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OPTIMAL CONTROL OF EPIDEMIOLOGICAL SEIR MODELS WITH L^1 -OBJECTIVES AND CONTROL-STATE CONSTRAINTS

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ABSTRACT. Optimal control is an important tool to determine vaccination policies for infectious diseases. For diseases transmitted horizontally, SEIR compartment models have been used. Most of the literature on SEIR models deals with cost functions that are quadratic with respect to the control variable, the rate of vaccination. In this paper, we consider L^1 -type objectives that are linear with respect to the control variable. Various pure control, mixed control-state and pure state constraints are imposed. For all constraints, we discuss the necessary optimality conditions of the Maximum Principle and determine optimal control strategies that satisfy the necessary optimality conditions with high accuracy. Since the control variable appears linearly in the Hamiltonian, the optimal control is a concatenation of bang-bang arcs, singular arcs and boundary arcs. For pure bang-bang controls, we are able to check second-order sufficient conditions.

1. Introduction. The annual WHO report on infectious diseases points to the fact that infectious diseases continue to be one of the most important health problems worldwide. "Mathematical models have become an important tool in describing the dynamics of the spread of a disease and the effect of vaccination and treatment" (Ledzewicz, Schättler [18]). A survey of epidemic models may be found in Hethcote Hethcote [15, 16]. Based on such dynamic models, various vaccination and treatment policies have been studied using optimal control techniques [11, 19, 32]. Most of these papers assume a control-quadratic objective to measure the cost of vaccination or treatment. It has been argued in [?] that such L^2 -type objectives are not appropriate in biological and biomedical application.

In this paper, we study the optimal vaccination policies in epidemiological SEIR models with L^1 -type objectives which are linear in the control (vaccination) variable. SEIR models consider four compartments, where S, E, I and R denote the number of individuals in the susceptible, exposed, infectious and recovered compartment, respectively. The control variable is the number of vaccines applied in the susceptible compartment. We shall consider various control, mixed control-state and pure state constraints. Our objective of L^1 -type to be minimized represents

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the integral over the weighted sum of the number of infectious individuals and the (linear) cost of vaccines. We discuss necessary optimality conditions (Maximum Principle) for all classes of control and state constraints and present numerical solutions that satisfy the necessary optimality conditions with high accuracy. In some cases, we are able to check sufficient optimality conditions.

In Section 2, several optimal control problems for SEIR models are discussed which differ in the type of imposed control or state constraint. In the following sections, we compare numerical solutions for two different weights in the control cost. Section 3 gives a brief account of numerical methods that we use in our computations. Mainly, we focus on discretization and nonlinear programming methods for which efficient implementations have been developed in the literature. In Section 4, we consider the basic SEIR control problem with a simple control constraint. Since the control variable appears linearly in the Hamiltonian, the Maximum Principle leads to either bang-bang or singular controls. We derive an expression of the singular control in terms of the state and adjoint variable. The numerical solution furnishes an optimal control with a bang-singular-bang structure. The solution exhibits a rather high total number W of vaccines. For that reason, in Section 5 we consider a terminal constraint for the total number $W(T)$ at the terminal time T . For a small weight in the control cost, the control has a bang-singular-bang structure, whereas for a larger weight the control is bang-bang with only one switch. Section 6 considers a mixed control-state constraint which was introduced by Biswas, Paiva, do Pinho [1]. The constraint is motivated by the observation that it is a more realistic scenario to consider a limited supply of vaccines at each instant of time than to merely limit the total amount of vaccines. Finally, in Section 6 we study the basic control problem with a pure state constraint, where an upper bound is imposed on the number of susceptibles.

2. Optimal Control Problems for SEIR models with L^1 -objectives. In SEIR models, the population is divided into four compartments. An individual is in the S compartment if susceptible (vulnerable) to the disease. Those infected but not able to transmit the disease are in the E compartment of exposed individuals. Infected individuals capable of spreading the disease are in the I compartment and those who are immune are in the R compartment. In SEIR models everyone is assumed to be susceptible to the disease by birth and the disease is transmitted to the individual by horizontal incidence, i.e., a susceptible individual becomes infected when in contact with infectious individuals. Let $S(t)$, $E(t)$, $I(t)$, and $R(t)$ denote the number of individuals in the susceptible, exposed, infectious and recovered compartments at time t respectively. The total population is

$$N(t) = S(t) + E(t) + I(t) + R(t).$$

The disease transmission in a certain population is described by the parameters, e , the rate at which the exposed individuals become infectious, g the rate at which infectious individuals recover and a denotes the death rate due to the disease. Also b is the natural birth rate and d denotes the natural death rate. Let c be the incidence coefficient of horizontal transmission. Then the rate of transmission of the disease is $cS(t)I(t)$. For simplicity the parameters are assumed constants although they may vary in reality if the time horizon is large. The control $u(t)$ represent the percentage of susceptible individuals being vaccinated per unit of time. In the following, we consider the dynamical system as in Neilan, Lenhart [26] Here, we assume that the

vaccine is effective so that all vaccinated susceptible individuals become immune. Then we are led to the following dynamical system:

$$\{\text{Sdifeq}\} \quad \dot{S}(t) = bN(t) - dS(t) - cS(t)I(t) - u(t)S(t), \quad S(0) = S_0, \quad (1)$$

$$\{\text{Edifeq}\} \quad \dot{E}(t) = cS(t)I(t) - (e + d)E(t), \quad E(0) = E_0, \quad (2)$$

$$\{\text{Idifeq}\} \quad \dot{I}(t) = eE(t) - (g + a + d)I(t), \quad I(0) = I_0, \quad (3)$$

$$\{\text{Ndifeq}\} \quad \dot{N}(t) = (b - d)N(t) - aI(t), \quad N(0) = N_0 \quad (4)$$

Remark: Since we are dealing with a variable population size N , it would be more realistic to replace the term $cS(t)I(t)$ in the equation for S by the term $cS(t)I(t)N(0)/N(t)$. However, the computations show that we get the same optimal control structure for both terms and only a small difference in the numerical results. Hence we keep the term $cS(t)I(t)$ which also simplifies the computation of singular controls in the next section.

Since $u(t)$ represent the fraction of susceptible individuals being vaccinated, we have the control constraints

$$0 \leq u(t) \leq 1 \text{ a.e. } t \in [0, T]. \quad (5) \quad \{\text{control-constraint}\}$$

The recovered population is related to the total population by

$$N(t) = S(t) + E(t) + I(t) + R(t).$$

Hence, $R(t) = N(t) - S(t) - E(t) - I(t)$ which gives the differential equation

$$\dot{R}(t) = gI(t) - dR(t) + u(t)S(t), \quad R(0) = R_0. \quad (6) \quad \{\text{Rdifeq}\}$$

To keep track of the number of vaccinated individuals we introduce an extra variable W that satisfies the equation

$$\dot{W}(t) = u(t)S(t), \quad W(0) = 0. \quad (7) \quad \{\text{Wdifeq}\}$$

The papers by Biswas et al. [1], Neilan, Lenhart [26] and Gaff and Schaefer [11] consider control quadratic cost functionals of L^2 -type. It has been argued in Schättler et al. [31] that a control quadratic cost is not appropriate for problems with a biological or biomedical background. Therefore, we consider a L^1 cost functional that is linear with respect to the control variable u (cf. also [31]):

$$J(x, u) = \int_0^T (I(t) + Bu(t)) dt \quad (B > 0). \quad (8) \quad \{\text{L1-objective}\}$$

Our *basic optimal control problem* then consists of determining a piecewise continuous control function $u : [0, T] \rightarrow \mathbb{R}$ that minimizes the L^1 -type functional (8) subject to the dynamic constraints (1)–(4) and control constraint (5). We shall consider several extensions of the basic control problem. Firstly, as in Neilan and Lenhart [26] we impose the terminal constraint

$$W(T) \leq W_T \quad W_T > 0. \quad (9) \quad \{\text{terminal-WT}\}$$

Biswas et al. [1] argue that it is more realistic to limit the supply of vaccines at each time t rather than limiting the total number of vaccines as in the boundary condition (9). This leads to a mixed control-state constraint of the form:

$$u(t)S(t) \leq V_0 \text{ a.e. } t \in [0, T], \quad (10) \quad \{\text{mixed}\}$$

where $V_0 > 0$ is an upper bound on vaccines available at each instant t . The inequality (10) is also known in the literature as state dependent control constraint. The constraint (9) will be satisfied only at the terminal time T , whereas the mixed

constraint (10) should hold at all times during the whole vaccination program. Furthermore, we shall consider the following pure state inequality constraint

$$\{\text{state}\} \quad S(t) \leq S_{max} \quad \forall t \in [0, T]. \quad (11)$$

Since the control u appears linearly in the system dynamics and the objective, the necessary optimality condition of the Maximum Principle show that any optimal control is a concatenation of bang-bang and singular arcs. The notion "bang-bang arc" or "singular arc" even refers to the mixed or state constraint itself which will be made clear in section 6..

In Table 1 we present the values of the initial conditions, parameters and constants which have been used in our computations. Apart from the weights A and B they coincide with those in [26]. Note that in the following computations we shall

TABLE 1. Parameters with their clinically approved values and constants as in [26, 1].

Parameter	Description	Value
b	natural birth rate	0.525
d	natural death rate	0.5
c	incidence coefficient	0.001
e	exposed to infectious rate	0.5
g	recovery rate	0.1
a	disease induced death rate	0.2
B	weight parameter	$\in [2, 10]$
T	number of years	20
S_0	initial susceptible population	1000
E_0	initial exposed population	100
I_0	initial infected population	50
R_0	initial recovered population	15
N_0	initial population	1165
W_0	initial vaccinated population	0

keep the rather high birth rate $b = 0.525$ and death rate $d = 0.5$. However, using smaller values of birth and death rate we obtain the same optimal control structure of, i.e., the same sequence of bang-bang and singular arcs.

3. Numerical methods: verification of necessary and sufficient conditions.

We obtain numerical solutions of the SEIR control problems by applying direct optimization method, i.e., we discretize the control problem and use nonlinear programming methods. The discretized optimal control problem can be conveniently formulated as a nonlinear programming problem (NLP) with the help of the Applied Modeling Programming Language AMPL ceated by Fourer et al. [10]. AMPL can be interfaced to the Interior-Point optimization solver IPOPT, which was developed by Wächter and Biegler [33] for solving large scale optimization problems. The task of formulating and solving the discretized control problem can be facilitated by employing the Imperial College London Optimal Control Software ICLOCS ([9]). This is an optimal control interface, implemented in Matlab, that also calls the solver IPOPT. For a study of different optimal control solvers see [28]. In our

computations, we mostly use $N = 10000$ or $N = 20000$ grid points and the Implicit Euler Scheme to compute the solution with an error tolerance less than 10^{-8} . Alternatively, we use the control package NUDOCCCS developed by C. Büskens [3] (cf. also [4]) which provides another approach to solving discretized control problems using nonlinear programming methods. Since high-order adaptive integration methods are implemented in NUDOCCCS, one needs less than 1000 grid points to obtain a highly accurate solution.

Although we do not show in all cases that the numerical solution is indeed a (local) optimum, we do however validate our findings. Using the Lagrange multipliers provided by the optimization solver IPOPT or by NUDOCCCS, we can validate our numerical solution by showing that it satisfies the necessary condition of optimality with high accuracy. In the special case that the control is bang-bang, we can do better by showing that second-order sufficient conditions (SSC) are satisfied. Here, we solve the so-called *Induced Optimization Problem*, where switching times are directly optimized, and show that (SSC) are satisfied for the Induced Optimization Problem and that the strict bang-bang property holds; cf. Maurer, Büskens, Kim, Kaya [22] and Osmolovskii, Maurer [27]. The test of SSC can be conveniently carried out implementing the *arc-parametrization method* [22] in the control package NUDOCCCS. This approach also allows to perform a sensitivity analysis of the optimal solution with respect to changes in the parameters.

4. Solution of the Basic Optimal Control Problem.

4.1. Necessary optimality conditions: Maximum Principle. The basic optimal control problem is written in a compact form as

$$(OCP) \left\{ \begin{array}{l} \text{Minimize } J_1(x, u) = \int_0^T (I(t) + Bu(t)) dt \\ \text{subject to} \\ \dot{S}(t) = bN(t) - dS(t) - cS(t)I(t) - u(t)S(t), \quad S(0) = S_0, \\ \dot{E}(t) = cS(t)I(t) - (e + d)E(t), \quad E(0) = E_0, \\ \dot{I}(t) = eE(t) - (g + a + d)I(t), \quad I(0) = I_0, \\ \dot{N}(t) = (b - d)N(t) - aI(t), \quad N(0) = N_0, \\ u(t) \in [0, 1] \text{ for a.e. } t \in [0, T], \end{array} \right.$$

The state vector is given by $x = (S, E, I, N)$. Since the control variable appears linearly in the dynamics, the right hand side of the ODEs has the form

$$\dot{x} = f(x) + g(x)u, \quad f(x) = \begin{pmatrix} bN - dS - cSI \\ cSI - (e + d)E \\ eE - (g + a + d)I \\ (b - d)N - aI \end{pmatrix}, \quad g(x) = \begin{pmatrix} S \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (12) \quad \{\text{dynamics-f}\}$$

The integrand of the objective is denoted by $L(x, u) = I + Bu$.

In the following, we shall evaluate the necessary optimality condition of the *Maximum Principle* for problem (OCP). Since we are *maximizing* $-J(x, u)$, the standard Hamiltonian function is given by

$$H(x, p, u) = -\lambda L(x, u) + \langle p, f(x) + g(x)u \rangle, \quad \lambda \in \mathbb{R}, \quad (13) \quad \{\text{Hamiltonian-unlimited}\}$$

where $p = (p_S, p_E, p_I, p_N) \in \mathbb{R}^4$ denotes the adjoint variable.

Let $(x_*, u_*) \in W^{1,\infty}([0, T], \mathbb{R}^4) \times L^\infty([0, T], \mathbb{R})$ be an optimal state and control pair. Then the Maximum Principle (cf. [29, 14, 7]) asserts the existence of a scalar

$\lambda \geq 0$, an absolutely continuous function $p : [0, T] \rightarrow \mathbb{R}^4$ such that the following conditions are satisfied almost everywhere, where the time argument $[t]$ denotes the evaluation along the optimal solution:

(i) $\max\{|p(t)| : t \in [0, T]\} + \lambda > 0$,

(ii) (adjoint equation and transversality condition)

$$\begin{aligned}\dot{p}(t) &= -H_x[t] = \lambda L_x[t] - \langle p(t), f_x[t] - g_x[t]u_*(t) \rangle, \\ p(T) &= (0, 0, 0, 0),\end{aligned}$$

(iii) (maximum condition for Hamiltonian H)

$$H(x_*(t), p(t), u_*(t)) = \max_u \{ H(x_*(t), p(t), u) \mid 0 \leq u \leq 1 \}.$$

The adjoint equations in (ii) for the adjoint variable $p = (p_S, p_E, p_I, p_N)$ are explicitly given by

$$\dot{p}_S(t) = p_S(t)(d + cI_*(t) + u_*(t)) - p_E(t)cI_*(t), \quad (14)$$

$$\dot{p}_E(t) = p_E(t)(e + d) - p_I(t)e, \quad (15)$$

$$\dot{p}_I(t) = 1 + p_S(t)cS_*(t) - p_E(t)cS_*(t) + p_I(t)(g + a + d) + p_N(t)a, \quad (16)$$

$$\dot{p}_N(t) = -p_S(t)b - p_N(t)(b - d). \quad (17)$$

To evaluate the maximum condition (iii) for the Hamiltonian H , we consider the *switching function*

$$\{\text{switching-function}\} \quad \phi(x, p) = H_u(x, u, p) = -B - p_S S, \quad \phi(t) = \phi(x(t), p(t)). \quad (18)$$

Then the condition (iii) is equivalent to the maximum condition

$$\{\text{maximum}\} \quad \phi(t)u_*(t) = \max_u \{ \phi(t)u \mid 0 \leq u \leq 1 \}. \quad (19)$$

which gives the control law

$$u_*(t) = \left\{ \begin{array}{ll} 1 & , \quad \text{if } \phi(t) > 0 \\ 0 & , \quad \text{if } \phi(t) < 0 \\ \text{singular} & , \quad \text{if } \phi(t) = 0 \text{ } [t_1, t_2] \subset [0, T] \end{array} \right\}. \quad (20)$$

Any isolated zero of the switching function $\phi(t)$ yields a switch of the control from 1 to 0 or vice versa. The control u is called *bang-bang* on an interval $[t_1, t_2] \subset [0, T]$ if the switching function $\phi(t)$ has only isolated zeros on $[t_1, t_2]$. The control u is called *singular* on an interval $[t_1, t_2] \subset [0, T]$, if the switching function $\phi(t)$ vanishes identically on $[t_1, t_2]$. The optimal control then is a concatenation of bang-bang and singular arcs.

Our computations in the next section show indeed that singular control arcs may occur. Hence, the singular case needs further analysis. To compute an expression for the singular control, we differentiate the relation $\phi(t) = -B - p_S(t)S(t) = 0$ holding on a time interval $[t_1, t_2] \subset [0, T]$. The derivatives can be computed LIE-brackets. Here, we compute the derivatives directly using the state and adjoint equations. For the first derivative we get omitting the time argument:

$$\{\text{dot-phi}\} \quad \dot{\phi} = p_E c I S - p_S b N = 0. \quad (21)$$

In agreement with the theory, the control variable u does not appear in the first derivative. From $\phi = -B - p_S S = 0$ we get $p_S = -B/S$. Substituting this

expression into $\dot{\phi} = 0$ and multiplying with S , we obtain the relation

$$\{\text{dot-phi-2}\} \quad \dot{\phi} \cdot S = BbN + p_E c I S^2 = 0. \quad (22)$$

The total time derivative of this expression yields $0 = d(\dot{\phi} S)/dt = \ddot{\phi} S$. Using the state and adjoint equations we get

$$0 = Bb((b-d)N - aI) + (p_E(e+d) - p_I e)c I S^2 + p_E c(eE - (g+a+d)I)S^2 + p_E c I 2S(bN - dS - cSI - uS) \quad (23) \quad \{\text{ddot-phi}\}$$

Thus the *Generalized Legendre-Clebsch Condition* GLC requires that the following inequality holds:

$$\frac{\partial \ddot{\phi}}{\partial u} = -p_E 2c I S \geq 0. \quad (24) \quad \{\text{GLC}\}$$

It is easy to show that even the *strict* GLC condition holds. Namely, it follows from (22) that $p_E c I S = -BbN/S < 0$ in view of $N(t) > 0$ and $S(t) > 0$. This allows to determine an expression of the singular control $u_{sing} = u_{sing}(x, p)$ from (23):

$$u_{sing}(x, p) = Bb((b-d)N - aI)/(p_E c I 2S^2) + 0.5(e+d) - 0.5 e p_I / p_E + 0.5(eE/I - (g+a+d)) + bN/S - d - cI. \quad (25) \quad \{\text{singular-control-1}\}$$

4.2. Comparison of solutions for $B = 2$ and $B = 10$. For both weights $B = 2$ and $B = 10$, AMPL/IPOPT and NUDOCCS furnish the control structure

$$u(t) = \left\{ \begin{array}{ll} 1 & \text{for } 0 \leq t < t_1 \\ u_{sing}(x(t), p(t)) & \text{for } t_1 \leq t \leq t_2 \\ 0 & \text{for } t_2 < t \leq T \end{array} \right\}. \quad (26) \quad \{\text{control-unlimited-B=10}\}$$

The optimal state and control variables are shown in Figure 1. We do not exhibit the corresponding adjoint variables $p = (p_S, p_E, p_I, p_N)$ but only list the computed initial values $p(0)$.

Numerical results for $B = 2$:

$$\begin{aligned} J &= 207.5697, & t_1 &= 6.575, & t_2 &= 12.825, \\ S(T) &= 1849.261, & E(T) &= 1.128632, & I(T) &= 0.6565977, \\ N(T) &= 1862.710, & R(T) &= 11.66398, & W(T) &= 4880.123, \\ p_S(0) &= -0.0282156, & p_E(0) &= -0.904113, & p_I(0) &= -1.90956, \\ p_N(0) &= -0.0649112, \end{aligned} \quad (27) \quad \{\text{results-unlimited-B=2}\}$$

Numerical results for $B = 10$:

$$\begin{aligned} J &= 262.6049, & t_1 &= 1.487, & t_2 &= 10.18, \\ S(T) &= 1849.140, & E(T) &= 3.811063, & I(T) &= 2.214767, \\ N(T) &= 1858.069, & R(T) &= 2.902986, & W(T) &= 3196.985, \\ p_S(0) &= -0.0324129, & p_E(0) &= -1.00455, & p_I(0) &= -2.03571, \\ p_N(0) &= -0.130821. \end{aligned} \quad (28) \quad \{\text{results-unlimited-B=10}\}$$

The bottom row of Figure 1 clearly exhibits a significant difference of the controls for $B = 2$ and $B = 10$, since the bang-bang arc with $u(t) = 1$ for $B = 10$ is much smaller than that for $B = 2$. Note, however, that infectious population $I(t)$ is nearly the same for both weights and, hence, the total population N is nearly identical in view of equation (4). We are not aware in the literature on epidemiological models that singular controls have actually been computed though a theoretical analysis of singular controls in SIR models may be found in Ledzewicz, Schättler [18].

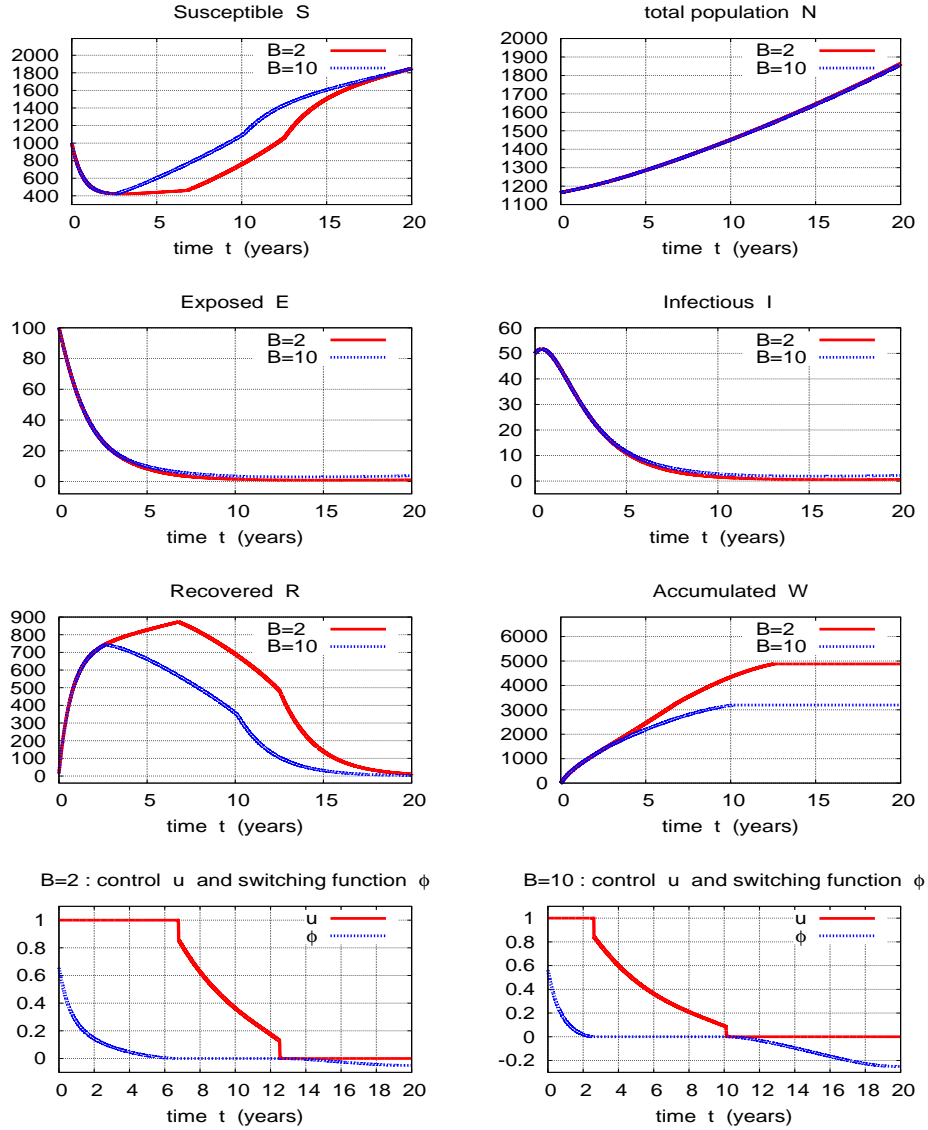


FIGURE 1. State and control variables for basic control problem with control constraint $0 \leq u(t) \leq 1$: comparison for weights $B = 2$ and $B = 10$. *Top row:* (left) susceptible population S , (right) total population N ; *Row 2:* (left) exposed population E , (right) infectious population I ; *Row 3* (left) recovered population R , (right) accumulated vaccinated W ; *Bottom row:* (left) control u and (scaled) switching function ϕ for $B = 2$, (right) control u and (scaled) switching function ϕ for $B = 10$.

For practical reasons it is convenient to approximate the bang-singular-bang by the following simpler control protocol, where the singular arc is replaced by a constant control u_c :

$$u(t) = \begin{cases} 1 & \text{for } 0 \leq t < t_1 \\ u_c & \text{for } t_1 \leq t \leq t_2 \\ 0 & \text{for } t_2 < t \leq T \end{cases} \quad (29) \quad \{\text{control-unlimited-B=10-a}$$

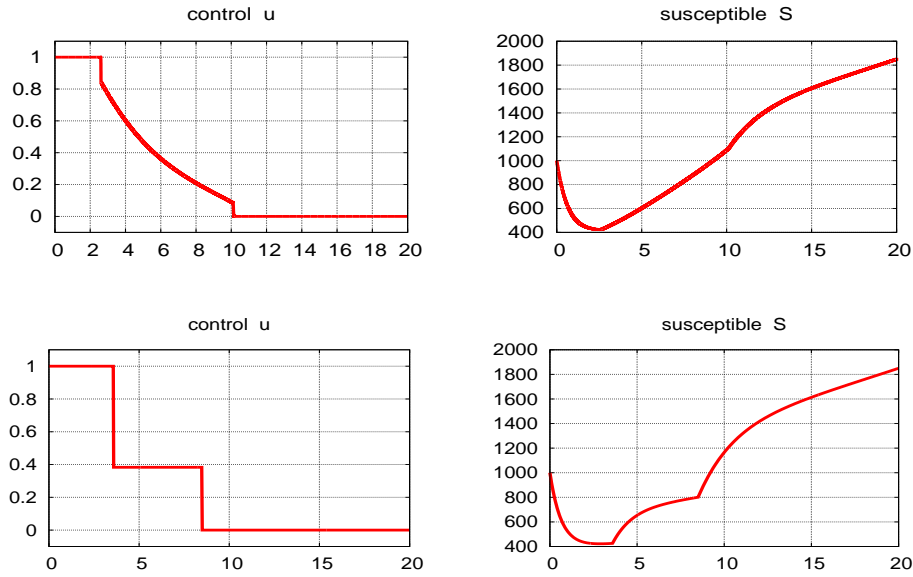


FIGURE 2. Weight $B = 10$: Comparison of basic and approximate control problem for control constraint $0 \leq u(t) \leq 1$. *Top row*: (left) control u , (right) susceptible population S . *Bottom row*: (left) approximate control u , (right) susceptible population S for approximate control u in (29)

To optimize the constant control u_c and the switching times t_1 and t_2 we use the arc-parametrization method described in [22] and implement the code NUDOC-CCS.

Numerical results for the approximating control with $B = 10$:

$$\begin{aligned} J &= 262.9807, & u_c &= 0.3831101 \\ t_1 &= 3.589273 & t_2 &= 8.501096, \\ S(T) &= 1849.624, & E(T) &= 3911697, & I(T) &= 2.272755, \\ N(T) &= 1858.109, & R(T) &= 2.3010733, & W(T) &= 3144.5782, \\ p_S(0) &= -0.0323784, & p_E(0) &= -1.00419, & p_I(0) &= -2.03474, \\ p_N(0) &= -0.131664. \end{aligned} \quad (30) \quad \{\text{results-unlimited-B=10-a}$$

It is remarkable how that the optimal value $J = 262.9807$ of the approximate control problem is very close to the optimal value $J = 262.6049$ in (28). In Figure 2, the

optimal and approximate control and susceptible population S are compared for $B = 10$.

5. Solution with Terminal Constraints $W(T) = W_T$. For the basic optimal control problem (OCP) we obtained the terminal value $W(T) = 488.123$ for $B = 2$ and $W(T) = 3196.985$ for $B = 10$. In order to reduce the total number of vaccinated people, we prescribe as in [26] the much smaller terminal value $W(T) = 2500$. Then the necessary optimality conditions slightly change, since we have to take into account the equation (7) for W ,

$$\dot{W}(t) = u(t) S(t), \quad W(0) = 0.$$

Now the state vector is $x = (S, E, I, N, W) \in \mathbb{R}^5$, while the adjoint variable is $p = (p_S, p_E, p_I, p_N, p_W) \in \mathbb{R}^5$. Then the adjoint equations $\dot{p}(t) = -H_x[t]$ are explicitly:

$$\begin{aligned} \dot{p}_S(t) &= p_S(t)(d + cI_*(t) + u_*(t)) - p_E(t)cI_*(t) - p_W(t)u_*(t), \\ \dot{p}_E(t) &= p_E(t)(e + d) - p_I(t)e, \\ \dot{p}_I(t) &= 1 + p_S(t)cS_*(t) - p_E(t)cS_*(t) + p_I(t)(g + a + d) + p_N(t)a, \\ \dot{p}_N(t) &= -p_S(t)b - p_N(t)(b - d), \\ \dot{p}_W(t) &= 0. \end{aligned} \quad (31)$$

The transversality condition is $(p_S, p_E, p_I, p_N)(T) = (0, 0, 0, 0)$, whereas no terminal condition is prescribed for the (constant) adjoint variable $p_W(t)$. The modified switching function ϕ becomes

$$\phi(x, p) = H_u(x, u, p) = -B - p_S S + p_W S, \quad \phi(t) = \phi(x(t), p(t)). \quad (32)$$

Then maximization of the Hamiltonian with respect to the control u gives the control law

$$u_*(t) = \left\{ \begin{array}{ll} 1 & , \quad \text{if } \phi(t) > 0 \\ 0 & , \quad \text{if } \phi(t) < 0 \\ \text{singular} & , \quad \text{if } \phi(t) = 0 \forall t \in [t_1, t_2] \subset [0, T] \end{array} \right\}. \quad (33)$$

For $B = 10$ we get the bang-singular-bang control

$$u(t) = \left\{ \begin{array}{lll} 1 & \text{for} & 0 \leq t < t_1 \\ u_{sing}(x(t), p(t)) & \text{for} & t_1 \leq t \leq t_2 \\ 0 & \text{for} & t_2 < t \leq T \end{array} \right\}. \quad (34)$$

$B = 10$: Numerical results:

$$\begin{aligned} J &= 265.9739, & t_1 &= 2.554, & t_2 &= 6.58, \\ S(T) &= 1844.584, & E(T) &= 5.948392, & I(T) &= 3.460670, \\ N(T) &= 1855.266, & R(T) &= 1.273017, & W(T) &= 2500.0, \\ p_S(0) &= -0.0420880, & p_E(0) &= -1.06128, & p_I(0) &= -2.09929, \\ p_N(0) &= -0.198856, & p_W(t) &\equiv 0.01122669. \end{aligned} \quad (35)$$

However, for $B = 2$ the control does not have a singular arc and is a bang-bang with one switch:

$$u(t) = \left\{ \begin{array}{lll} 1 & \text{for} & 0 \leq t < t_1 \\ 0 & \text{for} & t_1 < t \leq T \end{array} \right\}. \quad (36)$$

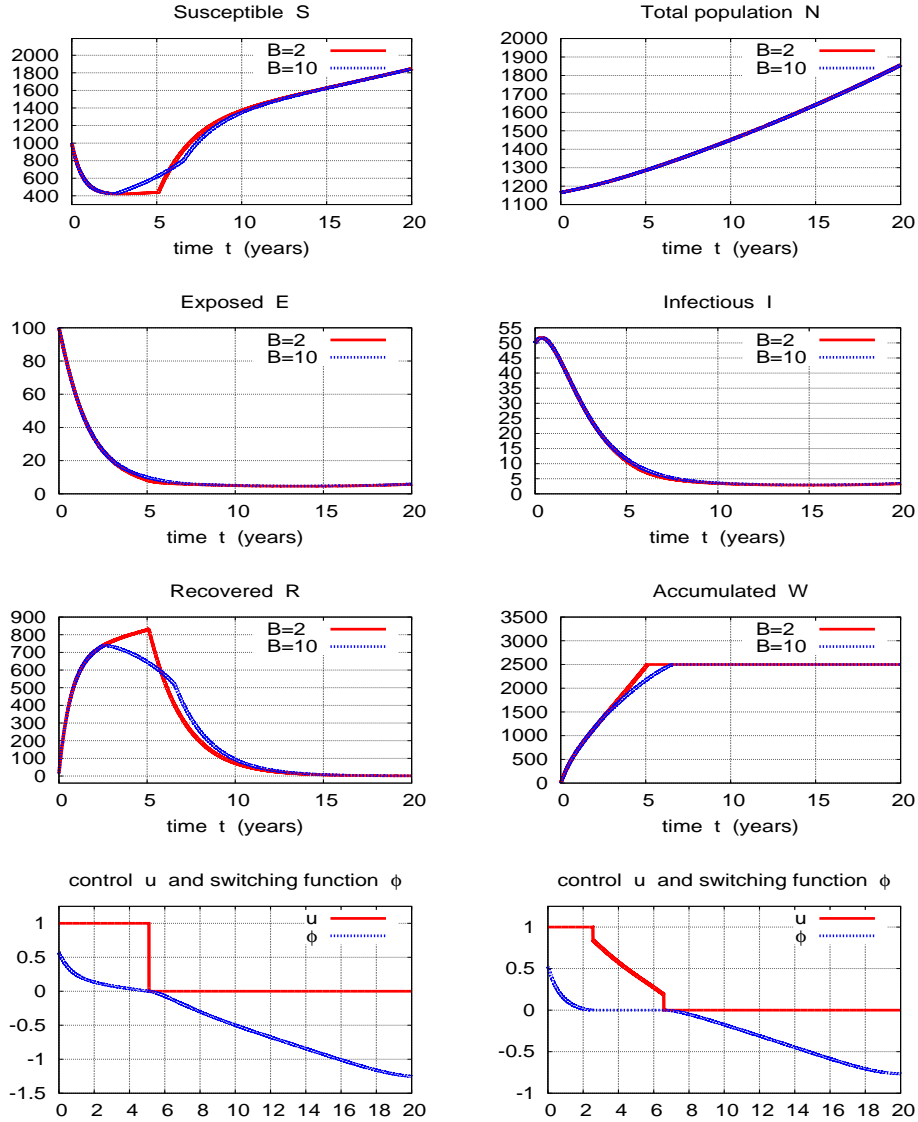


FIGURE 3. State and control variables for the basic control problem with control constraint $0 \leq u(t) \leq 1$ and terminal constraint $W(T) = 2500$: comparison for weights $B = 2$ and $B = 10$. *Top row:* (left) susceptible population S , (right) total population N ; *Row 2:* (left) exposed population E , (right) infectious population I ; *Row 3:* (left) recovered population R , (right) accumulated vaccinated W ; *Bottom row:* (left) control u and (scaled) switching function ϕ for $B = 2$, (right) control u and (scaled) switching function ϕ for $B = 10$.

Thus the Induced Optimization Problem has only one optimization variable t_1 . The arc-parametrization method [22] gives the numerical results

$$\begin{aligned} J &= 226.71703, & t_1 &= 5.105562, \\ S(T) &= 1846.065, & E(T) &= 5.783559, & I(T) &= 3.363232, \\ N(T) &= 1856.3226, & R(T) &= 1.1106551, & W(T) &= 2500.0, \\ p_S(0) &= -0.0507087, & p_E(0) &= -1.03623, & p_I(0) &= -2.05823, \\ p_N(0) &= -0.026091, & p_W(t) &\equiv -0.0260915. \end{aligned} \quad (37)$$

A comparison of optimal state and control variables is presented in Figure 3

6. Solution for mixed control-state constraint $u S \leq 125$. In this section, we consider the pointwise mixed control-state constraint (10)

$$\text{\{mixed-2\}} \quad u(t)S(t) \leq V_0 \text{ a.e. } t \in [0, T], \quad (38)$$

instead of the terminal condition $W(T) = W_T = 2500$. Since the time horizon is $T = 20$, a convenient choice of the bound is $V_0 = W_T/20 = 125$. We write the mixed control-state constraint in the form

$$\text{\{mixed-m\}} \quad m(x, u) = u S - V_0 \leq 0. \quad (39)$$

On every *boundary arc* of the mixed constraint with $m(x(t), u(t)) = 0$, the following *regularity condition* holds:

$$\text{\{mixed-regularity\}} \quad m_u(x(t), u(t)) = S(t) \neq 0. \quad (40)$$

6.1. Evaluation of the Maximum Principle. Let the pair (x_*, u_*) be a local minimum. We shall evaluate the necessary optimality condition of the *Maximum Principle* as given in [7] (cf. also [14, 24]). The standard Hamiltonian function (??) is defined by

$$H(x, p, u) = -\lambda L(x, u) + \langle p, f(x) + g(x)u \rangle, \quad p \in \mathbb{R}^5,$$

where $p = (p_S, p_E, p_I, p_N) \in \mathbb{R}^4$ denotes the adjoint variable and $\lambda \geq 0$. Then the *augmented* Hamiltonian is obtained by adjoining the mixed constraint by a multiplier $q \in \mathbb{R}$ to the Hamiltonian:

$$\mathcal{H}(x, p, q, u) = H(x, p, u) - q m(x, u).$$

Here, the minus sign is due to the fact that the *Maximum Principle* assumes that the control-state constraint is written in the form $-m(x, u) = V_0 - u S \geq 0$. In view of the regularity condition (40), Theorem 7.1 in [7] (cf. also [14, 24]) asserts the existence of a scalar $\lambda \geq 0$, an absolutely continuous function $p : [0, T] \rightarrow \mathbb{R}^4$ and an integrable function $q : [0, T] \rightarrow \mathbb{R}$ such that the following conditions are satisfied almost everywhere:

(i) $\max\{|p(t)| : t \in [0, T]\} + \lambda > 0$,

(ii) (adjoint equation and transversality condition)

$$\begin{aligned} \dot{p}(t) &= -\mathcal{H}_x[t] = \lambda L_x[t] - \langle p(t), f_x[t] + g_x[t]u_*(t) \rangle + \langle q(t), m_x[t] \rangle, \\ p(T) &= (0, 0, 0, 0), \end{aligned}$$

(iii) (maximum condition for Hamiltonian H)

$$H(x_*(t), p(t), u_*(t)) = \max_u \{ H(x_*(t), p(t), u) \mid 0 \leq u \leq 1, m(x_*(t), u) \leq 0 \},$$

(iv) (local maximum condition for augmented Hamiltonian \mathcal{H})

$$\mu(t) = \mathcal{H}_u[t] = -L_u[t] + \langle p(t), g[t] \rangle - q(t) m_u[t] \in N_{[0,1]}(u_*(t)),$$

(v) (complementarity condition)

$$q(t)m(x_*(t), u_*(t)) = q(t)(u_*(t)S_*(t) - V_0) = 0 \quad \text{and} \quad q(t) \geq 0.$$

In (iv), $N_{[0,1]}(u_*(t))$ stands for the normal cone from convex analysis to $[0, 1]$ at the optimal control $u_*(t)$ (see e.g. [5]) and it reduces to $\{0\}$ when $u^*(t) \in]0, 1[$. Since the terminal state $x(T)$ is free, it is easy to prove that the above necessary conditions hold with $\lambda = 1$; for a complete discussion see [1]. Hence, our problem is *normal*. We can further prove the existence of a constant K_q^1 such that

$$|q(t)| \leq K_q^1 |p(t)| \tag{41} \quad \{\text{boundq}\}$$

for almost every $t \in [0, T]$ (see [7]).

Now we want to extract information from the conclusions (i)–(v) with $\lambda = 1$ that later will be used to validate our numerical solution. The adjoint equations in (ii) for the adjoint variable $p = (p_S, p_E, p_I, p_N)$ read explicitly:

$$\dot{p}_S(t) = p_S(t)(d + cI_*(t) + u_*(t)) - p_E(t)cI_*(t) + q(t)u_*(t), \tag{42}$$

$$\dot{p}_E(t) = p_E(t)(e + d) - p_I(t)e, \tag{43}$$

$$\dot{p}_I(t) = 1 + p_S(t)cS_*(t) - p_E(t)cS_*(t) + p_I(t)(g + a + d) + p_N(t)a, \tag{44}$$

$$\dot{p}_N(t) = -bp_S(t) - (b - d)p_N(t). \tag{45}$$

Next, we evaluate the maximum condition (iii) for the Hamiltonian H . The *switching function* ϕ is defined by

$$\phi(x, p) = H_u(x, u, p) = -B - p_S S, \quad \phi(t) = \phi(x(t), p(t)) \tag{46} \quad \{\text{switching-function-m}\}$$

and agrees with that in (18). Then the condition (iii) is equivalent to the maximum condition

$$\phi(t)u_*(t) = \max_u \{ \phi(t)u \mid 0 \leq u \leq 1, uS_*(t) \leq V_0 \}. \tag{47} \quad \{\text{maximum}\}$$

This yields the control law

$$u_*(t) = \left\{ \begin{array}{ll} \min \left\{ 1, \frac{V_0}{S_*(t)} \right\} & , \quad \text{if } \phi(t) > 0 \\ 0 & , \quad \text{if } \phi(t) < 0. \end{array} \right\}. \tag{48}$$

Any isolated zero of the switching function $\phi(t)$ yields a switch of the control from $\min\{1, V_0/S_*(t)\}$ to 0 or vice versa. If, however, $\phi(t) = 0$ holds on an interval $[t_1, t_2] \subset [0, T]$, then we have a *singular control*. We do not enter here into a detailed discussion of singular controls as in Section 4.1, since singular controls never appeared in our computations. Moreover, we always have $0 < u_*(t) < 1$ on a boundary arc of the mixed constraint $uS \leq V_0$, i.e., whenever $u_*(t) = V_0/S_*(t)$. Hence, the control is determined by

$$u_*(t) = \left\{ \begin{array}{ll} V_0/S_*(t) & , \quad \text{if } \phi(t) > 0 \\ 0 & , \quad \text{if } \phi(t) < 0. \end{array} \right\}. \tag{49}$$

Due to $0 < u_*(t) < 1$ the multiplier $\mu(t)$ in (iv) vanishes, which yields the relation

$$0 = \mu(t) = \mathcal{H}_u[t] = -B - p_S(t)S_*(t) - q(t)S_*(t).$$

This allows us to compute the multiplier $q(t)$ for which we get in view of the complementarity condition (v)

$$q(t) = \begin{cases} -\frac{B}{S_*(t)} - p_s(t) = \phi(t)/S_*(t), & \text{if } u_*(t) = V_0/S_*(t) \\ 0, & \text{if } u_*(t) < V_0/S_*(t) \end{cases}. \quad (50)$$

6.2. Comparison of optimal solutions for weights $B = 2$ and $B = 10$. For both weights $B = 2$ and $B = 10$ we find the following control structure with one boundary arc $u(t)S(t) = V_0$ in $[0, t_1]$:

$$u_*(t)S_*(t) = \begin{cases} V_0 & , \quad \text{for } 0 \leq t < t_1 \\ 0 & , \quad \text{for } t_1 \leq t \leq T \end{cases}. \quad (51)$$

Thus the *new control variable* v defined by $v = uS$ is a *bang-bang control* with one switching at t_1 . This transformation of control variables has been studied in [23]. Hence, the Induced Optimization Problem for the bang-bang control problem (cf. [22, 27]) has the single optimization variable t_1 . Thus, the cost functional becomes a function $J = J(t_1)$. The arc-parameterization method in [22] and the code NUDOCCS provide the following results for $B = 2$,

$$\begin{aligned} \{ \text{results-mixed-B=2} \} \quad & J = 338.4225, & t_1 = 17.88659, \\ & S(T) = 1723.866, & E(T) = 7.702982, & I(T) = 4.703767, \\ & N(T) = 1824.171, & R(T) = 87.89877, & W(T) = 2235.823 \\ & p_S(0) = -0.126224, & p_E(0) = -1.51088, & p_I(0) = -2.93143, \\ & p_N(0) = -0.453975, & & \end{aligned} \quad (52)$$

and for $B = 10$,

$$\begin{aligned} \{ \text{results-mixed-B=10} \} \quad & J = 353.3593, & t_1 = 15.23429, \\ & S(T) = 1785.273, & E(T) = 9.080961, & I(T) = 5.394489, \\ & N(T) = 1823.897, & R(T) = 24.14828, & W(T) = 1904.283, \\ & p_S(0) = -0.125173, & p_E(0) = -1.51988, & p_I(0) = -2.94704, \\ & p_N(0) = -0.449700. & & \end{aligned} \quad (53)$$

The optimal state variables for $B = 2$ and $B = 10$ are shown in Figure 4. Figure 5 displays the controls u and switching functions ϕ as well as the constraint functions $u(t)S(t)$ in relation to the multiplier q in (50). It can be seen from Figure 5 that the following strict bang-bang property (cf. the definition in [27, 22]) holds for $B = 2$ and $B = 10$;

$$\phi(t) > 0 \quad \text{for } 0 \leq t < t_1, \quad \dot{\phi}(t_1) < 0, \quad \phi(t) < 0 \quad \text{for } t_1 < t \leq T.$$

Recall that the objective $J = J(t_1)$ is a function of the single optimization variable t_1 . Then the Hessian of J is the second derivative which is computed as:

$$B = 10 : \quad J''(t_1) = 0.350589 > 0; \quad B = 2 : \quad J''(t_1) = 0.174805 > 0.$$

Hence, it follows from [27], Chapter 7, and [22] that the solutions shown in Figures 4 and 5 provide a strict strong minimum.

7. Optimal solution for state constraint $S(t) \leq S_{max} = 1100$ and terminal constraint $W(T) \leq W_T$. We infer from Figure 1 that the susceptible population $S(t)$ assumes rather large values, when only control constraints $u(t) \in [0, 1]$ are present. Imposing smaller terminal value $S(T)$ does not prevent the solution from

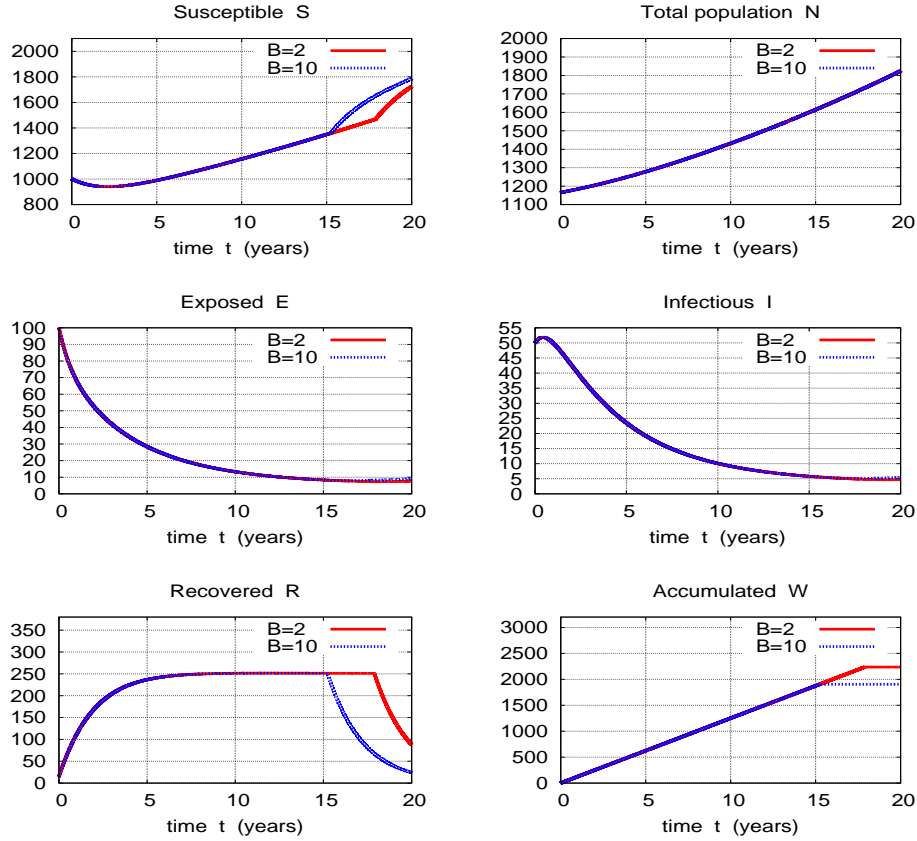


FIGURE 4. State variables for basic control problem with mixed control-state constraint $u(t)S(t) \leq 125$: comparison for weights $B = 2$ and $B = 10$. *Top row*: (left) susceptible population S , (right) total population N ; *Middle row*: (left) exposed population E , (right) infectious population I ; *Bottom row* (left) recovered population R , (right) accumulated vaccinated W .

reaching large intermediate values $S(t)$. For that reason we require the point-wise state constraint (11)

$$\{\text{state-2}\} \quad S(t) \leq S_{max} \quad \forall t \in [0, T]. \quad (54)$$

with an appropriate value S_{max} specified below. Let us first write the state constraint in the form

$$s(x) = S - S_{max} \leq 0. \quad (55) \quad \{\text{state-MP}\}$$

This is a state constraint of *order one*, since the control variable u appears in the first total time derivative of $s(x)$, cf. [13, 21]:

$$s^{(1)}(x, u) = \frac{d}{dt} s(x) = \dot{S} = bN - dS - cSI - uS.$$

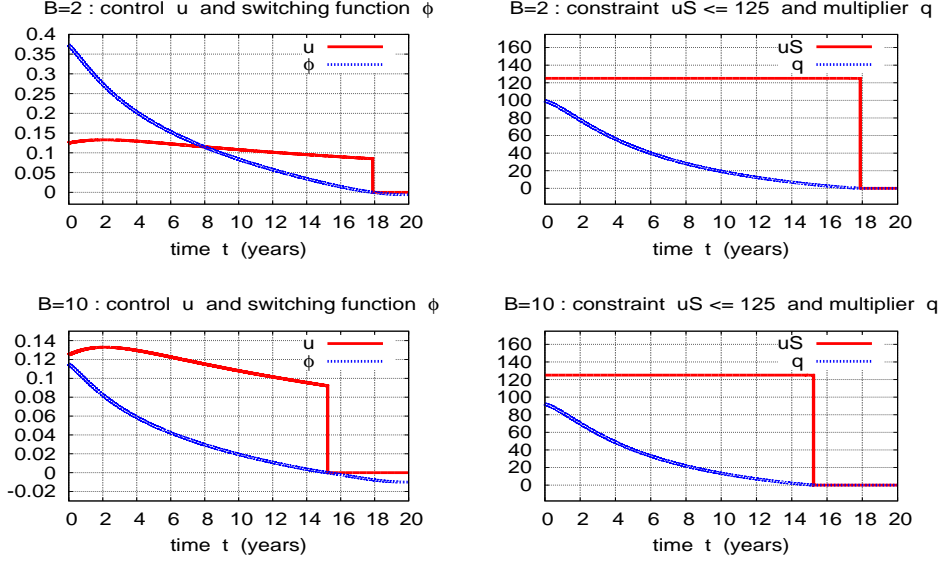


FIGURE 5. State and control variables for basic control problem with mixed control-state constraint $u(t)S(t) \leq 125$: comparison for weights $B = 2$ and $B = 10$. *Top row:* Weight $B = 2$: (left) control u and (scaled) switching function ϕ , (right) function uS and multiplier q in (50); *Bottom row:* Weight $B = 10$: (left) control u and (scaled) switching function ϕ , (right) function uS and multiplier q in (50).

The state constraint satisfies the *regularity condition*

$$\frac{\partial}{\partial u} s^{(1)}(x(t), u(t)) = S(t) \neq 0 \quad (56) \quad \{\text{state-regularity}\}$$

on every boundary arc $[t_1, t_2] \subset [0, T]$ with $S(t) = S_{max}$. Then the *boundary control* $u = u_b(x)$ is determined by the equation $s^{(1)}(x, u) = 0$ as

$$\{\text{control-boundary}\} \quad u = u_b(x) = bN/S - d - cI. \quad (57)$$

When we choose small values for the upper bound $S_{max} \geq S(0)$, then the terminal value $W(T)$ can attain rather large values. For that reason we impose, as in Section 5, the constraint (9),

$$W(T) \leq W_T,$$

and take into account the equation

$$\dot{W}(t) = u(t)S(t), \quad W(0) = 0.$$

7.1. Evaluation of the Maximum Principle. Now we shall evaluate the necessary optimality condition of the *Maximum Principle* as given in [13, 21]. In view of the regularity condition (56) the multiplier associated with the state constraint has a density ν which is a differentiable function on the boundary arc ([21]). Using

again the standard Hamiltonian function (13),

$H(x, p, u) = -\lambda L(x, u) + \langle p, f(x) + g(x)u \rangle$, $\lambda \in \mathbb{R}$, $p = (p_S, p_E, p_I, p_N, p_W) \in \mathbb{R}^5$, the *augmented* Hamiltonian is defined by adjoining the state constraint $-s(x) = S_{max} - S \geq 0$ to the Hamiltonian H by a multiplier q , cf. [13]:

$$\mathcal{H}(x, p, q, u) = H(x, p, u) - q s(x) = H(x, p, u) - q (S_{max} - S).$$

Let the pair (x_*, u_*) be a local minimum. In view of the regularity condition (56), the Maximum (Minimum) Principle in [13, 21] asserts the existence of a scalar $\lambda \geq 0$, an absolutely continuous function $p : [0, T] \rightarrow \mathbb{R}^4$ and an absolutely continuous function $\nu : [0, T] \rightarrow \mathbb{R}$, and jump parameters γ_s at any junction or contact time t_s with the state boundary, such that the following conditions are satisfied almost everywhere:

(i) $\max\{|p(t)| : t \in [0, T]\} + \lambda > 0$,

(ii) (adjoint equation, jump conditions and transversality condition)

$$\begin{aligned} \dot{p}(t) &= -\mathcal{H}_x[t] = \lambda L_x[t] - \langle p(t), f_x[t] + -g_x[t]u_*(t) \rangle + q(t) s_x[t], \\ p(t_s+) &= p(t_s-) - \gamma_s s_x(x(t_s)), \quad \gamma_s \geq 0, \\ p(T) &= (0, 0, 0, 0) \quad \text{if } S_*(T) < S_{max}, \\ p(T) &= (p_S(T), 0, 0, 0) \quad \text{if } S_*(T) = S_{max}. \end{aligned}$$

(iii) (maximum condition for Hamiltonian H)

$$H(x_*(t), p(t), u_*(t)) = \max_{0 \leq u \leq 1} H(x_*(t), p(t), u)$$

(iv) (complementarity condition)

$$q(t) s(x_*(t)) = q(t) (S_*(t) - S_{max}) = 0 \quad \text{and} \quad q(t) \geq 0.$$

We assume that the problem is normal so that we can put $\lambda = 1$ in the necessary conditions. This assumption will be verified by the numerical results. The adjoint equations in (ii) for the adjoint variable $p = (p_S, p_E, p_I, p_N, p_W)$ read explicitly:

$$\dot{p}_S(t) = p_S(t)(d + cI_*(t) + u_*(t)) - p_E(t)cI_*(t) + p_W(t)u_*(t) + q(t), \quad (58)$$

$$\dot{p}_E(t) = p_E(t)(e + d) - p_I(t)e, \quad (59)$$

$$\dot{p}_I(t) = 1 + p_S(t)cS_*(t) - p_E(t)cS_*(t) + p_I(t)(g + a + d) + ap_N(t), \quad (60)$$

$$\dot{p}_N(t) = -p_S(t)b - p_N(t)(b - d). \quad (61)$$

$$\dot{p}_W(t) = 0. \quad (62)$$

To evaluate the maximum condition (iii) for the Hamiltonian H , we need the *switching function*

$$\phi(x, p) = H_u(x, u, p) = -B - p_S S + p_W S, \quad \phi(t) = \phi(x(t), p(t)) \quad (63) \quad \{\text{switching-function-sc}\}$$

which agrees with the switching function (32) Then the maximum condition (iii) gives

$$u_*(t) = \begin{cases} 1 & , \quad \text{if } \phi(t) > 0, \\ 0 & , \quad \text{if } \phi(t) < 0, \\ \text{singular / boundary control} & , \quad \text{if } \phi(t) = 0 \quad \text{on } [t_1, t_2] \subset [0, T]. \end{cases} \quad (64)$$

An expression of a singular control on interior arcs with $S(t) < S_{max}$ were already discussed in Section 4.1. Recall that the boundary control is given by $u_b(x) = bN/S - d - cI$ in (57). Computations show that $0 < u_b(x(t)) < 1$ holds along a boundary arc. This implies that $\phi(t) = 0$ holds on a boundary arc $[t_1, t_2]$. Hence,

in view of (64) the boundary control behaves formally like a singular control; cf. Maurer [20]. Differentiating the relation $\phi = -B - p_S S + p_W S = 0$, using the modified adjoint equations (58) and noting that $p_W(t)S(t)$ is constant on a boundary arc, we get

$$\dot{\phi} = -p_S b N + p_E c I S - q S = 0.$$

This equation gives the multiplier for the state constraint as a function of the state and adjoint variables:

$$\{\text{multiplier-state}\} \quad q = q(x, p) = -p_S b N / S + p_E c I. \quad (65)$$

7.2. Optimal solution $S_{max} = 1300$, $W(T) = 3000$ and weight $B = 10$. We choose the upper bound $S_{max} = 1300$ in (54) and the terminal constraint $W(T) \leq W_T = 3000$. For both weights $b = 2$ and $B = 10$, the solutions are nearly identical. Therefore, we show only the solution for $B = 10$. The optimal control has two bang-bang arcs followed by a terminal boundary arc:

$$u_*(t) = \left\{ \begin{array}{ll} 1 & , \quad \text{for } 0 \leq t < t_1 \\ 0 & , \quad \text{for } t_1 \leq t < t_2 \\ u_b(x(t)) & , \quad \text{for } t_2 \leq t \leq T \end{array} \right\}. \quad (66)$$

The boundary control $u_b(x)$ is given by the expression (57). Using this structure the Induced Optimization Problem consists of determining the two switching times t_1 and t_2 such that the conditions $S(t_2) = S_{max} = 1300$ and $W(T) = 3000$ are satisfied. The arc-parametrization method [?] and the control package NUDOCSS yield the following numerical results:

$$\{\text{ts-state-constraint-B=10}\} \quad \begin{array}{lll} J = 335.0512, & t_1 1.348069, & t_2 = 7.597497, \\ S(T) = 1300.0, & E(T) = 6.164159, & I(T) = 4.326310, \\ N(T) = 1833.282, & R(T) = 522.7910, & W(T) = 3000.0, \\ p_S(0) = -0.0918380, & p_E(0) = -1.24484, & p_I(0) = -2.24107, \\ p_N(0) = -1.09261, & p_S(T) = -0.0824279, & \end{array} \quad (67)$$

Figure 6, top row, left, shows that the control is *discontinuous* at the junction t_2 of the singular arc with the boundary arc. Then it follows from the junction theorems in Maurer [?] that the adjoint variable $p_S(\cdot)$ does not have a jump (??) at t_2 and, hence, the adjoint variable $p(\cdot)$ is *continuous* on $[0, T]$; cf. Figure 6, bottom row, left. We can check that the solution shown in Figure 6 satisfies second-order sufficient conditions (SSC) by applying the test of SSC in [25]. The Jacobian of the equality constraints $S(t_1) = S_{max}$ and $W(T) = 3000$ with respect to the optimization variables t_1, t_2 is a regular 2×2 -matrix. Moreover, the switching function $\phi(t)$ satisfies the following strict bang-bang property in relation to the boundary arc, where $\phi(t) = 0 \quad \forall t_2 \leq t \leq T$:

$$\phi(t) > 0 \quad \forall 0 \leq t < t_1, \quad \dot{\phi}(t_1) < 0; \quad \phi(t) < 0 \quad \forall t_1 < t < t_2, \quad \dot{\phi}(t_2-) > 0.$$

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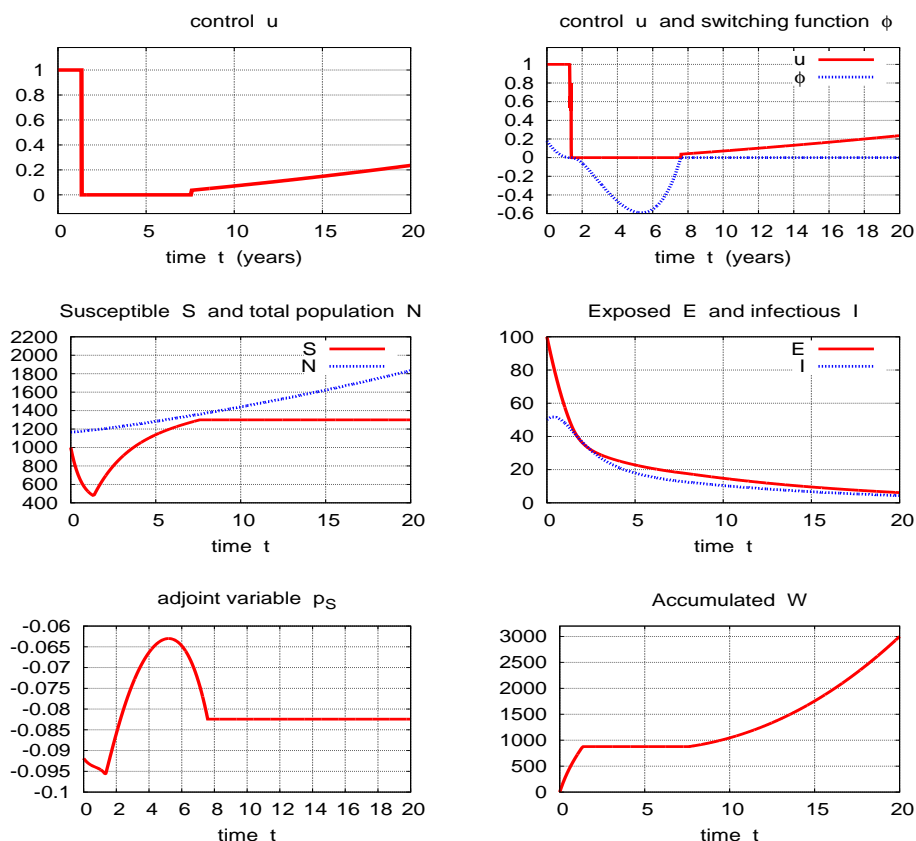


FIGURE 6. State variables for basic control problem with state constraint $S(t) \leq S_{max} = 1300$ and terminal constraint $W(T) \leq 3000$, weight $B = 10$: *Top row*: (left) control u , (right) control u and (scaled) switching function ϕ in (63). *Middle row*: (left) susceptible population S and total population N , (right) exposed population E and infectious population I ; *Bottom row* (left) continuous adjoint variable p_S , (right) accumulated vaccinated W .

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