



**HAL**  
open science

# A Continuous Exact $l_0$ penalty (CEL0) for least squares regularized problem

Emmanuel Soubies, Laure Blanc-Féraud, Gilles Aubert

► **To cite this version:**

Emmanuel Soubies, Laure Blanc-Féraud, Gilles Aubert. A Continuous Exact  $l_0$  penalty (CEL0) for least squares regularized problem. 2015. hal-01102492v1

**HAL Id: hal-01102492**

**<https://inria.hal.science/hal-01102492v1>**

Preprint submitted on 12 Jan 2015 (v1), last revised 22 May 2015 (v2)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# A Continuous Exact $\ell_0$ penalty (CEL0) for least squares regularized problem.

Emmanuel Soubies<sup>†</sup>      Laure Blanc-Féraud<sup>†</sup>      Gilles Aubert<sup>‡</sup>

January 12, 2015

## Abstract

Within the framework of the  $\ell_0$  regularized least squares problem, we focus, in this paper, on nonconvex *continuous* penalties to approximate the  $\ell_0$ -norm which usually lead to better results than the  $\ell_1$  convex relaxation since they are more “ $\ell_0$ -like”. Based on some results in one dimension and in the orthogonal case, we propose the *Continuous Exact  $\ell_0$  penalty* (CEL0) leading to a relaxation of the  $\ell_2 - \ell_0$  problem. Global minimizers of the objective function using CEL0 are proved to be the same as for the  $\ell_2 - \ell_0$  problem in a given sense. We also demonstrate that all local minimizers of this functional are local minimizers for  $\ell_2 - \ell_0$ . However, the reciprocal is false and some local minimizers of the initial functional are not critical points for the relaxed one. Then solving the initial  $\ell_2 - \ell_0$  problem is equivalent, in a sense, to solve it by replacing the  $\ell_0$ -norm with the CEL0 penalty which provides better properties for the objective function although the problem remains nonconvex. Finally, recent nonsmooth nonconvex algorithms are used to address this relaxed problem within a macro algorithm ensuring the convergence to a critical point of the relaxed functional which is also a (local) optimum of the initial problem.

**Key words** inverse problems,  $\ell_0$  regularization, sparse modelling, underdetermined linear systems, global minimizers, local minimizers, minimizers equivalence, Continuous Exact  $\ell_0$  penalty, nonconvex nonsmooth penalty

## 1 Introduction

In many applications such as coding (to reduce data storage), compressed sensing (to recover a signal from fewer measurements), source separation, variable selection, image decomposition and many others, one aims to compute a *sparse* solution of an underdetermined linear system of equations. In other words these problems search for an approximation of a signal as a linear combination of *redundant dictionary atoms* also known as *synthesis* approach. The underlying idea relies on the existence of a *representation* of the unknown signal involving only few atoms of the dictionary. This can be modeled using sparsity constraints or penalties. The problem reads as a least-squares loss function  $\|Ax - d\|$  defined from a matrix  $A \in \mathbb{R}^{M \times N}$  and data  $d \in \mathbb{R}^M$  plus a sparsity prior usually provided by the  $\ell_0$ -norm<sup>1</sup> defined by

$$\|x\|_0 = \# \{x_i, i = 1, \dots, N : x_i \neq 0\} . \quad (1.1)$$

---

<sup>†</sup>Université de Nice Sophia Antipolis, CNRS, Laboratoire I3S UMR 7271, 06903 Sophia Antipolis, France ([soubies@i3s.unice.fr](mailto:soubies@i3s.unice.fr), [blancf@i3s.unice.fr](mailto:blancf@i3s.unice.fr))

<sup>‡</sup>Université de Nice Sophia Antipolis, CNRS, Laboratoire J.A. Dieudonné UMR 7351, 06100 Nice, France ([gilles.aubert@unice.fr](mailto:gilles.aubert@unice.fr))

<sup>1</sup>Note that even if it is not a norm since the absolute homogeneity is not verified we will refer to it as  $\ell_0$ -norm.

This problem can be stated under different formulations. The constrained forms are used when a prior is known either on the sparsity of the solution or on the noise variance. When we seek a solution  $\hat{x}$  under the constraint  $\|\hat{x}\|_0 \leq k \in \mathbb{N}$ , the problem is usually formulated as follows,

$$\hat{x} = \arg \min_{x \in \mathbb{R}^N} \|Ax - d\|_2^2 \quad \text{subject to} \quad \|x\|_0 \leq k. \quad (1.2)$$

Otherwise, a knowledge about the variance of the noise leads to the formulation

$$\hat{x} = \arg \min_{x \in \mathbb{R}^N} \|x\|_0 \quad \text{subject to} \quad \|Ax - d\|_2^2 \leq \epsilon, \quad (1.3)$$

where  $\epsilon > 0$  is determined with respect to the statistic of the data  $d \in \mathbb{R}^M$ . Finally, when no prior about the noise or the sparsity of the solution is known (*i.e.* we do not know  $k$  and  $\epsilon$ ) the penalized form

$$\hat{x} = \arg \min_{x \in \mathbb{R}^N} G_{\ell_0}(x) := \frac{1}{2} \|Ax - d\|_2^2 + \lambda \|x\|_0, \quad (1.4)$$

is used. Here,  $\lambda > 0$  is an hyperparameter characterizing the trade-off between data fidelity and sparsity. These problems are well known to be *NP-Hard* [15, 26]. Note that in general, because of the nonconvexity, they are not equivalent [28]. Over the last decade, numerous methods/principles have been proposed to find a good approximate solution. We can group them into four categories.

**Convex relaxation** One common alternative is the convex relaxation of this combinatorial problem. It consists in replacing the  $\ell_0$ -norm by the convex  $\ell_1$ -norm to take advantage of well-known algorithms in convex optimization. The *Basis Pursuit* (BP) principle or its *De-Noising* (BPDN) adaptation, proposed by Chen *et al* [11], deals with the  $\ell_1$  relaxation of problems (1.3) and (1.4) using linear and quadratic programming algorithms (*e.g.* simplex, active set or interior-point methods). Iterative thresholding algorithms such as ISTA (Daubechies *et al* [14]) and its accelerated version FISTA (Beck *et al* [3]), belonging to the family of *forward-background* algorithms [13], are now well-known to be very efficient for such  $\ell_1$  relaxed problems. The fact that this convex relaxation provides sparse solutions comes from the singularity of the  $\ell_1$ -norm at zero [17]. For illustration, one can see that in the orthogonal case, solving the relaxed  $\ell_1$  problem leads to a soft-thresholding which sets small coefficients to zero. In fact, a necessary and sufficient condition to obtain a thresholding rule as estimator (in the orthogonal case) is that the associated nonconvex penalty be singular at the origin [17]. Sufficient conditions have been provided by several authors [16, 8, 33] on the matrix  $A$  under which it is shown that sufficiently sparse signals can be exactly recovered from  $\ell_1$  minimization. These conditions are usually too restrictive for practical applications.

**Greedy algorithms** Another class of algorithms dedicated to these problems are the *greedy methods* based on the following idea: starting from a null solution, the most relevant components (with respect to some criteria) are added iteratively one by one. The *Matching Pursuit* (MP), introduced in [23] by Mallat *et al*, is an iterative algorithm selecting one atom per iterate that matches at best the residual — maximizing the correlation — until the desired accuracy is reached. The residual is proved to converge exponentially to zero. A revision of this algorithm, the *Orthogonal Matching Pursuit* (OMP), have been proposed by Pati *et al* [30]. OMP improves the convergence performances of MP by adding, at each iterate, a least-squares minimization allowing to get the best approximation over the selected atoms. We can also cite the *Orthogonal Least Squares* (OLS) introduced previously by Chen *et al* [10] and similar to OMP. Tropp [33] gives a theoretical result which states that under a sufficient incoherence assumption, OMP recovers the sparsest representation of the input signal. He showed that this result also stand for BP. However some authors (see [11, §2.3.2] and references therein) have

provided examples where these greedy algorithms fail to provide sparse solutions. An adaptation of OMP and OLS has been proposed as *Single Best Replacement* (SBR) by Soussen *et al* [31]. At each iteration an atom can be *selected or de-selected*. Then a wrong selected atom in the first few iterates, which is a major drawback of classical greedy algorithms, can be latter removed. A survey of greedy algorithms can be found in [32].

**Graduated Non Convexity (GNC) algorithms** GNC approach have been initially introduced by Blake and Zisserman in [4] for the minimization of nonconvex functions. The main idea is to minimize a sequence of functions starting with a convex one and introducing progressively the nonconvexity. Mohimani *et al* [25] used this principle in the context of sparse approximation where they proposed the SL0 (*Smoothed  $\ell_0$* ) algorithm to deal with problem (1.3). The  $\ell_0$ -norm is approximated by a suitable continuous function depending on a positive parameter and tending to the  $\ell_0$ -norm when the parameter tends to zero. They proved the convergence to the sparsest solution under some conditions on  $\|A\|$  [24].

**Continuous nonsmooth nonconvex penalties** This last point is more a principle than a method although algorithms have also been proposed in this context. Replacing the noncontinuous nonconvex  $\ell_0$ -norm by a nonsmooth nonconvex but *continuous* penalty have been widely investigated in the statistic community in the context of variable selection. The main motivation was to overstep the bias introduced by the  $\ell_1$  penalty on large coefficients [17, 36]. Zou [36] proposed the *Adaptive Lasso* assigning different weights to each term of the sum defining the  $\ell_1$  penalty. Based on an initial estimation (for instance the least-squares solution), the main idea is to set the weights in a way to penalize more small coefficients rather than higher ones. The associated minimization problem remains convex. The *nonnegative garrote*, proposed by Breiman [7], is a procedure which, from the ordinary least-squares estimate, sets some variables to zero and shrinks others. This defines a thresholding rule which can be associated to the minimization of a penalty function [1]. For the nonnegative garrote this associated penalty is nonsmooth and nonconvex. Candès *et al* [9] proposed a *reweighted  $\ell_1$  algorithm* (IRL1) which performs a series of weighted  $\ell_1$  minimization updating the weights at each step. Experimental results showed that IRL1 significantly improves the performance of BP in many applications. This algorithm belongs to the family of *Majorization-Minimization* (MM) algorithms [21] for minimizing a nonsmooth nonconvex function which is the *log-sum penalty* under linear or quadratic constraints. Like the penalty associated to the nonnegative garrote, the log-sum penalty is more “ $\ell_0$ -like” than the convex  $\ell_1$ -norm that can explain its better behaviour on many applications. Fan and Li [17] argued that a “good” penalty function for variable selection should provide an *unbiased* solution for large coefficients, *sparse* and *continuous* in data. Thus they concluded that such a penalty must be singular at the origin and admit an horizontal asymptote at infinity. They proposed the *Smoothly Clipped Absolute Deviation* (SCAD) penalty verifying these conditions. Similarly, Zhang [35] (see also the discussion [34]) introduced a notion of *sparse convexity* to compare performances of penalties (in term of unbiasedness) and proposed the *Minimax Concave Penalty* (MCP) which, according to this sparse convexity condition, is the best penalty for unbiasedness (over penalties satisfying the conditions of Fan and Li). This notion offers a new point of view for the characterization of penalty functions not based on the classical orthogonal case. Both MCP and SCAD are quadratic spline penalties. Fourcart et Lai [18] investigated the use of  $\ell_p$ -norms ( $0 < p < 1$ ) instead of the  $\ell_0$  one (see also references in [18]). Comparable guaranties as for the  $\ell_1$  convex relaxation, in term of exact signal recovery, have been shown. Finally the *Iterative Hard Thresholding* (IHT) is a well known algorithm which deals directly with problems (1.2) or (1.4). It has been initially introduced by Blumensath and Davies in [5] under two variants dedicated respectively to problem (1.4) and (1.2). They proved the convergence of both algorithms under the condition  $\|A\| < 1$ . As an illustration in a general context of descent methods for semi-algebraic problems, Attouch *et al* [2] showed that IHT algorithm (including

an additional step size) for the regularized problem (1.4) actually converges for any matrix  $A$ .

**Contributions and outline** The bulk of this paper is to present and study a new nonsmooth nonconvex penalty for the regularized problem (1.4). Starting in §2 from the computation of the *convex hull* of  $G_{\ell_0}$  in the one dimensional case and in the orthogonal case (see §3), we propose a new relaxation of problem (1.4) by introducing a nonsmooth nonconvex but *continuous* penalty term that we call *Continuous Exact  $\ell_0$*  (CEL0) penalty. This penalty can be viewed as a generalization of the penalty associated to the hard thresholding. In §4 we study the objective function of (1.4) where the  $\ell_0$ -norm is replaced by the CEL0 penalty for any matrix  $A \in \mathbb{R}^{M \times N}$ . Particularly, using the description of the minimizers of  $G_{\ell_0}$  given by Nikolova [27], we prove two main theoretical results concerning the relations between minimizers of the initial function  $G_{\ell_0}$  and its approximation using the CEL0 penalty. They share the *same global minimizers* and, local minimizers for this approximation are local minimizers for  $G_{\ell_0}$ . Then numerical examples show that local minimizers of  $G_{\ell_0}$  are not necessarily critical points for the approximated function. Consequently this relaxed function seems to eliminate a significant number of local minimizers of  $G_{\ell_0}$  although this is not quantified theoretically. Hence, one can solve problem (1.4) by minimizing the sum of the quadratic data term and the CEL0 penalty. This can be addressed using numerous recent nonsmooth nonconvex algorithms converging to a critical point of the objective function. A small review of such algorithms will be outlined in §5. Based on this kind of algorithms and the properties of the CEL0 penalty, we propose in §5 a macro algorithm by adding an outer loop to ensure the convergence to a point which is both a critical point of the relaxed functional and a (local) minimizer of  $G_{\ell_0}$ .

## 2 The convex envelope in one dimension

In this section we will focus on the one dimensional problem defined by

$$\hat{u} = \arg \min_{u \in \mathbb{R}} g(u) := \frac{1}{2}(au - d)^2 + \lambda|u|_0, \quad (2.1)$$

where  $a > 0$  is the equivalent, in one dimension, to the matrix  $A \in \mathbb{R}^{M \times N}$  in (1.4) and  $d \in \mathbb{R}$ . Note that we can consider  $a > 0$  without loss of generality since  $g$  remains identical by multiplying both  $a$  and  $d$  by  $-1$ . Here  $|u|_0$  denotes the  $\ell_0$ -norm in one dimension,

$$\forall u \in \mathbb{R}, |u|_0 = \begin{cases} 0 & \text{if } u = 0, \\ 1 & \text{if } u \neq 0. \end{cases} \quad (2.2)$$

It is well known that the *convex envelope* or *convex hull* of a function  $f : \mathbb{R} \rightarrow \mathbb{R} \cup \{+\infty\}$  is also its biconjugate which is given by applying twice the Legendre-Fenchel transformation,

$$f^{**}(u) = \sup_{u \in \mathbb{R}} u^*u - f(u). \quad (2.3)$$

By applying the Legendre-Fenchel transformation to  $g$  we obtain the conjugate  $g^*$  (see Appendix A for details),

$$g^*(u^*) = \begin{cases} \frac{1}{2a^2}(u^* + ad)^2 - \lambda & \mathbb{1}_{\{|u^* + ad| > \sqrt{2\lambda}a\}} \\ -\frac{d^2}{2} & \end{cases}. \quad (2.4)$$

Then, we can determine the biconjugate  $g^{**}$  which is the conjugate of  $g^*$ . As shown in Appendix A we have

$$g^{**}(u) = \begin{cases} -adu + |u|\sqrt{2\lambda}a + \frac{d^2}{2} & \text{if } |u| \leq \frac{\sqrt{2\lambda}}{a}, \\ \frac{1}{2}(au - d)^2 + \lambda & \text{if } |u| > \frac{\sqrt{2\lambda}}{a}. \end{cases} \quad (2.5)$$

This convex envelope is the largest lower-semi-continuous (*l.s.c*) convex function less than  $g$ . It is of major importance in global optimization. The first line of (2.5) can be rewritten as

$$\frac{1}{2}(au - d)^2 - \frac{a^2 u^2}{2} + |u|\sqrt{2\lambda}a = \frac{1}{2}(au - d)^2 - \frac{a^2}{2} \left( |u| - \frac{\sqrt{2\lambda}}{a} \right)^2 + \lambda, \quad (2.6)$$

and then,

$$g^{**}(u) = \frac{1}{2}(au - d)^2 + \phi(a, \lambda; u), \quad (2.7)$$

where

$$\phi(a, \lambda; u) = \lambda - \frac{a^2}{2} \left( |u| - \frac{\sqrt{2\lambda}}{a} \right)^2 \mathbb{1}_{\{|u| \leq \frac{\sqrt{2\lambda}}{a}\}}. \quad (2.8)$$

Finally, the convex envelope of  $g$  can be obtained by replacing the  $\ell_0$  regularization term in (2.1) by the penalty given in (2.8). We can remark that the penalty term (2.8) is nonsmooth, nonconvex but *continuous* and when we combine it with the quadratic data term, the global objective function (2.7) is convex and gives the convex hull of  $g$ . Figure 1 presents two examples of the convex hull of  $g$  where the global minimum is respectively 0 and strictly positive. We also plot (Figure 1 right) the penalty (2.8) on the same graph as the  $\ell_0$ -norm. We can see that this penalty admits an horizontal asymptote which is a characteristic for unbiased solution (*i.e.* large coefficients are not shrunk) and is singular at the origin which is a necessary condition to provide sparse solutions according to Fan and Li [17].

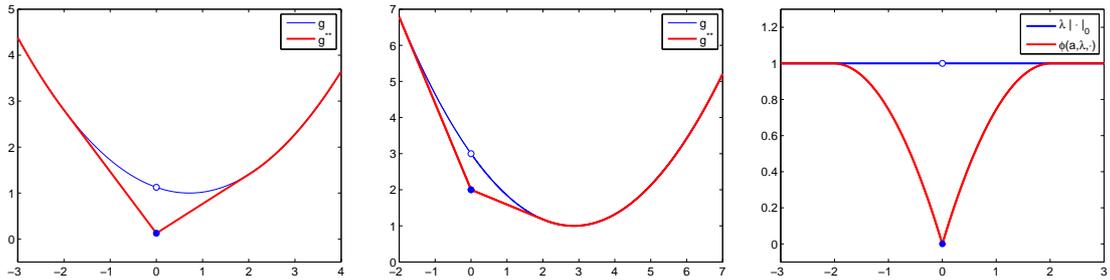


Figure 1: Plot of  $g$  (blue) and  $g^{**}$  (red) for  $a = 0.7$ ,  $\lambda = 1$  and  $d = 0.5$  (left) or  $d = 2$  (center). Right: plot of  $\lambda|\cdot|_0$  (blue) and  $\phi(a, \lambda; \cdot)$  for  $a = 0.7$  and  $\lambda = 1$ .

### 3 A family of functions underestimating $G_{\ell_0}$

Let us first introduce some notations (we will use the same as in [27]):

- $\mathbb{I}_N = \{1, \dots, N\}$ ,
- $a_i \in \mathbb{R}^M$ , the  $i$ th column of  $A \in \mathbb{R}^{M \times N}$ . We assume that  $a_i \neq 0_{\mathbb{R}^M}$ ,  $\forall i \in \mathbb{I}_N$ ,
- $\|\cdot\| = \|\cdot\|_2$  the  $\ell_2$ -norm. Otherwise we will precise the norm with a subscript,
- $e_i \in \mathbb{R}^N$ , the unitary vector of the standard basis of  $\mathbb{R}^N$ ,
- $\sigma(x) = \{i \in \mathbb{I}_N; x_i \neq 0\} \subseteq \mathbb{I}_N$ , the support of  $x \in \mathbb{R}^N$ ,

- $\mathbf{x}^{(i)} = (x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_N) \in \mathbb{R}^N$ ,
- $A_\omega = (a_{\omega[1]}, \dots, a_{\omega[\#\omega]}) \in \mathbb{R}^{M \times \#\omega}$  for  $\omega \subseteq \mathbb{I}_N$ , the restriction of  $A \in \mathbb{R}^{M \times N}$  to the columns indexed by the elements of  $\omega \subseteq \mathbb{I}_N$ ,
- $\mathbf{x}_\omega = (x_{\omega[1]}, \dots, x_{\omega[\#\omega]}) \in \mathbb{R}^{\#\omega}$  for  $\omega \subseteq \mathbb{I}_N$ , the restriction of  $\mathbf{x} \in \mathbb{R}^N$  to the entries indexed by the elements of  $\omega \subseteq \mathbb{I}_N$ .

Extending the results of the previous section to the dimension  $N \in \mathbb{N}$  is very tricky. Indeed, the conjugate of the function  $G_{\ell_0} : \mathbb{R}^N \rightarrow \mathbb{R}$  (eq. 1.4) is solution of the problem

$$G_{\ell_0}^*(\mathbf{x}^*) = \sup_{\mathbf{x} \in \mathbb{R}^N} \langle \mathbf{x}^*, \mathbf{x} \rangle_{\mathbb{R}^N} - \frac{1}{2} \|A\mathbf{x} - d\|^2 - \lambda \|\mathbf{x}\|_0, \quad (3.1)$$

which is also a combinatorial problem and probably as hard as the initial one given in (1.4). However, in the case where the matrix  $A$  is orthogonal (*i.e.*  $A^T A$  is diagonal), problem (3.1) can be solved analytically. Let  $D \in \mathbb{R}^{N \times N}$  be a diagonal matrix whose diagonal entries are given by  $d_i = \|a_i\| \forall i \in \mathbb{I}_N$ . Note that  $D^{-1}$  is well defined since by hypothesis  $\|a_i\| \neq 0 \forall i \in \mathbb{I}_N$ . Let  $\hat{d} = AD^{-2}A^T d$  and  $\tilde{z} = D^{-1}A^T d$ , then the quadratic term of (3.1) can be written as

$$\frac{1}{2} \|A\mathbf{x} - d\|^2 = \frac{1}{2} \|d - \hat{d}\|^2 + \frac{1}{2} \|D\mathbf{x} - \tilde{z}\|^2. \quad (3.2)$$

Combining (3.1) and (3.2) we obtain,

$$\begin{aligned} G_{\ell_0}^*(\mathbf{x}^*) &= \sup_{\mathbf{x} \in \mathbb{R}^N} \langle \mathbf{x}^*, \mathbf{x} \rangle_{\mathbb{R}^N} - \frac{1}{2} \|d - \hat{d}\|^2 - \frac{1}{2} \|D\mathbf{x} - \tilde{z}\|^2 - \lambda \|\mathbf{x}\|_0, \\ &= -\frac{1}{2} \|d - \hat{d}\|^2 + \sup_{\mathbf{x} \in \mathbb{R}^N} \sum_{i \in \mathbb{I}_N} x_i^* x_i - \frac{1}{2} (\|a_i\| |x_i - \tilde{z}_i|)^2 - \lambda |x_i|_0, \\ &= -\frac{1}{2} \|d - \hat{d}\|^2 + \sum_{i \in \mathbb{I}_N} \sup_{x_i \in \mathbb{R}} x_i^* x_i - \frac{1}{2} (\|a_i\| |x_i - \tilde{z}_i|)^2 - \lambda |x_i|_0. \end{aligned} \quad (3.3)$$

Finally, in the orthogonal case, solving (3.1) is equivalent to solve  $N$  independent one dimensional problems. Using the one dimensional conjugate  $g^*$  given by (2.4) it comes that

$$G_{\ell_0}^*(\mathbf{x}^*) = -\frac{1}{2} \|d - \hat{d}\|^2 + \sum_{i \in \mathbb{I}_N} \left\{ \frac{1}{2\|a_i\|^2} (x_i^* + \|a_i\| \tilde{z}_i)^2 - \lambda \right\} \mathbb{1}_{\{|x_i^* + \|a_i\| \tilde{z}_i| > \sqrt{2\lambda} \|a_i\|\}} - \frac{\tilde{z}_i^2}{2}, \quad (3.4)$$

which is additively separable and allows to use the expression of the one dimensional convex hull  $g^{**}$  given by (2.7) to get,

$$\begin{aligned} G_{\ell_0}^{**}(\mathbf{x}) &= \frac{1}{2} \|d - \hat{d}\|^2 + \sum_{i \in \mathbb{I}_N} \frac{1}{2} (\|a_i\| |x_i - \tilde{z}_i|)^2 + \phi(\|a_i\|, \lambda; x_i), \\ &= \frac{1}{2} \|d - \hat{d}\|^2 + \frac{1}{2} \|D\mathbf{x} - \tilde{z}\|^2 + \sum_{i \in \mathbb{I}_N} \phi(\|a_i\|, \lambda; x_i), \\ &\stackrel{(3.2)}{=} \frac{1}{2} \|A\mathbf{x} - d\|^2 + \sum_{i \in \mathbb{I}_N} \phi(\|a_i\|, \lambda; x_i). \end{aligned} \quad (3.5)$$

Note that, as in the one dimensional case, the convex hull of  $G_{\ell_0}$  for an orthogonal matrix  $A$  is obtained by replacing the non continuous  $\ell_0$ -norm by a continuous penalty defined as in (3.5) with  $\phi$

given by (2.8).

Since finding the convex hull for a general case with an arbitrary matrix  $A$  seems to be as hard as the initial problem (1.4) we will not address it directly. Based on the results presented for the one dimensional problem and the orthogonal case, we will consider, in the sequel, the family of functions  $\{G_\alpha\}_{\alpha \in \mathcal{A}}$  underestimating  $G_{\ell_0}$  for a fixed  $\lambda > 0$  where

$$\mathcal{A} = \{\alpha \in \mathbb{R}^N; \forall i \in \mathbb{I}_N, 0 < \alpha_i \leq \|a_i\|\}, \quad (3.6)$$

and

$$\forall x \in \mathbb{R}^N, G_\alpha(x) = \frac{1}{2} \|Ax - d\|^2 + \sum_{i \in \mathbb{I}_N} \phi(\alpha_i, \lambda; x_i). \quad (3.7)$$

with  $\phi$  given by (2.8). It is not hard to see that functions of  $\{G_\alpha\}_{\alpha \in \mathcal{A}}$  underestimate  $G_{\ell_0}$ . For  $\alpha \in \mathcal{A}$ ,  $\lambda > 0$  we have  $0 \leq \phi(\alpha_i, \lambda; u) \leq 1 \forall u \in \mathbb{R}, \forall i \in \mathbb{I}_N$ . Then, since  $\phi(\alpha_i, \lambda; 0) = 0$  it comes that  $\phi(\alpha_i, \lambda; u) \leq |u|_0, \forall u \in \mathbb{R}, \forall i \in \mathbb{I}_N$ , which proves the above affirmation. When  $\alpha_i = \|a_i\| \forall i \in \mathbb{I}_N$  and  $A$  orthogonal, we have  $G_\alpha(x) = G_{\ell_0}^*(x)$  but in the general case this result is false. Moreover  $x \rightarrow G_\alpha(x)$  is nonconvex.

We will also denote by  $\Phi(\alpha, \lambda; \cdot)$ ,  $\alpha \in \mathcal{A}$ ,  $\lambda > 0$ , the penalty function used to define  $G_\alpha$ , that is

$$\forall x \in \mathbb{R}^N, \Phi(\alpha, \lambda; x) = \sum_{i \in \mathbb{I}_N} \phi(\alpha_i, \lambda; x_i). \quad (3.8)$$

This continuous penalty is thus used to replace the  $\ell_0$ -norm in the initial problem (1.4). Then recent nonsmooth nonconvex optimization algorithms can be used to minimize the approximated function  $G_\alpha$ . This point will be developed in §5.

In §4, the study of the particular case  $\alpha_i = \|a_i\| \forall i \in \mathbb{I}_N$  will be developed in details. Here, we focus on general properties of the function family  $\{G_\alpha\}_{\alpha \in \mathcal{A}}$  for fixed  $\lambda > 0$ . We characterize the *critical points* of  $G_\alpha$  ( $\alpha \in \mathcal{A}$ ). Let us start with the expression of the *generalized gradient* (see Appendix B.1 for definition) of the penalty  $\Phi(\alpha, \lambda; \cdot)$ , denoted  $\partial\Phi$ , which will be useful in the sequel. For  $\alpha \in \mathcal{A}$ ,  $\lambda > 0$ ,

$$\partial\Phi(\alpha, \lambda; x) = \prod_{i \in \mathbb{I}_N} [-\sqrt{2\lambda}\alpha_i, \sqrt{2\lambda}\alpha_i] \mathbb{1}_{\{x_i=0\}} + \alpha_i \left\{ \text{sign}(x_i) \sqrt{2\lambda} - \alpha_i x_i \right\} \mathbb{1}_{\{0 < |x_i| \leq \frac{\sqrt{2\lambda}}{\alpha_i}\}}. \quad (3.9)$$

This expression comes directly from the *generalized derivative* of the one dimensional penalty  $\phi$  (see Appendix B.1 for details). The following lemma gives a characterization of the critical points of the functional  $G_\alpha$ . Such points  $x \in \mathbb{R}^N$  verify  $0_{\mathbb{R}^N} \in \partial G_\alpha(x)$  (by definition) where  $\partial G_\alpha$  denotes the generalized gradient of  $G_\alpha$ .

**Lemma 3.1. Critical points of  $G_\alpha$ :** Let  $G_\alpha$  be defined by (3.7) for  $d \in \mathbb{R}^M$ ,  $\alpha \in \mathcal{A}$ ,  $\lambda > 0$  and let  $s_i = \text{sign}(\langle a_i, A\hat{x}^{(i)} - d \rangle)$ . Then  $\hat{x}$  is a critical point of  $G_\alpha$  (i.e.  $0_{\mathbb{R}^N} \in \partial G_\alpha(\hat{x})$ ) if and only if  $\forall i \in \mathbb{I}_N$ ,

- if  $\alpha_i < \|a_i\|$ ,

$$\begin{aligned} \hat{x}_i &= 0 && \text{if } |\langle a_i, A\hat{x}^{(i)} - d \rangle| \leq \sqrt{2\lambda}\alpha_i, \\ \hat{x}_i &= \frac{-\langle a_i, A\hat{x}^{(i)} - d \rangle + s_i \sqrt{2\lambda}\alpha_i}{\|a_i\|^2 - \alpha_i^2} && \text{if } \sqrt{2\lambda}\alpha_i < |\langle a_i, A\hat{x}^{(i)} - d \rangle| \leq \frac{\sqrt{2\lambda}}{\alpha_i} \|a_i\|^2, \\ \hat{x}_i &= \frac{-\langle a_i, A\hat{x}^{(i)} - d \rangle}{\|a_i\|^2} && \text{if } |\langle a_i, A\hat{x}^{(i)} - d \rangle| > \frac{\sqrt{2\lambda}}{\alpha_i} \|a_i\|^2, \end{aligned} \quad (3.10)$$

- if  $\alpha_i = \|a_i\|$ ,

$$\begin{aligned}
\hat{x}_i &= 0 & \text{if } |\langle a_i, A\hat{x}^{(i)} - d \rangle| < \sqrt{2\lambda}\|a_i\|, \\
\hat{x}_i &= -s_it, \quad t \in \left[0, \frac{\sqrt{2\lambda}}{\|a_i\|}\right] & \text{if } |\langle a_i, A\hat{x}^{(i)} - d \rangle| = \sqrt{2\lambda}\|a_i\|, \\
\hat{x}_i &= -\frac{\langle a_i, A\hat{x}^{(i)} - d \rangle}{\|a_i\|^2} & \text{if } |\langle a_i, A\hat{x}^{(i)} - d \rangle| > \sqrt{2\lambda}\|a_i\|.
\end{aligned} \tag{3.11}$$

*Proof.* The proof is outlined in Appendix B.2.  $\square$

Lemma 3.1 does not provide an exact expression for the critical points of  $G_\alpha$  but gives some conditions verified by critical points. These conditions will be used in the sequel to provide bounds on the nonzero entries of (local) minimizers.

## 4 The CEL0 penalty

In this section we will study the special case of

$$\alpha = (\|a_1\|, \dots, \|a_n\|) \in \mathcal{A}. \tag{4.1}$$

The associated penalty  $\Phi(\alpha, \lambda; \cdot)$ ,  $\lambda > 0$  is called *Continuous Exact  $\ell_0$*  (CEL0) penalty and denoted  $\Phi_{\text{CEL0}}(\cdot)$  because of the two main results on its associated objective function,

$$G_{\text{CEL0}}(x) = \frac{1}{2}\|Ax - d\|^2 + \Phi_{\text{CEL0}}(x) = \frac{1}{2}\|Ax - d\|^2 + \sum_{i \in \mathbb{I}_N} \phi(\|a_i\|, \lambda, x_i),$$

which are presented in the following. The first one, given in Theorem 4.2, states that global minimizers of  $G_{\ell_0}$  are also global minimizers for  $G_{\text{CEL0}}$  and that the converse is true in a given sense. The second one, given in Theorem 4.4, partially extends this result to local minimizers and numerical illustrations show that, in general, a large number of local minimizers of  $G_{\ell_0}$  are not critical points of  $G_{\text{CEL0}}$ . Therefore  $G_{\text{CEL0}}$  admits less local minimizers than  $G_{\ell_0}$ . We introduce a notation used in the following,

$$\forall x \in \mathbb{R}^N, \quad \sigma^-(x) := \left\{ i \in \mathbb{I}_N : 0 < |x_i| \leq \frac{\sqrt{2\lambda}}{\|a_i\|} \right\}, \tag{4.2}$$

which is a subset of  $\sigma(x)$ , the support of  $x$ .

### 4.1 On the minimizers of $G_{\text{CEL0}}$

Let us first recall two results given in [27] which are used in the sequel. The first one provides a lower bound on the nonzero coefficients of the global minimizers of  $G_{\ell_0}$  while the second one characterizes the (local) minimizers of  $G_{\ell_0}$ .

**Proposition 4.1.** *M. Nikolova [27, Proposition 4.1]: For  $d \in \mathbb{R}^M$  and  $\lambda > 0$ , let  $G_{\ell_0}$  have a global minimum at  $\hat{x}$ . Then*

$$i \in \sigma(\hat{x}) \implies |\hat{x}_i| \geq \frac{\sqrt{2\lambda}}{\|a_i\|}, \tag{4.3}$$

*Proof.* The proof is given in [27, Appendix 8.2].  $\square$

**Corollary 4.2.** *M. Nikolova [27, Corollary 2.5]: For  $d \in \mathbb{R}^M$  and  $\lambda > 0$ , let  $\hat{x}$  be a (local) minimizer of  $G_{\ell_0}$ . Set  $\hat{\sigma} = \sigma(\hat{x})$ . Then*

$$\hat{x}_{\hat{\sigma}} \text{ solves } (A_{\hat{\sigma}})^T A_{\hat{\sigma}} \hat{x}_{\hat{\sigma}} = (A_{\hat{\sigma}})^T d. \quad (4.4)$$

*Conversely, if  $\hat{x} \in \mathbb{R}^N$  verifies (4.4) for  $\hat{\sigma} = \sigma(\hat{x})$ , then  $\hat{x}$  is a (local) minimizer of  $G_{\ell_0}$ .*

*Proof.* The proof follows directly from [27, Lemma 2.4] for (4.4) and [27, Proposition 2.3] for the reciprocal.  $\square$

For the function  $G_{\text{CELO}}$  we have a result similar to the one given in Proposition 4.1 which is provided by the following lemma.

**Lemma 4.1.** *For  $d \in \mathbb{R}^M$  and  $\lambda > 0$ , let  $G_{\text{CELO}}$  have a minimum (local or global) at  $\hat{x}$ . If  $\sigma^-(\hat{x}) \neq \emptyset$ , then*

$$\forall i \in \sigma^-(\hat{x}), \forall t \in \left[0, \frac{\sqrt{2\lambda}}{\|a_i\|}\right], (\hat{x}^{(i)} - s_i e_i t) \text{ is also a minimizer of } G_{\text{CELO}}. \quad (4.5)$$

where  $s_i = \text{sign}(\langle a_i, A\hat{x}^{(i)} - d \rangle)$ .

*Proof.* The proof comes directly from Lemma 3.1 which gives a characterization of the critical points of  $G_{\text{CELO}}$ . Indeed, since  $\hat{x}$  is a minimizer (local or global) of  $G_{\text{CELO}}$  it has to verify the conditions given in (3.11) which are equivalent (see equivalence between (B.10) and (B.11) in the proof of Lemma 3.1) to,

$$\begin{cases} |\langle a_i, A\hat{x}^{(i)} - d \rangle| \leq \sqrt{2\lambda} \|a_i\| & \text{if } \hat{x}_i = 0, \\ \langle a_i, A\hat{x}^{(i)} - d \rangle = -\text{sign}(\hat{x}_i) \sqrt{2\lambda} \|a_i\| & \text{if } 0 < |\hat{x}_i| \leq \frac{\sqrt{2\lambda}}{\|a_i\|}, \\ \hat{x}_i = -\frac{\langle a_i, A\hat{x}^{(i)} - d \rangle}{\|a_i\|^2} & \text{if } |\hat{x}_i| > \frac{\sqrt{2\lambda}}{\|a_i\|}. \end{cases} \quad (4.6)$$

Then from (4.6) we get

$$\forall i \in \sigma^-(\hat{x}), |\langle a_i, A\hat{x}^{(i)} - d \rangle| = \sqrt{2\lambda} \|a_i\|. \quad (4.7)$$

Moreover, Lemma 3.1 ensures that

$$\forall i \in \sigma^-(\hat{x}), \hat{x}_i = -s_i \hat{t}, \text{ for } \hat{t} \in \left[0, \frac{\sqrt{2\lambda}}{\|a_i\|}\right], \quad (4.8)$$

where  $s_i = \text{sign}(\langle a_i, A\hat{x}^{(i)} - d \rangle)$ . Let  $i \in \sigma^-(\hat{x})$  and  $f$  be the restriction of  $G_{\text{CELO}}$  defined by

$$f(t) = G_{\text{CELO}}(\hat{x}^{(i)} - s_i e_i t), \forall t \in \left[0, \frac{\sqrt{2\lambda}}{\|a_i\|}\right]. \quad (4.9)$$

We can rewrite  $f$  as follows,  $\forall t \in \left[0, \frac{\sqrt{2\lambda}}{\|a_i\|}\right]$

$$\begin{aligned} f(t) &= \frac{1}{2} \|A\hat{x}^{(i)} - s_i a_i t - d\|^2 + \sum_{j \in \mathbb{1}_N, j \neq i} \phi(\|a_j\|, \lambda; \hat{x}_j) + \phi(\|a_i\|, \lambda; -s_i t) \\ &= C - s_i t \langle a_i, A\hat{x}^{(i)} - d \rangle + \frac{\|a_i\|^2}{2} t^2 + \lambda - \frac{\|a_i\|^2}{2} \left( |s_i t| - \frac{\sqrt{2\lambda}}{\|a_i\|} \right)^2 \\ &= C - s_i t \langle a_i, A\hat{x}^{(i)} - d \rangle + t \sqrt{2\lambda} \|a_i\|, \end{aligned}$$

where  $C = \frac{1}{2}\|A\hat{x}^{(i)} - d\|^2 + \sum_{j \in \mathbb{I}_N, j \neq i} \phi(\|a_j\|, \lambda; \hat{x}_j)$  is a constant independent of  $t$ . Then,

$$\forall t \in \left[0, \frac{\sqrt{2\lambda}}{\|a_i\|}\right], \quad f'(t) = -s_i \langle a_i, A\hat{x}^{(i)} - d \rangle + \sqrt{2\lambda}\|a_i\| = -|\langle a_i, A\hat{x}^{(i)} - d \rangle| + \sqrt{2\lambda}\|a_i\| = 0.$$

Finally it comes

$$\forall t \in \left[0, \frac{\sqrt{2\lambda}}{\|a_i\|}\right], \quad f(t) = G_{\text{CELO}}(\hat{x}^{(i)} - s_i e_i t) = \text{cste} = f(\hat{t}) = G_{\text{CELO}}(\hat{x}). \quad (4.10)$$

Since  $\hat{x}$  is a minimizer (local or global) of  $G_{\text{CELO}}$ , (4.10) completes the proof.  $\square$

This lemma ensures that all strict minimizers of  $G_{\text{CELO}}$  — *i.e.*  $\hat{x} \in \mathbb{R}^N$  such that there exists a neighborhood  $\mathcal{V} \subset \mathbb{R}^N$  containing  $\hat{x}$  for which  $\forall y \in \mathcal{V} \setminus \{\hat{x}\}, G_{\text{CELO}}(\hat{x}) < G_{\text{CELO}}(y)$  — verify  $\sigma^-(\hat{x}) = \emptyset$ . Indeed, suppose that  $\hat{x}$  is a strict minimizer with  $\sigma^-(\hat{x}) \neq \emptyset$ . Then Lemma 4.1 states that  $\forall i \in \sigma^-(\hat{x}), \forall t \in [0, \sqrt{2\lambda}/\|a_i\|], \hat{x}^{(i)} - s_i e_i t$  is also a minimizer of  $G_{\text{CELO}}$  and contradicts the fact that  $\hat{x}$  is strict.

Moreover, for a nonstrict minimizer  $\hat{x} \in \mathbb{R}^N$ , fixing all its nonzero components indexed by  $\sigma^-(\hat{x})$  to zero will result, from Lemma 4.1, in another minimizer denoted  $\hat{x}^0$  and

$$G_{\text{CELO}}(\hat{x}^0) = G_{\text{CELO}}(\hat{x}). \quad (4.11)$$

Note that  $\hat{x}^0$  verifies  $\sigma^-(\hat{x}^0) = \emptyset$  but is not strict by definition. Based on this lemma, the following theorem gives a relation between the *global* minimizers of  $G_{\ell_0}$  and  $G_{\text{CELO}}$ .

**Theorem 4.2. Link between global minimizers of  $G_{\ell_0}$  and  $G_{\text{CELO}}$ :** Let  $d \in \mathbb{R}^M$  and  $\lambda > 0$ ,

- (i) if  $\hat{x} \in \mathbb{R}^N$  is a global minimizer of  $G_{\ell_0}$ , then it is also a global minimizer of  $G_{\text{CELO}}$ ,
- (ii) conversely if  $\hat{x} \in \mathbb{R}^N$  is a global minimizer of  $G_{\text{CELO}}$ , let  $\hat{x}^0$  be defined by

$$\forall i \in \mathbb{I}_N, \quad \hat{x}_i^0 = \hat{x}_i \mathbf{1}_{\{\|\hat{x}_i\| > \frac{\sqrt{2\lambda}}{\|a_i\|}\}}, \quad (4.12)$$

then  $\hat{x}^0$  is a global minimizer of  $G_{\ell_0}$  and

$$G_{\text{CELO}}(\hat{x}) = G_{\text{CELO}}(\hat{x}^0) = G_{\ell_0}(\hat{x}^0). \quad (4.13)$$

*Proof.* The proof is detailed in Appendix C.  $\square$

The following lemma characterizes some critical points of  $G_{\text{CELO}}$  as (local) minimizers of  $G_{\ell_0}$ .

**Lemma 4.3. Link between critical points of  $G_{\text{CELO}}$  and minimizers of  $G_{\ell_0}$ :** Let  $\hat{x} \in \mathbb{R}^N$  be a critical point of  $G_{\text{CELO}}$  verifying  $\sigma^-(\hat{x}) = \emptyset$ . Then it is a (local) minimizer of  $G_{\ell_0}$  and  $G_{\text{CELO}}(\hat{x}) = G_{\ell_0}(\hat{x})$ .

*Proof.*  $\hat{x}$  verifies conditions (3.11) which can be rewritten, since  $\sigma^-(\hat{x}) = \emptyset$ , as follows

$$\forall i \in \mathbb{I}_N, \quad \begin{cases} \hat{x}_i = 0 & \text{if } |\langle a_i, A\hat{x}^{(i)} - d \rangle| \leq \sqrt{2\lambda}\|a_i\|, \\ \hat{x}_i = -\frac{\langle a_i, A\hat{x}^{(i)} - d \rangle}{\|a_i\|^2} & \text{if } |\langle a_i, A\hat{x}^{(i)} - d \rangle| > \sqrt{2\lambda}\|a_i\|, \end{cases} \quad (4.14)$$

and from which, setting  $\hat{\sigma} = \sigma(\hat{x})$ , it is clear that  $\hat{x}$  verifies

$$(A_{\hat{\sigma}})^T A_{\hat{\sigma}} \hat{x}_{\hat{\sigma}} = (A_{\hat{\sigma}})^T d. \quad (4.15)$$

Then from Corollary 4.2 it comes that  $\hat{x}$  is a (local) minimizer for  $G_{\ell_0}$ . The fact that  $G_{\text{CELO}}(\hat{x}) = G_{\ell_0}(\hat{x})$  follows from the same arguments as in the proof of Theorem 4.2. This completes the proof.  $\square$

From this lemma we can derive a similar result as Theorem 4.2 for *local* minimizers.

**Theorem 4.4. Link between local minimizers of  $G_{\ell_0}$  and  $G_{\text{CEL0}}$ :** Let  $d \in \mathbb{R}^M$ ,  $\lambda > 0$ , and  $G_{\text{CEL0}}$  have a local minimum at  $\hat{x} \in \mathbb{R}^N$ . Then  $\hat{x}^0$  (defined as in (4.12)) is a local minimizer of  $G_{\ell_0}$  and (4.13) is verified.

*Proof.* From Lemma 4.1 we get that since  $\hat{x}$  is a local minimizer of  $G_{\text{CEL0}}$ ,  $\hat{x}^0$  is also a local minimizer of  $G_{\text{CEL0}}$ . Thus,  $\hat{x}^0$  is a critical point of  $G_{\text{CEL0}}$  such that  $\sigma^-(\hat{x}^0) = \emptyset$  and, from Lemma 4.3, it is a (local) minimizer of  $G_{\ell_0}$  and (4.13) is verified. Moreover, according to Theorem 4.2, it can not be global for  $G_{\ell_0}$  since it is local for  $G_{\text{CEL0}}$ . This completes the proof.  $\square$

In other words, Theorem 4.4 states that the second point of Theorem 4.2 is also true for local minimizers of  $G_{\text{CEL0}}$ . However, the reciprocal is not verified for all the local minimizers of  $G_{\ell_0}$ . Indeed, let us consider  $N = 1$  (i.e. problem 2.1) with  $a = 1$  and  $d > \sqrt{2\lambda}$ . Here,  $g$  reaches a global minimum at  $u = d$  and a local one at  $u = 0$  while  $g^{**}$  has a unique global minimizer at  $u = d$ . Then all the local minimizers of  $g$  are not local minimizer for  $g^{**}$ . This remark can be easily extended to the  $N$ -dimensional orthogonal case since, in this case, replacing the  $\ell_0$ -norm by the CEL0 penalty leads to the convex hull of  $G_{\ell_0}$  (showed in §3) which cannot have local minimizers.

In a more general framework, for any matrix  $A \in \mathbb{R}^{M \times N}$ , let  $\hat{x}$  be a local minimizer of  $G_{\ell_0}$ . Then from Corollary 4.2,  $\hat{x}_{\hat{\sigma}}$  solves

$$(A_{\hat{\sigma}})^T A_{\hat{\sigma}} \hat{x}_{\hat{\sigma}} = (A_{\hat{\sigma}})^T d, \quad (4.16)$$

where  $\hat{\sigma} = \sigma(\hat{x})$ . To be a critical point of  $G_{\text{CEL0}}$ , such a  $\hat{x}$  has to verify conditions (3.11). Then  $\hat{x}$  is a critical point of  $G_{\text{CEL0}}$  if, in addition to solve (4.16), it verifies the following conditions,

$$\begin{cases} |\langle a_i, A\hat{x}^{(i)} - d \rangle| \leq \sqrt{2\lambda} \|a_i\| & \forall i \notin \sigma(\hat{x}), \\ |\langle a_i, A\hat{x}^{(i)} - d \rangle| = \sqrt{2\lambda} \|a_i\| & \forall i \in \sigma^-(\hat{x}), \\ |\langle a_i, A\hat{x}^{(i)} - d \rangle| > \sqrt{2\lambda} \|a_i\| & \forall i \in \sigma(\hat{x}) \setminus \sigma^-(\hat{x}). \end{cases} \quad (4.17)$$

From (3.11), we get (4.17) using the same kind of arguments as in the proof of Lemma 3.1 (to go from (B.10) to (B.11)). Consequently, if  $\hat{x}$ , a local minimizer of  $G_{\ell_0}$ , does not verify (4.17) it is not a critical point of  $G_{\text{CEL0}}$  and then  $G_{\text{CEL0}}$  does not admit a local minimum at  $\hat{x}$ . With regard to the previous one-dimensional example, the local minimizer  $u = 0$  of  $g$  does not verify  $|a(au - d)| \leq \sqrt{2\lambda}a \Leftrightarrow |d| \leq \sqrt{2\lambda}$  and then it is not a local minimizer for  $g^{**}$  as stated previously.

While from each local minimizer of  $G_{\text{CEL0}}$  Theorem 4.4 provides a local minimizer for  $G_{\ell_0}$ , conditions (4.17) can discard some local optima of  $G_{\ell_0}$  to be local optima for  $G_{\text{CEL0}}$ . Indeed, if  $\hat{x}$  verifies (4.16) then  $\forall i \in \sigma(\hat{x})$ ,  $\hat{x}_i = -\langle a_i, A\hat{x}^{(i)} - d \rangle / \|a_i\|^2$  and we get,

$$\forall i \in \sigma(\hat{x}), \begin{cases} i \in \sigma(\hat{x}) \setminus \sigma^-(\hat{x}) \quad \left( \text{i.e. } |\hat{x}_i| > \frac{\sqrt{2\lambda}}{\|a_i\|} \right) & \Rightarrow |\langle a_i, A\hat{x}^{(i)} - d \rangle| > \sqrt{2\lambda} \|a_i\|, \\ i \in \sigma^-(\hat{x}) \quad \left( \text{i.e. } 0 < |\hat{x}_i| \leq \frac{\sqrt{2\lambda}}{\|a_i\|} \right) & \Rightarrow 0 < |\langle a_i, A\hat{x}^{(i)} - d \rangle| \leq \sqrt{2\lambda} \|a_i\|. \end{cases}$$

Therefore, a minimizer  $\hat{x}$  of  $G_{\ell_0}$  which necessarily verifies (4.16) is not ensured to verify the two first lines of (4.17) and then to be a critical point of  $G_{\text{CEL0}}$ . This will be numerically illustrated on low dimensional examples in §4.2.

A consequence of Lemma 4.1 and Theorem 4.4 is given by the following corollary.

**Corollary 4.3.** All strict (local) minimizers  $\hat{x}$  of  $G_{\text{CEL0}}$  verify  $\sigma^-(\hat{x}) = \emptyset$  and are strict (local) minimizers of  $G_{\ell_0}$ .

*Proof.* From Lemma 4.1 we get that all strict (local) minimizers of  $G_{\text{CELO}}$  verify  $\sigma^-(\hat{x}) = \emptyset$ . Then since they are (local) minimizers of  $G_{\text{CELO}}$  and verify  $\hat{x} = \hat{x}^0$ , Theorem 4.4 (or Theorem 4.2 for the global ones) ensures that they are (local) minimizers of  $G_{\ell_0}$  and that  $G_{\text{CELO}}(\hat{x}) = G_{\ell_0}(\hat{x})$ . Let us now suppose that one of them, denoted  $\hat{x}$ , is not strict for  $G_{\ell_0}$ . Then there exists a neighbourhood  $\mathcal{V}_0 \subset \mathbb{R}^N$  containing  $\hat{x}$  such that for all  $\mathcal{V} \subset \mathcal{V}_0$  containing  $\hat{x}$ , there exists  $\bar{x} \in \mathcal{V}$  verifying  $G_{\ell_0}(\bar{x}) = G_{\ell_0}(\hat{x})$ . Since by definition  $G_{\text{CELO}}$  underestimates  $G_{\ell_0}$  we have

$$G_{\text{CELO}}(\bar{x}) \leq G_{\ell_0}(\bar{x}) = G_{\ell_0}(\hat{x}) = G_{\text{CELO}}(\hat{x}), \quad (4.18)$$

which contradicts the fact that  $\hat{x}$  is a *strict* (local) minimizer of  $G_{\text{CELO}}$  and completes the proof.  $\square$

Finally, in the following proposition we give a characteristic of the function  $G_{\text{CELO}}$  which could be useful to design specific algorithms.

**Proposition 4.4.** *The function  $G_{\text{CELO}}$  is convex with respect to each variable.*

*Proof.* Let  $i \in \mathbb{I}_N$  be fixed. Then  $G_{\text{CELO}}^i$ , the restriction of  $G_{\text{CELO}}$  to the  $i$ th variable, can be written as,  $\forall t \in \mathbb{R}$

$$\begin{aligned} G_{\text{CELO}}^i(t) &= G_{\text{CELO}}(\mathbf{x}^{(i)} + te_i) \\ &= \frac{1}{2} \|A\mathbf{x}^{(i)} + a_i t - d\|^2 + \sum_{j \in \mathbb{I}_N \setminus \{i\}} \phi(\|a_j\|, \lambda; \mathbf{x}_j) + \phi(\|a_i\|, \lambda; t) \\ &= \frac{1}{2} \|A\mathbf{x}^{(i)} - d\|^2 + t \langle a_i, A\mathbf{x}^{(i)} - d \rangle + \frac{1}{2} \|a_i\|^2 t^2 + \sum_{j \in \mathbb{I}_N \setminus \{i\}} \phi(\|a_j\|, \lambda; \mathbf{x}_j) + \phi(\|a_i\|, \lambda; t) \\ &= \underbrace{\frac{1}{2} \|A\mathbf{x}^{(i)} - d\|^2 + \sum_{j \in \mathbb{I}_N \setminus \{i\}} \phi(\|a_j\|, \lambda; \mathbf{x}_j)}_{\text{Independent of } t} - \frac{(\tilde{d})^2}{2} + \frac{1}{2} (\|a_i\|t - \tilde{d})^2 + \phi(\|a_i\|, \lambda; t), \end{aligned}$$

where  $\tilde{d} = \langle a_i, d - A\mathbf{x}^{(i)} \rangle / \|a_i\| \in \mathbb{R}$ . Then by identification with (2.7) it is straightforward that  $G_{\text{CELO}}^i$  is convex. This completes the proof.  $\square$

## 4.2 Numerical illustrations of the reciprocal of Theorem 4.4

In this part, we experimentally study the fact that  $G_{\text{CELO}}$  has “less” local minima than  $G_{\ell_0}$ .

### 4.2.1 Two dimensional examples

Here, we consider the case  $N = M = 2$ . On Figure 2 we present the isolevels of both  $G_{\ell_0}$  and  $G_{\text{CELO}}$  for different  $A \in \mathbb{R}^{2 \times 2}$ ,  $d \in \mathbb{R}^2$  and  $\lambda > 0$ . In all cases, global minimizers of  $G_{\ell_0}$  are also global minimizers of  $G_{\text{CELO}}$  as stated by Theorem 4.2. For the example given in 2(a) and 2(b),  $\forall \mathbf{x}^* \in [0, 1]^2$ ,  $\mathbf{x}^*$  is a global minimizer of  $G_{\text{CELO}}$ . Let  $\hat{x} \in ]0, 1[$ , which is thus a global optimum of  $G_{\text{CELO}}$ , then it is clear on this example that  $\hat{x}^0$  defined by (4.12) is a global minimizer for both  $G_{\text{CELO}}$  and  $G_{\ell_0}$  as stated by the second point of Theorem 4.2. This example also illustrates the remark given in the proof of Theorem 4.2: when a global minimizer  $\hat{x}$  of  $G_{\text{CELO}}$  verifies  $\sigma^-(\hat{x}) \neq \emptyset$ ,  $G_{\text{CELO}}$  is constant on the hypercube defined by (C.2). Here,  $G_{\text{CELO}}$  is constant on  $[0, 1]^2$ .

Other examples of Figure 2 illustrate that  $G_{\text{CELO}}$  has in general less local minima than  $G_{\ell_0}$ . When  $A \in \mathbb{R}^{2 \times 2}$ ,  $G_{\ell_0}$  has four minima (local or global). On the example presented by Figures 2(c) and 2(d) we can see that  $G_{\text{CELO}}$  has only two global minima and when we consider the example given on Figures 2(e)

and 2(f),  $G_{\text{CELO}}$  has an unique global minimum. For these two examples the use of the CELO penalty allows to *eliminate all the local minimizers of  $G_{\ell_0}$*  while the global ones are preserved. However this is not always the case as we can see on the example of Figures 2(g) and 2(h) where  $G_{\text{CELO}}$  has one global and one local minimizer. In this example  $G_{\text{CELO}}$  has two local minimizers less than  $G_{\ell_0}$ .

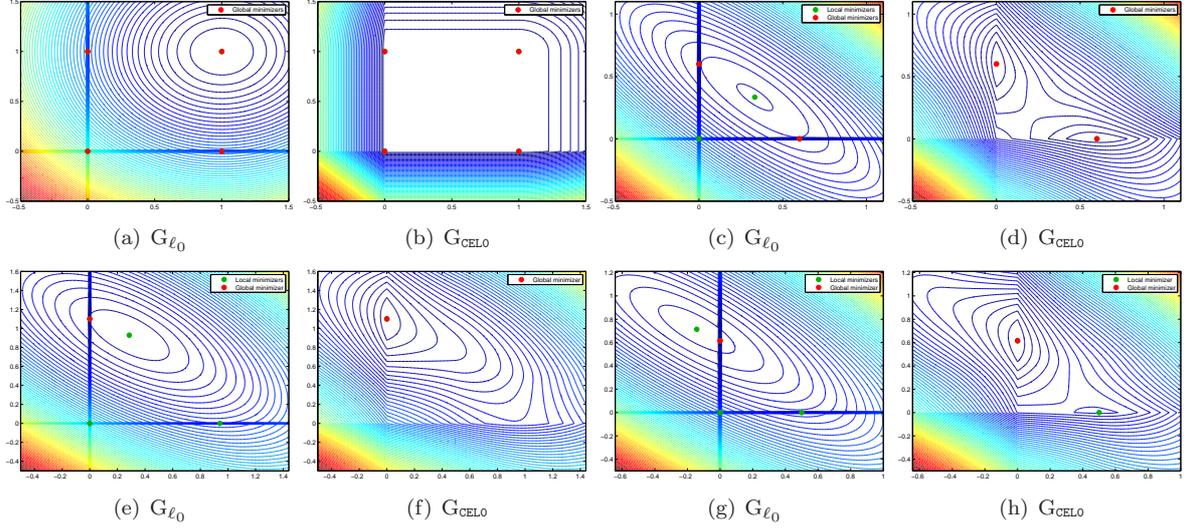


Figure 2: Isolevels of  $G_{\ell_0}$  and  $G_{\text{CELO}}$  for four examples where  $N = M = 2$ . (a)-(b)  $A = [1, 0; 0, 1]$ ,  $b = [1; 1]$  and  $\lambda = 0.5$  (c)-(d)  $A = [1, 2; 2, 1]$ ,  $b = [1; 1]$  and  $\lambda = 0.5$  (e)-(f)  $A = [0.5, 2; 2, 1]$ ,  $b = [2; 1.5]$  and  $\lambda = 0.5$  (g)-(h)  $A = [3, 2; 2, 3]$ ,  $b = [1; 2]$  and  $\lambda = 1$ . Green points are local minimizers and red ones are global.

#### 4.2.2 For an arbitrary dimension

Let us now consider examples in higher dimension. Following the numerical illustrations of [27, §6.2], we define  $G_{\ell_0}$  with  $M = 5$  and  $N = 10$  using

$$A = \begin{pmatrix} 7 & 2 & 4 & 9 & 0 & 3 & 3 & 6 & 6 & 7 \\ 3 & 4 & 9 & 3 & 3 & 9 & 1 & 3 & 1 & 5 \\ 5 & 4 & 2 & 4 & 0 & 7 & 1 & 9 & 2 & 9 \\ 8 & 4 & 0 & 9 & 6 & 0 & 4 & 2 & 3 & 7 \\ 6 & 3 & 6 & 5 & 0 & 9 & 0 & 0 & 3 & 8 \end{pmatrix}, \quad (4.19)$$

and

$$d = Ax^* \quad \text{with} \quad x^* = (0, 1, 8, 0, 3, 0, 0, 0, 0, 9)^T. \quad (4.20)$$

In the sequel, we will study the strict (local) minimizers of  $G_{\ell_0}$  and  $G_{\text{CELO}}$ . As stated by [27, Theorem 4.4], the global minimizers of  $G_{\ell_0}$  are strict. That makes strict minimizers interesting. Following [27, Corollary 3.3] a strict (local) minimizer of  $G_{\ell_0}$  can be easily computed by choosing a support  $\omega \in \Omega_{\max}$ , where  $\Omega_{\max}$  is defined by [27, Definition 3.1]

$$\Omega_{\max} = \bigcup_{r=0}^M \Omega_r \quad \text{where} \quad \Omega_r = \{\omega \subset \mathbb{I}_N : \#\omega = r = \text{rank}(A_\omega)\}, \quad (4.21)$$

and then solve the normal equations restricted to this support (as in (4.4)). In other words, strict (local) minimizers are the ones with a support belonging to  $\Omega_{max}$ .

Then one can compute all strict (local) minimizers by solving the restricted normal equations for all the supports of  $\Omega_{max}$  which is a finite set. This has been done for  $G_{\ell_0}$  defined according to (4.19) and (4.20). Results have been displayed on Figure 3(a) in the same way as in [27] where the x-axis lists all the strict (local) minimizers according to their support length and the y-axis gives the associated value of  $G_{\ell_0}$ . Among all these 638 (local) minimizers we can keep only those verifying (4.17) which are critical points of  $G_{\text{CEL0}}$ . They are represented on Figure 3(c). Only 283 of the 638 (local) minimizers of  $G_{\ell_0}$  are thus preserved. Following Corollary 4.3 these 283 critical points of  $G_{\text{CEL0}}$  contain all the strict (local) minimizers of  $G_{\text{CEL0}}$ . Indeed suppose that  $\hat{x} \in \mathbb{R}^N$ , a strict (local) minimizer of  $G_{\text{CEL0}}$ , is not represented on Figure 3(c). Then by construction it is neither represented on Figure 3(a). But, according to Corollary 4.3, it is also a strict (local) minimizer of  $G_{\ell_0}$ . This contradicts the fact that Figure 3(a) contains all strict (local) minimizers of  $G_{\ell_0}$  and prove the previous affirmation. However strict minimizers of  $G_{\ell_0}$  which are preserved on Figure 3(c) are not necessary minimizers for  $G_{\text{CEL0}}$  (we only know that they are critical points for  $G_{\text{CEL0}}$ ) and those which are minimizers for  $G_{\text{CEL0}}$  are not necessarily strict. This is illustrated with the global minima of the example given on Figures 2(a) and 2(b) which are strict for  $G_{\ell_0}$  and not strict for  $G_{\text{CEL0}}$ .

Using the CEL0 penalty instead of the  $\ell_0$ -norm seems to *remove a significant number of strict local minimizers of  $G_{\ell_0}$* . This is interesting since algorithms will have a “lower probability” to get trapped into a local minimizer. We can also observe on the zooms 3(b) and 3(d) that the unique global minimizer (see [27] for the uniqueness) of  $G_{\ell_0}$  is preserved which illustrates Theorem 4.2. Finally one can see that among all the strict local minimizers of  $G_{\ell_0}$ ,  $G_{\text{CEL0}}$  seems to preserve the ones with the lower value of  $G_{\ell_0}$  (or  $G_{\text{CEL0}}$  since we have equality for these points).

In Table 1 we have computed the number of strict (local) minimizers of  $G_{\ell_0}$  for different values of  $M$  and  $N$ . For each couple  $(M, N)$ , 1000 random matrix  $A \in \mathbb{R}^{M \times N}$  and  $d \in \mathbb{R}^M$  have been generated from an uniform distribution and the number of strict (local) minimizers of  $G_{\ell_0}$  preserved with  $G_{\text{CEL0}}$  (we denote this number by  $P$ ) have been determined. We report in Table 1 the minimal, maximal and mean values of  $P$  with respect to the 1000 realizations. These experiments are in agreement with the fact that  $G_{\text{CEL0}}$  has significantly less strict local minimizers than  $G_{\ell_0}$ <sup>2</sup>. Following Theorem 4.4, from every local minimizer of  $G_{\text{CEL0}}$ , one can obtain a local minimizer of  $G_{\ell_0}$ . Then the experiment of Table 1 shows that the reciprocal seems to be false for a considerable number of strict local minimizers of  $G_{\ell_0}$ . We can also remark, in Table 1, that the number of removed strict minimizer with  $G_{\text{CEL0}}$  is higher with a larger value of  $\lambda$ .

### 4.3 The case where $A$ is orthogonal

As it has been shown in §3, when  $A$  is orthogonal,  $G_{\text{CEL0}}$  is the convex hull of  $G_{\ell_0}$ . Thus  $G_{\text{CEL0}}$  is convex and all its critical points are global minimizers of  $G_{\text{CEL0}}$  from which, following Theorem 4.2, we can deduce global minimizers of  $G_{\ell_0}$ . Using the orthogonality of  $A$  we can rewrite the critical point characterization (3.11) given by Lemma 3.1 as follows

$$\forall i \in \mathbb{I}_N, \begin{cases} \hat{x}_i = 0 & \text{if } |\langle a_i, d \rangle| < \sqrt{2\lambda} \|a_i\|, \\ \hat{x}_i = -s_i t, t \in \left[ 0, \frac{\sqrt{2\lambda}}{\|a_i\|} \right] & \text{if } |\langle a_i, d \rangle| = \sqrt{2\lambda} \|a_i\|, \\ \hat{x}_i = \frac{\langle a_i, d \rangle}{\|a_i\|^2} & \text{if } |\langle a_i, d \rangle| > \sqrt{2\lambda} \|a_i\|. \end{cases} \quad (4.22)$$

<sup>2</sup>Theorem 4.2 ensures that removed minimizers from  $G_{\ell_0}$  to  $G_{\text{CEL0}}$  are local.

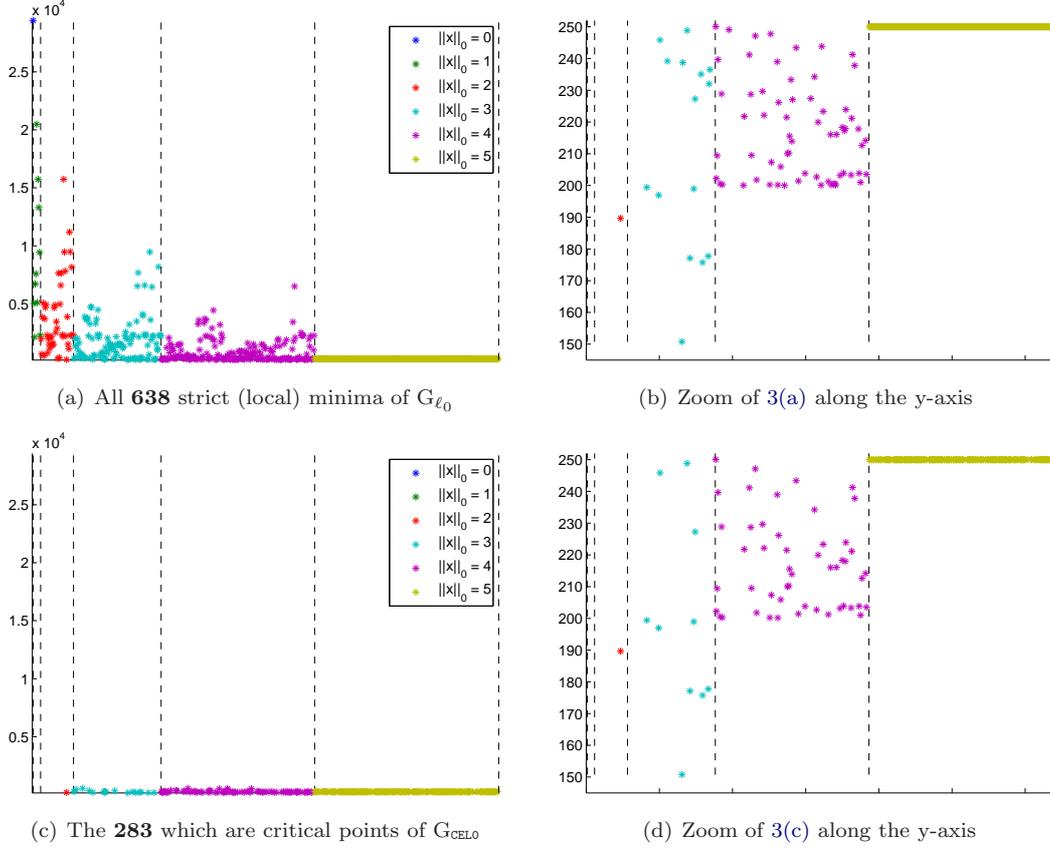


Figure 3: (a)-(b) Strict (local) minima of  $G_{\ell_0}$  defined according to equations (4.19) and (4.20) for  $\lambda = 50$ . (c)-(d) The ones verifying (4.17) which are critical points of  $G_{\text{CELO}}$ .

Table 1: Number of strict (local) minimizers ( $\#\Omega_{max}$ ) of  $G_{\ell_0}$  and the number  $P$  of them which are critical points of  $G_{\text{CELO}}$ . This is computed for 1000 matrices with entries randomly generated from a uniform distribution.  $P_{min}$ ,  $P_{max}$  and  $\bar{P}$  correspond respectively to the minimal, the maximal and the mean value of  $P$  over the 1000 realizations. This experiment is repeated for different matrix sizes and two different  $\lambda$ .

		$\#\Omega_{max}$	$P_{min}$	$P_{max}$	$\bar{P}$
$\lambda = 0.5$	$M = 3, N = 5$	26	1	20	9
	$M = 5, N = 10$	638	39	347	206
	$M = 7, N = 15$	16384	2237	7374	5037
$\lambda = 0.1$	$M = 3, N = 5$	26	4	23	13
	$M = 5, N = 10$	638	173	444	306
	$M = 7, N = 15$	16384	5370	9431	7570

Let  $z = A^T d$ , then the sparsest minimizer of  $G_{\text{CELO}}$ , denoted  $\hat{x}^0$ , is given by the following thresholding rule

$$\forall i \in \mathbb{I}_N, \hat{x}_i^0 = \frac{z_i}{\|a_i\|^2} \mathbb{1}_{\{|z_i| > \sqrt{2\lambda} \|a_i\|\}}. \quad (4.23)$$

Finally, in the orthogonal case, replacing the  $\ell_0$ -norm by the CEL0 penalty transforms the initial problem into a convex problem (its convex envelop) for which we have an analytic expression (4.23) of the sparsest solution. Note that in (4.23) *the threshold is different for each components*  $x_i$  ( $i \in \mathbb{I}_N$ ) and involves the norm of the columns of  $A$ . When the columns of  $A$  are normalized (*i.e.*  $\|a_i\| = 1, \forall i \in \mathbb{I}_N$ ) we get the well known hard thresholding rule.

#### 4.4 Conclusions about the CEL0 penalty

In this section, we made an analytical study of the *Continuous Exact  $\ell_0$  penalty* obtained previously by computing the convex hull of  $G_{\ell_0}$  in the case where  $A$  is orthogonal. It leads to a *continuous* approximation of the objective function  $G_{\ell_0}$  in (1.4), denoted by  $G_{\text{CEL0}}$ , which *has the same global minimizers as  $G_{\ell_0}$*  (Theorem 4.2). This equivalence is verified in a sense that all global minimizers of  $G_{\ell_0}$  are global minimizers of  $G_{\text{CEL0}}$  and conversely, from each global optimum of  $G_{\text{CEL0}}$  we obtain, with (4.12), a global minimizer of  $G_{\ell_0}$ . Then Theorem 4.4 extends the second point of this result to local minimizers: from all local minimizers of  $G_{\text{CEL0}}$ , (4.12) provides a local minimizer of  $G_{\ell_0}$ . We showed that the converse is false and that experimentally, *a large number of strict local minimizers of  $G_{\ell_0}$  are not critical points of  $G_{\text{CEL0}}$  and thus not local minimizers*. We also have observed that local minimizers of  $G_{\ell_0}$  which are not critical points of  $G_{\text{CEL0}}$  seem to be the ones corresponding to the higher values of  $G_{\ell_0}$  (among all the strict local minimizers of  $G_{\ell_0}$ ). Although  $G_{\text{CEL0}}$  remains nonconvex, it is *convex with respect to each variable* and its *continuity* offers the ability to use recent nonsmooth nonconvex algorithms to deal with the  $\ell_2 - \ell_0$  problem as we will see in the following paragraph.

### 5 Algorithms to deal with the $\ell_2$ -CEL0 problem

The analysis provided in the previous section on the link between the  $\ell_2 - \ell_0$  and  $\ell_2$ -CEL0 problems allows us to address problem (1.4) by minimizing  $G_{\text{CEL0}}$  which has the nice property to be *continuous* unlike  $G_{\ell_0}$ . Numerous recent algorithms have been dedicated to such nonsmooth nonconvex functional. Let us mention three of them which are well adapted to the minimization of  $G_{\text{CEL0}}$ . In the framework of *Difference of Convex* (DC) functions programming, Gasso *et al* [19] propose a generic algorithm for solving problem (1.4) where the  $\ell_0$ -norm is replaced by a nonconvex penalty belonging to a particular family of functions. Note that the CEL0 penalty belongs to this family. They propose a suitable decomposition for these penalties as the difference of two convex functions to formulate the problem as a DC program. In this particular decomposition, one function is the  $\ell_1$ -norm which allows them to use efficient schemes to solve each step of the DC algorithm. A decrease of the cost function and a convergence to a critical point is ensured by [19, Theorem 1]. More recently, Ochs *et al* [29] study a sub-class of Majorization-Minimization (MM) algorithms to minimize energies belonging to a family of nonsmooth nonconvex functions containing in particular  $G_{\text{CEL0}}$ . Under the *Kurdyka-Lojasiewicz* (KL) property of the objective function, they prove the convergence of the generated sequence to a critical point by showing that all the requirement of [2, Theorem 2.9] hold. They detail four particular instances of this general sub-class of algorithms. Among them, the Iterative Reweighted  $\ell_1$  algorithm (IRL1) is the more appropriated for the minimization of our objective function  $G_{\text{CEL0}}$  since it majorizes the function using tangents. Such a majorization is well adapted to approximate the singularity of the CEL0 penalty at zero. An experimental analysis shows the ability of the algorithm to avoid local minimizers on some examples. Whenever this can not be generalized, it illustrates an interesting behaviour of the proposed algorithm. Finally we can cite the General Iterative Shrinkage and Thresholding (GIST) algorithm proposed by Gong *et al* [20] to solve nonconvex penalized problems. It deals with a large class of nonconvex penalties including CEL0. GIST is an iterative forward-backward splitting algorithm combined with a line search to select a good step size a each iteration. Convergence to a critical point of the objective function is proved. Like the algorithm proposed by Ochs *et al*, GIST belongs to the family

of MM algorithms. One can see this algorithm as an extension of the Iterative Hard Thresholding to other nonconvex penalties than the  $\ell_0$ -norm. In a recent work, Kowalski [22] shows the convergence of ISTA with any thresholding rule which leads to similar numerical schemes as GIST.

Note that all the mentioned algorithms have been proposed during the last years and this is not an exhaustive list. Optimization of such nonsmooth nonconvex functional is thus a very active research topic which suggests that new improvements will be done in the next years.

## 5.1 A macro algorithm to find (local) minimizers of $G_{\ell_0}$

Let  $\text{Alg}(x^{init}, \lambda)$  be an algorithm, initialized with  $x^{init} \in \mathbb{R}^N$  (first guess) and  $\lambda > 0$  (hyperparameter), producing a sequence of iterates  $(x^k)_{k \in \mathbb{N}}$  which

H1. converges to a critical point of  $G_{\text{CELO}}$ ,

H2. verifies

$$\forall k \in \mathbb{N}, G_{\text{CELO}}(x^{k+1}) \leq G_{\text{CELO}}(x^k) - \beta \|x^{k+1} - x^k\|^2, \quad (5.1)$$

where  $\beta$  is a positive constant.

Condition H2 is also called *sufficient-decrease condition*. It is usually used to prove convergence of descent algorithms, hence such algorithms generally verify this condition. In the sequel, we denote  $\sigma_k^- = \sigma^-(x^k)$ .

Based on such an algorithm  $\text{Alg}$ , for example one of the three algorithms mentioned above, we propose the “macro” algorithm detailed in Algorithm 1. The idea of this macro algorithm is to move iteratively from a critical point of  $G_{\text{CELO}}$  to another one, decreasing the cost function, until reaching a critical point of  $G_{\text{CELO}}$ ,  $\hat{x}$ , verifying  $\sigma^-(\hat{x}) = \emptyset$ . From Lemma 4.3, such a critical point is a (local) minimizer of  $G_{\ell_0}$ .

**Input:** Alg satisfying H1 and H2,  $x^{init} \in \mathbb{R}^N$ ,  $\lambda > 0$

- 1 Set  $k = 1$ ;
- 2  $x^1 \leftarrow \text{Alg}(x^{init}, \lambda)$ ;
- 3 Compute  $\sigma_1^-$ ;
- 4 **while**  $\sigma_k^- \neq \emptyset$  **do**
- 5     Set  $x^{temp} = x^k$ ;
- 6     **while**  $x^{temp}$  is a critical point of  $G_{\text{CELO}}$  and  $\sigma_k^- \neq \emptyset$  **do**
- 7         Select  $i \in \sigma_k^-$ ;
- 8         Set  $x_i^{temp} = 0$ ;
- 9         Set  $\sigma_k^- = \sigma_k^- \setminus \{i\}$ ;
- 10      $x^{k+1} \leftarrow \text{Alg}(x^{temp}, \lambda)$ ;
- 11      $k = k + 1$ ;
- 12     Compute  $\sigma_k^-$ ;

**Algorithm 1:** A macro algorithm to find critical points of  $G_{\text{CELO}}$  being (local) minimizer for  $G_{\ell_0}$ .

Starting with a *critical point*  $x^1$  of  $G_{\text{CELO}}$  obtained by  $\text{Alg}$  (line 2), Algorithm 1 sets (at iteration  $k$ ) one by one the components indexed by  $i \in \sigma_k^-$  to zero while the resulting point (defined by line 8) remains a critical point of  $G_{\text{CELO}}$ . If all the components belonging to  $\sigma_k^-$  are thus set to zero then the algorithm stop and returns this new point which is a critical point of  $G_{\text{CELO}}$  verifying  $\sigma_k^- = \emptyset$ . Otherwise if the setting of one  $x_i^k$ ,  $i \in \sigma_k^-$ , to zero results in a point which is not a critical point of  $G_{\text{CELO}}$ , the

next iterate  $x^{k+1}$  is computed by Alg initialized with this non stationary point (line 10). Following the hypothesis H1 on Alg,  $x^{k+1}$  is a critical point of  $G_{\text{CELO}}$ .

Note that the algorithm is ensured to stop since from all global minimizer  $\hat{x} \in \mathbb{R}^N$ , Theorem 4.2 allows to define another global optimum  $\hat{x}^0$  (defined by (4.12)) verifying  $\sigma^-(\hat{x}^0) = \emptyset$ . Therefore the set of critical points  $\hat{x}$  of  $G_{\text{CELO}}$  verifying  $\sigma^-(\hat{x}) = \emptyset$  is nonempty. The following theorem gives a more formal convergence result on the generated sequence  $(x^k)_{k \in \mathbb{N}}$ .

**Theorem 5.1. Convergence of the macro algorithm:** *Let  $(x^k)_{k \in \mathbb{N}}$  be the sequence generated by Algorithm 1. There exists  $k^* \in \mathbb{N}$  such that  $\sigma_{k^*}^- = \emptyset$  and  $x^{k^*}$  is a (local) minimizer of  $G_{\ell_0}$ .*

*Proof.* The proof is given in Appendix D. □

This macro algorithm allows, from any state of the art Alg satisfying H1 and H2, to find a (local) minimizer of  $G_{\ell_0}$  by minimizing  $G_{\text{CELO}}$ . Note that we can not ensure this point to be also a (local) minimizer for  $G_{\text{CELO}}$ . Indeed, Algorithm 1 converges to a critical point of  $G_{\text{CELO}}$  which is a (local) minimizer of  $G_{\ell_0}$  from Lemma 4.3 but we do not have a complete characterization of (local) minimizers of  $G_{\text{CELO}}$ <sup>3</sup>. This is still an open issue. Consequently Algorithm 1 can converge to a point which is not (local) minimizer for  $G_{\text{CELO}}$  and from Theorem 4.2 such a point can not be a global optimum of  $G_{\ell_0}$ . Indeed, from Theorem 4.2 we get that being a minimizer of  $G_{\text{CELO}}$  is a *necessary condition* to be a global optimum of  $G_{\ell_0}$ .

**Remark 5.1.** *The proof of Theorem 5.1, detailed in Appendix D, is based on the fact that the image of the set of Clarke critical points of  $G_{\text{CELO}}$  is finite. The number of elements of this set can be huge which suggests that the macro algorithm can stop after a large number of iteration  $k^* \in \mathbb{N}$ . However in the numerical examples presented in §5.3, the macro algorithm generally stops after one or two iterations. Indeed performing many iterates in the outer loop of Algorithm 1 means that Alg reaches “unstable” critical points (i.e. which are not minimizers). Even if Alg can converge to such points, it is less frequent than getting trapped into a (local) minimizer.*

**Remark 5.2.** *The selection at line 7 is arbitrary but one can imagine to find an optimal strategy for this selection.*

## 5.2 Note on the Iterative Hard Thresholding (IHT) algorithm

Introduced by Blumensath and Davies in [5], the IHT algorithm performs iterates as

$$x^{k+1} = H_\lambda(x^k - A^T(Ax - d)), \quad (5.2)$$

where  $H_\lambda(x_i) = x_i \mathbb{1}_{\{|x_i| > \sqrt{2\lambda}\}}$  denotes the hard thresholding function. It has been proved to converge to a (local) minimizer of  $G_{\ell_0}$  under the condition that  $\|A\| < 1$  [5]. More recently some authors showed that the convergence of this forward-backward splitting algorithm actually holds for any matrix  $A \in \mathbb{R}^{M \times N}$  [2]. Note that they consider a slightly different scheme from (5.2) including an additional step size  $\gamma^k$ ,

$$x^{k+1} = H_{\lambda\gamma^k}(x^k - \gamma^k A^T(Ax - d)). \quad (5.3)$$

Using such a forward-backward iteration [2] on  $G_{\text{CELO}}$  gives,

$$x^{k+1} \in \text{prox}_{\gamma^k \Phi_{\text{CELO}}(\cdot)}(x^k - \gamma^k A^T(Ax^k - d)), \quad (5.4)$$

---

<sup>3</sup>From Lemma 3.1 we have only a characterization of its critical points

where  $0 < \underline{\gamma} < \gamma^k < \bar{\gamma} < \frac{1}{L}$  with  $L$  the Lipschitz constant of the gradient of the quadratic term and  $\text{prox}$  denotes the proximity operator defined by,

$$\text{prox}_{\gamma f(\cdot)}(y) := \arg \min_{x \in \mathbb{R}^N} \frac{1}{2\gamma} \|x - y\|^2 + f(x). \quad (5.5)$$

Since  $\Phi_{\text{CELO}}$  is additively separable we obtain that for  $y = (y_1, \dots, y_N) \in \mathbb{R}^N$

$$\text{prox}_{\gamma \Phi_{\text{CELO}}(\cdot)}(y) = \left( \text{prox}_{\gamma \phi(\|a_1\|, \lambda; \cdot)}(y_1), \dots, \text{prox}_{\gamma \phi(\|a_N\|, \lambda; \cdot)}(y_N) \right), \quad (5.6)$$

where for  $a > 0$ ,  $\lambda > 0$  and  $\gamma > 0$

$$\text{prox}_{\gamma \phi(a, \lambda; \cdot)}(u) = \begin{cases} \text{sign}(u) \min \left( |u|, (|u| - \sqrt{2\lambda\gamma}a)_+ / (1 - a^2\gamma) \right) & \text{if } a^2\gamma < 1, \\ u \mathbb{1}_{\{|u| > \sqrt{2\lambda\gamma}\}} + \{0, u\} \mathbb{1}_{\{|u| = \sqrt{2\lambda\gamma}\}} & \text{if } a^2\gamma \geq 1. \end{cases} \quad (5.7)$$

Following [2], since  $\frac{1}{2} \|Ax - d\|^2$  is a polynomial function and  $\Phi_{\text{CELO}}$  has a piecewise polynomial graph,  $G_{\text{CELO}}$  is semi-algebraic and then verifies the *Kurdyka-Lojasevicz* (KL) property. Therefore  $G_{\text{CELO}}$  is a proper lower semicontinuous KL function bounded from below and [2, Theorem 5.1] ensures the convergence of the sequence  $(x^k)_{k \in \mathbb{N}}$  generated by the scheme (5.4) to a critical point of  $G_{\text{CELO}}$ . Note that when  $a^2\gamma \geq 1$  we obtain exactly the hard thresholding rule given in [2] which admits a discontinuity. However, by setting

$$\bar{\gamma} = \min \left( \min_{i \in \mathbb{I}_N} \frac{1}{\|a_i\|^2}, \frac{1}{L} \right), \quad (5.8)$$

the proximity operator (5.7) define a continuous threshold function (*i.e.* the first line of (5.7)). This can prevent some unstability problems which can arise from the discontinuity of the classical hard thresholding function. Note that the resulting thresholding function differs between each components. Moreover, one can see that even by taking  $\bar{\gamma} < \frac{1}{L}$  we obtain, from (5.7), a thresholding rule which can be continuous or not for each component. This depends on the values of  $\|a_i\|$ ,  $i \in \mathbb{I}_N$ . To obtain exactly the hard thresholding scheme, one must impose that

$$\underline{\gamma} \geq \max_{i \in \mathbb{I}_N} \frac{1}{\|a_i\|^2}, \quad (5.9)$$

which can violate the upper bound  $1/L$ . Then, the IHT algorithm solves  $G_{\text{CELO}}$  if  $\underline{\gamma}$  verifies (5.9), otherwise it is not ensured to converge to a critical point of  $G_{\text{CELO}}$ . Consequently some local minimizers of  $G_{\ell_0}$  can be avoided using the thresholding rule (5.7) instead of the hard one  $H_{\gamma\lambda}$  according to our numerical illustrations of §4.2.

### 5.3 Numerical illustrations

In this section, we illustrate on low dimensional examples the behaviour of the proposed macro algorithm (Algorithm 1) and compare its performances with the IHT algorithm (in its version defined in [2]). For the macro algorithm, we will consider as inner algorithm Alg, the IRL1 algorithm [29] and the Forward-Backward splitting (FB) defined by the scheme (5.4).

Figure 4 presents the behaviour of the proposed macro algorithm on a 2D example. Red cross correspond to the inner iterates of Alg while green points are the outer ones. The green cross represent the temporary variable  $x^{\text{temp}}$  in Algorithm 1. Since Algorithm 1 minimizes  $G_{\text{CELO}}$ , these iterates are represented with the isolevels of  $G_{\text{CELO}}$ . When the IRL1 algorithm is used (Figure 4(a)), the macro algorithm (green points) converges in one iteration to the global minimizer. With the same starting point, the FB algorithm converges to an “unstable” critical point of  $G_{\text{CELO}}$  such that  $\sigma^-(\hat{x}) = \{1, 2\}$

(i.e. the two components of  $\hat{x}$  belongs to  $\sigma^-(\hat{x})$ ). Then, in this case, the macro algorithm performs an outer iteration. On Figures 4(b) and 4(c) we can see that, depending on the selected component (line 7 of Algorithm 1), the macro algorithm converges either to the global minimizer or to a local one. This reflects remark 5.2 on the way to define an “optimal” selection leading to the better (local) minimizer (in terms of minimization).

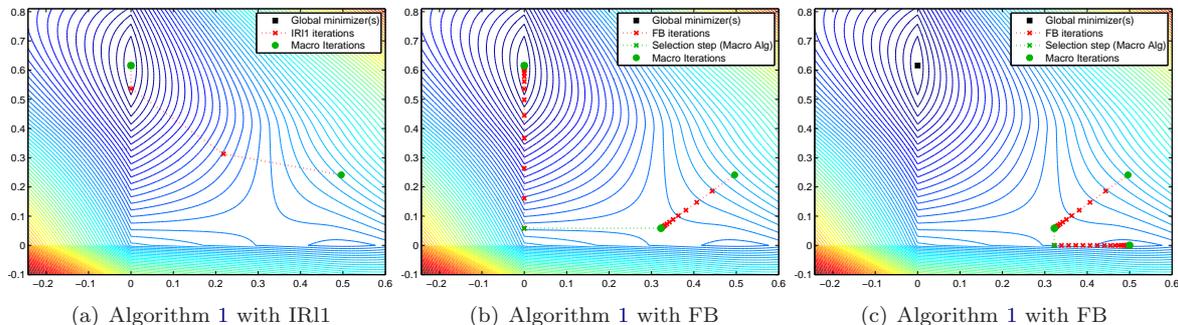


Figure 4: Iterates evolution of Algorithm 1 on the example defined by Figure 2(h). The difference between (b) and (c) holds in the selection process of line 7 in Algorithm 1.

Note that on the examples of Figure 4, the starting point has been defined in order to have two iterates in the outer loop of the macro algorithm with FB. This illustrates remark 5.1 since we can see that very specific initial conditions are required to reach an “unstable” critical point and thus perform many iterates with the macro algorithm. Hence we can pretend that, in general, Algorithm 1 will converge in a few number of iterates.

We will now consider examples in higher dimension to compare performances of the IHT algorithm with the proposed macro algorithm. The comparison will be done in terms of minimization of the objective function  $G_{\ell_0}$ . In other words we will say that an algorithm found a “better” (local) minimizer  $\hat{x}$  if it leads to a lower function value. To this end, we will not consider very high dimensional problems as the ones usually encountered in image processing but smaller ones (e.g.  $M = 7$  and  $N = 15$ ) since, in this case, we are able to do a combinatorial search (as in §4.2) in a reasonable amount of time to find a global minimizer  $x^*$  and the associated energy value  $G_{\ell_0}(x^*)$ . Then we can compute the error  $|G_{\ell_0}(\hat{x}) - G_{\ell_0}(x^*)|$  as a measure of performance. The lower the error the better the estimated solution  $\hat{x}$ .

Figure 5 presents cumulative histograms of the errors  $|G_{\ell_0}(\hat{x}) - G_{\ell_0}(x^*)|$  obtained from 1000 random generation of  $A \in \mathbb{R}^{7 \times 15}$  and  $d \in \mathbb{R}^7$  from an uniform distribution. The IHT algorithm and the macro algorithm with IR1 or FB are used with both an initialization with  $A^T d$  or the solution of the  $\ell_1$  relaxed problem. Different values of the hyper-parameter  $\lambda$  are considered and the steps size of the IHT, FB and FISTA<sup>4</sup> algorithms are fixed to  $\frac{1}{L}$  where  $L$  is the Lipschitz constant of the data term in  $G_{\ell_0}$ .

Such figures represent the proportion of the 1000 achievements where the algorithm have produced an error smaller than the value given on the  $x$ -axis. Then the  $y$  value at zero corresponds to the proportion of the 1000 realizations where the algorithm reached a global optimum. The smallest  $x$  for which the curve attains 1 provides the largest error obtained over all the generated problems. Therefore the faster the curve tends to 1, the better the algorithm behaves.

We can see on Figure 5 that the macro algorithm always performs better than the IHT algorithm. This agrees with the fact that  $G_{\text{CEL0}}$  has less local minimizer than  $G_{\ell_0}$  which makes the macro algorithm

<sup>4</sup>FISTA [3] is used for the inner loop of IR1 algorithm.

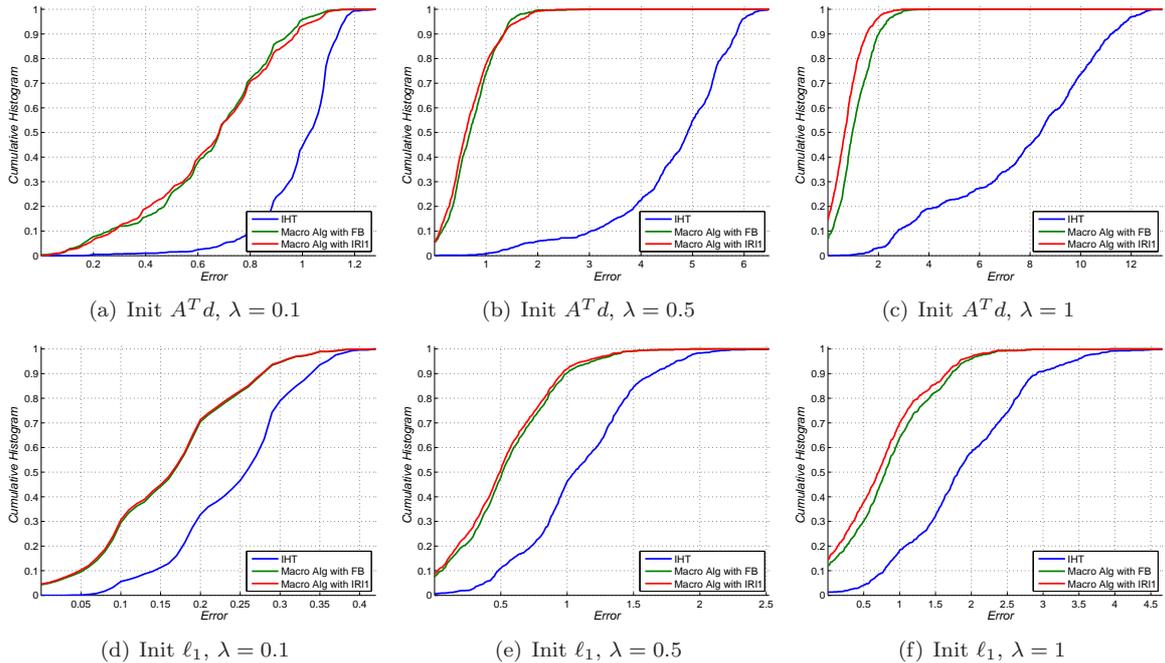


Figure 5: Cumulative histograms of the error  $|G_{\ell_0}(\hat{x}) - G_{\ell_0}(x^*)|$  where  $\hat{x}$  is the estimated solution and  $x^*$  is a global minimizer of  $G_{\ell_0}$ . The histograms are computed from 1000 random matrix  $A \in \mathbb{R}^{7 \times 15}$  and  $d \in \mathbb{R}^7$  generated from a uniform distribution. Two different initializations are considered:  $x^0 = A^T d$  (Init  $A^T d$ ) and  $x^0 = x_{\ell_1}$  (Init  $\ell_1$ ) the solution of the  $\ell_1$  relaxed problem. The experiment is repeated for three values of  $\lambda$  (0.1, 0.5 and 1). For each configuration the estimation is performed using the IHT algorithm (blue) and the macro algorithm combined with IR1 (red) or FB (green). The random seed is fixed at the same value for the six configurations in order to generate the same sequence of problems.

more “robust” against local minimizers of  $G_{\ell_0}$ . A slightly difference appears between the use of the IR1 or FB as inner algorithm. It can illustrate a point outlined by the authors in [29] that is the ability of IR1 to avoid local minimizers. The initialization has a huge effect on the solution obtained with the IHT algorithm. Initializing with the solution obtained by  $\ell_1$  relaxation clearly improves the performance of the IHT algorithm. However this is not true for Algorithm 1 for  $\lambda = 0.5$  and  $\lambda = 1$  which is robust toward the initialization on these experiments. For the case  $\lambda = 0.1$  we also observe an improvement of the performance of the macro algorithm by initializing with the  $\ell_1$  solution. One can note that the difference between IHT and macro algorithm is less important for low values of  $\lambda$ . This can be explained by the results presented in Table 1 where it was outlined that more strict local minimizer of  $G_{\ell_0}$  are removed with  $G_{\text{CEL0}}$  when  $\lambda$  is large. Results of Figure 5 show an interesting behaviour of the proposed macro algorithm which can be a good alternative to the IHT for the minimization of  $G_{\ell_0}$ .

## 6 Conclusions and perspectives

In this paper we provided an analysis of the CEL0 penalty used as a *continuous* alternative to the  $\ell_0$ -norm in problem (1.4). The definition of this penalty follows from the study of the one dimensional

case and the orthogonal one. For these special cases, we showed (§2 and §3) that the objective function  $G_{\text{CEL0}}$ , composed of a quadratic data term and the CEL0 penalty, is the *convex hull* of the functional,  $G_{\ell_0}$ , of the initial  $\ell_0$  penalized least squares problem (1.4). Then we studied the links between the minimizers of  $G_{\text{CEL0}}$  and  $G_{\ell_0}$  for any  $A \in \mathbb{R}^{M \times N}$ ,  $d \in \mathbb{R}^M$  and  $\lambda > 0$ . In this context, we proved two main results which are that  $G_{\ell_0}$  and  $G_{\text{CEL0}}$  *share the same global minimizers* (in the sense of Theorem 4.2) and that *from local minimizers of  $G_{\text{CEL0}}$  we can define a local minimizer for  $G_{\ell_0}$*  (Theorem 4.4). Moreover, we showed that the reciprocal of Theorem 4.4 is false and numerical experiments on low dimensional examples revealed that *a large number of strict local minimizers of  $G_{\ell_0}$  are not critical points of  $G_{\text{CEL0}}$* . Furthermore, on these experiments, these points are the ones providing the higher values of  $G_{\ell_0}$  among all strict local minimizers of  $G_{\ell_0}$ .

The analysis provided in §4 shows that it is preferable to address the  $\ell_0$  penalized least squares problem (1.4) by minimizing  $G_{\text{CEL0}}$  instead of  $G_{\ell_0}$ . Moreover,  $G_{\text{CEL0}}$  has the nice property to be *continuous* unlike  $G_{\ell_0}$  and then nonsmooth nonconvex algorithms can be used to minimize  $G_{\text{CEL0}}$  and indirectly  $G_{\ell_0}$ .

In the context of such algorithms, we defined in §5 a macro algorithm for which we proved the convergence of the generated sequence to a (local) minimizer of  $G_{\ell_0}$  (Theorem 5.1). From *any state of the art algorithm* verifying a *sufficient decrease condition* and the *convergence to a critical point of  $G_{\text{CEL0}}$* , the proposed macro algorithm add an outer loop ensuring the convergence to a point that is both a critical point of  $G_{\text{CEL0}}$  and a (local) minimizer of  $G_{\ell_0}$ . Numerical illustrations showed an interesting behaviour of this macro algorithm in terms of  $G_{\ell_0}$  minimization.

As perspectives, it seems interesting to further investigate several theoretical and practical issues concerning the CEL0 penalty and associated algorithms.

- Lemma 3.1 provides a characterization of critical points of  $G_{\text{CEL0}}$ . What about a characterization of (local) minimizers of  $G_{\text{CEL0}}$ ? As outlined previously, being a (local) minimizer of  $G_{\text{CEL0}}$  is a *necessary condition* to be a global optimum of  $G_{\ell_0}$  which makes this question relevant.
- In §3 we defined a family of functions underestimating  $G_{\ell_0}$ . Then we only focused on the special case of  $\Phi_{\text{CEL0}}$ . It would be of interest to study in details this family. One can see that among these functions some of them are convex and we can exhibit the subset of  $\mathcal{A}$  associated to such convex functions. In the orthogonal case, the largest one is given by  $G_{\text{CEL0}}$  which defines the convex hull of  $G_{\ell_0}$ . Then, the question of exhibiting the largest convex function of this family for a general case seems interesting.
- Is there a similar way to define a penalty with the same properties as  $\Phi_{\text{CEL0}}$  when we consider  $\|Dx\|_0$  where  $D$  is a linear operator? This formulation, often encountered in image processing applications, need to be investigated.
- How can we define a smart rule for the selection step of the macro algorithm (line 7 of Algorithm 1)?

Finally, this work provides a new way to address the  $\ell_0$  regularized least squares problem (1.4). Then one may focus on the development of new algorithms (or the improvement of existing ones) dedicated to the minimization of  $G_{\text{CEL0}}$  which, unlike  $G_{\ell_0}$ , has the nice property to be *continuous* and *convex with respect to each variable*.

## A The 1D biconjugate of $g$ (eq. 2.1)

Let us first determine the expression (2.4) of the conjugate of  $g$  which is the solution of the problem (2.3). In the following we will denote by  $h_{u^*}$  the objective function of (2.3) (with  $f = g$ ) that is

$$h_{u^*}(u) = u^*u - g(u) = u^*u - \frac{1}{2}(au - d)^2 - \lambda|u|_0. \quad (\text{A.1})$$

From the definition of  $|u|_0$  in (2.2), it is clear that  $h_{u^*}$  has two local maxima on  $\mathbb{R}$ . The first one is reached for  $u_0 = 0$  and corresponds to

$$h_{u^*}(u_0) = -\frac{d^2}{2}. \quad (\text{A.2})$$

The second one, reached at  $u_1 \neq 0$ , corresponds to the continuous part of  $h_{u^*}$  (*i.e.* when  $|u|_0 = 1$ ). Then  $u_1$  solves

$$h'_{u^*}(u_1) = 0, \quad u_1 \neq 0 \iff u^* - a^2u_1 + ad = 0, \quad u_1 \neq 0, \quad (\text{A.3})$$

which is equivalent to

$$u_1 = \frac{1}{a^2}(u^* + ad), \quad u^* \neq -ad, \quad (\text{A.4})$$

and we have

$$h_{u^*}(u_1) = \frac{(u^*)^2}{2a^2} + \frac{u^*d}{a} - \lambda = \frac{1}{2a^2}(u^* + ad)^2 - \frac{d^2}{2} - \lambda, \quad u^* \neq -ad. \quad (\text{A.5})$$

If  $u^* = -ad$  then  $u_0 = u_1 = 0$  and the associated value of  $h_{u^*}$  is given by (A.2). Otherwise we have to compare the values given in (A.2) and (A.5) to know if  $u_0$  or  $u_1$  is global maximizer,

$$\begin{aligned} \frac{1}{2a^2}(u^* + ad)^2 - \frac{d^2}{2} - \lambda + \frac{d^2}{2} &> 0 \\ \iff |u^* + ad| &> \sqrt{2\lambda a} \end{aligned}$$

Finally, the conjugate  $g^*$  is given by,

$$g^*(u^*) = \left\{ \frac{1}{2a^2}(u^* + ad)^2 - \lambda \right\} \mathbb{1}_{\{|u^* + ad| > \sqrt{2\lambda a}\}} - \frac{d^2}{2}. \quad (\text{A.6})$$

We can now calculate the biconjugate of  $g$  which is given by

$$g^{**}(u) = \sup_{v \in \mathbb{R}} uv - g^*(v). \quad (\text{A.7})$$

Again we will denote by  $h_u^*$  the objective function of (A.7),

$$h_u^*(v) = uv - g^*(v) = uv - \left\{ \frac{1}{2a^2}(v + ad)^2 - \lambda \right\} \mathbb{1}_{\{|v + ad| > \sqrt{2\lambda a}\}} + \frac{d^2}{2}. \quad (\text{A.8})$$

We have to consider two cases:

1. if  $|v + ad| \leq \sqrt{2\lambda a}$ ,

$$h_u^*(v) = uv + \frac{d^2}{2}. \quad (\text{A.9})$$

Since  $h_u^*$  is linear and that we want to maximize it on  $[-ad - \sqrt{2\lambda}a, -ad + \sqrt{2\lambda}a]$ , the entire interval is solution if  $u = 0$  and otherwise the maximizer  $v_1$  is one of the bounds of this interval depending on the sign of  $u$ . Then,  $v_1$  can be written as

$$v_1 = -ad + \text{sign}(u)\sqrt{2\lambda}a, \quad (\text{A.10})$$

and we have

$$h_u^*(v_1) = -adu + \text{sign}(u)u\sqrt{2\lambda}a + \frac{d^2}{2} = -adu + |u|\sqrt{2\lambda}a + \frac{d^2}{2}. \quad (\text{A.11})$$

2. if  $|v + ad| > \sqrt{2\lambda}a$ ,

$$h_u^*(v) = uv - \frac{1}{2a^2}(v + ad)^2 + \lambda + \frac{d^2}{2}, \quad (\text{A.12})$$

and the maximizer  $v_2$  solves

$$(h_u^*)'(v_2) = 0 \iff u - \frac{1}{a^2}(v_2 + ad) = 0, \quad (\text{A.13})$$

which is equivalent to

$$v_2 = a^2u - ad. \quad (\text{A.14})$$

Then we must insure that

$$|v_2 + ad| > \sqrt{2\lambda}a \iff |u| > \frac{\sqrt{2\lambda}}{a}. \quad (\text{A.15})$$

Finally, if  $u$  verifies the condition (A.15) then  $v_2$  given by (A.14) is the maximizer of  $h_u^*$  on  $\mathbb{R} \setminus [-ad - \sqrt{2\lambda}a, -ad + \sqrt{2\lambda}a]$  and

$$h_u^*(v_2) = \frac{a^2u^2}{2} - aud + \frac{d^2}{2} + \lambda = \frac{1}{2}(au - d)^2 + \lambda. \quad (\text{A.16})$$

Otherwise the supremum of  $h_u^*$  on  $\mathbb{R} \setminus [-ad - \sqrt{2\lambda}a, -ad + \sqrt{2\lambda}a]$  is given by (A.11).

From the two previous cases it is straightforward that

$$g^{**}(u) = \begin{cases} -adu + |u|\sqrt{2\lambda}a + \frac{d^2}{2} & \text{if } |u| \leq \frac{\sqrt{2\lambda}}{a}, \\ \frac{1}{2}(au - d)^2 + \lambda & \text{if } |u| > \frac{\sqrt{2\lambda}}{a}. \end{cases} \quad (\text{A.17})$$

## B Generalized gradient and proof of Lemma 3.1 (Critical points of $G_{\text{CELO}}$ )

### B.1 Generalized Gradient

In this section we recall the definition and properties of the *generalized gradient* introduced by Clarke [12]. This generalized gradient extends the notion of subdifferential for convex functions to nonconvex ones.

**Definition B.1. Local Lipschitz condition:** A function  $f$  is said to be locally Lipschitz at  $x$  if,

$$\exists \varepsilon > 0, \forall (y, y') \in \mathcal{B}(x, \varepsilon)^2 |f(y) - f(y')| \leq K \|y - y'\|, \quad (\text{B.1})$$

where  $K$  is a nonnegative scalar and  $\mathcal{B}(x, \varepsilon)$  is a  $\varepsilon$ -neighborhood of  $x$ .

**Definition B.2. Generalized gradient (Clarke [12]):** The generalized gradient of a locally Lipschitz function  $f : \mathbb{R}^N \rightarrow \mathbb{R}$  at  $x$ , denoted  $\partial f(x)$ , is defined by,

$$\partial f(x) := \{ \xi \in \mathbb{R}^N : f^0(x, v) \geq \langle v, \xi \rangle \forall v \in \mathbb{R}^N \}, \quad (\text{B.2})$$

where  $f^0(x, v)$  denotes the generalized directional derivative of  $f$  at  $x$  in the direction  $v$ ,

$$f^0(x, v) = \limsup_{\substack{y \rightarrow x \\ \eta \downarrow 0}} \frac{f(y + \eta v) - f(y)}{\eta}. \quad (\text{B.3})$$

When  $f$  is smooth,  $\partial f(x)$  reduces to the singleton  $\{\nabla f(x)\}$  [12, Proposition 2.2.4 and related Corollary] and, in the convex case, the Clarke generalized gradient coincides with the classical subdifferential of convex functions [12, Proposition 2.2.7].

**Proposition B.3. Local Extrema (Clarke [12, Proposition 2.3.2]):** If  $f$  attains a local minimum or maximum at  $x$ , then  $0 \in \partial f(x)$ .

The condition  $0 \in \partial f(x)$  is thus *necessary* for  $x$  being a local minimizer or maximizer of  $f$ . We call *critical points* of  $f$  the  $x \in \mathbb{R}^N$  verifying  $0 \in \partial f(x)$ .

From definition B.2 we can compute  $\partial \phi$  where  $\phi$  is defined by (2.8). Since for all  $u \neq 0$ ,  $\phi$  is differentiable, we have

$$\forall u \neq 0, \partial \phi(u) = \{ \phi'(u) \} = \left\{ a \{ \text{sign}(u) \sqrt{2\lambda} - au \} \mathbf{1}_{\{0 < |u| \leq \frac{\sqrt{2\lambda}}{a}\}} \right\}. \quad (\text{B.4})$$

The remaining case is  $u = 0$ . Let us start with the computation of the generalized directional derivative (B.3) at  $u = 0$ ,

$$\begin{aligned} \forall v \in \mathbb{R}, f^0(0, v) &= \limsup_{\substack{y \rightarrow 0 \\ \eta \downarrow 0}} \frac{\phi(y + \eta v) - \phi(y)}{\eta} \\ &= \limsup_{\substack{y \rightarrow 0 \\ \eta \downarrow 0}} \frac{1}{2\eta} \left[ \left( a|y| - \sqrt{2\lambda} \right)^2 - \left( a|y + \eta v| - \sqrt{2\lambda} \right)^2 \right]. \end{aligned}$$

Then one can see that,

$$f^0(0, v) = \begin{cases} av\sqrt{2\lambda} & \text{if } v \geq 0, \\ -av\sqrt{2\lambda} & \text{if } v < 0, \end{cases} \quad (\text{B.5})$$

which writes also as  $f^0(0, v) = a|v|\sqrt{2\lambda}$ . Therefore, from (B.2),  $\partial \phi(0)$  contains all  $\xi \in \mathbb{R}^N$  verifying  $a|v|\sqrt{2\lambda} \geq \xi v \forall v \in \mathbb{R}^N$  that is,

$$\partial \phi(0) = [-a\sqrt{2\lambda}, a\sqrt{2\lambda}]. \quad (\text{B.6})$$

## B.2 Proof of Lemma 3.1

First of all, from the definition of  $\mathcal{A}$  (eq (3.6)), we have

$$\alpha \in \mathcal{A} \implies \forall i \in \mathbb{I}_N, \alpha_i > 0, \quad (\text{B.7})$$

and then equations (3.10) and (3.11) are well-defined. Since the data term in  $G_\alpha$  is differentiable, we have  $\forall \mathbf{x} \in \mathbb{R}^N$

$$\partial G_\alpha(\mathbf{x}) = A^T(A\mathbf{x} - d) + \partial\Phi(\alpha, \lambda; \mathbf{x}). \quad (\text{B.8})$$

Let  $\hat{\mathbf{x}} \in \mathbb{R}^N$  be a critical point of  $G_\alpha$ , *i.e.*

$$0_{\mathbb{R}^N} \in \partial G_\alpha(\hat{\mathbf{x}}), \quad (\text{B.9})$$

which, following (3.9) and (B.8), is equivalent to

$$\begin{aligned} 0_{\mathbb{R}^N} \in \prod_{i \in \mathbb{I}_N} [A^T(A\hat{\mathbf{x}} - d)]_i + [-\sqrt{2\lambda}\alpha_i, \sqrt{2\lambda}\alpha_i] \mathbb{1}_{\{\hat{x}_i=0\}} + \alpha_i \left\{ \text{sign}(\hat{x}_i)\sqrt{2\lambda} - \alpha_i\hat{x}_i \right\} \mathbb{1}_{\{0 < |\hat{x}_i| \leq \frac{\sqrt{2\lambda}}{\alpha_i}\}}, \\ \iff \forall i \in \mathbb{I}_N, \begin{cases} 0 \in [\langle a_i, A\hat{\mathbf{x}} - d \rangle - \sqrt{2\lambda}\alpha_i, \langle a_i, A\hat{\mathbf{x}} - d \rangle + \sqrt{2\lambda}\alpha_i] & \text{if } \hat{x}_i = 0, \\ 0 = \langle a_i, A\hat{\mathbf{x}} - d \rangle + \alpha_i \left\{ \text{sign}(\hat{x}_i)\sqrt{2\lambda} - \alpha_i\hat{x}_i \right\} \mathbb{1}_{\{|\hat{x}_i| \leq \frac{\sqrt{2\lambda}}{\alpha_i}\}} & \text{if } \hat{x}_i \neq 0, \end{cases} \\ \iff \forall i \in \mathbb{I}_N, \begin{cases} |\langle a_i, A\hat{\mathbf{x}} - d \rangle| \leq \sqrt{2\lambda}\alpha_i & \text{if } \hat{x}_i = 0, \\ 0 = \langle a_i, A\hat{\mathbf{x}} - d \rangle - \alpha_i^2\hat{x}_i + \text{sign}(\hat{x}_i)\sqrt{2\lambda}\alpha_i & \text{if } 0 < |\hat{x}_i| \leq \frac{\sqrt{2\lambda}}{\alpha_i}, \\ 0 = \langle a_i, A\hat{\mathbf{x}} - d \rangle & \text{if } |\hat{x}_i| > \frac{\sqrt{2\lambda}}{\alpha_i}, \end{cases} \\ \iff \forall i \in \mathbb{I}_N, \begin{cases} |\langle a_i, A\hat{\mathbf{x}}^{(i)} - d \rangle| \leq \sqrt{2\lambda}\alpha_i & \text{if } \hat{x}_i = 0, \\ 0 = \langle a_i, A\hat{\mathbf{x}}^{(i)} - d \rangle + (\|a_i\|^2 - \alpha_i^2)\hat{x}_i + \text{sign}(\hat{x}_i)\sqrt{2\lambda}\alpha_i & \text{if } 0 < |\hat{x}_i| \leq \frac{\sqrt{2\lambda}}{\alpha_i}, \\ 0 = \langle a_i, A\hat{\mathbf{x}}^{(i)} - d \rangle + \|a_i\|^2\hat{x}_i & \text{if } |\hat{x}_i| > \frac{\sqrt{2\lambda}}{\alpha_i}, \end{cases} \end{aligned}$$

where  $\hat{\mathbf{x}}^{(i)} = (\hat{x}_1, \dots, \hat{x}_{i-1}, 0, \hat{x}_{i+1}, \dots, \hat{x}_N)$ . Now we distinguish two cases in Lemma 3.1. Let  $i \in \mathbb{I}_N$  then,

1. if  $\alpha_i = \|a_i\|$  we have

$$\begin{cases} |\langle a_i, A\hat{\mathbf{x}}^{(i)} - d \rangle| \leq \sqrt{2\lambda}\|a_i\| & \text{if } \hat{x}_i = 0, \\ \langle a_i, A\hat{\mathbf{x}}^{(i)} - d \rangle = -\text{sign}(\hat{x}_i)\sqrt{2\lambda}\|a_i\| & \text{if } 0 < |\hat{x}_i| \leq \frac{\sqrt{2\lambda}}{\|a_i\|}, \\ \hat{x}_i = -\frac{\langle a_i, A\hat{\mathbf{x}}^{(i)} - d \rangle}{\|a_i\|^2} & \text{if } |\hat{x}_i| > \frac{\sqrt{2\lambda}}{\|a_i\|}. \end{cases} \quad (\text{B.10})$$

$$\iff \begin{cases} \hat{x}_i = 0 & \text{if } |\langle a_i, A\hat{\mathbf{x}}^{(i)} - d \rangle| < \sqrt{2\lambda}\|a_i\|, \\ \hat{x}_i = -s_i t, t \in \left[0, \frac{\sqrt{2\lambda}}{\|a_i\|}\right] & \text{if } |\langle a_i, A\hat{\mathbf{x}}^{(i)} - d \rangle| = \sqrt{2\lambda}\|a_i\|, \\ \hat{x}_i = -\frac{\langle a_i, A\hat{\mathbf{x}}^{(i)} - d \rangle}{\|a_i\|^2} & \text{if } |\langle a_i, A\hat{\mathbf{x}}^{(i)} - d \rangle| > \sqrt{2\lambda}\|a_i\|, \end{cases} \quad (\text{B.11})$$

where  $s_i = \text{sign}(\langle a_i, A\hat{\mathbf{x}}^{(i)} - d \rangle)$ .

To show the equivalence between (B.10) and (B.11) we proceed by proving both implications.

$\Rightarrow$  From equation (B.10) we see that  $\hat{x}_i$  can be equal to 0,  $x^+ = -\langle a_i, A\hat{x}^{(i)} - d \rangle / \|a_i\|^2$  or be such that  $|\hat{x}_i| \in ]0, \sqrt{2\lambda} / \|a_i\|]$ .

- If  $|\langle a_i, A\hat{x}^{(i)} - d \rangle| < \sqrt{2\lambda} \|a_i\|$  we proceed by contradiction. If  $|\hat{x}_i| \in ]0, \sqrt{2\lambda} / \|a_i\|]$  then thanks to the second line of (B.10),  $|\langle a_i, A\hat{x}^{(i)} - d \rangle| = \sqrt{2\lambda} \|a_i\|$  which contradicts our hypothesis. Then either  $\hat{x}_i = 0$  or  $|\hat{x}_i| > \sqrt{2\lambda} / \|a_i\|$ . This later case is impossible thanks to the third line of (B.10) and then  $\hat{x}_i = 0$ .
- If  $|\langle a_i, A\hat{x}^{(i)} - d \rangle| = \sqrt{2\lambda} \|a_i\|$ , suppose that  $\hat{x}_i = x^+$  then

$$\hat{x}_i = -s_i \frac{\sqrt{2\lambda}}{\|a_i\|}. \quad (\text{B.12})$$

Moreover, the second line of (B.10) gives that  $\text{sign}(\hat{x}_i) = -s_i$  which, with (B.12), proves the second line of (B.11).

- If  $|\langle a_i, A\hat{x}^{(i)} - d \rangle| > \sqrt{2\lambda} \|a_i\|$ , suppose that  $|\hat{x}_i| \in [0, \sqrt{2\lambda} / \|a_i\|]$ , then from (B.10) we have  $|\langle a_i, A\hat{x}^{(i)} - d \rangle| \leq \sqrt{2\lambda} \|a_i\|$  which contradicts the previous condition. Thus in this case we have necessarily  $\hat{x}_i = x^+$  which prove the last line of (B.11).

$\Leftarrow$  We now prove the reciprocal implication,

- If  $\hat{x}_i = 0$ , suppose that  $|\langle a_i, A\hat{x}^{(i)} - d \rangle| > \sqrt{2\lambda} \|a_i\|$ , then from (B.11) we get

$$|\hat{x}_i| = |x^+| > \sqrt{2\lambda} / \|a_i\|, \quad (\text{B.13})$$

which contradicts the fact that  $\hat{x}_i = 0$ . Then in this case we have necessarily  $|\langle a_i, A\hat{x}^{(i)} - d \rangle| \leq \sqrt{2\lambda} \|a_i\|$  which prove the first line of (B.10).

- If  $0 < |\hat{x}_i| \leq \sqrt{2\lambda} / \|a_i\|$ , suppose that  $|\langle a_i, A\hat{x}^{(i)} - d \rangle| < \sqrt{2\lambda} \|a_i\|$  then, from (B.11), we get that  $\hat{x}_i = 0$  which contradicts the previous condition. Similarly, suppose that  $|\langle a_i, A\hat{x}^{(i)} - d \rangle| > \sqrt{2\lambda} \|a_i\|$  then, from (B.11), we get again (B.13) which also contradicts  $0 < |\hat{x}_i| \leq \sqrt{2\lambda} / \|a_i\|$ . Finally, in this case we have necessarily  $|\langle a_i, A\hat{x}^{(i)} - d \rangle| = \sqrt{2\lambda} \|a_i\|$  which, combined with the fact that from (B.11) we have  $\text{sign}(\hat{x}_i) = -s_i$ , proves the second line of (B.10).
- If  $|\hat{x}_i| > \sqrt{2\lambda} / \|a_i\|$ , suppose that  $|\langle a_i, A\hat{x}^{(i)} - d \rangle| \leq \sqrt{2\lambda} \|a_i\|$  then, from (B.11), we get that  $|\hat{x}_i| \in [0, \sqrt{2\lambda} / \|a_i\|]$  which contradicts the previous condition. Then we have necessary  $|\langle a_i, A\hat{x}^{(i)} - d \rangle| > \sqrt{2\lambda} \|a_i\| \Rightarrow \hat{x}_i = x^+$  and prove the third line of (B.10).

2. otherwise if  $\alpha_i < \|a_i\|$  we have

$$\hat{x}_i = \begin{cases} |\langle a_i, A\hat{x}^{(i)} - d \rangle| \leq \sqrt{2\lambda} \alpha_i & \text{if } \hat{x}_i = 0, \\ \frac{-\langle a_i, A\hat{x}^{(i)} - d \rangle + \sqrt{2\lambda} \alpha_i}{\|a_i\|^2 - \alpha_i^2} & \text{if } -\frac{\sqrt{2\lambda}}{\alpha_i} \leq \hat{x}_i < 0, \\ \frac{-\langle a_i, A\hat{x}^{(i)} - d \rangle - \sqrt{2\lambda} \alpha_i}{\|a_i\|^2 - \alpha_i^2} & \text{if } 0 < \hat{x}_i \leq \frac{\sqrt{2\lambda}}{\alpha_i}, \\ \frac{-\langle a_i, A\hat{x}^{(i)} - d \rangle}{\|a_i\|^2} & \text{if } |\hat{x}_i| > \frac{\sqrt{2\lambda}}{\alpha_i}, \end{cases} \quad (\text{B.14})$$

$$\Leftrightarrow \hat{x}_i = \begin{cases} |\langle a_i, A\hat{x}^{(i)} - d \rangle| \leq \sqrt{2\lambda} \alpha_i & \text{if } \hat{x}_i = 0, \\ \frac{-\langle a_i, A\hat{x}^{(i)} - d \rangle + \sqrt{2\lambda} \alpha_i}{\|a_i\|^2 - \alpha_i^2} & \text{if } \sqrt{2\lambda} \alpha_i < \langle a_i, A\hat{x}^{(i)} - d \rangle \leq \frac{\sqrt{2\lambda}}{\alpha_i} \|a_i\|^2, \\ \frac{-\langle a_i, A\hat{x}^{(i)} - d \rangle - \sqrt{2\lambda} \alpha_i}{\|a_i\|^2 - \alpha_i^2} & \text{if } \frac{-\sqrt{2\lambda}}{\alpha_i} \|a_i\|^2 \leq \langle a_i, A\hat{x}^{(i)} - d \rangle < -\sqrt{2\lambda} \alpha_i, \\ \frac{-\langle a_i, A\hat{x}^{(i)} - d \rangle}{\|a_i\|^2} & \text{if } |\langle a_i, A\hat{x}^{(i)} - d \rangle| > \frac{\sqrt{2\lambda}}{\alpha_i} \|a_i\|^2. \end{cases} \quad (\text{B.15})$$

$$\Leftrightarrow \begin{cases} \hat{x}_i = 0 & \text{if } |\langle a_i, A\hat{x}^{(i)} - d \rangle| \leq \sqrt{2\lambda}\alpha_i, \\ \hat{x}_i = \frac{-\langle a_i, A\hat{x}^{(i)} - d \rangle + s_i\sqrt{2\lambda}\alpha_i}{\|a_i\|^2 - \alpha_i^2} & \text{if } \sqrt{2\lambda}\alpha_i < |\langle a_i, A\hat{x}^{(i)} - d \rangle| \leq \frac{\sqrt{2\lambda}}{\alpha_i}\|a_i\|^2, \\ \hat{x}_i = \frac{-\langle a_i, A\hat{x}^{(i)} - d \rangle}{\|a_i\|^2} & \text{if } |\langle a_i, A\hat{x}^{(i)} - d \rangle| > \frac{\sqrt{2\lambda}}{\alpha_i}\|a_i\|^2. \end{cases} \quad (\text{B.16})$$

The equivalence between (B.14) and (B.15) is based on the same kind of arguments used previously to show the equivalence between (B.10) and (B.11). This completes the proof.

## C Proof of Theorem 4.2 (Link between global minimizers of $G_{\ell_0}$ and $G_{\text{CELO}}$ )

We start with the second point of Theorem 4.2 which will be used, in the following, to prove the first affirmation.

- (ii) Let  $G_{\text{CELO}}$  reach a global minimum at  $\hat{x} \in \mathbb{R}^N$ . If  $\sigma^-(\hat{x}) = \emptyset$ , then  $\hat{x} = \hat{x}^0$  where  $\hat{x}^0$  is defined by (4.12) and

$$G_{\text{CELO}}(\hat{x}^0) = G_{\text{CELO}}(\hat{x}). \quad (\text{C.1})$$

Otherwise, if  $\sigma^-(\hat{x}) \neq \emptyset$ , then Lemma 4.1 ensures that  $\hat{x}^0$  defined by (4.12) is also a global minimizer of  $G_{\text{CELO}}$ . Indeed setting one by one the nonzero components of  $\hat{x}$  indexed by the elements of  $\sigma^-(\hat{x})$  to zero, results, from Lemma 4.1, in another global minimizer of  $G_{\text{CELO}}$ . This means that when  $\sigma^-(\hat{x}) \neq \emptyset$ ,  $G_{\text{CELO}}$  is constant on the hypercube,

$$\prod_{i \in \sigma^-(\hat{x})} I_i \quad \text{where } I_i = \begin{cases} \left[0, \frac{\sqrt{2\lambda}}{\|a_i\|}\right] & \text{if } \hat{x}_i > 0, \\ \left[-\frac{\sqrt{2\lambda}}{\|a_i\|}, 0\right] & \text{if } \hat{x}_i < 0. \end{cases} \quad (\text{C.2})$$

Thus (C.1) also holds when  $\sigma^-(\hat{x}) \neq \emptyset$ . Then, by definition,  $G_{\text{CELO}}$  underestimates  $G_{\ell_0}$ , *i.e.*

$$\forall x \in \mathbb{R}^N, G_{\text{CELO}}(x) \leq G_{\ell_0}(x), \quad (\text{C.3})$$

and

$$G_{\text{CELO}}(\hat{x}^0) = G_{\ell_0}(\hat{x}^0) \quad (\text{C.4})$$

since  $\hat{x}^0$  lives in the subset of  $\mathbb{R}^N$  on which  $G_{\text{CELO}}$  and  $G_{\ell_0}$  are identical (see from definition of  $\Phi_{\text{CELO}}$  in (3.8) and (4.1) that  $\forall x \in \mathbb{R}^N$  such that  $\sigma^-(x) = \emptyset$ , we have  $\Phi_{\text{CELO}}(x) = \lambda\|x\|_0$ ). Let us now suppose that  $\hat{x}^0$  is not a global minimizer for  $G_{\ell_0}$ , then

$$\exists \bar{x} \in \mathbb{R}^N \quad \text{such that } G_{\ell_0}(\bar{x}) < G_{\ell_0}(\hat{x}^0). \quad (\text{C.5})$$

Combining (C.5) with (C.1),(C.3) and (C.4) it comes

$$G_{\text{CELO}}(\bar{x}) \leq G_{\ell_0}(\bar{x}) < G_{\ell_0}(\hat{x}^0) = G_{\text{CELO}}(\hat{x}^0) = G_{\text{CELO}}(\hat{x}), \quad (\text{C.6})$$

which contradicts the fact that  $\hat{x}$  is a global minimizer for  $G_{\text{CELO}}$  and completes the proof of the second affirmation of the theorem.

- (i) To prove the first point of Theorem 4.2 let  $G_{\ell_0}$  have a global minimum at  $\hat{x} \in \mathbb{R}^N$ . Then from Proposition 4.1, all the nonzero components of  $\hat{x}$  verify (4.3) and by definition of  $G_{\text{CELO}}$  we have

$$G_{\text{CELO}}(\hat{x}) = G_{\ell_0}(\hat{x}). \quad (\text{C.7})$$

Let us now suppose that  $\hat{x}$  is not a global minimizer for  $G_{\text{CELO}}$ , then

$$\exists \bar{x} \in \mathbb{R}^N \text{ such that } G_{\text{CELO}}(\bar{x}) < G_{\text{CELO}}(\hat{x}). \quad (\text{C.8})$$

Choosing  $\bar{x}$  as global minimizer of  $G_{\text{CELO}}$ , the second point of Theorem 4.2 states that  $\bar{x}^0$ , defined by (4.12), is also a global minimizer for  $G_{\ell_0}$ . Thus combining (4.13), (C.7) and (C.8) we obtain

$$G_{\ell_0}(\bar{x}^0) = G_{\text{CELO}}(\bar{x}^0) = G_{\text{CELO}}(\bar{x}) < G_{\text{CELO}}(\hat{x}) = G_{\ell_0}(\hat{x}), \quad (\text{C.9})$$

which contradicts the fact that  $\hat{x}$  is a global minimizer for  $G_{\ell_0}$  and completes the proof.

## D Proof of Theorem 5.1 (Convergence of the macro algorithm)

In this proof we consider the restriction of  $G_{\text{CELO}}$  defined by,

$$G_{\text{CELO}}^{\mathcal{E}} : \mathcal{E} \longrightarrow \mathbb{R} \quad (\text{D.1})$$

where  $\mathcal{E} := \{x \in \mathbb{R}^N, G_{\text{CELO}}(x) < G_{\text{CELO}}(x^{\text{init}})\} \subset \mathbb{R}^N$ . Note that since the macro algorithm is a descent algorithm it works only on the restriction  $G_{\text{CELO}}^{\mathcal{E}}$  and then we can limit the proof to this restriction.

Let us denote  $\mathcal{C}$  the set of Clarke critical points of  $G_{\text{CELO}}^{\mathcal{E}}$ . We start the proof by showing that the image of  $\mathcal{C}$  by  $G_{\text{CELO}}^{\mathcal{E}}$ , denoted  $G_{\text{CELO}}^{\mathcal{E}}(\mathcal{C})$ , is finite. Let  $\hat{x} \in \mathcal{C}$ ,  $\hat{\sigma} = \sigma(\hat{x})$  and  $\hat{\sigma}^- = \sigma^-(\hat{x})$ . Then from equation (B.10) in Appendix B.2,  $\hat{x}$  solves

$$[(A_{\hat{\sigma}})^T A_{\hat{\sigma}} - D_{\hat{\sigma}^-}] \hat{x}_{\hat{\sigma}} = (A_{\hat{\sigma}})^T d - \sqrt{2\lambda} d_{\hat{\sigma}^-}, \quad (\text{D.2})$$

where  $(d_{\hat{\sigma}^-})_i = \text{sign}(\hat{x}_i) \|a_i\| \mathbf{1}_{\{i \in \hat{\sigma}^-\}}$  and  $D_{\hat{\sigma}^-}$  is a diagonal matrix whose diagonal entries are given by  $(D_{\hat{\sigma}^-})_{ii} = \|a_i\|^2 \mathbf{1}_{\{i \in \hat{\sigma}^-\}}$ . Therefore, belonging to  $\mathcal{E}$  and solving (D.2) is a *necessary and sufficient* condition to be a critical point of  $G_{\text{CELO}}^{\mathcal{E}}$ .

Let us now take  $\omega \subset \mathbb{I}_N$ ,  $\omega^- \subset \omega$  and let  $x \in \mathbb{R}^N$  solve

$$[(A_{\omega})^T A_{\omega} - D_{\omega^-}] x_{\omega} = (A_{\omega})^T d - \sqrt{2\lambda} d_{\omega^-}, \quad (\text{D.3})$$

where  $(d_{\omega^-})_i = \pm \|a_i\| \mathbf{1}_{\{i \in \omega^-\}}$ . Then such a  $x$  is a critical point of  $G_{\text{CELO}}^{\mathcal{E}}$  *if and only if*  $x \in \mathcal{E}$  and solves (D.2) (*i.e.*  $\sigma(x) = \omega$ ,  $\sigma^-(x) = \omega^-$  and  $d_{\sigma^-(x)} = d_{\omega^-}$ ). A finite number of systems as (D.3) can be constructed by choosing,  $\omega$ ,  $\omega^-$  and the sign of the nonzero entries of  $d_{\omega^-}$ . Indeed the number of subset  $\omega \subset \mathbb{I}_N$  is equal to  $\sum_{k=1}^N \binom{N}{k}$  and similarly the number of subset  $\omega^- \subset \omega$  is equal to  $\sum_{l=1}^{\#\omega} \binom{\#\omega}{l}$ . Then we have the choice of the sign for the nonzero entries of  $d_{\omega^-}$  that is  $2^{\#\omega^-}$  possibilities. Finally one can generate

$$\sum_{k=1}^N \binom{N}{k} \times \sum_{l=1}^k \binom{k}{l} \times (2l), \quad (\text{D.4})$$

different systems. This number is *huge* but *finite*. We recall that solutions of these systems are critical points of  $G_{\text{CELO}}^{\mathcal{E}}$  *if and only if* they belong to  $\mathcal{E}$  and solve (D.2). Thus the number of systems as (D.3) leading to critical points of  $G_{\text{CELO}}^{\mathcal{E}}$  is smaller than (D.4).

Systems as (D.3) can admit a unique solution if  $\text{rank}((A_{\omega})^T A_{\omega} - D_{\omega^-}) = \#\omega$  or, either none or an infinity of solutions if  $\text{rank}((A_{\omega})^T A_{\omega} - D_{\omega^-}) < \#\omega$ . When such a system admits an infinity of

solutions, these solutions define a *vector subspace* of  $\mathbb{R}^N$ . Among them, the ones which are critical points of  $G_{\text{CELO}}^{\mathcal{E}}$  belong to the intersection of this vector subspace with  $\mathcal{E}$  and

$$\left( \prod_{i \in \mathbb{I}_N} I_i \right) \subset \mathbb{R}^N \quad \text{where } I_i = \begin{cases} \{0\} & \text{if } i \notin \omega \\ \left[ -\frac{\sqrt{2\lambda}}{\|a_i\|}, 0 \right[ , & \text{if } i \in \omega^- \text{ and } (d_{\omega^-})_i < 0 \\ \left[ 0, \frac{\sqrt{2\lambda}}{\|a_i\|} \right] , & \text{if } i \in \omega^- \text{ and } (d_{\omega^-})_i > 0 \\ \left[ -\infty, -\frac{\sqrt{2\lambda}}{\|a_i\|} \right] \cup \left[ \frac{\sqrt{2\lambda}}{\|a_i\|}, +\infty \right[ & \text{if } i \in \omega \setminus \omega^- \end{cases}$$

which is composed of a *finite* number of connected components in  $\mathbb{R}^N$ . Then one system as (D.3) results at most in a *finite* number of connected component in  $\mathcal{C}$ . Combining that with (D.4) we get that  $\mathcal{C}$  contains a *finite number of connected components*. Following [2], since  $\frac{1}{2}\|Ax - d\|^2$  is a polynomial function and  $\Phi_{\text{CELO}}$  has a piecewise polynomial graph,  $G_{\text{CELO}}^{\mathcal{E}}$  is *semi-algebraic* and then *subanalytic*. Hence we have all the requirements of [6, Theorem 7] which states that  $G_{\text{CELO}}^{\mathcal{E}}$  is constant on each connected component of  $\mathcal{C}$ . Consequently  $G_{\text{CELO}}^{\mathcal{E}}(\mathcal{C})$  is a *finite set*.

Following the same arguments as in the proof of Lemma 4.1, setting one  $x_i^k$  such that  $i \in \sigma_k^-$  to zero does not change the value of  $G_{\text{CELO}}^{\mathcal{E}}$ . Hence  $G_{\text{CELO}}^{\mathcal{E}}(x^{\text{temp}}) = G_{\text{CELO}}^{\mathcal{E}}(x^k)$ . With hypothesis H1 and H2 on Alg we get

$$\forall k \in \mathbb{N}, x^k \in \mathcal{C} \quad \text{and} \quad G_{\text{CELO}}^{\mathcal{E}}(x^{k+1}) \leq G_{\text{CELO}}^{\mathcal{E}}(x^{\text{temp}}) - \beta \|x^{k+1} - x^{\text{temp}}\|^2, \quad (\text{D.5})$$

and that  $x^{k+1}$  is a critical point of  $G_{\text{CELO}}^{\mathcal{E}}$ . Then  $x^{k+1} = x^{\text{temp}}$  means that  $x^{\text{temp}}$  was a critical point of  $G_{\text{CELO}}$  and that the inner loop has been broken thanks to  $\sigma_k^- = \emptyset$ . In this case the algorithm stops. Otherwise, if  $x^{k+1} \neq x^{\text{temp}}$  we get from (D.5),

$$G_{\text{CELO}}^{\mathcal{E}}(x^{k+1}) < G_{\text{CELO}}^{\mathcal{E}}(x^{\text{temp}}) = G_{\text{CELO}}^{\mathcal{E}}(x^k), \quad (\text{D.6})$$

which means that the algorithm moves to another value  $G_{\text{CELO}}^{\mathcal{E}}(x^{k+1}) \in G_{\text{CELO}}^{\mathcal{E}}(\mathcal{C})$  *strictly lower* than  $G_{\text{CELO}}^{\mathcal{E}}(x^k)$ . Moreover, for all global minimizer  $\hat{x} \in \mathcal{C}$ , Lemma 4.1 ensures that  $\hat{x}^0$  is also a global minimizer verifying  $\sigma^-(\hat{x}^0) = \emptyset$ . Then there exists at least one element  $\bar{x} \in \mathcal{C}$  verifying  $\sigma^-(\bar{x}) = \emptyset$  and  $G_{\text{CELO}}^{\mathcal{E}}(\bar{x}) = \inf\{G_{\text{CELO}}^{\mathcal{E}}(\mathcal{C})\}$ . This last statement with (D.6) and the fact that  $G_{\text{CELO}}^{\mathcal{E}}(\mathcal{C})$  is finite allow to conclude that there exists  $k^* \in \mathbb{N}$  such that  $\sigma_{k^*}^- = \emptyset$ .

Finally, from Lemma 4.3 we get that  $x^{k^*}$  is a (local) minimizer of  $G_{\ell_0}$ . This completes the proof.

## References

- [1] ANESTIS ANTONIADIS ET AL., *Wavelet methods in statistics: Some recent developments and their applications*, Statistics Surveys, 1 (2007), pp. 16–55.
- [2] HEDY ATTOUCH, JÉRÔME BOLTE, AND BENAR FUX SVAITER, *Convergence of descent methods for semi-algebraic and tame problems: proximal algorithms, forwardbackward splitting, and regularized gaussseidel methods*, Mathematical Programming, 137 (2013), pp. 91–129.
- [3] AMIR BECK AND MARC TEOULLE, *A fast iterative shrinkage-thresholding algorithm for linear inverse problems*, SIAM Journal on Imaging Sciences, 2 (2009), pp. 183–202.
- [4] ANDREW BLAKE AND ANDREW ZISSERMAN, *Visual reconstruction*, vol. 2, MIT press Cambridge, 1987.
- [5] THOMAS BLUMENSATH AND MIKE E DAVIES, *Iterative thresholding for sparse approximations*, Journal of Fourier Analysis and Applications, 14 (2008), pp. 629–654.

- [6] JÉRÔME BOLTE, ARIS DANILIDIS, ADRIAN LEWIS, AND MASAHIRO SHIOTA, *Clarke critical values of subanalytic lipschitz continuous functions*, Ann. Polon. Math.(memorial issue for S. Lojasiewicz), (2005).
- [7] LEO BREIMAN, *Better subset regression using the nonnegative garrote*, Technometrics, 37 (1995), pp. 373–384.
- [8] EMMANUEL J CANDÈS, JUSTIN ROMBERG, AND TERENCE TAO, *Robust uncertainty principles: Exact signal reconstruction from highly incomplete frequency information*, Information Theory, IEEE Transactions on, 52 (2006), pp. 489–509.
- [9] EMMANUEL J CANDÈS, MICHAEL B WAKIN, AND STEPHEN P BOYD, *Enhancing sparsity by reweighted  $\ell_1$  minimization*, Journal of Fourier analysis and applications, 14 (2008), pp. 877–905.
- [10] SHENG CHEN, COLIN FN COWAN, AND PETER M GRANT, *Orthogonal least squares learning algorithm for radial basis function networks*, Neural Networks, IEEE Transactions on, 2 (1991), pp. 302–309.
- [11] SCOTT SHAOBING CHEN, DAVID L DONOHO, AND MICHAEL A SAUNDERS, *Atomic decomposition by basis pursuit*, SIAM journal on scientific computing, 20 (1998), pp. 33–61.
- [12] FRANK H CLARKE, *Optimization and nonsmooth analysis*, vol. 5, Siam, 1990.
- [13] PATRICK L COMBETTES AND VALÉRIE R WAJS, *Signal recovery by proximal forward-backward splitting*, Multiscale Modeling & Simulation, 4 (2005), pp. 1168–1200.
- [14] INGRID DAUBECHIES, MICHEL DEFRISE, AND CHRISTINE DE MOL, *An iterative thresholding algorithm for linear inverse problems with a sparsity constraint*, Communications on Pure and Applied Mathematics, 57 (2004), pp. 1413–1457.
- [15] GEOFF DAVIS, STEPHANE MALLAT, AND MARCO AVELLANEDA, *Adaptive greedy approximations*, Constructive approximation, 13 (1997), pp. 57–98.
- [16] DAVID L DONOHO, *For most large underdetermined systems of linear equations the minimal  $\ell_1$ -norm solution is also the sparsest solution*, Communications on Pure and Applied Mathematics, 59 (2006), pp. 797–829.
- [17] JIANQING FAN AND RUNZE LI, *Variable selection via nonconcave penalized likelihood and its oracle properties*, Journal of the American Statistical Association, 96 (2001), pp. 1348–1360.
- [18] SIMON FOUCART AND MING-JUN LAI, *Sparsest solutions of underdetermined linear systems via  $\ell_q$ -minimization for  $0 < q \leq 1$* , Applied and Computational Harmonic Analysis, 26 (2009), pp. 395–407.
- [19] GILLES GASSO, ALAIN RAKOTOMAMONJY, AND STÉPHANE CANU, *Recovering sparse signals with a certain family of nonconvex penalties and DC programming*, Signal Processing, IEEE Transactions on, 57 (2009), pp. 4686–4698.
- [20] PINGHUA GONG, CHANGSHUI ZHANG, ZHAOSONG LU, JIANHUA HUANG, AND JIEPING YE, *A General Iterative Shrinkage and Thresholding Algorithm for Non-convex Regularized Optimization Problems*, in Proceedings of The 30th International Conference on Machine Learning, 2013, pp. 37–45.
- [21] DAVID R HUNTER AND KENNETH LANGE, *A tutorial on MM algorithms*, The American Statistician, 58 (2004), pp. 30–37.

- [22] MATTHIEU KOWALSKI, *Thresholding rules and iterative shrinkage/thresholding algorithm: A convergence study*, in ICIP-International Conference on Image Processing, 2014.
- [23] STÉPHANE G MALLAT AND ZHIFENG ZHANG, *Matching pursuits with time-frequency dictionaries*, Signal Processing, IEEE Transactions on, 41 (1993), pp. 3397–3415.
- [24] HOSEIN MOHIMANI, MASSOUD BABAIE-ZADEH, IRINA GORODNITSKY, AND CHRISTIAN JUTTEN, *Sparse recovery using smoothed  $\ell_0$  (SLO): Convergence analysis*, arXiv preprint arXiv:1001.5073, (2010).
- [25] HOSEIN MOHIMANI, MASSOUD BABAIE-ZADEH, AND CHRISTIAN JUTTEN, *A fast approach for overcomplete sparse decomposition based on smoothed  $\ell_0$  norm*, Signal Processing, IEEE Transactions on, 57 (2009), pp. 289–301.
- [26] BALAS KAUSIK NATARAJAN, *Sparse approximate solutions to linear systems*, SIAM journal on computing, 24 (1995), pp. 227–234.
- [27] MILA NIKOLOVA, *Description of the minimizers of least squares regularized with  $\ell_0$ -norm. Uniqueness of the global minimizer*, SIAM Journal on Imaging Sciences, 6 (2013), pp. 904–937.
- [28] ———, *Relationship between the optimal solutions of least squares regularized with  $\ell_0$ -norm and constrained by  $k$ -sparsity*, hal-00944006v1 Preprint, (2014).
- [29] P. OCHS, A. DOSOVITSKIY, T. BROX, AND T. POCK, *An iteratively reweighted Algorithm for Non-smooth Non-convex Optimization in Computer Vision*, Technical Report, (2014).
- [30] YAGYENSH CHANDRA PATI, RAMIN REZAIIFAR, AND PS KRISHNAPRASAD, *Orthogonal matching pursuit: Recursive function approximation with applications to wavelet decomposition*, in Signals, Systems and Computers, 1993. 1993 Conference Record of The Twenty-Seventh Asilomar Conference on, IEEE, 1993, pp. 40–44.
- [31] CHARLES SOUSSEN, JÉRÔME IDIER, DAVID BRIE, AND JUNBO DUAN, *From Bernoulli–Gaussian deconvolution to sparse signal restoration*, Signal Processing, IEEE Transactions on, 59 (2011), pp. 4572–4584.
- [32] VLADIMIR N TEMLYAKOV, *Greedy approximation*, Acta Numerica, 17 (2008), pp. 235–409.
- [33] JOEL A TROPP, *Greedy is good: Algorithmic results for sparse approximation*, Information Theory, IEEE Transactions on, 50 (2004), pp. 2231–2242.
- [34] CUN-HUI ZHANG, *Discussion: One-step sparse estimates in nonconcave penalized likelihood models*, The Annals of Statistics, 36 (2008), pp. 1553–1560.
- [35] ———, *Nearly unbiased variable selection under minimax concave penalty*, The Annals of Statistics, (2010), pp. 894–942.
- [36] HUI ZOU, *The adaptive lasso and its oracle properties*, Journal of the American statistical association, 101 (2006), pp. 1418–1429.