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Differential Games of Competition in Online Content Diffusion

Francesco De Pellegrini[◇], Alexandre Reiffers[⋆] and Eitan Altman[⋆]

***Abstract*—Access to online contents represents a large share of the Internet traffic. Most such contents are multimedia items which are user-generated, i.e., posted online by the contents' owners. In this paper we focus on how those who provide contents can leverage online platforms in order to profit from their large base of potential viewers.**

Actually, platforms like Vimeo or YouTube provide tools to accelerate the dissemination of contents, i.e., recommendation lists and other re-ranking mechanisms. Hence, the popularity of a content can be increased by paying a cost for advertisement: doing so, it will appear with some priority in the recommendation lists and will be accessed more frequently by the platform users.

Ultimately, such acceleration mechanism engenders a competition among online contents to gain popularity. In this context, our focus is on the structure of the acceleration strategies which a content provider should use in order to optimally promote a content given a certain daily budget. Such a best response indeed depends on the strategies adopted by competing content providers. Also, it is a function of the potential popularity of a content and the fee paid for the platform advertisement service.

We formulate the problem as a differential game and we solve it for the infinite horizon case by deriving the structure of certain Nash equilibria of the game.

***Index Terms*—Content Popularity, Acceleration, Differential Games, Best Response, Nash Equilibria**

I. Introduction

Online content delivery represents an ever increasing fraction of Internet traffic. In the case of online videos, the support for content distribution is provided by commercial platforms such as Vimeo or YouTube. In many cases, such contents are also delivered by means of social network platforms. One core feature of such systems is the delivery of user-generated content (UGC): platform users become often producers of the contents which populate those systems.

Reference figures for UGC platforms are indeed those of YouTube, with over 6 billion hours of videos watched each month, which averages as an hour for every person on Earth a month. New UGCs are continuously created and 100 hours of video uploaded every minute by the platform's users[1].

A relevant parameter for the UGC platforms owners is the *viewcount*, i.e., the number of times an item has been accessed. Viewcount in fact represents one of the possible metrics to measure content popularity. In turn, a popular video becomes a source of revenue because of click-through rates of linked advertisements: those are actually part of the YouTube's business model.

Among several research works in the field, many efforts have been spent to characterize the dynamics of popularity of online media contents [1], [2], [3], [4], [5], [6]. The ultimate target there would be indeed to provide models able to perform the early-stage prediction of a content's popularity [7].

Such studies have highlighted certain phenomena that are typical of UGC delivery. A key study in [7] shows that the dynamics of popularity of online contents experiences two phases. In the initial phase, a content gains popularity through advertisement and other marketing tools. Afterwards, UGC platform mechanisms induce users to access contents by re-ranking mechanisms. Those also appear to be main drivers of popularity.

Motivated by such findings, we model the behavior of those who create content – shortly content providers in the rest of the paper – as a dynamic game. Once they generate a content, in particular, they leverage on UGC platforms to diffuse it. We note that, by paying a fee for the advertisement service of the UGC platform, a content provider is able to receive a preferential treatment to her content such in a way that the rate of propagation is increased. Clearly, this engenders a competition among content providers to capture the attention of potential viewers at faster rate than other contents.

To this respect, the notion of *acceleration* is a key concept. An example reported in Fig. 1 explains how a video can be accelerated by the UGC platform. In our example we performed a generic search *"Hakusai"* which produces a series of output results for matching contents. The one reported in the figure is one with viewcount 568507 that has been listed by the main YouTube.

In particular, Fig. 1 represents the viewer's screen: in the central part of the window it stands the video. However, there exists a recommendation list on the right as provided by the platforms' search engine.

It is important to observe the two videos recommended on the top of the list. The first one is an advertisement of a known commercial activity. In order to appear in the top position of the list, that content has been paying a fee to the UGC platform owner. In the second position, a link appears to a video which is tagged *featured*. The meaning of the term featured is that the video linked there was placed high on the recommendation list either because it is a very popular video or because it is a partner video. A partner video, as in the case of advertisements, is from someone who pays

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[1]See: http://www.youtube.com/t/press_statistics/

a fee to rank higher in the recommendation list. The other videos in the recommendation list are ranked according to the default order, i.e., that induced by the recommendation system [8]. Another advertisement with the suggestion to buy a product is appearing at the bottom of the figure. In this paper we focus on the acceleration of featured videos. In fact, in order to accelerate a video, customers perform a *promoted video campaign* on YouTube; to do so, content providers are required four steps: choose a video, attach promotional text, some keywords by which the promotion is performed, and set the *daily budget* amount allowed.

With respect to the acceleration cost, it is important to note that the so called pay-per-view model is applied. I.e., the YouTube pay-per-view policy for acceleration is meant to charge the content provider a fixed amount each time a viewer has accessed the content. Charging is triggered by a click-through on the icons of the promoted content which appear in the recommendation list. However, for the platform owner it is best that the customer's daily budget is attained. Then, the total cost paid in order to increment the number of views would increase linearly in time. A linear cumulative cost for acceleration is also one of our assumptions in the rest of the paper.

Now, since the viewer' browser has finite size, only those who are able to appear in the higher end of the recommendation list are visible without scrolling. Thus, those are accessed with higher probability: the viewcount of a content is expected indeed to grow faster, i.e., to be accelerated, whenever it is showed higher in the list.

In this work, we consider a competition between several contents. The promotion fees, i.e., the cost to accelerate the viewcount, will depend on the content provider and may depend on the content itself. Even the rate of propagation, i.e., the rate at which viewers access the content, may depend on the content. Finally, each content provider may decide whether or not to purchase priority to accelerate the popularity of the content for a certain period.

The objective of this paper is to determine the best strategy for a content provider in order to accelerate a content and study the resulting equilibria of the system. To this aim, we propose a game theoretical framework rooted in differential games. The solution of the problem allows us to provide guidelines for the advertisement strategies of content providers.

A brief outline of the paper follows. In Sec. II we revise the main results in literature for online content diffusion. In Sec. III we introduce the system model and the differential game subject of this paper. In Sec. IV we derive the analysis of best responses whereas symmetric Nash equilibria are characterized in Sec. V. In Sec. VI we tackle the limit case of small discounts. A section with conclusions, practical implications of our model and future research directions ends the paper.

II. Related works

The dynamics of popularity of online contents has been attracting attention from the research community. [3] proposed

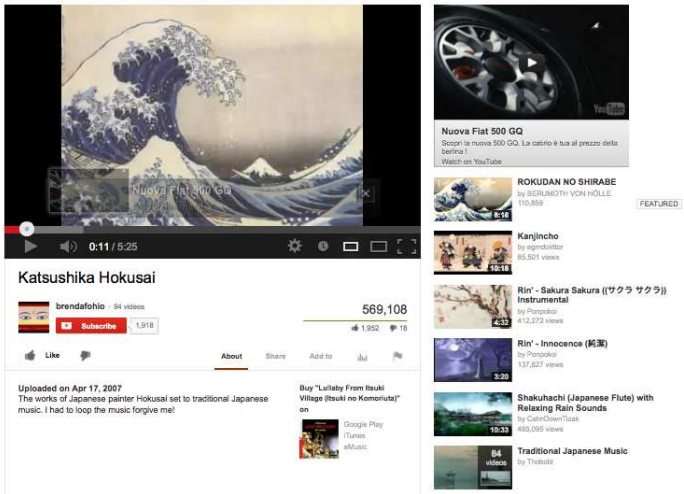

Figure 1: A sample video: the icons on the right of the main window is the recommendation list. The upper entry seen there is a commercial advertisement, whereas the second entry is tagged "Featured". The remaining entries are sorted according to their viewcount.

an analysis of the YouTube system focusing on the characteristics of the traffic generated by that platform. [5] addressed the relation between metrics used to evaluate popularity, such as number of comments, ratings, or favorites. In this paper our analysis is restricted to the viewcount.

In [4] the authors study the ranking change induced by UGC online platforms. Bursty acceleration in content's viewcount is found to depend on the way how online platforms expose popular contents to users and on re-ranking of existing contents. Here, we model the competition that arises when several content providers leverage such acceleration tools.

In literature, competition in epidemic processes has been addressed with game theoretical tools. In [9], the authors focus on an economic game on graphs, where firms try to conquer the largest market share. They derive the complexity for the computation of the equilibria of the game; results for the price of anarchy in those games has been developed recently in [10].

Emergence of equilibria of the Wardrop type has been studied in [11] from viewer's perspective. Our objective here is to describe the content provider viewpoint in a dynamic game framework.

In [12] the authors consider information propagation through social networks. The question there is how the finite budget of attention of individuals influences the rate at which contents can be pushed into the other players' network. In our work, we limit our focus to the case of online content diffusion in UGC platforms.

Novel contributions: In this paper, we provide a complete framework for the analysis of dynamic games in UGC provision. Under a meanfield approximation for contents diffusion, differential games [13] provide the model for capturing the strategic behavior of competing content providers. Our main findings are:

Table I: Main notation used throughout the paper

Symbol	*Meaning*
N	number of players (content providers)
λ_i	intensity of views per second for content i
τ	time horizon
$x_i(t)$	fraction of viewers having viewed content i at time t
$x(t)$	summation $\sum_i x_i(t)$
$y(t)$	$y(t) := 1 - x(t)$
z_i	$:= x_i(0)$ will be taken 0 unless otherwise stated.
$u_i(t)$	acceleration control (strategy) for player i; $\mathbf{u} = (u_1, u_2, \ldots, u_N)$ $u_i(t) \in [u_{\min}, u_{\max}]$ $1 \leq u_{\min} < u_{\max} < \infty$
$\mathbf{u}_{-i}(t)$	strategy profile for all players not i
a	sum of the $\lambda_i u_i$
a_{-i}	sum of the $\lambda_j u_j$ for all players $j \neq i$
p_i	discount factor for player i, $p_i > 0$

- The structure of the best response of content providers and a method for calculating it;
- Conditions for the existence and uniqueness of symmetric Nash equilibria in threshold form;
- Approximated asymmetric Nash equilibria in the regime of small discounts.

To the best of the authors' knowledge, results on Nash equilibria for differential games in UGC provision have not been derived so far in literature.

III. System Model

The main symbols used in the paper are reported in Tab. I.

In our system model we assume that N competing content providers release a content each and viewers will access one of such contents at the earliest chance.

We assume a base of M potential viewers who can access each of the N contents. To this respect, we adopt a fluid approximation which is assumed to hold for large M, and let content viewers to access content i according to a point process with intensity λ_i. I.e., content i is accessed by a randomly picked viewer every λ_i^{-1} seconds.

In general $\lambda_i \neq \lambda_j$: in fact, contents may experience diverse popularity, and so different intensities. Every content provider will participate to the content diffusion in some time frame $[0, \tau]$. Also, $\tau = \infty$ in the development of the infinite horizon formulation of the game.

Using the advertisement options of the platform, content providers can pay a cost in order to accelerate the diffusion of their video: viewers will access the content according to the intensity $u_i \lambda_i$, where $u_i \geq 1$ is the acceleration control for player i. The maximum acceleration is bounded as $u_{\max}$, and the minimum acceleration is $u_{\min}$: $1 \leq u_{\min} < u_{\max}$.

Finally, there is a linear cost paid for the acceleration control: such a cost represents the ideal case, i.e., when a promoted content receives per day a certain number of new views per cent paid to the platform owner. Conversely, the case with no acceleration, namely, $u_i = u_{\min}$, falls back to the default intensity $u_{\min}\lambda_i$. Of course, this happens at zero cost. Also, we consider somehow an ideal platform where only the content owner who promotes the video can benefit from the acceleration.To do so, we assume that the platform owner acts such in a way to make the access exclusive, i.e., so that targeted viewers tend not acquire a competing content after accessing another one, e.g., by making recommendation sets disjoint [8].[2]

We introduce below the game model that we use to describe the competition among content providers in order to accelerate the dynamics of the viewcount. The formulation of the problem is initially provided in general to cover both in the finite horizon and in the infinite horizon case. Within the scope of the paper, our development is confined to the infinite horizon analysis. For a description of the finite horizon case, we refer the reader to [14].

A. Game model

The differential game model for online content diffusion is composed as follows.

Players: players are content providers, who compete in order to diffuse their content over a base of potential viewers. Since all players share the same base, the formulation will result in a competitive differential game.

Strategies: the strategy of each player is the acceleration control. The control is thus dynamic, since each player should determine at each point in time the acceleration $u_i(t)$.

Utilities: the utility for player i is linear and has two terms. First, there is a cost paid for accelerating the content. Second, there is a revenue represented by the number of copies. The total utility is defined, as customary in differential games, as the integral of an instantaneous utility.

We denote x_i the fraction of viewers who have accessed to the contents generated by the i-th content provider. The governing equation for the dynamics of the i-th content's viewcount is

$$\dot{x}_i = \lambda_i u_i (1 - x), \quad i = 1, 2, \ldots, N \tag{1}$$

where $x = \sum_i x_i$ is the total fraction of viewers who accessed some content; the initial condition is $x_i(0) = z_i$. Actually, (1) is a fluid approximation for the dynamics of the fraction of viewers of content i.

Remark 1. *The fluid approximation which we use in this context can be justified formally with the derivation proposed by [15]. In particular, let $\widehat{\mathbf{X}}^{(M)}(t)$ be a N dimensional vector whose components are $\widehat{X}_i^{(M)}(t)$, for $i = 1, \ldots, N$. Here, $\widehat{X}_i^{(M)}(t)$ stands for the fraction of the potential viewers that watched the content at time t, when the basin of users has size M: it represents the branching process of the i-th content being watched. Thus, when we refer to fluid approximations that describe the dynamics of the fraction viewers watching the content, we are referring the meanfield approximation of such process. In particular, for a formal explanation of the convergence for large M to the fluid approximations of the type used hereafter, the reader can refer to [16].*

The acceleration control u_i, namely the strategy of player i, belongs to the space of the piecewise continuous functions $\mathcal{U} = \{u \in \text{p.w.c. functions of } [u_{\min}, u_{\max}]^{[0,\tau]}\}$.

[2]To confirm this rationale, we observe that featured videos never appear together in the recommendation list presented to users.

Hence, because the control is upper bounded, the above ODE system (1) is Lipschitz continuous, and because it is lower bounded, it is so uniformly in the control, so that a solution at large is guaranteed to exist unique for a given strategy profile $\mathbf{u} = (u_1, \dots, u_N)$ ([17], pp. 99).

The cost function for the i-th player is given by

$$J_i(x,u) = \int_0^\infty e^{-p_i s}\big(- \dot{x}_i(s) + \gamma_i\big(u_i(s) - u_{min}\big)\big)ds$$
$$= \int_0^\infty e^{-p_i s}\ell(x_i, u_i)ds \quad (2)$$

where $\gamma_i > 0$, $p_i \geq 0$ is a discount factor; here $\ell(x_i, u_i) = -\dot{x}_i + \gamma_i(u_i - 1)$.

A cautionary remark: in the infinite horizon case, the discount factor has the role of ensuring the existence of a finite cost. Besides that, looking at (2), we observe that a large value of $p_i > 0$ characterizes an "impatient" player who aims at fast dissemination of the content. Conversely, a "patient" player would use a small value of p_i.

In particular, we note that for $p = 0$ the cost function has a more familiar expression $J_i(x,u) = -x_i(\tau) + \gamma_i(\int_0^\tau (u_i(s) - u_{min}))ds$ where the dependence on the number of copies appears with no discount. In Sec. VI we are studying an approximation of our differential game that provides closed form expression of the threshold type for the infinite horizon case for vanishing discounts. In that case, the first term of (2) can be approximated assuming very large values of τ so that

$$\int_0^\tau e^{-p_i s}\dot{x}_i(s)ds = \frac{e^{-p_i\tau}x_i(\tau) - z_i}{1 - p_i} \to x_i(\tau) - z_i \text{ as } p_i \downarrow 0$$

Finally, the problem we want to solve is thus to determine the optimal cost function, namely the *value function* $V_i(x)$

Problem 1 (Best response)**.** *For any strategy $\mathbf{u}_{-i}$ of the remaining players, determine the best response, i.e., the optimal control u_i^* of player i for which the value function is attained, i.e.,*

$$V_i(x) := \inf_{u_i \in \mathcal{U}} J_i(x, u_i) \quad (3)$$

We will solve the problem using the discounted formulation in the infinite horizon: $p_i > 0$ for all players, and $\tau = \infty$.[3].

IV. Best response analysis

Best response strategies are determined using the Hamilton-Jacobi-Bellman equation (HJB) for the infinite horizon.

A. Infinite horizon with positive discount $p > 0$

The existence of the optimal cost function bounded and uniformly continuous is immediate from [17] Prop. 2.8, since indeed the $\ell(\cdot,\cdot)$ is bounded and uniformly Lipschitz, since it holds:

$$|\ell(x,u_i) - \ell(y,u_i)| \leq \lambda_i u_{\max}|x-y|, \ell(x,u_i)| \leq \lambda_i u_{\max} \quad (4)$$

In particular, we can write the Hamiltonian for each one of the players with respect to the dynamical system (1) corresponding to the problem in (1). Before that, it is easy to see that the aggregated dynamics can be written as

$$\dot{x} = \sum \lambda_i u_i (1 - x) \quad (5)$$

which will let us develop the optimal control for each one of the players having fixed the control of the competing ones. In particular, the optimal control needs to maximize the Hamiltonian

$$H_i(x,\zeta) = \sup_{u \in \mathcal{U}} \Big\{ - \sum \lambda_i u_i (1-x)\zeta + \big(u_i\lambda_i(1-x) - \gamma_i(u_i - u_{min})\big)\Big\} \quad (6)$$

Maximization of (6) provides the closed loop solution of our problem. However, the optimal cost function V_i in turn is one solving for the HJB equation

$$p_i V_i + H_i\big(x, DV_i\big) = 0 \quad (7)$$

so that minimizing the cost function V_i is equivalent to maximize $H_i\big(x, DV_i\big)$ [17], where $DV_i = \frac{d}{dx}V_i(x)$. In turn, we can write in closed form based on (1) and (5).

$$H_i(x) = u_{min}\gamma_i - \sum_{j \neq i} u_j \lambda_j (1-x) DV_i + \sup_{u_i \in \mathcal{U}} \Big\{ u_i \Big[\lambda_i(1-x)\Big(1 - DV_i\Big) - \gamma_i\Big]\Big\} \quad (8)$$

Now, since we observe that the Hamiltonian is linear in u_i, if a maximum is attained at some control $u \in \mathcal{U}$, the intuition is that it may assume only extremal values as it is often seen in the case of open-loop type of solutions [13]. This is actually true: we can allow the control u to take values only on the vertices of the codomain polyhedron, i.e., $[u_{\min}, u_{\max}]^N$. The value function will be the same of the original problem where such restriction does not hold (see [17], pp.113). This is due to the fact that (1) is of the type $\dot{x} := f_1(x) + f_2(x)u$ and of the running cost function ℓ which is linear in the control.

Motivated by this observation, we are interested in a class of best responses, namely

Definition 1. *A strategy u_i is of the bang-bang type if it takes extremal values $u_{\min}$ and $u_{\max}$ only.*

Also, we denote $t_{ik}, k = 1, 2, \dots$ the switching times associated to best response u_i, and $[t_{ik}, t_{ik+1})$ represents the corresponding k-th switching period of player i. If we limit our analysis to the best responses of bang-bang type, then $u_i \in \{u_{\min}, u_{\max}\}$, $i = 1, \dots, N$: best responses are in fact piecewise constants.

In particular, for the case of bang-bang strategies, the condition for optimality, i.e., a best response, writes from (8)

$$u_i(x) = \begin{cases} u_{\min} & \text{if} \quad \lambda_i(1-x)\Big(1 - DV_i\Big) - \gamma_i < 0 \\ u_{\max} & \text{if} \quad \lambda_i(1-x)\Big(1 - DV_i\Big) - \gamma_i > 0 \end{cases} \quad (9)$$

We can denote *switching interval* the interval of time between two consecutive switching instants: within such interval, the best responses are constant, i.e., $\mathbf{u}$ is constant and only

[3] A similar derivation in the case of the finite horizon is possible either with or without discount [14]

assumes values in $\{u_{\min}, u_{\max}\}^N$. We can resume our findings above with the following

Theorem 1. *The value function V_i corresponding to the best response of the game can be attained by a strategy u_i of the bang-bang type.*

It is worth noting that in general the solution of HJB equations requires to search for a *viscosity solution* [17]. This is due to the fact that the classical solutions assuming the differentiability of the value function may not exist in general and thus require to solve for a general notion of differentiation. Here, it is the structure of the system that spares us this step, since we know apriori that the best response is of the bang-bang type. The main difference with respect to the general case, is that strategies u_is draw values in the finite set $u_i \in \{u_{\min}, u_{\max}\}$; again, compared to the general problem, this fundamental simplification is due to the linear structure of the content provider game.

B. Infinite horizon for $p_i > 0$

In order to decide on the sign of the above terms, we need to solve for the HJB equation in V_i. This is in fact possible, once we notice that (7) can be written as the following ODE

$$p_i V_i - a(1-x)DV_i + b_i(1-x) + c = 0 \tag{10}$$

where V_i is the value function that solves for the best response.

Here we simplify the notation by letting

$$\begin{aligned} c_i &= \gamma_i(u_{min} - u_i) \\ b_i &= \lambda_i u_i \\ a &= \sum \lambda_i u_i \end{aligned} \tag{11}$$

Whenever convenient, for the sake of clarity, we will denote $a = a(u_i)$ to stress the fact that strategy profile a depends on the best response of player i, and we will also resort sometimes to the notation $a_{-i} := \sum_{j \neq i} \lambda_i u_i$ the sum of piecewise constant controls played by all the remaining players. It is important to note that a and b_i are assumed constant during each switching interval.

Also, note that since we do not know the optimal control, the expression solving for (10) here depends on the specific switching interval. Hence, V_i will have a specific dependence on the control which we need to maximize aposteriori since we know that (3) holds.

The solution of the HJB equation above will result can be solved as (see App. A)

$$V_i(x) = K \cdot (1-x)^{-\frac{p_i}{a}} - \frac{p_i b_i(1-x) + (p_i + a)c_i}{p_i(p_i + a)} \tag{12}$$

where K is a real constant.

In the following considerations we need the closed form of the function that is maximized by the control u_i in (8)

$$T_i(u_i, a_{-i})(x) = u_i \left[\left(1 - \frac{b_i}{p_i + a}\right)(1-x) - \frac{K p_i}{a}(1-x)^{-\frac{p_i}{a}} - \frac{\gamma_i}{\lambda_i} \right] \tag{13}$$

As a first step, from (12) we can obtain information on the structure of the value function. In particular we resume some basic facts in the following

Lemma 1. *i. The best response u_i^* has a finite number of switches.*
ii. There exists a threshold value of x_∞ such that $u_i^*(x) = u_{\min}$ for all $x > x_\infty$ for every player i.

Proof: i. By contradiction: assume an infinite number of switches for player i and define constants K_r, $r = 1, \ldots, \infty$ for each such switching interval. By the continuity of x and the continuity of $V_i(x)$, together with (12), there exist an infinite sequence of K_rs which are non zero. Hence, since $x \uparrow 1$, by the continuity of x, we can find sequence $\{x_r\} \uparrow 1$ where x_r belongs to the r-th switching interval. Due again to (12), we hence found a subsequence of values of V_i which diverges. This is a contradiction since the value function is bounded.
ii. Denote $x_{i,\infty}$ the value of x above which the control switches to $u_{\min}$ for good: indeed the constant appearing in (12) is zero. If it was not, again, V_i would grow unbounded as $x \to 1$. Hence, DV_i is bounded for $x > x_{i,\infty}$, so that by inspection of (6), the control needs to be $u_{\min}$ for values of x close to 1, and on the rest of the last switching interval as well since it is constant there. Finally, we can define $x_\infty = \max(x_{i,\infty}, 1 \leq i \leq N)$. ■

Furthermore, we can characterize immediately a class of problems where players have no incentive to accelerate anyway: in particular we see that the following sufficient condition holds

Lemma 2 (Degenerate Nash Equilibrium)**.** *i. Let $\lambda_i < \gamma_i$, then the best response for the player i is $u_i^* = u_{\min}$ irrespective of the other players strategies. ii. If $\lambda_i < \gamma_i$ for $i = 1, \ldots, N$, then $u_i^* = u_{\min}$, $i = 1, \ldots, n$ is the unique Nash equilibrium.*

Proof: i. This is a consequence of the statement in Lemma.1: in the last switching interval, since $K = 0$, from (13)

$$T_i(u_i, a_{-i})(x) \leq \left(1 - \frac{\frac{b_i}{a(u_{\min})}}{1 + \frac{p_i}{a(u_{\min})}}\right) - \frac{\gamma_i}{\lambda_i} < 0$$

The rightmost inequality is equivalent to

$$\frac{a_{-i} + p_i}{a(u_{\min}) + p_i} < \frac{\gamma_i}{\lambda_i}$$

which writes also $(\lambda_i - \gamma_i)(a_{-i} + p_i) < \gamma_i \lambda_i u_{\min}$. The statement follows once we observe that the previous result is independent of a_{-i}, i.e., the strategies played by all the other players.
ii. Follows immediately from i. ■

Remark 2. *From the above results, we can see that the general form of the best response for player i against the strategy profile $\mathbf{u}_{-i}$ for the remaining players can be determined by proceeding backwards from the latest switching value $x_{i,\infty}$ which is calculated first. Then, by continuity, the constant K appearing in* (12) *for the switching interval before the last one*

(where now player i would use $u_{\max}$) can be determined by imposing the continuity of the value function. In fact, at the switching time, the expression (12) *has different values of b_i and a_i in the two adjacent switching intervals. The procedure can be iterated backwards to determine all the threshold values of x when player i switches.*

In the case of a symmetric game, i.e., when the content providers have all the same parameters, the procedure described above can solve the game in closed form, as showed in the next section where symmetric equilibria are described.

V. Symmetric Nash Equilibrium

We consider now the symmetric case: $\lambda_i = \lambda$, $\gamma_i = \gamma$, $p_i = p$ for all $i = 1, \ldots, N$. The proofs of the statements hereafter are deferred to the Appendix. Let us consider tagged content provider i, and assume that all the remaining players use the same threshold type of strategy, i.e., of the type

$$u_j(x) = \begin{cases} u_{\max} & \text{if } x < \hat{x} \\ u_{\min} & \text{if } x > \hat{x} \end{cases} \tag{14}$$

for some $0 \leq \hat{x} \leq 1$. Denote x^* the last switch of player i: we are now ready to show that there exists a symmetric equilibrium where also player i will use $\hat{x} = x^*$, i.e., the threshold type strategy (14) is the best response to itself when all content providers play it. Furthermore, it is the unique symmetric equilibrium of the game.

In particular such a Nash equilibrium is given by a threshold x^* which is derived by the form of the value function in the last switching interval (recall that $K = 0$ in that interval)

$$V_i(x) = -\frac{u_{min}\lambda(1-x)}{(p + N\lambda u_{min})} \tag{15}$$

by imposing that $T(u_{\min}, u_{\min}(N-1))(x^*) = 0$ (switching condition). These results are detailed formally in the statements below.

Lemma 3. *Let $x^* \geq \hat{x}$, then the following holds:*
i. Player i can switch at some $0 < x^* < 1$ iff $\lambda(1 - \frac{u_{min}\lambda}{(p+N\lambda u_{min})}) - \gamma > 0$; moreover

$$x^* = 1 - \frac{\gamma(p + N\lambda u_{min})}{\lambda(p + (N-1)u_{min}\lambda)} \tag{16}$$

ii. Let all players switch at x^*: the constant K^* which ensures the continuity of the value function $V_i(x)$ at the switching threshold x^* is positive and it holds the following relation

$$K^* = (u_{\max} - u_{\min})(1-x^*)^{\frac{p}{\lambda n u_{\max}}} \left\{ \frac{(1-x^*)^{\frac{p}{\lambda u_{\min}}}}{\left(N + \frac{p}{\lambda u_{\max}}\right)\left(N + \frac{p}{\lambda u_{\min}}\right)} + \frac{\gamma}{p} \right\}$$

Theorem 2 (Symmetric Nash Equilibrium)**.** *Let $\lambda(1 - \frac{u_{min}\lambda}{(p+N\lambda u_{min})}) - \gamma > 0$, then the threshold type strategy* (14) *where $\hat{x} = x^*$ and x^* is as defined in Lemma 3 is the unique symmetric Nash equilibrium of the game.*

Proof: We first need to ensure that when content provider i plays against (14) with $\hat{x} = x^*$ for all the remaining players, the switch for player i is unique. Indeed for $x < x^*$ it holds

$$DT(u_i, a_{-i})(x) = \frac{-\lambda(1 - \frac{b_i}{a+p})(1-x)^{\frac{p}{a}+1} - \frac{\lambda^2 p K^*}{a^2}}{(1-x)^{\frac{p}{a}+1}}$$

However, we note that we are in the assumptions of Lemma 3, so that $K^* > 0$. Thus, for $x < x^*$, indeed $DT(u_i, a_{-i})(x) < 0$ by inspection of the above equation. This ensures that there is not any other switch for player i, so the strategy of player i is also threshold with $\hat{x} = x^*$. Indeed, threshold strategy (14) with $\hat{x} = x^*$ for all players is a best reply to itself for all players, so that it defines a Nash equilibrium for the game. The uniqueness of the equilibrium is obtained by the fact that (15) has a unique zero. ■

The existence of equilibria in the non symmetric case is the next question that we are answering. In particular, we obtain certain asymmetric equilibria which are ϵ-approximated Nash equilibria. I.e., the unilateral deviation from those strategy profiles may provide some improvement to the utility of a content provider. But, such improvement can be made arbitrarily small by choosing an appropriate value of the discount p_i.

VI. Vanishing discount regime

Hereafter, we consider the cases of small discount factors. As described in Sec. III-A, we can consider the case when p_i has a very small value. This means that player i does not pose much of a constraint on the time taken in order to make the content popular. In particular, we would consider the case of vanishing discounts sequences: $p_i(r) = o(1)$, $i = 1, \ldots, N$ and consider the form of the best replies in the regime of vanishing discounts. This provides further insight into the structure of the equilibria for the content providers game.

Corollary 1. *Let $p_i(r) = o(1)$: there exists a best reply in threshold form for the i-th player that is arbitrarily close to the best reply of the game for a small enough discount factor.*

Proof: Let $\zeta_r = \frac{p_i(r)}{a}$, we can write (13) as

$$\begin{aligned} T(u_i, a_{-i}) &= u_i \left[\left(1 - \frac{\frac{\lambda_i u_i}{\sum_i \lambda_i u_i}}{1+\zeta_n}\right) y - K\zeta_r y^{-\zeta_r} - \frac{\gamma_i}{\lambda_i} \right] \\ &= u_i \left[\left(1 - \frac{\lambda_i u_i}{\sum_i \lambda_i u_i}\right)(1 - \zeta_r + o(\zeta_r))y - \frac{\gamma_i}{\lambda_i} + \right. \\ &\quad \left. K\zeta_r(yu_i - \zeta_n + o(\zeta_r)) \right) \Big] \\ &= u_i \left[\left(1 - \frac{\lambda_i u_i}{\sum_i \lambda_i u_i}\right) y - \frac{\gamma_i}{\lambda_i} \right] + \zeta_r f(y) + o(\zeta_r) \end{aligned} \tag{17}$$

where $y = 1 - x$ and $f(y) = u_i\big(y\frac{\lambda_i u_i}{\sum_i \lambda_i u_i} - 1/y^{\zeta_r}\big)$. We already noticed that there exists x_∞ above which every player switches to $u_{\min}$: we can hence restrict our discussion to the range $y \in [1 - x_\infty, 1]$. Indeed, $f(y)$ is bounded therein: denote $\tilde{T}(u_i, a_{-i}) = u_i\Big(\frac{\lambda_i u_i}{\sum_i \lambda_i u_i} - \frac{\gamma_i}{\lambda_i}\Big)$.

Hence, we can fix $\epsilon > 0$ and consider $r > r_o$ such that $|\tilde{T}(u_i) - T(u_i)| < \epsilon$ uniformly in y. The best response of the user i will at most produce a value function that differs by ϵ

from the one which maximizes (8). Hence, we search for the solution of the maximization problem

$$\tilde{u}_i^* = \underset{u_i \in \{u_{\min}, u_{\max}\}}{\operatorname{argmax}} \quad \tilde{T}(u_i)$$

which corresponds to a modified game where the cost function is $\tilde{T}$. We hence need to state when $\tilde{T}(u_{\min}, a_{-i})$ is larger or smaller than $\tilde{T}(u_{\max}, a_{-i})$: this turns out to be equivalent to the condition for the state x to exceed or not the threshold

$$x_{0,i} := 1 - \frac{\gamma_i}{\lambda_i} \frac{1}{\left(1 + \frac{\lambda_i u_{\min}}{a_{-i}}\right)\left(1 + \frac{\lambda_i u_{\max}}{a_{-i}}\right)} \tag{18}$$

so that the final control law that governs the best reply of the i-th player. is

$$\tilde{u}_i^*(x) = \begin{cases} u_{\max} & \text{if} \quad x \leq x_{0,i} \\ u_{\min} & \text{if} \quad x > x_{0,i} \end{cases} \tag{19}$$

which concludes the proof. ■

Remark 3. *Because of the above result, we can always find a discount factor small enough so as to find a threshold type strategy which approximates the cost function of the best response within an arbitrarily small positive additive constant $\epsilon > 0$. In turn, this also means that if a Nash equilibrium exists under the modified utility function $\tilde{T}(\cdot)$, then it is an ϵ-approximated Nash equilibrium in threshold policy for the original game. It is hence interesting to study the existence of a Nash equilibrium for the modified game.*

Theorem 3 (Asymmetric ϵ- approximated Nash Equilibrium)**.** *Let $\lambda_i = \lambda$ for $i = 1, \ldots, N$ and $\lambda > \gamma_1 > \gamma_2 > \ldots > \gamma_N$. Then, there exists an ϵ-approximated Nash equilibrium in threshold form for the original game.*

Proof: We assume the regime of vanishing discounts such in a way that ϵ is defined in the sense of Thm. 1. The proof is based on the following observation: at time 0, indeed $u_i^*(0) = u_{\max}$ because of (18) and $\lambda > \gamma_i$ for all is. Clearly, since $\gamma_1 > \gamma_i$ for $i > 1$, then $x_{0,1} < x_{0,i}$ for $i > 1$, and t_1 corresponds to the switch of node 1 from $u_{\max}$ to $u_{\min}$. Also, $a_{-i}(t_1^-) = u_{\max} N > u_{\max}(N-1) + u_{\min} = a_{-i}(t_1^+)$ for all players $i > 1$, so that $x_{0,i}(t_1^-) < x_{0,i}(t_1^+)$. Thus, $u_i^*(t_1^-) = u_i^*(t_1^+) = u_{\max}$. Finally, until $x < x_{0,2}(a_{-i}(t_1^+))$, all players not i will use $u_{\max}$.

By induction: assume that first $k-1$ players that switched from $u_{\max}$ to $u_{\min}$ did not switch back and prove that under the conditions in the assumptions even the k-th player will never switch from $u_{\min}$ to $u_{\max}$.

In order to proceed further with the proof we need to precise some notation

- $x_{0,i}(k)$ is the threshold (18) for player i when k players already switched to $u_{\min}$;
- $a_{-i}(k)$ is the sum of the other players $\lambda_j u_j$ when k of them switched to $u_{\min}$.

At the time when player $k+1$ switches, it holds $x(t_{k+1}) = x_{0,k}(k)$. Hence, in order for player $k+1$ not to switch back to $u_{\max}$, it must hold $x(t_{k+1}) > x_{0,k}(k+1)$. However, we know that the dynamics in the k-th switching period is

$$x(t_{k+1}) = 1 - (1 - x_{0,k}(k-1)))e^{-a_k(t_{k+1} - t_k)}$$

Also, by inductive assumption, $x_{0,k}(k-1) = x_{0,k}(k)$ because no player switched back to $u_{\max}$. Then, since a_k is constant in the k-th switching period, the dynamics in that interval is governed by condition

$$\frac{1 - x_{0,k}(k+1)}{1 - x_{0,k}(k)} > e^{-a_k \Delta t_{k+1}} \tag{20}$$

Moreover, we have a condition on $\Delta t_{k+1} = t_{k+1} - t_k$:

$$\begin{aligned} x_{0,k}(k) &= 1 - (1 - x_{0,k}(k))e^{-a_k(t_{k+1} - t_k)} \\ &\Rightarrow \Delta t_{k+1} = \log\left(\frac{1 - x_{0,k+1}(k)}{1 - x_{0,k}(k)}\right) \end{aligned} \tag{21}$$

Now, combining (20) and (21) we obtain

$$\frac{1 - x_{0,k}(k+1)}{1 - x_{0,k}(k)} > \frac{1 - x_{0,k+1}(k)}{1 - x_{0,k}(k)} \Rightarrow x_{0,k}(k+1) < x_{0,k+1}(k) \tag{22}$$

We can now express the condition above by considering the explicit expression

$$x_{0,k}(k+1) = \frac{\gamma_k}{\lambda} \frac{1}{\left(1 + \frac{u_{\min}}{v_k}\right)\left(1 + \frac{u_{\max}}{v_k}\right)}$$

where $v_k := \frac{a_{-k}(k+1)}{\lambda_k}$. Also, in the same way,

$$x_{0,k+1}(k) = \frac{\gamma_k}{\lambda} \frac{1}{\left(1 + \frac{u_{\min}}{v_{k+1}}\right)\left(1 + \frac{u_{\max}}{v_{k+1}}\right)}$$

where $v_{k+1} := \frac{a_{-(k+1)}(k)}{\lambda_k}$. Finally, let us observe that

$$v_k = (N - 1 - k)u_{\max} + k u_{\min} = v_{k+1}$$

so the condition in (22) becomes $\gamma_k > \gamma_{k+1}$ which is true according to our assumptions. Hence the inductive step is complete and the statement is true. ■

Remark 4. *We note that the constructive proof of the ϵ-approximated Nash equilibrium states that if some player stops accelerating, it will not have incentives to accelerate later for larger values of the state x, because the increment in the state x is decreasing. Overall, the above result suggests that in the fully asymmetric case, the presence of diverse costs induces an equilibrium in threshold form where even if a content provider has an incentive in deviating from the given strategy profile, and so change strategy, the incentive that the player has in deviating can be made small at wish by choosing an appropriate value of the discount.*

VII. Numerical Results

We provide hereafted a numerical description of the results for the content providers' best response. It is interesting to visualize the best reply of a certain content provider facing different strategies of the remaining ones. In Fig. 2a) and b) we reported on the best response in the vanishing discount regime as a function of a_{-i} in the homogeneous scenario, i.e.,

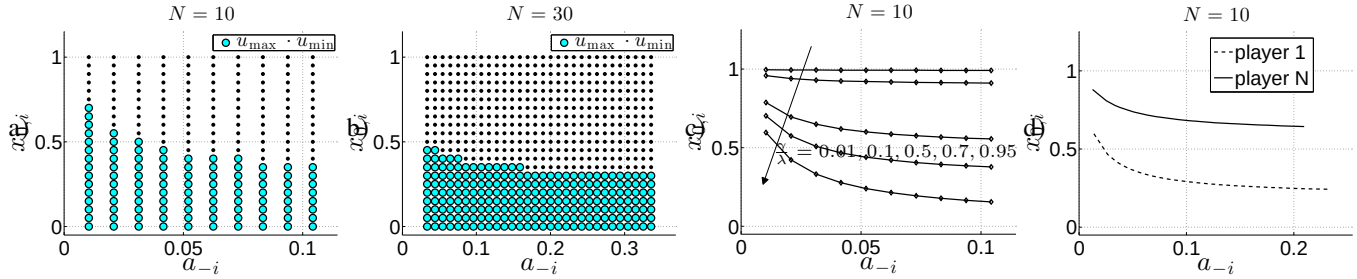

Figure 2: a) and b): best reply for $\lambda_i = \lambda_0 = 100$ views/day, $u_{\max} = 10$, $u_{\min} = 1$, $p = \lambda$, and $\gamma_i = \gamma = 0.7\lambda$; a) $N = 10$ and b) $N = 30$; c): impact of cost on the best response; d) case for heterogeneous scenario: $\lambda_i = \lambda_0$ for $i = 1, \ldots, N/2$ and $\lambda_i = 2\lambda_0$ for $i = N/2 + 1, \ldots, N$, $N = 10$.

$\gamma_i = \gamma$ and $\lambda_i = \lambda_0$ for $i = 1, \ldots, N$. Here, $u_{\max} = 10$ and $u_{\min} = 1$.

The graphs of Fig. 2a) and b) refer to two values $N = 10$ and $N = 30$, respectively. The value of λ_0 is settled to 100 views per day, whereas $\gamma = 0.7 \cdot \lambda$.

The best response is depicted using different markers for $u_{\min}$ and $u_{\max}$: for a fixed value of a_{-i} the best response starts with $u_{\max}$ and switches to $u_{\min}$ above the threshold $x_{0,i}$. In both cases a) and b) the threshold value decreases with a_{-i}. This is competition's effect: larger a_{-i} values mean more players using $u_{\max}$. Hence, the number of views that can be expected are small so that the cost for accelerating takes over. Hence the switch occurs at lower values of x.

Finally, we observe the floor at $1 - \frac{\gamma}{\lambda}$: this is the limit best response for large number of players as from (18). For a very large number of competing contents, the best response of the single player should depend on γ and λ only. When this happens, in fact, the game becomes singular: a unique best response for every player exists which defines the only equilibrium of the system.

In Fig. 2c) we observe the best response for $N = 10$ and for increasing values of γ/λ: clearly, larger values of the ratio makes players accelerate less because of increased cost. Again, we observe that the effect of competition is to reduce the acceleration for larger values of players using $u_{\max}$, i.e., larger values of a_{-i}. However, for small values of the cost, not only the best reply is to use a large threshold, but the strategy of competing content providers becomes less and less relevant so that the threshold becomes almost constant in a_{-i}.

In the last figure, i.e., Fig. 2d), we considered an heterogeneous scenario. In this case, contents $i = 1, \ldots, 5$ have $\lambda_i = \lambda_0$, whereas $i = 6, \ldots, 10$ have $\lambda_i = 2\lambda_0$. In this case we expect that a_{-1} and a_{-10} in the example may be different. Hence, we are interested in the relative behavior of best response of the two type of players.

As seen in the figure, the switching order $x_{0,1} < x_{0,10}$ is maintained for increasing values of a_{-i}: players with higher value of λ_i always switch before. Actually, even for a small number of players as in this example, the range of values taken by a_{-i} is basically the same for both types of players. As a consequence, the ratio γ_i/λ_i is the main parameter characterizing the relative behavior of the two classes of players, i.e., the switching order.

VIII. Conclusions

We have introduced models for advertisement in online content diffusion where content providers compete to make contents popular. They leverage on acceleration tools of online platforms in order to increase the viewcount of their videos. By doing so, contents occupy positions that are more visible on the web pages of potential viewers. But, a fee is paid in order to profit from re-ranking mechanisms of online platforms.

Hence, each player, e.g., each content provider, needs to decide over time if it is worth to accelerate or not. And, this choice is dynamic depending on what other content providers do. We introduced a model based on differential games: players' best responses are determined by solving an ODE involving the Hamiltonian of the system they observe.

We showed that in the infinite horizon case, the closed loop best replies of players are of the bang-bang type in the state x. We have identified a unique threshold-type Nash equilibrium in the symmetric case and we found that dual counterparts exist in the fully asymmetric case for small discounts.

Practical implications. We derived the existence of Nash equilibria in threshold form in the symmetric and (ϵ-approximate) in the asymmetric case. However, we conjecture that Nash equilibria in threshold form do exist also for any choice of the λ_is and the γ_is. Thus, content providers would only pay when the total fraction of views is below a certain threshold and stop promoting above it. Now, we can observe (16) closer, and draw the following conclusions when the equilibrium is reached:

a) *all-or-nothing effect*: content i with low potential, i.e., very small λ_i, will not be accelerated at all when $\lambda_i < \gamma_i$. Hence, content providers compete by promoting first contents likely to become most popular, e.g., with larger values of λ_i.

b) *best response and promotion*: the best response is of the threshold type, so that content providers are able to maximize the number of views while minimizing the cost by using the maximum promotion budget per day until the threshold is reached.

c) *daily budget:* the daily budget γ determines the threshold x^* such in a way that the larger the cost which is paid per day, the lower x^*. From the platform owner perspective, for a given cumulative budget paid by a customer, the smaller the threshold, the lesser the promotion time will last. This is indeed convenient for the sake of system's resources: larger

acceleration fares will lead to shorter promotion campaigns with smaller load for the platform's promotion mechanisms.

Future works. The results of our paper indicate that these game models can lead to new tools for the pricing of online content advertisement and for the prediction of content popularity. There are several interesting research directions that are left for future work. First, the dynamic setting in the finite horizon case appears the most immediate extension. In future work, we plan to study the effect of the horizon duration onto the equilibria and the effect that time constraints have on content providers' strategies. Another aspect relates to the number of competitors: in the case of large N, the strategy of single players does not change significantly other players' utility and the strategy profile $\mathbf{u}_{-i}$. To this respect, the dynamic game formulation could be reduced in the limit of large N to a static formulation which could be studied using Wardrop-like equilibria [11].

APPENDIX

ODE solution

The solution of (10) is equivalent to the solution of $DV_i - \frac{p_i}{a(1-x)}V_i = \frac{b_i}{a} + \frac{c}{a(1-x)}$, so that it is sufficient to observe that the integrating factor for this first order ODE is $(1-x)^{\frac{p_i}{a}}$. Hence the solution follows from

$$V_i(x) = (1-x)^{-\frac{p_i}{a}} a^{-1} \int (1-x)^{\frac{p_i}{a}} \Big(b_i + \frac{c_i}{1-x}\Big) dx$$
$$= K(1-x)^{-\frac{p_i}{a}} - \frac{b_i}{a+p_i}(1-x) - \frac{c}{p_i} \tag{23}$$

where K is an arbitrary real constant.

Proof of Lemma 3

Proof: (i) The proof is made in three steps. In the first step we prove that $T(u_{\min}, (N-1)u_{\min})(x)$ is decreasing for $x > x^*$ (we use notation $T(x)$ when it does not generate confusion). In the second step we derive the sufficient condition in the assumptions. Finally, in the third step we compute x^* and the corresponding constant K^*.

Step 1. From (15), in the last switching interval we have

$$DV_i = \frac{u_{\min}\lambda}{(p + N\lambda u_{\min})}$$

If we plug in DV_i in $T(x)$ we finally have:

$$T(x) = (1-x)(1 - \frac{u_{\min}\lambda}{(p + N\lambda u_{\min})}) - \frac{\gamma}{\lambda}$$

so that $DT(x) < 0$.

Step 2. Since $T(x)$ is decreasing in the last switching interval, a threshold when player i switches to $u_{\min}$ exists if and only if $T(0) > 0$, which is the assumption in the statement, namely $1 - \frac{u_{\min}\lambda}{p+N\lambda u_{\min}} > \frac{\gamma}{\lambda}$.

Step 3. The threshold x^* for player i is obtained by solving

$$T(x^*) = 0 \Leftrightarrow x^* = 1 - \frac{\gamma}{\lambda} \cdot \frac{p + \lambda\, N\, u_{\min}}{p + \lambda\,(N-1)\, u_{\min}}$$

Now we can assume for player i a switch occurs in x^*. Hence, we impose the continuity of the value function [17]. Because it is continuous on both sides of the threshold x^*, the limit values $V_i(x^-)$ for $x \uparrow x^*$ when $(u_i, a_{-i}) = (u_{\max}, (n-1)u_{\max})$ and $V_i(x^+)$ for $x \downarrow x^*$ when $(u_i, a_{-i}) = (u_{\min}, (n-1)u_{\min})$ need to be the same. This will determine constant K^*. The equation to be solved is thus

$$K^*(1-x^*)^{-\frac{p}{\lambda n u_{\max}}} - \frac{\lambda u_{\max}(1-x^*)}{p + \lambda N u_{\max}} - \frac{\gamma}{p}(u_{\max} - u_{\min})$$
$$= -\frac{u_{\min}\lambda(1-x^*)}{p + N\lambda u_{\min}} \tag{24}$$

and the expression for K^* writes as in the statement. Finally, we observe from (24) that indeed $K > 0$: in fact

$$\frac{u_{\max}\lambda}{p + N\lambda u_{\max}} > \frac{u_{\min}\lambda}{p + N\lambda u_{\min}}$$

which concludes the proof. ■

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