of Theorem ??] Let  $G$  be a connected unweighted graph with  $n$  edges,  $g$  be its geometric genus and  $d_{max}$  be its maximum degree. We may embed  $G$  in a genus  $g$  surface such that each component of the complement of  $G$  is a topological disk [?]. Note however that the closure of these faces might not be topological disks. To avoid this, we triangulate each component by adding edges in such a way that the degree of each vertex is multiplied by at most a constant, which is always possible. Letting  $G'$  be the obtained graph, the maximum degree and the number of edges obviously are within a constant factor in  $G$  and in  $G'$ . Also, the eigenvalues of  $G$  are at most a constant times the ones of  $G'$  by the characterization of the eigenvalues of the normalized Laplacian as generalized eigenvalues of the standard Laplacian together with the according diagonal matrix. Hence, without loss of generality, we may assume that  $G$  is the 1-skeleton of a triangulation of a genus  $g$  surface.

We next perform a subdivision of each triangle by adding a vertex and three *cone edges* connecting it to the vertices of the triangle. This produces a triangulated surface such that each triangle is incident

to exactly one graph edge. As a consequence, the open stars of the vertices of the graph form a 2-fold covering  $\mathcal{U}$  of the surface.

For reasons that will soon be clearer, we set the length of each graph edge to be 1 and the length of cone edges to be  $\cos(\pi/d_{max})^{-1}/2$ . The angles of all triangles at graph vertices are thus equal to  $\pi/d_{max}$ . Call  $M$  the resulting surface, equipped with the locally flat metric  $\mathfrak{h}$  induced by the triangles. The metric  $\mathfrak{h}$  has only conical singularities, and we denote by  $\lambda_k(M) = \lambda_k(M_{\mathfrak{h}})$ , the eigenvalues of the surface  $M$  as defined in the preceding subsection: thus  $\Lambda_k(M)$  is the  $k$ th eigenvalue of the measured metric surface  $(M_{\mathfrak{g}}, \mu = \mu_{\mathfrak{h}})$ , where  $\mathfrak{g}$  is a metric of constant curvature in the conformal class of  $\mathfrak{h}$ .

A result of Hassannezhad [?] states that there is a universal constant  $A$  such that for each  $k$ :

$$\lambda_k(M)\mu(M) \leq A(g + k).$$

Note that this result is not explicitly stated in the framework of measured metric surfaces in [?], however the proof given in [?] works also in this setting and gives the above statement. Using the result in the previous section, we can relate the eigenvalues of  $M$  with those of  $G$ . First note that the non zero entries of the weight matrix  $W_{\mathcal{U}}$  correspond to the edges of the graph. Also, they are all equal, hence  $\mathcal{L}_{\mathcal{U}}$  equals the normalized Laplacian of  $G$ . To apply the transfer result, we only need to lower bound the Neumann value of the open stars of graph vertices. We do so by again applying the transfer result to a specific 2-fold covering. Let  $O$  be the open star of a graph vertex. Thanks to the choice of edge lengths, graph vertices have non negative curvature, so by cutting  $O$  along a cone edge we may unfold it to the plane without overlap. The cutting operation can only decrease the Neumann value so it is sufficient to bound the one of the unfolded version  $O'$ . The planar set  $O'$  is the union of the stars of the graph edges, which are kites. Each of these kites will be part of the cover. Now cut each kite along the diagonal opposite to the graph edge. This cuts  $O'$  into a polygon  $P$  containing the graph vertex, together with one triangle for each kite. These components together with the kites form a 2-fold cover  $\mathcal{U}'$  of  $O'$ .

Kites and triangles in  $\mathcal{U}'$  are planar convex sets with at most unit diameter, hence their Neumann value is lower bounded by a constant [?]. Thanks to the choice of edge length, the cone angle at the graph vertex of polygon  $P$  is at most  $\pi$  so  $P$  is convex as well and the same conclusion applies to its Neumann value. Also, the non zero entries in  $W_{\mathcal{U}'}$  are equal, and correspond to a once subdivided star graph. Hence  $\mathcal{L}_{\mathcal{U}'}$  equals the normalized Laplacian of such a graph, which has lower bounded Neumann value. As a result, the Neumann value of  $O'$  is also lower bounded. It now suffices to notice that  $\mu(M) \simeq n/d_{max}$  to conclude.  $\square$

## 4 Eigenvalues of the Laplacian on metric graphs

We quickly recall the basic set-up concerning the spectral theory of metric graphs, and refer to [?, ?] for more details and basic facts on harmonic analysis on metric graphs.

Consider a graph  $G = (V, E)$  where each edge  $e$  in  $E$  has been assigned a length  $\ell_e > 0$ . The metric graph  $\Gamma$  is the geometric realization of  $(G, \ell)$  where each edge is replaced by an interval  $[0, \ell_e]$  of length  $\ell_e$ . Note that different pairs  $(G, \ell)$  can result to the same metric graph. The pair  $(G, \ell)$  is called a model of  $\Gamma$ . A model of  $\Gamma$  where  $G$  is a simple graph is called a simple graph model of  $\Gamma$ .

Consider a continuous function  $f : \Gamma \rightarrow \mathbb{R}$ , and let  $G = (V, E)$  be a simple graph model fo  $\Gamma$ . The directional derivative of  $f$  at any vertex  $v$  of  $G$  along an edge  $e = \{v, u\}$  in  $E$  is denoted by  $d_e f(v)$  and is defined by restricting the function  $f$  to the segment  $[0, \ell_e] \simeq [v, u]$ , and taking the derivative of  $f$  along the unit tangent vector at 0 on the segment.

The space of piecewise smooth function  $S(\Gamma)$  on  $\Gamma$  consists of all continuous functions  $f : \Gamma \rightarrow \mathbb{R}$  for which there exists a simple graph model  $G$  of  $\Gamma$  such that the restriction of  $f$  to the interior of all segments

$[0, \ell_e]$ , for any edge  $e$  of  $G$ , lies in  $C^2((0, \ell_e))$  and all the directional derivatives  $d_e f(v)$  are defined, for any vertex  $v$  in  $G$  and any edge  $e$  in  $G$  incident to  $v$ .

The metric graph  $\Gamma$  has a natural Lebesgue measure denoted by  $dx$ . The Laplacian of  $\Gamma$  is the (measure valued) operator  $\Delta$  on  $\Gamma$  which to a function  $f \in S(\Gamma)$  associated the measure

$$\Delta(f) := -f'' dx - \sum_p \sigma_p \delta_p,$$

where  $dx$  is the Lebesgue measure on  $\Gamma$ ,  $\delta_p$  is the Dirac measure at  $p$ , and  $\sigma_p$  is the sum of directional derivatives of  $f$  along incident edges:

$$\sigma_p = \sum_{e \ni p} d_e f(p),$$

in a simple graph model  $G$  for which  $f$  is smooth in the interior of every edge. Note that this does not depend on the choice of the model.

Define the space  $\text{Zh}(\Gamma)$  as the space of all functions  $f \in S(\Gamma)$  such that  $f'' \in L^1(\Gamma, dx)$ . The inner product  $(\cdot, \cdot)$  and the Dirichlet pairing  $(\cdot, \cdot)_{\text{Dir}}$  on  $\text{Zh}(\Gamma)$  are defined as follows.

$$(f, g) = \int_{\Gamma} f g dx.$$

$$(f, g)_{\text{Dir}} := \int_{\Gamma} f \Delta(g) = \int_{\Gamma} g \Delta(f) = \int_{\Gamma} f' g' dx = (f', g').$$

A function  $f$  in  $\text{Zh}(\Gamma)$  is an eigenfunction on  $\Gamma$  with eigenvalue  $\lambda$  if for any function  $g \in \text{Zh}(\Gamma)$ , we have  $(f, g)_{\text{Dir}} = \lambda(f, g)$ . The eigenvalues of  $\Delta$  are all nonnegative and, assuming  $\Gamma$  is connected, they form a discrete subset  $0 = \lambda_0 < \lambda_1 < \lambda_2 < \dots < \lambda_n < \dots$  of  $\mathbb{R}$ . In addition,  $\lambda_k$  has the following (usual) variational characterization:

$$\lambda_k = \inf_{\substack{\Lambda_{k+1} \subset \text{Zh}(\Gamma) \\ \dim(\Lambda_{k+1}) = k+1}} \sup_{f \in \Lambda_{k+1}} \frac{(f, f)_{\text{Dir}}}{(f, f)} \quad (4)$$

For a metric graph with a simple graph model  $(G, \ell)$ , and for a positive number  $\beta$ , the metric graph  $\beta\Gamma$  is the metric graph with the model  $(G, \beta\ell)$ , where the length of the edge  $e \in E(G)$  in  $\beta\Gamma$  is equal to  $\beta\ell_e$ . It is quite straightforward to see that the eigenvalues of  $\beta\Gamma$  are related to those of  $\Gamma$  by  $\lambda_k(\beta\Gamma) = \frac{1}{\beta^2} \lambda_k(\Gamma)$ .

Let  $(G, \ell)$  be a simple graph model of  $\Gamma$ . For any vertex  $v$  of  $G$ , denote by  $S_G(v)$  the metric star with center  $v$  in  $G$ , which is by definition the metric graph with the vertex set  $v \cup N_G(v)$  and with edge set the set of all edges in  $G$  incident on  $v$ , where the length of each edge  $e \in S_v$  is equal to  $\ell_e$ . Here  $N_G(v)$  refers to the set of neighbor vertices of  $v$  in  $G$ . The family of all the stars  $S_v$ , for  $v \in V(G)$ , forms a 2-fold cover  $\mathcal{S}$  of  $\Gamma$ . Denote by  $\lambda_k^{nr}(G, \ell)$  the  $k$ -th eigenvalue of  $\mathcal{L}_{\mathcal{S}}$ . Applying Theorem ??, we get

**Corollary 4.1** *Let  $l_{max}$  be the length of the longest edge in  $G$ . There is a universal constant  $C$  such that for any  $k$ , we have*

$$\lambda_k^{nr}(G, \ell) \leq C l_{max}^2 \lambda_k(\Gamma).$$

**Proof:** Denote by  $\eta_{(G, \ell)}$  the minimum Neumann value of all the stars  $S_v$  for  $v \in V(G)$ . We have to show that  $\eta_{(G, \ell)} \geq c/l_{max}^2$  for some constant  $c > 0$ . To this end, we use the convergence theorem of Faber [?]. Assume for simplicity that the branches of  $S_v$  have rational lengths. We can then form a regular subdivision of  $S_v$  with  $N$  points, for arbitrary large  $N$ . Denote by  $G_v$  the obtained graph. By the main theorem of Faber [?]:

$$\lim_{N \rightarrow \infty} N^2/l^2 \lambda_1(G_v) = \lambda_1(S_v)$$

where  $l$  is the total length of  $S_v$ . Now, since the best way to cut a metric star, in terms of the Cheeger ratio, is to cut (possibly part of) its longest branch, we see that the Cheeger constant of  $G_v$  is at least  $cl/(Nl_{max})$ , where  $c$  is a universal constant. Hence, by Cheeger's inequality,  $\lambda_1^{nr}(G_v) \geq cl^2/(N^2l_{max}^2)$ . Passing to the limit, we get the desired bound.  $\square$

We now show that under certain natural conditions, it is possible to achieve eigenvalue lower bounds closely matching the upper bound of the above corollary. For a simple graph model  $(G, \ell)$  of  $\Gamma$  denote by  $\ell_{\min, G}$  the minimum length of edges  $e$  in  $E(G)$ . A simple graph model of  $\Gamma$  is called length-balanced if the length of all edges  $e \in E(G)$  satisfy  $\ell_e \leq 2\ell_{\min, G}$ . We have the following stronger theorem.

**Theorem 4.2** *There are absolute constants  $c_1, c_2$  such that for any length-balanced simple graph model  $(G, \ell)$  of  $\Gamma$  on  $n$  vertices, and for any  $k \leq n - 1$ , we have*

$$\frac{c_2}{d_{\max}} \ell_{\min, G}^2 \lambda_k(\Gamma) \leq \lambda_k^{nr}(G, \ell) \leq c_1 \ell_{\min, G}^2 \lambda_k(\Gamma).$$

Define now  $\ell_{\min}$  as the supremum of  $\ell_{\min, G}$  over all length-balanced simple graph models of  $\Gamma$ . It is easy to see that there is a length-balanced simple graph model  $G$  of  $\Gamma$  such that  $\ell_{\min} = \ell_{\min, G}$ . For such a simple graph model  $(G, \ell)$ , define the model  $(G_k, \ell)$  as the  $k$ -th subdivision of  $G$  where each edge  $e$  is subdivided into  $k$  edges of equal lengths  $\ell_e/k$ . Note that  $G_k$  is length-balanced, has at least  $k + 1$  vertices, and has minimum edge length equal to  $\ell_{\min}/k$ . Thus as a consequence of Theorem ??, we get

**Corollary 4.3** *With the notations as above, there are absolute constants  $c_1$  and  $c_2$  such that for any metric graph  $\Gamma$ , we have*

$$\frac{c_2}{d_{\max}} \ell_{\min}^2 \lambda_k(\Gamma) \leq k^2 \lambda_k^{nr}(G_k, \ell) \leq c_1 \ell_{\min}^2 \lambda_k(\Gamma).$$

Our results, especially corollary ?? above, should be viewed as a quantitative complement to Faber's theorem [?] on the spectral convergence of finite graphs to metric graphs, in the sense they provide uniform upper and lower bounds on the eigenvalues of  $\Gamma$  in terms of eigenvalues of simple graph models of  $\Gamma$ . By contrast, Faber's methods are purely existential.

**Proof:** [Proof of Theorem ??] First note that since  $\lambda_k(\beta\Gamma) = \frac{1}{\beta^2} \lambda_k(\Gamma)$  and since  $\lambda_k^{nr}(G, \beta\ell) = \lambda_k^{nr}(G, \ell)$ , by the very definition, it will be enough to prove the theorem for  $\ell_{\min} = 1$ .

The right hand side inequality follows from Corollary ?? and the well-balanced property of the simple graph model  $G$  of  $\Gamma$ . We now prove the other inequality, namely the existence of  $c_2$  such that for any  $k \leq n - 1$ ,  $c_2 \lambda_k(\Gamma) \leq d_{\max} \lambda_k^{nr}(G, \ell)$  (still under the assumption that  $\ell_{\min} = 1$  and the length-balanced property of the model  $(G, \ell)$ ). Since the lengths of all edges are between 1 and 2, we get  $\lambda_k^{nr}(G, \ell) \geq \frac{1}{2d_{\max}} \lambda_k(G)$ . Indeed, letting  $g = D_S^{1/2} f$ , we have the following expression for the Rayleigh quotient

$$\frac{(g, \mathcal{L}_S g)}{(g, g)} = \frac{\sum_{e=\{u,v\} \in E} \ell(e)(f(u) - f(v))^2}{\sum_v d_v^\ell f(v)^2} \geq \frac{1}{2d_{\max}} \frac{\sum_{e=\{u,v\} \in E} (f(u) - f(v))^2}{\sum_v f(v)^2}$$

(where  $d_v^\ell = \sum_{u: u \sim v} \ell(\{u, v\})$ ), which using the variational characterization of the eigenvalues proves the claim. So it will be enough to show the existence of a constant  $c'_2$  such that

$$c'_2 \lambda_k(\Gamma) \leq \lambda_k(G).$$

Consider  $W_{k+1}$  the vector space of dimension  $k + 1$  generated by the first  $k + 1$  eigenfunctions  $g_0, \dots, g_k \in C(G)$  associated to  $\lambda_i(G)$ , for  $i = 0, \dots, k$ . Note that in particular

$$\lambda_k(G) \geq \sum_{\substack{u, v \in V(G) \\ u \sim v}} \frac{(g(u) - g(v))^2}{\sum_v g(v)^2}$$

for any  $g \in W_{k+1} \setminus \{0\}$ . We construct an injective linear map  $\Psi : C(G) \rightarrow \text{Zh}(\Gamma)$  such that for any  $g \in C(G) \setminus \{0\}$ , we have

$$\frac{(\Psi(g), \Psi(g))_{\text{Dir}}}{(\Psi(g), \Psi(g))} \leq 8 \sum_{\substack{u, v \in V(G) \\ u \sim v}} \frac{(g(u) - g(v))^2}{\sum_v g(v)^2}.$$

Applying the variational characterization of  $\lambda_k(\Gamma)$ , given in Equation (??), to the test space  $\Psi(W_k)$ , for  $k \leq n - 1$ , will then give the result.

Consider an edge  $e = \{u, v\}$  of  $G$ , and denote by  $u_e$  and  $v_e$  the two points at distance  $\frac{1}{4d_u}$  and  $\frac{1}{4d_v}$  from  $u$  and  $v$  on  $e$ , respectively, where  $d_u$  and  $d_v$  denote the valence of the vertices  $u$  and  $v$  in  $G$ , respectively. Note that the length of each segment  $[u_e, v_e]$  in  $\Gamma$  is at least  $\frac{1}{2}$ .

For any vertex  $v$  of  $G$ , denote by  $B_v$  the union of all segments  $[v, v_e]$  on the edges  $e$  adjacent to  $v$  in  $G$  (i.e.,  $B_v$  is the ball of radius  $\frac{1}{4d_v}$  around  $v$  in  $\Gamma$ ). For any function  $g \in C(G)$ , defined on the set of vertices of  $G$ , let  $\Psi(g)$  be the function on  $\Gamma$  which takes value equal to  $g(v)$  on each ball  $B_v$ , and which is affine linear of slope  $(g(v) - g(u))/\ell([u_e, v_e])$  on each segment  $[u_e, v_e]$ , for any edge  $e \in E(G)$ . Obviously,  $\Psi$  is an injective linear map from  $C(G)$  to  $\text{Zh}(\Gamma)$ .

Let now  $g \in C(G) \setminus \{0\}$  and denote  $f = \Psi(g)$ . We have

$$(f, f)_{\text{Dir}} = \int_{\Gamma} f'^2 dx = \sum_{e=\{u,v\} \in E(G)} \frac{1}{\ell([u_e, v_e])} (g(u) - g(v))^2 \leq 2 \sum_{\{u,v\} \in E(G)} (g(u) - g(v))^2.$$

Denote by  $B$  the union  $\cup_{v \in V(G)} B_v$ . Since each ball  $B_v$  has total length equal to  $1/4$ , we have

$$\int_{\Gamma} f^2 dx \geq \int_B f^2 dx = \frac{1}{4} \sum_{v \in V(G)} g(v)^2.$$

It thus follows from the two above estimates that for any  $g \in C(G) \setminus \{0\}$ , we have

$$\frac{(\Psi(g), \Psi(g))_{\text{Dir}}}{(\Psi(g), \Psi(g))} \leq 8 \sum_{\substack{u, v \in V(G) \\ u \sim v}} \frac{(g(u) - g(v))^2}{\sum_v g(v)^2},$$

and the theorem follows. □

## 5 Anisotropic mesh partitioning

In this final section we discuss a practical application of our transfer theorem to the mesh partitioning problem in scientific computing. Parallelizing finite elements computations requires to split the base mesh in such a way that communication between different pieces is minimized. This is naturally formalized as a (possibly multi-way) sparsest cut problem, which we may want to solve using spectral clustering. Guarantees for such methods in this setting were proved by Miller-Teng-Thurston-Vavasis and Spielman-Teng [?, ?]. More precisely, these papers show that spectral partitioning provides good cuts for meshes in  $d$ -dimensional Euclidean space provided that all  $d$ -simplices in the mesh are well-shaped, i.e. not too far from being equilateral.

It is not hard to design a 2-fold cover of a general mesh such that our transfer result provides guarantees for spectral clustering applied to anisotropic meshes. Specifically, let  $T$  be a triangulation of a domain  $D \subset \mathbb{R}^d$ . Performing a barycentric subdivision of all  $d$ -simplices gives a triangulation  $T'$ . For a  $d$ -simplex  $\sigma$  of  $T$ , let now  $U_\sigma$  be the interior of the union of  $\sigma$  with the  $d + 1$   $d$ -simplices of  $T'$  that

share a facet with  $\sigma$ . The collection of  $U_\sigma$  forms a 2-fold cover  $\mathcal{U}$  of the domain, and the corresponding Laplacian  $\mathcal{L}_\mathcal{U}$  is defined using weights  $w_{\sigma_1, \sigma_2}$  that are proportional to the sum of the volumes of  $\sigma_1$  and  $\sigma_2$ . Hence, assuming that neighboring  $d$ -simplices in  $T$  have volumes within a ratio of  $\kappa > 1$ , we see that the eigenvalues of  $\mathcal{L}_\mathcal{U}$  and those of the normalized Laplacian of the dual graph of  $T$  are also within a ratio of  $\kappa$ .

**Proposition 5.1** *The Neumann value of  $U_\sigma$  is at least  $C^{-1}\kappa^{-1}\epsilon^{-2}$  for some universal constant  $C > 0$ , where  $\epsilon$  is the maximum diameter of simplices in  $T$ .*

**Proof:** Let  $\tau_i, i = 1 \dots d+1$ , be the  $d$ -simplices in  $T'$  that share a facet with  $\sigma$ , and  $\sigma_i$  be the  $d$ -simplex in  $T'$  that is included in  $\sigma$  and shares a facet with  $\tau_i$ . The interiors of  $\sigma$ ,  $\tau_i$ , and of  $\tau_i \cup \sigma_i$  form a 2-fold cover of  $U_\sigma$ . The entries of the corresponding Laplacian are within a factor  $\kappa$  of those of the normalized Laplacian of the intersection graph of the elements of the cover, which is a once subdivided star graph. Such a star graph has Fiedler value lower bounded by a constant. Now each element in the cover is a convex set with diameter at most  $2\epsilon$ , so by [?] their Neumann value is lower bounded by a constant times  $\epsilon^{-2}$ . The claim then follows from theorem ??  $\square$

Therefore, Theorem ?? applied to the cover  $\mathcal{U}$  yields that the Fiedler value of the dual graph of  $T$  is at most  $2C\kappa^2\lambda_1(D)\epsilon^2$ . By Cheeger's inequality, a suitable spectral partitioning algorithm gives a balanced cut of size at most  $\kappa C' \sqrt{\lambda_1(D)}/\epsilon$ , for some constant  $C'$ . We note that if  $d$ -simplices in  $T$  are nearly equilateral, then  $\epsilon \simeq (\text{vol}(D)/n)^{1/d}$ , where  $n$  is the number of simplices in  $T$ . Hence in this case we recover the  $n^{1/d}$  behaviour proved in [?, ?] for the size of the cut, since the assumption that simplices are well-shaped implies an upper bound on  $\kappa$ . However, the methods used in those works do not seem to apply to the case of general anisotropic meshes.

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