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► **To cite this version:**

Julio Araujo, Nicolas Nisse, Stéphane Pérennes. Weighted Coloring in Trees. *SIAM Journal on Discrete Mathematics*, 2014, 28 (4), pp.2029 - 2041. 10.1137/140954167 . hal-01109194

**HAL Id: hal-01109194**

**<https://inria.hal.science/hal-01109194v1>**

Submitted on 25 Jan 2015

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# WEIGHTED COLORING IN TREES\*

JULIO ARAUJO<sup>†</sup>, NICOLAS NISSE<sup>‡</sup>, AND STÉPHANE PÉRENNES<sup>‡</sup>

**Abstract.** A proper coloring of a graph is a partition of its vertex set into stable sets, where each part corresponds to a *color*. For a vertex-weighted graph, the *weight of a color* is the maximum weight of its vertices. The *weight of a coloring* is the sum of the weights of its colors. Guan and Zhu defined the *weighted chromatic number* of a vertex-weighted graph  $G$  as the smallest weight of a proper coloring of  $G$  (1997). If vertices of a graph have weight 1, its weighted chromatic number coincides with its chromatic number. Thus, the problem of computing the weighted chromatic number, a.k.a. Max Coloring Problem, is NP-hard in general graphs. It remains NP-hard in some graph classes as bipartite graphs. Approximation algorithms have been designed in several graph classes, in particular, there exists a PTAS for trees. Surprisingly, the time-complexity of computing this parameter in trees is still open.

The Exponential Time Hypothesis (ETH) states that 3-SAT cannot be solved in sub-exponential time. We show that, assuming ETH, the best algorithm to compute the weighted chromatic number of  $n$ -node trees has time-complexity  $n^{\Theta(\log n)}$ . Our result mainly relies on proving that, when computing an optimal proper weighted coloring of a graph  $G$ , it is hard to combine colorings of its connected components.

**1. Introduction.** Given a loop-less graph  $G = (V, E)$ , a (*proper*)  $k$ -*coloring* of  $G$  is a surjective function  $c : V \rightarrow \{1, \dots, k\}$  that assigns to each vertex  $v \in V$  a *color*  $c(v) \in \{1, \dots, k\}$ , such that, for any  $\{u, v\} \in E$ ,  $c(u) \neq c(v)$ . Equivalently, a  $k$ -coloring of  $G$  is a partition  $c = (S_1, \dots, S_k)$  of  $V$  such that, for any  $1 \leq i \leq k$ ,  $S_i$  is a non-empty independent set of vertices that have the same color  $i$ . One of the most studied problems in Graph Theory consists in minimizing the number of colors of a proper coloring of a graph. Namely, GRAPH COLORING aims at computing the *chromatic number* of a graph  $G$ , denoted by  $\chi(G)$ , which is the minimum  $k$  for which  $G$  has a  $k$ -coloring. This is one of the Karp's NP-hard problems [8].

In [6], Guan and Zhu generalized GRAPH COLORING to vertex-weighted graphs. A (*vertex*) *weighted graph*  $(G, w)$  consists of a loop-less graph  $G = (V, E)$  and a weight function  $w : V \rightarrow \mathbb{R}_+$  over the vertices of  $G$ . Given a  $k$ -coloring  $c = (S_1, \dots, S_k)$  of a weighted graph  $(G, w)$ , the *weight of color*  $i$  ( $1 \leq i \leq k$ ) is defined by  $w(i) = \max_{v \in S_i} w(v)$ . The *weight of coloring*  $c$  is  $w(c) = \sum_{i=1}^k w(i)$ . The *weighted chromatic number* of  $(G, w)$ , denoted by  $\chi_w(G)$ , is the minimum weight of a proper coloring of  $(G, w)$ . The WEIGHTED COLORING Problem (also known as Max-coloring [15, 12, 13, 14, 11]) takes a weighted graph  $(G, w)$  and  $k \in \mathbb{R}_+$  as inputs and asks whether  $\chi_w(G) \leq k$  [6].

Observe that if the weight of each of the vertices of a graph  $(G, w)$  is equal to one, then the weight of a coloring is the number of its colors and thus,  $\chi_w(G) = \chi(G)$ . Therefore, WEIGHTED COLORING generalizes GRAPH COLORING to weighted graphs, and, as a consequence, this problem is NP-hard in general graphs. Moreover, WEIGHTED COLORING has been shown NP-hard in bipartite graphs [3], where GRAPH COLORING is trivial. In the last years, the WEIGHTED COLORING Problem has been addressed several times, however the complexity of this problem is surprisingly still unknown in the class of trees.

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\*This work was partly funded by the ANR project GRATEL, and promoted by the Inria/FUNCAP project ALERTE and the Inria associate-team AIDyNet and CNPq-Brazil (contract PDE 202049/2012-4). An extended abstract of this work has been accepted in STACS 2014.

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Here, we show that, if 3-SAT cannot be solved in sub-exponential time (Exponential Time Hypothesis), then WEIGHTED COLORING in trees is not in P.

**Related work.** WEIGHTED COLORING has been shown to be NP-hard in the classes of split graphs, interval graphs, triangle-free planar graphs with bounded degree, and bipartite graphs [3, 14, 2, 5, 15]. On the other hand, the weighted chromatic number of cographs and of some subclasses of bipartite graphs can be found in polynomial-time [3, 2]. Constant-factor approximation algorithms have been designed for various graph classes such as interval graphs, perfect graphs, etc. [14, 11, 12, 13, 4]. In particular, it is known that WEIGHTED COLORING can be approximated by a factor  $\frac{8}{7}$  in bipartite graphs and cannot be approximated by a factor  $\frac{8}{7} - \epsilon$  for any  $\epsilon > 0$  in this graph class unless  $P = NP$  [13].

Guan and Zhu showed that, given a fixed parameter  $r \in \mathbb{N}$ , the minimum weight of a coloring using at most  $r$  colors can be computed in polynomial-time<sup>1</sup> in the class of bounded treewidth graphs (a.k.a. partial  $k$ -trees) [6]. They left open the question of the time-complexity of the WEIGHTED COLORING Problem in this class (partial  $k$ -trees) and, in particular, in trees. In [13], a sub-exponential algorithm and a polynomial-time approximation scheme to compute the weighted chromatic number of trees are presented. Later on, Escoffier et al. proposed a polynomial-time approximation scheme to compute the weighted chromatic number of bounded treewidth graphs [5]. Kavitha and Mestre recently presented polynomial-time algorithms for subclasses of trees [9]. They show that computing the weighted chromatic number can be done in linear time in the class of trees where nodes with degree at least three induce a stable set [9].

In the last years, many studies have been done on the WEIGHTED COLORING Problem, however the complexity of this problem was still unknown on trees. WEIGHTED COLORING in trees has some intriguing properties. On the one hand, a reduction from another NP-hard problem was unlikely to exist due to the existence of a sub-exponential algorithm [13] (see also Section 2). Indeed, the problem cannot be NP-hard (under Karp reduction) unless problems in NP can be solved in time  $n^{\text{poly} \log n}$  - since the problem is solvable in  $n^{O(\log n)}$  (where  $n$  is the size of the input). On the other hand, all the classical methods to derive polynomial-time algorithms on trees failed [5, 9]. We provide here some explanation for these facts.

**Our results.** We show that, under the Exponential Time Hypothesis (ETH) (see Section 2), the best algorithm to compute the weighted chromatic number of trees has time-complexity  $n^{\Theta(\log n)}$ , where  $n$  is the number of vertices of the input tree. The existence of an algorithm that solves the WEIGHTED COLORING Problem in time  $n^{\Theta(\log n)}$  in bounded treewidth graphs follows easily from previous results. The difficulty is to prove that it is optimal under ETH. For this, we show that computing the weighted chromatic number of an  $n$ -node tree is as hard as deciding whether a 3-SAT formula with size  $\log^2 n$  can be satisfied, where the size of a formula is its number of variables<sup>2</sup>. So, our reduction is rather technical, but we hope that it contains ideas that may be used in other contexts. Along the line of our reduction, one will discover another surprising aspect: the difficulty of the problem not only comes from the graph structure, but rather relies on the way weights are structured. This implies that choosing the right color for a node is hard. We indeed use non-binary constraint

<sup>1</sup>We emphasize that this algorithm is exponential in  $r$

<sup>2</sup>Note that, in this paper, the number of clauses of the instances of SAT will always be polynomially bounded in the number of variables.

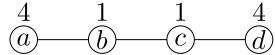


FIGURE 1. *The unique optimal weighted coloring of  $(P_4, w)$  uses strictly more than  $\chi(P_4)$  colors.*

satisfaction formulae (i.e., constraint satisfaction formulae over positive integers) as main tool. Lastly, our reduction also proves that computing an optimal weighted coloring of a disconnected graph may be hard even if optimal colorings of each of its components can be done in polynomial-time.

**Organization of the paper.** The remainder of the paper is organized as follows. In Section 2, we formally state the main results of the paper: in Section 2.1, an  $n^{\mathcal{O}(\log n)}$ -time algorithm is derived from previous works, and in Section 2.2 we prove our main result assuming a technical reduction (Proposition 2.2). The remaining part of the paper is devoted to the proof of Proposition 2.2. In Section 3, we give the main ideas of its proof. Finally, in Section 4, we prove a technical result (Proposition 3.3) which allows us to prove Proposition 2.2.

## 2. Preliminaries.

**2.1. Sub-exponential algorithm.** In this section, we show that there exists a sub-exponential algorithm to solve the WEIGHTED COLORING Problem in the class of bounded treewidth graphs (including trees). This is an almost trivial consequence of previous works that mainly relies on the number of colors used by weighted colorings in these graphs.

There exist weighted graphs  $G$  for which any optimal weighted coloring uses strictly more than  $\chi(G)$  colors: let us consider the 4-node path  $P_4$  with  $V(P_4) = \{a, b, c, d\}$ ,  $w(a) = w(d) = 4$  and  $w(b) = w(c) = 1$  (see Figure 1). Any coloring of  $P_4$  with  $2 = \chi(P_4)$  colors has weight 8, and the optimal weighted coloring  $\{\{a, d\}, \{b\}, \{c\}\}$  of  $(P_4, w)$  has weight  $\chi_w(P_4) = 6$  but uses 3 colors.

Luckily, the number of colors used by optimal weighted colorings can be bounded by  $\mathcal{O}(\log n)$  in the class of bounded treewidth graphs with  $n$  nodes. Indeed, Guan and Zhu studied the number of colors used by an optimal weighted coloring [6]. More precisely, they proved that the maximum number of colors of an optimal weighted coloring of a weighted graph  $(G, w)$  is its first-fit chromatic number  $\chi_{FF}(G)$  (a.k.a., *Grundy number*) [6]. This is tight since, for any graph  $G$ , there exists a weight function  $w$  such that an optimal weighted coloring of  $(G, w)$  uses  $\chi_{FF}(G)$  colors. On the other hand, for any  $n$ -node graph  $G$  with tree-width at most  $k$ ,  $\chi_{FF}(G) = \mathcal{O}(k \log n)$  [10]. In particular, this implies that, for any  $n$ -node tree, there is an optimal weighted coloring using  $\mathcal{O}(\log n)$  colors. Finally, in the class of bounded treewidth graphs and when the number  $r \in \mathbb{N}$  of colors is fixed, there is an algorithm (using dynamic programming on the tree-decomposition) that computes the minimum weight of a coloring using at most  $r$  colors in time polynomial in  $\mathcal{O}(n^r)$  where  $n$  is the number of vertices of the input graph [6].

By combining these results, the following proposition is straightforward:

**PROPOSITION 2.1.** *There exists an algorithm that solves the WEIGHTED COLORING Problem in time  $n^{\mathcal{O}(\log n)}$  in the class of bounded treewidth graphs (including trees), where  $n$  is the number of vertices of the input graph.*

**2.2. Main Result.** We now formalize our main result. Recall that an instance of the 3-SAT Problem is any Boolean formula  $\Phi(v_1, \dots, v_n)$  over the variables  $v_1, \dots, v_n$

in the conjunctive normal form (CNF) where each clause involves at most three variables. The *size* of  $\Phi$  is its number of variables, denoted by  $\eta$ . The 3-SAT Problem asks whether there exists a truth assignment to the variables of  $\Phi$  such that  $\Phi(v_1, \dots, v_\eta)$  is true. It is well known that the 3-SAT Problem is NP-complete [1]. A fundamental question is to know whether it can be solved in sub-exponential time. Note that, otherwise,  $P \neq NP$ .

**CONJECTURE 1. Exponential Time Hypothesis (ETH)[7].**

*3-SAT cannot be solved in time  $2^{o(\eta)}$  where  $\eta$  is the size of the instance.*

The main part of this paper is devoted to proving the following result.

**PROPOSITION 2.2.** *For any Boolean formula  $\Phi$  of size  $\eta$ , there exist a weighted tree  $(T, w)$  with  $n = 2^{\mathcal{O}(\sqrt{\eta})}$  vertices and  $M \in \mathbb{R}$  such that  $\Phi$  is satisfiable if and only if  $\chi_w(T) \leq M$ . Moreover,  $(T, w)$  and  $M$  are computable in time polynomial in  $n$ .*

Proposition 2.2 allows us to prove that there is no polynomial-time algorithm to solve the WEIGHTED COLORING Problem in trees, unless ETH fails.

**THEOREM 2.3.** *If ETH is true, then the best algorithm to compute the weighted chromatic number of an  $n$ -node tree  $T$  has time-complexity  $n^{\Theta(\log n)}$ .*

*Proof.* The existence of such an algorithm directly follows from Proposition 2.1. For purpose of contradiction, let us assume that there exists an algorithm  $\mathcal{A}$  that solves the WEIGHTED COLORING Problem in time  $n^{o(\log n)}$  in the class of trees, where  $n$  is the number of vertices of the input tree. Let  $\Phi$  be any Boolean formula of size  $\eta$ . By Proposition 2.2, there exists a weighted tree  $(T, w)$  with  $n = 2^{\mathcal{O}(\sqrt{\eta})} = 2^{o(\eta)}$  vertices and  $M \in \mathbb{R}$  such that  $\Phi$  is satisfiable if and only if  $\chi_w(T) \leq M$ . Consider the following algorithm to solve 3-SAT. For any Boolean formula  $\Phi$  of size  $\eta$ , first compute  $(T, w)$  and  $M$  in time  $2^{o(\eta)}$ , then use Algorithm  $\mathcal{A}$  to compute  $\chi_w(T)$  in time  $n^{o(\log n)} = 2^{o(\eta)}$ . By definition,  $\Phi$  is satisfiable if and only if  $\chi_w(T) \leq M$ . Therefore, the above algorithm solves the 3-SAT Problem in time  $2^{o(\eta)}$  where  $\eta$  is the size of the instance, contradicting ETH.  $\square$

Note that our result is actually stronger since we prove that if the WEIGHTED COLORING Problem can be solved in time  $n^{o(\log n)}$  in  $n$ -node trees then 3-SAT is solvable in time  $2^{\mathcal{O}(\sqrt{\eta})}$ .

The remaining part of the paper is devoted to the proof of Proposition 2.2.

**3. From boolean variables to integral variables.** Proposition 2.2 establishes a link between the WEIGHTED COLORING Problem and 3-SAT. Informally, to evaluate the time-complexity of the WEIGHTED COLORING Problem, the ideal way would be to reduce any 3-SAT formula  $\Phi$  to a weighted tree  $(T, w)$  such that (1) there is a correspondence between truth assignments of the variables of  $\Phi$  and the optimal colorings of  $T$ , and (2)  $\Phi$  is satisfiable if and only if  $\chi_w(T)$  is at most some pre-defined value  $M$  (depending on  $\Phi$ ). To do such a reduction, we would like to proceed as follows: given a boolean formula  $\Phi$  of size  $\eta$ , we build a weighted tree  $T$  such that any truth assignment of  $\Phi$  for which  $\Phi$  is satisfied, we have a coloring of  $T$  of bounded weight, where the weight of a color reflects the truth assignment of a variable. Hence, the desired weighted tree  $T$  must be such that any optimal coloring of  $T$  uses  $\eta$  colors. However, proceeding that way, since the number of colors in an optimal weighted coloring of an  $n$ -node tree is at most  $\mathcal{O}(\log n)$ ,  $T$  must have at least  $n = 2^\eta$  nodes. Hence, a polynomial-time algorithm to solve the WEIGHTED COLORING Problem in  $T$  would only lead to an exponential-time algorithm for deciding whether  $\Phi$  is satisfiable.

**3.1. From 3-SAT to INT-SAT.** To bypass the above problem, we will use an auxiliary formula. Intuitively, given a 3-SAT formula with  $\eta$  boolean variables, we will translate it into another logical formula with  $\sqrt{\eta}$  integral variables. Using this new

formula, we build a tree with  $2^{\sqrt{\eta}}$  nodes, where the weights of the colors in coloring of bounded weight will correspond to the integral values of the variables. Note that our method is close to the *Split and List* method of [16]. More formally,

**DEFINITION 3.1.** *Given a set of  $n \times m$  boolean variables  $(y_j^i)_{0 \leq i < n, 0 \leq j < m}$ , an integral assignment of these variables is a truth assignment such that, for any  $0 \leq i < n$ , exactly one variable  $y_j^i$ ,  $0 \leq j < m$ , receives value 1.*

*A boolean formula  $\Phi$  with  $n \times m$  boolean variables  $(y_j^i)_{i < n, j < m}$  is integrally satisfiable w.r.t.  $(y_j^i)_{0 \leq i < n, 0 \leq j < m}$  if there is an integral assignment of its variables that satisfies  $\Phi$ .*

*The INT-SAT Problem takes a formula  $\Phi$  with variables  $(y_j^i)_{0 \leq i < n, 0 \leq j < m}$  as input and asks whether  $\Phi$  is integrally satisfiable w.r.t.  $(y_j^i)_{0 \leq i < n, 0 \leq j < m}$ .*

In what follows, we widely use the fact that there is a one-to-one mapping between any integral assignment of a set of  $n \times m$  boolean variables  $(y_j^i)_{i < n, j < m}$  and the set of  $n$ -tuples  $(x_0, \dots, x_{n-1})$  of integers in  $\{0, \dots, m-1\}$ . Indeed, for any  $0 \leq i < n$ ,  $x_i = j$  where  $0 \leq j < m$  is the unique index such that  $y_j^i = 1$ .

We now show that 3-SAT can be sub-exponentially reduced to INT-SAT. This is an important ingredient of the proof of Proposition 2.2. We also think this result has its own interest and could be used in other contexts.

**THEOREM 3.2.** *For any instance  $\Phi$  of 3-SAT with size  $\eta$ , there is a Boolean formula  $\Phi_{int}$  of size  $n = 2^{\mathcal{O}(\sqrt{\eta})}$ , with variables  $(y_j^i)_{0 \leq i < \sqrt{\eta}, 0 \leq j < 2\sqrt{\eta}}$ , s.t.  $\Phi$  is satisfiable if and only if  $\Phi_{int}$  is integrally satisfiable w.r.t.  $(y_j^i)_{i,j}$ .  $\Phi_{int}$  can be computed in time  $\mathcal{O}(n)$  and it is a CNF formula where all variables appear positively.*

*Proof.* Let  $\Phi(u_0, \dots, u_{\eta-1})$  be an instance of 3-SAT of size  $\eta = N^2$  (if  $\eta \neq N^2$ , we can add dummy variables). For any two integers  $a < N$  and  $b < 2^N$ , let  $bit(a, b)$  correspond to the  $a$ -th bit of the binary representation of  $b$ .

Let  $\Phi_{int}$  be the formula obtained from  $\Phi$  by replacing each literal  $u_{iN+j}$ ,  $0 \leq i < N$  and  $0 \leq j < N$ , by  $\bigvee_{\{\ell | bit(j, \ell) = 1, 0 \leq \ell < 2^N\}} v_\ell^i$ . Then, each literal  $\bar{u}_{iN+j}$ ,  $0 \leq i < N$  and  $0 \leq j < N$  is replaced by  $\bigvee_{\{\ell | bit(j, \ell) = 0, 0 \leq \ell < 2^N\}} v_\ell^i$ . Hence,  $\Phi_{int}$  has  $N \cdot 2^N$  variables

$$(v_0^0, \dots, v_{2^N-1}^0, v_0^1, \dots, v_{2^N-1}^1, \dots, v_0^{N-1}, \dots, v_{2^N-1}^{N-1})$$

and  $poly(N)$  clauses of size  $\mathcal{O}(2^N)$ . Because  $\Phi$  is in CNF, it is also the case for  $\Phi_{int}$ . Moreover, all variables appear positively in  $\Phi_{int}$ .

It remains to show that  $\Phi_{int}$  is integrally satisfiable if and only if  $\Phi$  is satisfiable.

First, let us assume that  $\Phi$  is satisfiable. Let  $u_0, \dots, u_{\eta-1}$  be a valid assignment of its variables and, for any  $0 \leq i < N$ , let  $x_i$  be the integer with  $(u_{Ni}, \dots, u_{N(i+1)-1})$  as binary representation. Note that  $x_i \in \{0, \dots, 2^N - 1\}$ . Finally, for any  $0 \leq i < N$  and  $0 \leq j < 2^N$ , let us define  $v_j^i = 1$  if  $x_i = j$  and  $v_j^i = 0$  otherwise. By definition of  $\Phi_{int}$ ,  $(v_j^i)_{0 \leq i < N, 0 \leq j < 2^N}$  is a valid assignment and  $\Phi_{int}$  is therefore integrally satisfiable.

Conversely, let us assume that  $\Phi_{int}$  is integrally satisfiable and let  $(x_0, \dots, x_{N-1})$  be  $N$  integers representing a valid assignment for it. Let  $u_0, \dots, u_{\eta-1}$  be defined such that, for any  $0 \leq i < N$ ,  $(u_{Ni}, \dots, u_{N(i+1)-1})$  is the binary representation of  $x_i$ . Then,  $u_0, \dots, u_{\eta-1}$  is a satisfying assignment for  $\Phi$  which is satisfiable.  $\square$

**3.2. Proof of Proposition 2.2.** Theorem 3.2 allows us to reduce any 3-SAT instance  $\Phi$  of size  $\eta$  into an INT-SAT instance  $\Phi_{int}$  with size  $2^{\mathcal{O}(\sqrt{\eta})}$ . For simplicity of presentation, we assume that  $\sqrt{\eta}$  is an integer. The key point is that this reduction allows us to turn the choice of  $\eta$  boolean variables into the choice of  $\sqrt{\eta}$  integers in  $\{0, \dots, 2^{\sqrt{\eta}} - 1\}$ . Then, in further sections, we build a tree  $T$  with  $2^{\mathcal{O}(\sqrt{\eta})}$  vertices



from the formula  $\Phi_{int}$ , such that there is a one to one mapping between any optimal weighted coloring of  $T$  and the  $\sqrt{\eta}$ -tuples of integers in  $\{0, \dots, 2^{\sqrt{\eta}} - 1\}$ . Finally, our reduction ensures that the value of  $\chi_w(T)$  depends on the integral satisfiability of  $\Phi_{int}$  and therefore, on the satisfiability of  $\Phi$ . More formally, the next section is devoted to proving the following result:

**PROPOSITION 3.3.** *For any CNF Boolean formula  $\Phi_{int}$  of size  $\eta = n2^n$  where all variables  $(y_j^i)_{i,j}$  appear positively, there exist a weighted tree  $(T(\Phi_{int}), w(\Phi_{int}))$  with size polynomial in  $\eta$  and  $M \in \mathbb{R}$  s.t.  $\Phi_{int}$  is integrally satisfiable w.r.t.  $(y_j^i)_{i,j}$  if and only if  $\chi_w(T(\Phi_{int})) \leq M$ . The pair  $(T(\Phi_{int}), w(\Phi_{int}))$  and  $M$  are computable in time polynomial in  $\eta$ .*

The proof of Proposition 2.2 is straightforward from Theorem 3.2 and Proposition 3.3.

**4. Proof of Proposition 3.3.** This section is devoted to the proof of Proposition 3.3.

Let us introduce some notations. Let  $n \in \mathbb{N}$  and let  $m = 2^n$ . Let  $\Phi_{int}$  be a Boolean formula with  $n \times m$  variables  $\{y_i^j \mid 0 \leq i < n, 0 \leq j < m\}$  and  $L$  clauses, where  $L$  is polynomial in  $n$ . We assume that  $\Phi_{int}$  is a CNF formula and that each variable appears positively. Moreover, we may assume that each variable appears at least once. That is,  $\Phi_{int} = \bigwedge_{\ell \leq L} Q_\ell$  and, for any  $1 \leq \ell \leq L$ ,  $Q_\ell$  is the disjunction of  $p_\ell \geq 1$  positive variables.

Let  $\epsilon > 0$  such that  $n m \epsilon = o(\frac{1}{2^{4n}})$  and let

$$M = \sum_{i=0}^{4n+2} \frac{1}{2^i} + n(m-1)\epsilon < 2.$$

Let  $w_i^j = 1/2^i + j\epsilon$ , for any  $0 \leq i \leq 4n+3$  and  $0 \leq j \leq m$ . Let  $\mathcal{W} = \{w_i^j \mid 0 \leq i \leq 4n+3, 0 \leq j \leq m\}$  denote a set of weights. Note that the length of the encoding of these weights is polynomially bounded. For any  $0 \leq k \leq 3$ , let  $W_k = w_{4n+k}^0 = 1/2^{4n+k}$ . Finally, for any rooted tree  $T$ , let  $r(T)$  denote its root. A rooted tree  $S$  is a (*proper*) *subtree* of a rooted tree  $T$  if there is an edge  $e$  of  $T$  such that  $S$  is the connected component of  $T \setminus \{e\}$  that does not contain  $r(T)$ . We now define various subtrees required to build  $(T(\Phi_{int}), w)$ .

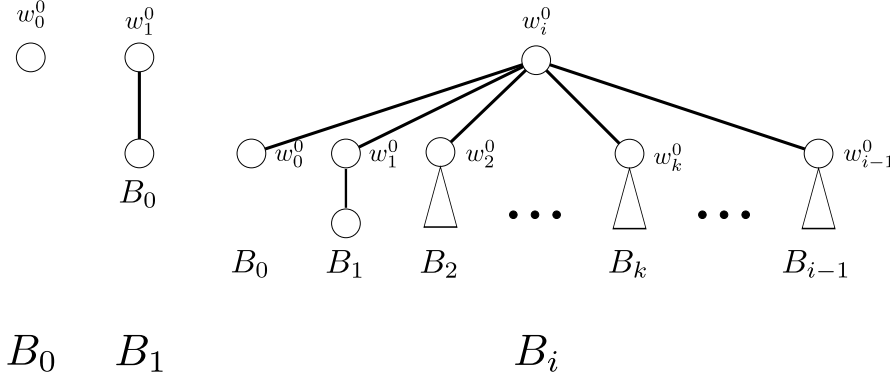
**4.1. Binomial trees.** We first define a particular family of *binomial trees*  $B_i$ ,  $0 \leq i \leq 4n+2$ . They will be used in the construction of  $T(\Phi_{int})$ . Their role is to force the color of most of the nodes in any coloring  $c$  of  $T(\Phi_{int})$  with  $w(c) \leq M$ . Note that, the notion of binomial trees has also been used in [2].

**DEFINITION 4.1.** *For any  $0 \leq i \leq 4n+2$ , let  $B_i$  be the weighted rooted tree defined recursively as follows (see Figure 2).*

- if  $i = 0$ , then  $B_0$  has a unique node with weight  $w_0^0$ ;
- otherwise,  $B_i$  has a root of weight  $w_i^0$  whose children are the roots of copies of  $B_0, B_1, \dots, B_{i-1}$ .

Note that  $B_i$  has  $2^i$  nodes and that it just contains nodes of weight  $w_j^0$ , for  $0 \leq j \leq i \leq 4n+2$ . We will use these binomial trees with two main goals in our reduction:

- enforce the number of used colors and the weights of these colors (up to an additive constant  $c\epsilon$ ) in any optimal weighted coloring of the tree we build from the 3-SAT formula;
- forbid the color  $i$  to appear in any vertex that is adjacent to a root of a binomial tree  $B_i$ .


 FIGURE 2. The construction of the binomial tree  $B_i$ .

We get these properties according to the following lemmas:

LEMMA 4.2. Let  $0 \leq i \leq 4n + 2$ . Let  $(T, w)$  be a weighted tree having  $B_i$  as subtree. If there exists a coloring  $c$  of  $(T, w)$  with  $w(c) \leq M$ , then, for any  $0 \leq k \leq i$ :

1. all vertices of  $B_i$  with weight in  $w_k^0$  receive the same color  $S_k$  of  $c$ ; and
2. there exists a unique color class  $S_k$  in  $c$  of weight in  $\{w_k^j \mid 0 \leq j \leq m\}$ .

*Proof.* The proof is by induction on the index  $i$ . In case  $i = 0$ , we prove both statements of the lemma at once by observing that any two vertices of  $(T, w)$  of weight in  $\{w_0^j \mid 0 \leq j \leq m\}$  must belong to the same color class  $S_0$ , otherwise we would conclude that  $w(c) \geq 2$ , that would be a contradiction to the fact that  $w(c) \leq M < 2$ .

Now, let  $0 \leq k \leq i$ , observe that the set of nodes of  $B_i$  with weight in  $w_k^0$  is an independent set that dominates the nodes of  $B_i$  with smaller weights (i.e., in  $\{w_{k'}^0 \mid k < k' \leq i\}$ ).

By induction hypothesis, for any  $0 \leq k < i$ , the set of nodes of  $B_i$  with weight in  $w_k^0$  receive the same color  $S_k$  of  $c$  and this color class is the unique with weight in  $\{w_k^j \mid 0 \leq j \leq m\}$ . Then, for any  $0 \leq k < i$ , the root of  $B_i$  cannot be colored  $S_k$ , since it has a neighbor with weight  $w_k^0$ . Let  $S_i$  be the color of the root of  $B_i$  in  $c$ . We proved that the color  $S_i$  cannot contain nodes with weight greater than  $w_i^{m-1}$  and that  $c$  cannot have another color  $S'_i \neq S_i$  with weight in  $\{w_i^j \mid 0 \leq j \leq m\}$ , because, otherwise the weight of  $c$  would be at least  $\frac{1}{2^i} + \sum_{k=0}^i \frac{1}{2^k} = 2 > M$  in both cases.  $\square$

COROLLARY 4.3. Let  $(T, w)$  be a weighted tree having  $B_{4n+2}$  as subtree. Let  $c$  be any coloring of  $(T, w)$  s.t.  $w(c) \leq M$ . Then,  $c = (S_0, \dots, S_k)$  with  $k \geq 4n + 2$  and, for any  $0 \leq i \leq 4n + 2$ ,  $S_i$  is the unique color with weight in  $\{w_i^j \mid 0 \leq j \leq m\}$ .

The trees we consider below will always satisfy the requirements of Corollary 4.3. Therefore, we are able to identify a color by its weight. In other words, in what follows, for any coloring  $c = (S_0, \dots, S_k)$  of weight at most  $M$  and for any  $i \leq 4n + 2$ ,  $S_i$  will be the unique color such that  $w(S_i) \in \{w_i^j \mid 0 \leq j \leq m\}$ .

Recall that we defined, for any  $0 \leq k \leq 3$ ,  $W_k = w_{4n+k}^0 = 1/2^{4n+k}$ . By a slight abuse of notation, for any  $0 \leq k \leq 3$ , we denote  $W_k = S_{4n+k}$  as the unique color with weight  $W_k$ .

**4.2. Auxiliary trees and Variable-trees.** This section is mainly devoted to the construction of subtrees that will represent the boolean variables. First, the family of *auxiliary trees*  $A_i^j$ ,  $0 \leq i < 4n$ ,  $0 \leq j \leq m$ , will be used to introduce some choice



when coloring  $T(\Phi_{int})$ .

DEFINITION 4.4. For any  $0 \leq i < 4n, 0 \leq j \leq m$ , let  $A_i^j$  be the weighted rooted tree defined as follows (see Figure 3). Note that  $A_i^j$  consists of  $2^{4n}$  nodes.

1. Let  $u$  be its root with weight  $w(u) = W_0$ , and connect it to a node  $v$  (its subroot) with weight  $w(v) = w_i^j$ ;
2.  $v$  is made adjacent to the root of a copy of  $B_\ell$ , for any  $0 \leq \ell < i - 1$ ;
3.  $u$  is made adjacent to the root of a copy of  $B_\ell$ , for any  $0 \leq \ell < 4n, \ell \neq i - 1$ .

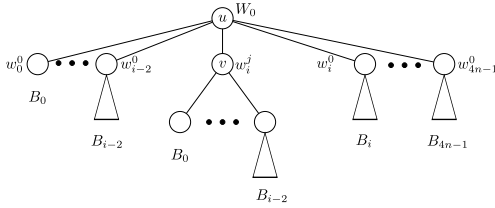


FIGURE 3. Auxiliary tree  $A_i^j$ .

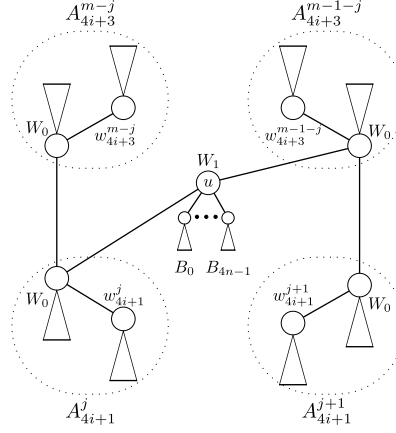


FIGURE 4. The variable tree  $T(y_i^j)$ .

LEMMA 4.5. Let  $0 \leq i < 4n$  and  $0 \leq j \leq m$ . Let  $(T, w)$  be any weighted tree having  $B_{4n+2}$  and  $A_i^j$  as subtrees. Let  $u$  and  $v$  be the root and the sub-root of  $A_i^j$ , respectively. For any coloring  $c$  of  $(T, w)$  with weight  $w(c) \leq M$ , then it holds:

- either  $v$  is colored  $S_{i-1}$  and  $u$  must be colored with the color  $W_0$ ;
- or  $v$  is colored  $S_i$  (therefore,  $w(S_i) \geq w_i^j$ ) and  $u$  is colored either with  $S_{i-1}$  or with  $W_0$ .

*Proof.* Recall that, by Corollary 4.3, we can identify the colors of  $c$  and their weights. By Lemma 4.2, the root of each subtree  $B_k$ ,  $0 \leq k < 4n$ , must be colored with  $S_k$  and then the sub-root  $v$  can be colored only with color  $S_{i-1}$  or  $S_i$ . Note that, if  $v$  is colored with color  $S_p$  for some  $p > i$ , then  $w(S_p) \geq w_i^j$ , contradicting Corollary 4.3. In the first case,  $u$  is adjacent to a node with color  $S_k$ , for any  $k < 4n$ . Therefore,  $u$  must be colored with color  $S_{4n} = W_0$ .

Otherwise,  $u$  is adjacent to a node with color  $S_k$ , for any  $k < 4n$  but  $k = i - 1$ . Moreover,  $u$  cannot be colored with  $S_k$  for  $k > 4n$  since otherwise,  $w(c) > M$  ( $S_k$  would have weight strictly more than  $w_k$ ). Hence,  $u$  can only be colored with  $S_{i-1}$  or  $W_0$ .  $\square$

Intuitively, the previous lemma states that, either we “pay”  $j\epsilon$  in the weight of color  $S_i$ , or  $u$  must be colored with the color  $W_0$ . We now define the *variable-trees*  $T(y_i^j)$  using the auxiliary trees.

DEFINITION 4.6. For any  $0 \leq i < n, 0 \leq j < m$ , let  $T(y_i^j)$  be the weighted rooted tree, representing the variable  $y_i^j$ , defined as follows (see Figure 4):

- let  $u$  be its root with weight  $w(u) = W_1$  and connected to the root of a copy of  $B_\ell$ , for any  $0 \leq \ell < 4n$ ;
- take one copy of  $A_{4i+1}^j, A_{4i+1}^{j+1}, A_{4i+3}^{m-j}$  and  $A_{4i+3}^{m-1-j}$  and:
  - connect  $r(A_{4i+1}^j)$  to  $r(A_{4i+3}^{m-j})$ , and  $r(A_{4i+1}^{j+1})$  to  $r(A_{4i+3}^{m-1-j})$ ;
  - connect  $u$  with  $r(A_{4i+1}^j)$  and  $r(A_{4i+3}^{m-j-1})$ .

Note that  $T(y_i^j)$  consists of  $\mathcal{O}(2^{4n})$  nodes (i.e. polynomial in  $nm$ ).

Intuitively, setting the color of the root of variable-tree  $T(y_i^j)$  to  $W_0$  or  $W_1$  will correspond to setting the corresponding variable  $y_i^j$  to true or false respectively. Moreover, next lemma shows that, for a fixed  $i$ , the weights of the classes  $S_{4i+3}$  and  $S_{4i+1}$  impose that at most one variable  $y_i^j$ ,  $0 \leq j < m$ , is set to true.

LEMMA 4.7. *Let  $(T, w)$  be any weighted tree having  $B_{4n+2}$  as subtree and containing  $T(y_i^j)$  as subtree, for all  $0 \leq i < n$  and  $0 \leq j < m$ . Let  $c$  be a coloring of  $T$  with weight  $w(c) \leq M$ .*

*Then, there are  $(j_0, \dots, j_{n-1}) \in \{0, \dots, m-1\}^n$  such that each root  $u$  of each subtree  $T(y_i^j)$ , for any  $0 \leq i < n$  and  $0 \leq j < m$ , satisfies:*

- *if  $j \neq j_i$ , then the color of  $u$  in  $c$  must be  $W_1$ ;*
- *otherwise, no neighbors of  $u$  in  $T(y_i^j)$  is colored with  $W_0$  and  $u$  is colored either  $W_0$  or  $W_1$ .*

*Proof.* Since  $T$  contains  $B_{4n+2}$ , by Corollary 4.3, a coloring  $c = (S_0, \dots, S_k)$  of weight  $w(c) \leq M$  is such that  $k \geq 4n+2$ , and, for any  $0 \leq i \leq 4n+2$ ,  $S_i$  is the unique color such that  $w(S_i) \in \{w_i^j \mid 0 \leq j \leq m\}$ . In particular,  $w(c) \geq \sum_{i=0}^{4n+2} 1/2^i = M - n(m-1)\epsilon$ .

For any  $0 \leq i < n$ , let  $0 \leq j_i \leq m$  be such that  $w(S_{4i+1}) = w_{4i+1}^{j_i}$ .

First, let us assume that  $j_i < m$ . In particular, this means that every sub-root of a subtree  $A_{4i+1}^r$ , for each  $j_i < r \leq m$ , is colored  $S_{4i}$  (recall that its color is either  $S_{4i}$  or  $S_{4i+1}$ , by Lemma 4.5). Consequently, any root of a subtree  $A_{4i+1}^r$ , for each  $j_i < r \leq m$ , must be colored  $W_0$ . Therefore, by the construction of the variable-trees, any root of a subtree  $A_{4i+3}^{m-r}$ , for each  $j_i < r \leq m$ , cannot be colored  $W_0$  because it is adjacent to a root of a subtree  $A_{4i+1}^r$ . Thus, by Lemma 4.5, it must be colored  $S_{4i+2}$  and the color of each sub-root of  $A_{4i+3}^{m-r}$  must be  $S_{4i+3}$ . Consequently,  $w(S_{4i+3}) \geq w_{4i+3}^{m-(j_i+1)}$ . Hence, for any  $0 \leq i < n$ , if  $j_i < m$ , we conclude that  $w(S_{4i+3}) + w(S_{4i+1}) \geq w_{4i+1}^{j_i} + w_{4i+3}^{m-(j_i+1)} = (m-1)\epsilon + 1/2^{4i+1} + 1/2^{4i+3}$ .

On the other hand, if  $j_i = m$ , it follows directly that  $w(S_{4i+3}) + w(S_{4i+1}) \geq m\epsilon + 1/2^{4i+1} + 1/2^{4i+3}$ .

Since  $w(c) \leq M$ , it implies that, for any  $0 \leq i < n$ ,  $j_i < m$  and  $w(S_{4i+3}) = w_{4i+3}^{m-j_i-1}$  and, for any  $0 \leq 2k < 4n$ ,  $w(S_{2k}) = w_{2k}^0$ .

Consequently, since  $w(S_{4i+3}) = w_{4i+3}^{m-j_i-1}$ , by a similar argument, the roots of all subtrees  $A_{4i+3}^{m-j}$ , for each  $0 \leq j \leq j_i$ , must be colored  $W_0$  and, then, the roots of all subtrees  $A_{4i+1}^j$ , for each  $0 \leq j \leq j_i$ , must be colored  $S_{4i}$ .

Let  $0 \leq i < n$  and  $0 \leq j < m$ . Consider a subtree  $T(y_i^j)$  of  $T$ . If  $j \neq j_i$ , then (exactly) one of the roots of  $A_{4i+1}^j$  and  $A_{4i+3}^{m-1-j}$  must be colored  $W_0$ . In that case, the color of the root  $u$  of  $T(y_i^j)$  must be  $W_1$ . Indeed,  $u$  is adjacent to the root of  $B_k$ ,  $0 \leq k < 4n$ , and therefore it cannot be colored  $S_k$ . Moreover, if  $u$  is colored  $W_2$ , then we have a contradiction as  $w(c) > M$ , because  $w(u) = W_1$ . On the other hand, if  $j = j_i$ , the root of  $A_{4i+1}^j$  is colored with  $S_{4i}$  and the root of  $A_{4i+3}^{m-1-j}$  is colored with  $S_{4i+2}$ . In particular, none of the roots of  $A_{4i+1}^j$  and  $A_{4i+3}^{m-1-j}$  are colored with  $W_0$ . Therefore,  $u$  is colored either  $W_0$  or  $W_1$  ( $u$  cannot have color  $S_k$  for  $k > 4n+1$  since otherwise  $w(c) > M$ ).  $\square$

**4.3. Clause-trees and definition of  $T(\Phi_{int})$ .** We define the subtrees representing the clauses and combine them to get  $T(\Phi_{int})$ .

DEFINITION 4.8. *Let  $1 \leq \ell \leq L$  and let  $Q_\ell = \bigvee_{1 \leq k \leq p_\ell} u_k$  be any clause of  $\Phi_{int}$  (recall that, for any  $1 \leq k \leq p_\ell$ ,  $u_k \in \{y_i^j \mid 0 \leq i < n, 0 \leq j < m\}$ ). For any*

$1 \leq k \leq p_\ell$ , let  $T(Q_\ell^k)$  be the rooted weighted tree defined recursively as follows:

1.  $T(Q_\ell^1) = T(u_1)$ ;
2. for any  $k > 1$ , start with one copy of  $T(Q_\ell^{k-1})$  with root  $a$  and one copy of  $T(u_k)$  with root  $b$ . Let  $g, d$  be two nodes with weight  $W_1$  and  $e, f$  be two nodes with weight  $W_2$ . For each node  $v \in \{g, d, e, f\}$ , and for any  $0 \leq i < 4n$ , add one copy of  $B_i$  and make its root adjacent to  $v$ . Add one copy of  $B_{4n+1}$  and make its root adjacent to  $e$ . Finally, we add the edges  $\{\{a, f\}, \{b, g\}, \{g, f\}, \{d, e\}, \{e, f\}\}$  and  $d$  is chosen as the root.

Let us note  $T(Q_\ell) = T(Q_\ell^{p_\ell})$  the clause-tree corresponding to  $Q_\ell$  and that consists of  $\mathcal{O}(p_\ell 2^{4n})$  nodes (i.e. polynomial in  $nm$ ).  $T(Q_\ell^k)$  is depicted in Figure 5.

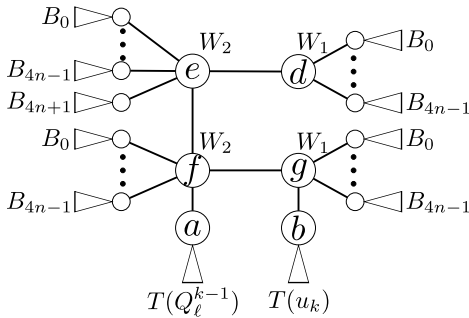


FIGURE 5. The clause tree  $T(Q_\ell^k)$ .

The next lemma describes how a coloring of a clause-tree  $T(Q_\ell^k)$  expresses the fact that the corresponding clause  $\bigvee_{1 \leq h \leq k} u_h$  is satisfied or not. Informally, the color of the root of  $T(Q_\ell^k)$  must be  $W_1$  if the clause is not satisfied, i.e., if  $\bigvee_{1 \leq h < k} u_h$  is not satisfied (in which case, the root  $a$  of  $T(Q_\ell^{k-1})$  is colored  $W_1$ ) and if  $u_k$  is assigned to false (in which case, the root  $b$  of  $T(u_k)$  is colored  $W_1$ ). Otherwise, it will be possible to color the root of  $T(Q_\ell^k)$  with  $W_0$ .

**LEMMA 4.9.** *Let  $(T, w)$  be any weighted tree having  $B_{4n+2}$  as subtree and containing  $T(Q_\ell^k)$  as a subtree ( $1 \leq \ell \leq L$ ,  $1 \leq k \leq p_\ell$ ) where  $Q_\ell^k = \bigvee_{1 \leq h \leq k} u_h$  is the prefix of a clause of  $\Phi_{int}$ . Let  $c$  be any coloring of  $T$  with weight  $w(c) \leq M$ . The color of  $d$  is either  $W_0$  or  $W_1$ . Moreover, if  $a$  and  $b$  are colored  $W_1$ , then the color of the root  $d$  of  $T(Q_\ell^k)$  must be  $W_1$ ;*

*Proof.* We prove it by induction on the number of variables  $k$  of  $Q_\ell^k$ . Observe that in case  $k = 1$ , then  $T(Q_\ell^k)$  is a variable-tree and the lemma trivially holds. Indeed, vertices  $a, g, d, e, f$  do not exist and the root is  $b$ , thus the first statement is trivially satisfied, and, by Lemma 4.7, the color of its root must be either  $W_0$  or  $W_1$ .

Now, consider that  $a$  and  $b$  are roots of a clause-tree on  $k - 1$  variables  $T(Q_\ell^{k-1})$  and of a variable-tree, respectively. By Lemma 4.7 and by the inductive hypothesis, the colors of  $a$  and  $b$  are either  $W_0$  or  $W_1$ .

In case  $c(a) = c(b) = W_1$ , by the hypothesis  $w(c) \leq M$ , by Lemma 4.2 and Corollary 4.3, we conclude that  $g$  is colored  $W_0$ ,  $f$  is colored  $W_2$ ,  $e$  is colored  $W_0$  (note that it is possible since no subtree  $B_{4n}$  is adjacent to  $e$ ) and  $d$  is forced to be colored  $W_1$ .

Finally, by the construction of  $T(Q_\ell^k)$ , by Lemma 4.2 and Corollary 4.3, the root  $d$  may be colored either  $W_0$  or  $W_1$ , since  $w(c) \leq M$ .  $\square$

**DEFINITION 4.10.** *Let  $T(\Phi_{int})$  be the weighted rooted tree obtained as follows (see Figure 6). Let  $r$  be the root with weight  $W_3$ . For any  $1 \leq \ell \leq L$ , the root of one copy of  $T(Q_\ell)$  is made adjacent to  $r$ . For any  $0 \leq i \leq 4n + 2$ ,  $i \neq 4n + 1$ ,  $r$  is made*

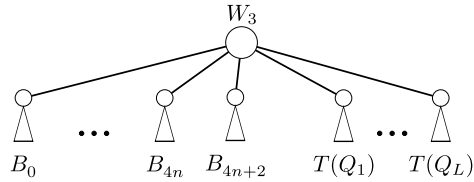


FIGURE 6. The final tree  $T(\Phi_{int})$ .

adjacent to the root of one copy of  $B_i$ .

LEMMA 4.11.  $T(\Phi_{int})$  has size polynomial in  $m = 2^n$ .

*Proof.* Observe that each clause-tree  $T(Q_\ell)$  has size  $\mathcal{O}(p_\ell 2^{4n})$  (see Definition 4.8), where  $p_\ell$  is polynomial in  $m$  (since  $p_\ell$  is at most the number  $nm$  of variables). Moreover, the number  $L$  of clauses is polynomial in  $m$  by the definition of  $\Phi_{int}$ .  $\square$

LEMMA 4.12. If  $\Phi_{int}$  is integrally satisfiable, then  $\chi_w(T(\Phi_{int})) \leq M$ .

*Proof.* Let  $(y_i^j)_{0 \leq i < n, 0 \leq j < m}$  be a valid integral assignment for  $\Phi_{int}$ . For any  $0 \leq i < n$ , let  $j_i$  be the (unique) index such that  $y_i^{j_i}$  is true. We construct a coloring  $c$  of  $(T(\Phi_{int}), w)$  such that  $w(c) \leq M$ . By Lemma 4.2, in any coloring  $c$  of  $T(\Phi_{int})$  such that  $w(c) \leq M$ , the colors of all nodes of the binomial subtrees of  $T(\Phi_{int})$  are forced. Consequently, we only need to decide the colors of the following nodes: the roots and sub-roots of any copy of  $A_i^j$ , the roots of the trees  $T(y_i^j)$ , and the nodes connecting the variables-trees into clause-trees (the nodes  $a, b, g, d, e, f$  in Figure 5), for any  $0 \leq i < n$  and  $0 \leq j < m$ .

We first set the weight of color  $S_i$  for any  $0 \leq i < 4n$ . In particular, for any  $0 \leq i < n$ , the color  $S_{4i+1}$  must have weight  $w_{4i+1}^{j_i}$ . As we observed in the proof of Lemma 4.7, this choice fixes the colors of all roots and sub-roots of all the  $A_i^j$  trees, i.e. all the nodes in the variable trees, except to the roots of the variable-trees  $T(y_i^{j_i})$ , by Lemma 4.7.

More precisely, for any  $0 \leq i < n$  and  $0 \leq j < m$ , let us consider a subtree  $T(y_i^j)$ . Let  $j' \in \{j, j+1\}$ . The sub-root of  $A_{4i+1}^{j'}$  receives color  $S_{4i+1}$  if  $j' \leq j_i$  and receives color  $S_{4i}$  otherwise. The root of  $A_{4i+1}^{j'}$  receives color  $S_{4i}$  if  $j' \leq j_i$  and receives color  $W_0$  otherwise. The sub-root of  $A_{4i+3}^{m-j'}$  receives color  $S_{4i+3}$  if  $j' > j_i$  and receives color  $S_{4i+2}$  otherwise. The root of  $A_{4i+3}^{m-j'}$  receives color  $S_{4i+2}$  if  $j' > j_i$  and receives color  $W_0$  otherwise. Finally, if  $j \neq j_i$ , the root of  $T(y_i^j)$  is colored  $W_1$ . On the other hand, if  $j = j_i$ , none of the neighbors of the root of  $T(y_i^j)$  is colored  $W_0$ , therefore, we can color it either  $W_0$  or  $W_1$ .

Now, let  $Q_\ell = \bigvee_{1 \leq k \leq p_\ell} u_k$  be any clause of  $\Phi_{int}$ . We show that we can extend the previous coloring such that the root of the clause-tree  $T(Q_\ell)$  is colored  $W_0$  and the weight of the coloring is  $\leq M$ . This is by induction on  $p_\ell$ . Indeed, if  $p_\ell = 1$ , then  $Q_\ell$  consists of a unique variable and this variable must be assigned to true (since the formula is true). Hence,  $Q_\ell = y_i^{j_i}$  for some  $0 \leq i < n$ . That is  $T(Q_\ell)$  is a subtree  $T(y_i^{j_i})$ . Hence, we can color the root of it with  $W_0$ .

Now, assume that the result is correct for any clause of length  $p \geq 1$  and let  $p_\ell = p+1$ . Thus,  $Q_\ell = u_{p+1} \vee (\bigvee_{1 \leq k \leq p} u_k)$ . Recall that  $T(Q_\ell)$  is built from a variable subtree  $T(u_{p+1})$  and a clause-subtree  $T(Q_\ell^p)$ . There are two cases to consider: either our assignment satisfies  $\bigvee_{1 \leq k \leq p} u_k$  or not. In the first case, the root of the clause-tree  $T(Q_\ell^p)$  (node  $a$  in Figure 5) is colored  $W_0$  by induction. Moreover, by above paragraphs, the root of  $T(u_{p+1})$  (node  $b$  in Figure 5) can be colored  $W_1$ . It is then easy to extend this coloring such that the root of  $T(Q_\ell)$  is colored  $W_0$ : in Figure 5, node  $f$  is colored  $W_1$ , node  $e$  is colored  $W_2$  and nodes  $g$  and  $d$  are colored  $W_0$ . If our assignment does not satisfy  $\bigvee_{1 \leq k \leq p} u_k$ , then it must satisfy  $u_{p+1}$ . That is,  $u_{p+1} = y_i^{j_i}$  for some  $0 \leq i < n$ . By a similar induction, we prove that the root of  $T(Q_\ell^p)$  can be colored  $W_1$ . Moreover, by above paragraphs, the root of  $T(u_{p+1}) = T(y_i^{j_i})$  can be colored  $W_0$ . This coloring can be extended such that the root of  $T(Q_\ell)$  is colored  $W_0$ : in Figure 5, node  $g$  is colored  $W_1$ , node  $e$  is colored  $W_2$  and nodes  $f$  and  $d$  are colored  $W_0$ .

Thus, we color the roots of all the clause-trees with color  $W_0$  and the root of  $T(\Phi_{int})$  with the color  $W_1$ .

Hence, the weight of this coloring  $c$  is  $w(c) = \sum_{i=0}^{4n+2} \frac{1}{2^i} + n(m-1)\epsilon = M$ .  $\square$

LEMMA 4.13. *If  $\Phi_{int}$  is not integrally satisfiable, then  $\chi_w(T(\Phi_{int})) > M$ .*

*Proof.*  $\Phi_{int}$  is not integrally satisfiable. For purpose of contradiction, let  $c$  be a coloring of  $T(\Phi_{int})$  with weight at most  $M$ . By Lemma 4.7, there are integers  $(j_0, \dots, j_{n-1})$  such that the color of the root of any subtree  $T(y_i^j)$  is forced to be  $W_1$ , if  $j \neq j_i$ . Let  $Y = (y_i^j)_{i < n, j < m}$  be the corresponding integral assignment. In other words, for any variable  $y_i^j$  ( $0 \leq i < n, 0 \leq j < m$ ),  $y_i^j = 0$  if  $j \neq j_i$ . Since  $\Phi_{int}$  is not integrally satisfiable, there is a clause  $Q$  that is not satisfied by this assignment. Let us consider the clause-subtree  $T(Q)$ . It has been built from variable-trees corresponding to the variables constituting the clause  $Q$ . Because all these variables are assigned to false, the roots of these variable-trees are all colored with  $W_1$ , by Lemma 4.7.

By induction on the length of  $Q$  and by Lemma 4.9, the color of the root of  $T(Q_\ell)$  must be  $W_1$ . Thus, the root of  $T(\Phi_{int})$  can only be colored  $W_3$ . Consequently, the coloring  $c$  has weight  $w(c) \geq \sum_{i=0}^{4n+3} \frac{1}{2^i} + n(m-1)\epsilon > M$ , a contradiction.  $\square$

Proposition 3.3 follows directly from Lemmas 4.11, 4.12 and 4.13.

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