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ON THE DETECTION OF SMALL MOVING DISKS IN A FLUID

ALEXANDRE MUNNIER* AND KARIM RAMDANI†

Abstract. We are interested in determining the positions and the velocities of moving rigid solids in a bounded cavity filled with a perfect fluid. We assume that the solids are small disks and that they move slowly. Using an integral formulation, we first derive the asymptotic expansion of the DtN map of the problem as the diameters of the disks tend to zero. Then, combining a suitable choice of exponential type data and the DORT method (which is usually used in inverse scattering for the detection of point-like scatterers), we propose a reconstruction method for the unknown positions and velocities.

Key words. Inverse problems, perfect fluid, integral equations, asymptotic analysis, DtN operator, time reversal, DORT Method

AMS subject classifications.

1. Introduction. The geometric inverse problem that we consider in this paper is to determine the positions and the velocities of rigid solids moving in a perfect fluid. More precisely, let Ω be an open and simply connected bounded domain of \mathbb{R}^2 with smooth boundary $\Gamma := \partial\Omega$. The domain Ω is supposed to be filled with a perfect fluid and it contains M rigid solids D_m , $m = 1, \dots, M$, where $D_m \subset \Omega$ is a closed disk with boundary γ_m . The domain occupied by the fluid is denoted by $\mathcal{F} := \Omega \setminus \left(\sum_{m=1}^M D_m\right)$. We also introduce the unit normal n to $\partial\mathcal{F}$ directed towards the exterior of the fluid and τ the unit tangent vector to $\partial\mathcal{F}$ such that $\tau = n^\perp$ (throughout the paper, we set $x^\perp := (-x_2, x_1)$ for all $x = (x_1, x_2) \in \mathbb{R}^2$). We assume that every solid moves with a velocity $V_m \in \mathbb{R}^2$. At every given time, we can consider the Eulerian velocity field $\mathbf{U}(x)$, $x \in \mathcal{F}$, of the fluid. We assume that the flow is irrotational

$$(1.1a) \quad \text{rot}(\mathbf{U}) = 0 \text{ in } \mathcal{F},$$

and circulation free, so that for every rectifiable Jordan curve $\gamma \subset \mathcal{F}$, there holds

$$(1.1b) \quad \int_{\gamma} \mathbf{U}(s) \cdot \tau(s) \, ds = 0.$$

The slip boundary conditions on $\partial\mathcal{F}$ read

$$(1.2a) \quad \mathbf{U} \cdot n = V_m \cdot n \quad \text{on } \gamma_m, \quad m = 1, \dots, M$$

$$(1.2b) \quad \mathbf{U} \cdot n = 0 \quad \text{on } \Gamma.$$

We shall address the inverse problem of determining the positions and the velocities of the solids by using actuators and sensors located on the outer boundary Γ . We assume that we can prescribe the normal velocity of the fluid on Γ , leading to the modification of equation (1.2b) into:

$$(1.3) \quad \mathbf{U} \cdot n = F \quad \text{on } \Gamma$$

where F is given. Regarding the available output, we assume that we can measure the tangential velocity $\mathbf{U} \cdot \tau$ on Γ . According to conditions (1.1a) and (1.1b), we

*Université de Lorraine and CNRS, Institut Élie Cartan de Lorraine, UMR 7502, Vandœuvre-lès-Nancy, F-54506, France. (alexandre.munnier@univ-lorraine.fr)

†Inria, Villers-lès-Nancy, F-54600, France. (karim.ramdani@inria.fr)

classically introduce the stream function $\psi : \mathcal{F} \rightarrow \mathbb{R}$ such that $\mathbf{U} = -\nabla^\perp \psi$ in \mathcal{F} . Then, equations (1.1a) and (1.2) read

$$(1.4a) \quad -\Delta \psi = 0 \quad \text{in } \mathcal{F}$$

$$(1.4b) \quad \psi = V_m^\perp \cdot x + c_m \quad \text{on } \gamma_m, \quad m = 1, \dots, M$$

$$(1.4c) \quad \psi = f \quad \text{on } \Gamma,$$

where $\partial_\tau f = F$ on Γ and the constants $c_m \in \mathbb{R}$, $k = 1, \dots, M$, are such that

$$(1.4d) \quad \int_{\gamma_m} \partial_n \psi(s) ds = 0.$$

With these settings, the measurement reads $\mathbf{U} \cdot \tau = -\partial_n \psi$. The detection problem under consideration is then to recover the centers, the diameters and the velocities of the solids from the DtN map $\Lambda : f \in H^{1/2}(\Gamma) \mapsto \partial_n \psi \in H^{-1/2}(\Gamma)$.

REMARK 1. *Instead of using the stream function, one can equivalently formulate the problem in terms of the potential function and obtain Neumann boundary conditions instead of Dirichlet in (1.4).*

Although such geometric inverse problems of detecting moving solids in a fluid appear in many applications, the associated literature is quite limited, as most contributions deal with the case of motionless solids (i.e. obstacles). In particular, the case of a fluid described by the Stokes equations has been investigated using optimization methods by Caubet *et al.* in [12, 5, 11, 10] and more recently by Bourgeois *et al.* in [6] using the quasi-reversibility method combined to a level set method. The case of small obstacles has been studied in Caubet and Dambrine [9]. Regarding moving obstacles, Conca *et al.* show in [15, 14] that the position and the velocity for a single disk moving in a perfect fluid can be recovered from one measurement of the velocity on part of the boundary. Linear stability estimates are also provided. In Conca *et al.* [16], the authors consider a moving rigid solid immersed in a potential fluid and provide examples of detectable (ellipses for instance) and undetectable shapes. Conca *et al.* obtained in [17] an identifiability result in the case of a rigid solid immersed in a viscous fluid.

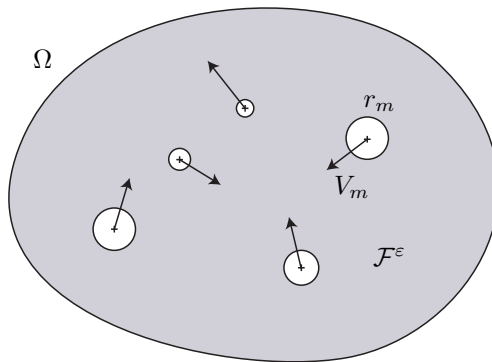


FIG. 1. The domain Ω containing M small rigid disks D_m^ε , $m = 1, \dots, M$ and filled with a perfect fluid occupying the domain $\mathcal{F}^\varepsilon = \Omega \setminus (\cup_{m=1}^M D_m^\varepsilon)$.

In this work, we restrict the analysis to the case where the solids are small disks (see Figure 1) of typical size ε , so that rotation plays no role. For such configurations,

we show an identifiability result and we provide a reconstruction method. More precisely, we denote each closed disk D_m^ε , $m = 1, \dots, M$ and we denote by γ_m^ε its boundary. We assume that D_m^ε is centered at r_m and is of radius εR_m , where the parameter ε is meant to tend to 0. Denoting by $\mathcal{F}^\varepsilon = \Omega \setminus (\cup_{m=1}^M D_m^\varepsilon)$ the domain occupied by the fluid and letting ψ^ε be the corresponding solution of (1.4), our inverse problem can be formulated as follows: *Knowing the DtN map*

$$(1.5) \quad \Lambda^\varepsilon : f \in H^{1/2}(\Gamma) \longmapsto \partial_n \psi^\varepsilon \in H^{-1/2}(\Gamma)$$

is it possible, and if so how, to recover the number M of disks, their positions r_m , $m = 1, \dots, M$, their (rescaled) radii R_m and their velocities V_m ?

We answer this question in several steps. In Section 2, we first derive the asymptotic expansion of the DtN operator Λ^ε as $\varepsilon \rightarrow 0^+$ (Theorem 2.1) using a boundary integral formulation of the forward problem. In Section 3, we combine this expansion with the DORT method¹ to recover the number of disks and their positions, provided they are distant enough. Initially introduced by Fink and Prada [18], this method has been justified mathematically and used in the framework of wave systems for the detection of distant point-like scatterers in acoustics [21, 26, 7] and electromagnetics [4]. Once the positions have been determined, the velocities and rescaled radii can be easily recovered using suitable data. Finally, we collect in Section 4 some numerical examples to illustrate the efficiency of the proposed reconstruction method.

2. Asymptotic expansion for small disks. Although the literature dealing with small inhomogeneities or small inclusions is quite rich (see for instance the review paper [2] and the book [3] by Ammari and Kang), the needed asymptotics is not, as far we know, directly available for our problem (1.4), nor for the equivalent Neumann conjugate problem satisfied by the velocity potential. Quoting only the tightly related works, let us mention Il'in [22] who studied an elliptic boundary value problem set in a three dimensional domain containing a small hole, Maz'ya, Nazarov and Plamenevskij who studied the Laplace problem with only one small hole (see [24, p. 59] for the Dirichlet case and [24, p. 291] for the Neumann case) and Friedman and Vogelius [19, Lemma 3.3] who obtained the first order asymptotics for several infinitely conducting small inhomogeneities (i.e. constant Dirichlet condition on the boundaries of the small holes and Neumann condition on the exterior boundary).

The next result provides the asymptotic expansion of the DtN map Λ^ε as $\varepsilon \rightarrow 0^+$.

THEOREM 2.1. *For every $f \in H^{1/2}(\Gamma)$, we denote by $U^f \in H^1(\Omega)$ the solution of the boundary value problem*

$$(2.1) \quad \begin{aligned} -\Delta U^f &= 0 && \text{in } \Omega \\ U^f &= f && \text{on } \Gamma \end{aligned}$$

Let $\Lambda^0 \in \mathcal{L}(H^{1/2}(\Gamma), H^{-1/2}(\Gamma))$ the DtN map $\Lambda^0 : f \in H^{1/2}(\Gamma) \longmapsto \partial_n U^f \in H^{-1/2}(\Gamma)$. Then, the DtN map $\Lambda^\varepsilon \in \mathcal{L}(H^{1/2}(\Gamma), H^{-1/2}(\Gamma))$ defined by (1.5) admits the following expansion as $\varepsilon \rightarrow 0^+$:

$$\Lambda^\varepsilon = \Lambda^0 + \varepsilon^2 \Lambda^2 + O(\varepsilon^3),$$

where for every $f, g \in H^{1/2}(\Gamma)$:

$$\langle \Lambda^2 f, g \rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)} = 2\pi \sum_{\ell=1}^M R_\ell^2 \nabla U^f(r_\ell) \cdot \nabla U^g(r_\ell) + 2\pi \sum_{\ell=1}^M R_\ell^2 \nabla U^g(r_\ell) \cdot V_\ell^\perp.$$

¹DORT is the French acronym for Decomposition of the Time-Reversal Operator.

The rest of this section is devoted to the proof of this result. The first ingredient is the following reciprocity identity, which can be easily obtained using Green's formula:

$$(2.2) \quad \langle (\Lambda^\varepsilon - \Lambda^0)f, g \rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)} = \sum_{\ell=1}^M [\psi^\varepsilon, U^g]_{\gamma_\ell^\varepsilon}$$

where we have set

$$(2.3) \quad [\phi_1, \phi_2]_{\gamma_\ell^\varepsilon} := \langle \partial_n \phi_2, \phi_1 \rangle_{H^{-1/2}(\gamma_\ell^\varepsilon), H^{1/2}(\gamma_\ell^\varepsilon)} - \langle \partial_n \phi_1, \phi_2 \rangle_{H^{-1/2}(\gamma_\ell^\varepsilon), H^{1/2}(\gamma_\ell^\varepsilon)}.$$

This formula shows in particular that the asymptotics of the bilinear form associated with Λ^ε can be obtained from the asymptotics of ψ^ε (which is given) and $\partial_n \psi^\varepsilon$ on γ_ℓ^ε .

In order to obtain this asymptotics, we use a superposition result, usually referred to as Kirchhoff principle in the context of fluid dynamics. Thus, we have

$$\psi^\varepsilon = u^\varepsilon + \sum_{m=1}^M V_m^\perp \cdot \begin{pmatrix} v_m^\varepsilon \\ w_m^\varepsilon \end{pmatrix},$$

where u^ε satisfies

$$(2.4) \quad \begin{aligned} -\Delta u^\varepsilon &= 0 && \text{in } \mathcal{F}^\varepsilon \\ u^\varepsilon &= a_m^\varepsilon && \text{on } \gamma_m^\varepsilon, (m = 1, \dots, M) \\ u^\varepsilon &= f && \text{on } \Gamma, \end{aligned}$$

and v_m^ε and w_m^ε solve for every $m = 1, \dots, M$ the following boundary value problems

$$\left| \begin{array}{ll} -\Delta v_m^\varepsilon = 0 & \text{in } \mathcal{F}^\varepsilon \\ v_m^\varepsilon = x_1 + b_{m,m}^\varepsilon & \text{on } \gamma_m^\varepsilon \\ v_m^\varepsilon = b_{m,k}^\varepsilon & \text{on } \gamma_k^\varepsilon, k \neq m \\ v_m^\varepsilon = 0 & \text{on } \Gamma, \end{array} \right| \quad \left| \begin{array}{ll} -\Delta w_m^\varepsilon = 0 & \text{in } \mathcal{F}^\varepsilon \\ w_m^\varepsilon = x_2 + c_{m,m}^\varepsilon & \text{on } \gamma_m^\varepsilon \\ w_m^\varepsilon = c_{m,k}^\varepsilon & \text{on } \gamma_k^\varepsilon, k \neq m \\ w_m^\varepsilon = 0 & \text{on } \Gamma, \end{array} \right.$$

where the constants a_m^ε , $b_{m,k}^\varepsilon$ and $c_{m,k}^\varepsilon$ are such that

$$(2.5) \quad \int_{\gamma_m^\varepsilon} \partial_n u^\varepsilon(s) ds = \int_{\gamma_m^\varepsilon} \partial_n v^\varepsilon(s) ds = \int_{\gamma_m^\varepsilon} \partial_n w^\varepsilon(s) ds = 0, \quad m = 1, \dots, M,$$

where these integrals should be understood in the sense of duality

$$\int_{\gamma_m^\varepsilon} \partial_n u^\varepsilon(s) ds = \langle \partial_n u^\varepsilon, 1 \rangle_{H^{-1/2}(\gamma_m^\varepsilon), H^{1/2}(\gamma_m^\varepsilon)}.$$

We detail in §.2.1 the derivation of the asymptotics of u^ε , based on a boundary integral formulation of problem (2.4)-(2.5). Then, as the proofs are quite similar, we only sketch in §.2.2 the proofs for v_m^ε and w_m^ε . These three asymptotic expansions are used to obtain the asymptotics of

$$(2.6) \quad [\psi^\varepsilon, U^g]_{\gamma_\ell^\varepsilon} = [u^\varepsilon, U^g]_{\gamma_\ell^\varepsilon} + \sum_{m=1}^M V_m^\perp \cdot \begin{pmatrix} [v_m^\varepsilon, U^g]_{\gamma_\ell^\varepsilon} \\ [w_m^\varepsilon, U^g]_{\gamma_\ell^\varepsilon} \end{pmatrix},$$

and the result follows then from (2.2). For the sake of clarity and for reader's convenience, we preferred to give a constructive proof of these expansions.

2.1. Asymptotic expansion for the function u^ε . The next result collects well-known properties of the single layer potential that we use in the sequel (see e.g. Mc Lean [25], Rjasanow and Steinbach [27] or Steinbach [28, Chapter 6]).

PROPOSITION 2.2. *For $x \in \mathbb{R}^2 \setminus \{0\}$, let $G(x) := -\frac{1}{2\pi} \log|x|$ denote the Green function of the operator $-\Delta$ in \mathbb{R}^2 . Let γ denote the smooth boundary of a bounded (possibly multiply connected) domain of \mathbb{R}^2 . We introduce the single layer potential associated to a given density p :*

$$(2.7) \quad (\mathcal{S}_\gamma p)(x) := \int_\gamma G(x-y) p(y) d\sigma_y, \quad x \in \mathbb{R}^2 \setminus \gamma.$$

Then, the following assertions hold true:

1. \mathcal{S}_γ defines a continuous operator from $H^{-1/2}(\gamma)$ onto $H^1(\mathbb{R}^2)$.
2. The trace of $\mathcal{S}_\gamma p$ on γ is given by the boundary integral operator

$$(2.8) \quad (\mathcal{S}_\gamma p)(x) := \int_\gamma G(x-y) p(y) d\sigma_y, \quad x \in \gamma.$$

Moreover, \mathcal{S}_γ defines an isomorphism from $H^{-1/2}(\gamma)$ to $H^1(\mathbb{R}^2)$ and the quantity $\|\cdot\|_{-\frac{1}{2}} := \langle \cdot, \mathcal{S}_\gamma \cdot \rangle_{H^{-1/2}(\gamma), H^1(\mathbb{R}^2)}^{1/2}$ defines on $H^{-1/2}(\gamma)$ a norm equivalent to the classical norm.

3. The normal derivative of $\mathcal{S}_\gamma p$ on γ is given by the so-called jump condition

$$(2.9) \quad p = \left[\frac{\partial(\mathcal{S}_\gamma p)}{\partial n} \right] := \frac{\partial(\mathcal{S}_\gamma p)}{\partial n} \Big|_- - \frac{\partial(\mathcal{S}_\gamma p)}{\partial n} \Big|_+$$

in which n denotes the exterior unit normal to γ (pointing from the interior bounded domain inside γ to the unbounded exterior) and the $+$ and $-$ signs refer respectively to the normal derivatives coming from the exterior or the interior of γ .

4. Assume that γ has only one connected component. Then, there exists a unique density $\psi^{\text{eq}} \in H^{-1/2}(\gamma)$, called the equilibrium density of γ , such that $\mathcal{S}_\gamma \psi^{\text{eq}}$ is constant on γ and satisfies the normalization condition $\int_\gamma \psi^{\text{eq}}(y) d\sigma_y = 1$.

We seek u^ε as a single layer potential:

$$(2.10) \quad u^\varepsilon(x) = \sum_{m=1}^M \mathcal{S}_{\gamma_m^\varepsilon} p_m^\varepsilon(x) + \mathcal{S}_\Gamma q^\varepsilon(x), \quad x \in \mathcal{F}^\varepsilon,$$

where p_m^ε and q^ε are densities in $H^{-1/2}(\Gamma)$ and $H^{-1/2}(\gamma_m^\varepsilon)$ to be determined.

REMARK 2. *Since by Green's formula $\int_{\gamma_\ell^\varepsilon} \frac{\partial}{\partial n} (\mathcal{S}_{\gamma_m^\varepsilon} p_m^\varepsilon) \Big|_- = 0$, relation (2.9) shows that: $\int_{\gamma_m^\varepsilon} p_m^\varepsilon(y) d\sigma_y = - \int_{\gamma_m^\varepsilon} \partial_n u^\varepsilon(y) d\sigma_y = 0$ for all $m = 1, \dots, M$.*

Taking into account Remark 2, system (2.4)-(2.5) satisfied by u^ε is equivalent to the following system of $(M+1)$ coupled integral equations on $\gamma_1^\varepsilon, \dots, \gamma_M^\varepsilon$ and Γ :

$$(2.11a) \quad \mathcal{S}_{\gamma_\ell^\varepsilon} p_\ell^\varepsilon + \sum_{\substack{m=1 \\ m \neq \ell}}^M (\mathcal{S}_{\gamma_m^\varepsilon} p_m^\varepsilon) \Big|_{\gamma_\ell^\varepsilon} + (\mathcal{S}_\Gamma q^\varepsilon) \Big|_{\gamma_\ell^\varepsilon} = a_\ell^\varepsilon, \quad \text{on } \gamma_\ell^\varepsilon, \ell = 1, \dots, M,$$

$$(2.11b) \quad \sum_{m=1}^M (\mathcal{S}_{\gamma_m^\varepsilon} p_m^\varepsilon) \Big|_\Gamma + \mathcal{S}_\Gamma q^\varepsilon = f, \quad \text{on } \Gamma,$$

$$(2.11c) \quad \int_{\gamma_\ell^\varepsilon} p_\ell^\varepsilon(y) d\sigma_y = 0, \quad \ell = 1, \dots, M.$$

Defining the curve $\gamma_\ell = r_\ell + \varepsilon^{-1}(\gamma_\ell^\varepsilon - r_\ell)$ and setting $r_{\ell m} = r_\ell - r_m$, a simple rescaling leads us to rewrite (2.11a)-(2.11b) in the equivalent form:

$$(2.12a) \quad G(\varepsilon) \int_{\gamma_\ell} \varepsilon p_\ell^\varepsilon(r_\ell + \varepsilon(y - r_\ell)) d\sigma_y + \int_{\gamma_\ell} G(x - y) \varepsilon p_\ell^\varepsilon(r_\ell + \varepsilon(y - r_\ell)) d\sigma_y \\ + \sum_{\substack{m=1 \\ m \neq \ell}}^M \int_{\gamma_m} G(r_{\ell m} + \varepsilon(x - y - r_{\ell m})) \varepsilon p_m^\varepsilon(r_m + \varepsilon(y - r_m)) d\sigma_y \\ + \int_{\Gamma} G(r_\ell - y + \varepsilon(x - r_\ell)) q^\varepsilon(y) d\sigma_y = a_\ell^\varepsilon, \quad x \in \gamma_\ell,$$

$$(2.12b) \quad \sum_{m=1}^M \int_{\gamma_m} G(x - r_m - \varepsilon(y - r_m)) \varepsilon p_m^\varepsilon(r_m + \varepsilon(y - r_m)) d\sigma_y \\ + \int_{\Gamma} G(x - y) q^\varepsilon(y) d\sigma_y = f(x), \quad x \in \Gamma.$$

The circulation-free condition (2.11c), once the moving disks are rescaled, becomes:

$$(2.13) \quad \int_{\gamma_m} \varepsilon p_m^\varepsilon(r_m + \varepsilon(y - r_m)) d\sigma_y = 0 \quad (m = 1, \dots, M).$$

For every $\varepsilon > 0$, we introduce the following a priori decomposition of the densities:

$$(2.14a) \quad \varepsilon p_\ell^\varepsilon(r_\ell + \varepsilon(y - r_\ell)) = p_{\ell,0}(y) + \varepsilon p_{\ell,1}(y) + \varepsilon^2 \mathcal{P}_\ell^\varepsilon(y), \quad y \in \gamma_\ell,$$

$$(2.14b) \quad q^\varepsilon(y) = q_0(y) + \varepsilon q_1(y) + \varepsilon^2 q_2(y) + \varepsilon^3 \mathcal{Q}^\varepsilon(y), \quad y \in \Gamma.$$

Likewise, we write the constants a_ℓ^ε as:

$$(2.15) \quad a_\ell^\varepsilon = a_{\ell,0} + \varepsilon a_{\ell,1} + \varepsilon^2 \mathcal{A}_\ell^\varepsilon$$

where the appearing in the right hand sides of (2.14)-(2.15) have to be determined. Classically, this can be achieved by plugging the above a priori expansions in (2.12) and identifying the terms of same order in the expansion. More precisely, (2.12) read

$$(2.16a) \quad G(\varepsilon) \int_{\gamma_\ell} [p_{\ell,0}(y) + \varepsilon p_{\ell,1}(y) + \varepsilon^2 \mathcal{P}_\ell^\varepsilon(y)] d\sigma_y \\ + \int_{\gamma_\ell} G(x - y) [p_{\ell,0}(y) + \varepsilon p_{\ell,1}(y) + \varepsilon^2 \mathcal{P}_\ell^\varepsilon(y)] d\sigma_y \\ + \sum_{\substack{m=1 \\ m \neq \ell}}^M \int_{\gamma_m} G(r_{\ell m} + \varepsilon(x - y - r_{\ell m})) [p_{m,0}(y) + \varepsilon p_{m,1}(y) + \varepsilon^2 \mathcal{P}_m^\varepsilon(y)] d\sigma_y \\ + \int_{\Gamma} G(r_\ell - y + \varepsilon(x - r_\ell)) [q_0(y) + \varepsilon q_1(y) + \varepsilon^2 q_2(y) + \varepsilon^3 \mathcal{Q}^\varepsilon(y)] d\sigma_y \\ = a_{\ell,0} + \varepsilon a_{\ell,1} + \varepsilon^2 \mathcal{A}_\ell^\varepsilon, \quad x \in \gamma_\ell, \ell = 1, \dots, M$$

$$(2.16b) \quad \sum_{m=1}^M \int_{\gamma_m} G(x - r_m - \varepsilon(y - r_m)) [p_{m,0}(y) + \varepsilon p_{m,1}(y) + \varepsilon^2 \mathcal{P}_m^\varepsilon(y)] d\sigma_y \\ + \int_{\Gamma} G(x - y) [q_0(y) + \varepsilon q_1(y) + \varepsilon^2 q_2(y) + \varepsilon^3 \mathcal{Q}^\varepsilon(y)] d\sigma_y = f(x), \quad x \in \Gamma,$$

while the circulation free condition (2.13) implies the following conditions:

$$(2.17) \quad \int_{\gamma_\ell} p_{\ell,0}(y) d\sigma_y = \int_{\gamma_\ell} p_{\ell,1}(y) d\sigma_y = \int_{\gamma_\ell} \mathcal{P}_\ell^\varepsilon(y) d\sigma_y = 0, \quad \ell = 1, \dots, M.$$

For every $X, Y \in \mathbb{R}^2$, $X \neq 0$, we have:

$$G(X + \varepsilon Y) = G(X) + \varepsilon \eta_1^\varepsilon(X, Y) = G(X) + \varepsilon \nabla G(X) \cdot Y + \varepsilon^2 \eta_2^\varepsilon(X, Y)$$

where

$$\begin{aligned} \eta_1^\varepsilon(X, Y) &= \int_0^1 \nabla G(X + \varepsilon s Y) \cdot Y ds, \\ \eta_2^\varepsilon(X, Y) &= \int_0^1 (1-s) \langle D^2 G(X + \varepsilon s Y), Y, Y \rangle ds. \end{aligned}$$

Using the above Green's function asymptotics in (2.16) and identifying the terms of order 0 in ε yields

$$(2.18a) \quad \mathcal{S}_{\gamma_\ell} p_{\ell,0}(x) + \mathcal{S}_\Gamma q_0(r_\ell) = a_{\ell,0}, \quad x \in \gamma_\ell, \ell = 1, \dots, M$$

$$(2.18b) \quad \mathcal{S}_\Gamma q_0(x) = f(x), \quad x \in \Gamma.$$

The last equation uniquely determines $q_0 = \mathcal{S}_\Gamma^{-1} f \in H^{-1/2}(\Gamma)$. In particular, recalling the notation U^f introduced in (2.1), we have $\mathcal{S}_\Gamma q_0 = U^f$ in Ω . Choosing $p_{\ell,0} = 0$ and $a_{\ell,0} = U^f(r_\ell)$ for every $\ell = 1, \dots, M$, the first equation in (2.18) is then fulfilled.

Now, we identify the terms of order 1 in ε of System (2.12), and get:

$$(2.19a) \quad \mathcal{S}_{\gamma_\ell} p_{\ell,1}(x) + \mathcal{S}_\Gamma q_1(r_\ell) + \nabla U^f(r_\ell) \cdot (x - r_\ell) = a_{\ell,1}, \quad x \in \gamma_\ell, \ell = 1, \dots, M$$

$$(2.19b) \quad \mathcal{S}_\Gamma q_1(x) = 0, \quad x \in \Gamma.$$

From the last equation, we have $q_1 = 0$. Taking the scalar product of the first equation of (2.19) with the equilibrium density ψ_ℓ^{eq} of γ_ℓ (see Proposition 2.2), we get that: $\nabla U^f(r_\ell) \cdot \left(\int_{\gamma_\ell} (x - r_\ell) \psi_\ell^{\text{eq}}(x) d\sigma_x \right) = a_{\ell,1}$. Since the moment of order 1 of the equilibrium density of a circle coincides with its center (this follows in particular from relation (B.4) in Lemma B.1 of the Appendix, with $V = 0$ and $c = (2\pi R_\ell)^{-1}$), we have $a_{\ell,1} = 0$. The first equation of (2.19) yields then $\mathcal{S}_{\gamma_\ell} p_{\ell,1}(x) = \nabla U^f(r_\ell) \cdot (r_\ell - x)$, and according to (B.4) in Lemma B.1, we have thus $p_{\ell,1}(x) = \frac{2}{R_\ell} \nabla U^f(r_\ell) \cdot (r_\ell - x)$.

Let us consider now the terms of order 2 in (2.12b) (this will be enough to derive the expected asymptotic expansion of Λ^ε). We easily obtain that

$$(2.20) \quad \mathcal{S}_\Gamma q_2(x) = \sum_{m=1}^M \nabla G(x - r_m) \cdot \left(\int_{\gamma_m} (y - r_m) p_{m,1}(y) d\sigma_y \right), \quad x \in \Gamma.$$

Applying once again Lemma B.1, with $V = -\nabla U^f(r_\ell)$ and $c = \nabla U^f(r_\ell) \cdot r_\ell$, it follows easily from (B.4) that for every $m \in \{1, \dots, M\}$:

$$(2.21) \quad \int_{\gamma_m} (y - r_\ell) p_{m,1}(y) d\sigma_y = -2\pi R_m^2 \nabla U^f(r_\ell).$$

Consequently, (2.20) shows that q_2 solves the boundary integral equation

$$(2.22) \quad \mathcal{S}_\Gamma q_2(x) = - \sum_{m=1}^M 2\pi R_m^2 \nabla U^f(r_m) \cdot \nabla G(x - r_m), \quad x \in \Gamma.$$

Let us focus now on the remainders $\mathcal{P}_\ell^\varepsilon \in H^{-1/2}(\gamma_\ell)$, $\mathcal{Q}^\varepsilon \in H^{-1/2}(\Gamma)$ and $\mathcal{A}^\varepsilon \in \mathbb{R}$ appearing in (2.14) and (2.15), and prove that they are bounded. This will provide a justification of the formal a priori expansion of p_ℓ^ε , q^ε and a_ℓ^ε . To do so, we need the following general result, proved in the Appendix.

LEMMA 2.3. *Let Γ_1 and Γ_2 be two rectifiable Jordan curves contained in two open, bounded and non intersecting sets \mathcal{V}_1 and \mathcal{V}_2 . Let $K : (x, y) \in \mathcal{V}_1 \times \mathcal{V}_2 \mapsto K(x, y) \in \mathbb{R}$ be a function of class C^∞ . Then there exists a constant $C > 0$ depending only on \mathcal{V}_1 , \mathcal{V}_2 and $\|K\|_{H^2(\mathcal{V}_1 \times \mathcal{V}_2)}$ such that for all $p \in H^{-1/2}(\Gamma_1)$ and $q \in H^{-1/2}(\Gamma_2)$:*

$$\left| \int_{\Gamma_1} \int_{\Gamma_2} K(x, y) p(x) q(y) d\sigma_y d\sigma_x \right| \leq C \|p\|_{H^{-1/2}(\Gamma_1)} \|q\|_{H^{-1/2}(\Gamma_2)}.$$

We are now in position to prove the following result.

LEMMA 2.4. *The remainders $\mathcal{P}_\ell^\varepsilon \in H^{-1/2}(\gamma_\ell)$, $\mathcal{Q}^\varepsilon \in H^{-1/2}(\Gamma)$ and $\mathcal{A}^\varepsilon \in \mathbb{R}$ appearing in (2.14) and (2.15) are uniformly bounded in ε .*

Proof. On Γ , we have:

$$S_\Gamma \mathcal{Q}^\varepsilon(x) = - \sum_{m=1}^M \int_{\gamma_m} \eta_1^\varepsilon(x - r_m, r_m - y) \mathcal{P}_m^\varepsilon(y) d\sigma_y + \int_{\gamma_m} \eta_2^\varepsilon(x - r_m, r_m - y) p_{m,1}(y) d\sigma_y.$$

Multiplying by \mathcal{Q}^ε , we get (recall that $\|\mathcal{Q}^\varepsilon\|_{-\frac{1}{2}}^2 = \langle \mathcal{Q}^\varepsilon, S_\Gamma \mathcal{Q}^\varepsilon \rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)}$):

$$\begin{aligned} \|\mathcal{Q}^\varepsilon\|_{-\frac{1}{2}}^2 &= - \sum_{m=1}^M \int_{\Gamma} \int_{\gamma_m} \eta_1^\varepsilon(x - r_m, r_m - y) \mathcal{P}_m^\varepsilon(y) \mathcal{Q}^\varepsilon(x) d\sigma_y d\sigma_x \\ &\quad + \int_{\Gamma} \int_{\gamma_m} \eta_2^\varepsilon(x - r_m, r_m - y) p_{m,1}(y) \mathcal{Q}^\varepsilon(x) d\sigma_y d\sigma_x. \end{aligned}$$

Lemma 2.3 implies the existence of a constant $C > 0$ independent of $\varepsilon > 0$ such that

$$(2.23) \quad \|\mathcal{Q}^\varepsilon\|_{-\frac{1}{2}} \leq C \left(1 + \sum_{m=1}^M \|\mathcal{P}_m^\varepsilon\|_{-\frac{1}{2}} \right).$$

On γ_ℓ , we have:

$$\begin{aligned} (2.24) \quad S_{\gamma_\ell} \mathcal{P}_\ell^\varepsilon(x) &+ \sum_{\substack{m=1 \\ m \neq \ell}}^M \int_{\gamma_m} \eta_1^\varepsilon(r_{\ell m}, x - y - r_{\ell m}) p_{m,1}(y) d\sigma_y \\ &+ \varepsilon \int_{\gamma_m} \eta_1^\varepsilon(r_{\ell m}, x - y - r_{\ell m}) \mathcal{P}_m^\varepsilon(y) d\sigma_y + \int_{\Gamma} \eta_2^\varepsilon(r_\ell - y, x - r_\ell) q_0(y) d\sigma_y \\ &+ \int_{\Gamma} G(r_\ell - y + \varepsilon(x - r_\ell)) q_2(y) d\sigma_y + \varepsilon \int_{\Gamma} G(r_\ell - y + \varepsilon(x - r_\ell)) \mathcal{Q}^\varepsilon(y) d\sigma_y = \mathcal{A}_\ell^\varepsilon. \end{aligned}$$

Multiplying by the equilibrium density ψ_ℓ^{eq} of γ_ℓ , integrating on γ_ℓ and applying Lemma 2.3, we get, since $\int_{\gamma_\ell} \mathcal{P}_\ell^\varepsilon(y) d\sigma_y = 0$:

$$(2.25) \quad |\mathcal{A}_\ell^\varepsilon| \leq C \left(1 + \varepsilon \sum_{\substack{m=1 \\ m \neq \ell}}^M \|\mathcal{P}_m^\varepsilon\|_{-\frac{1}{2}} + \varepsilon \|\mathcal{Q}^\varepsilon\|_{-\frac{1}{2}} \right),$$

where the constant $C > 0$ does not depend on ε . Multiplying now (2.24) by $\mathcal{P}_\ell^\varepsilon$, integrating on γ_ℓ and using once again Lemma 2.3, we get:

$$(2.26) \quad \|\mathcal{P}_\ell^\varepsilon\|_{-\frac{1}{2}} \leq C \left(1 + \varepsilon \sum_{\substack{m=1 \\ m \neq \ell}}^M \|\mathcal{P}_m^\varepsilon\|_{-\frac{1}{2}} + \varepsilon \|\mathcal{Q}^\varepsilon\|_{-\frac{1}{2}} \right).$$

Combining the estimates (2.23) and (2.26), we immediately obtain that

$$\|\mathcal{P}_\ell^\varepsilon\|_{-\frac{1}{2}} \leq C \left(1 + \varepsilon \sum_{\substack{m=1 \\ m \neq \ell}}^M \|\mathcal{P}_m^\varepsilon\|_{-\frac{1}{2}} \right),$$

which clearly implies that $\|\mathcal{P}_\ell^\varepsilon\|_{-\frac{1}{2}}$ is uniformly bounded, for every $\ell = 1, \dots, M$. The conclusion of the Lemma follows then from (2.23) and (2.25). \square

Summing up, we have proved the following result:

PROPOSITION 2.5. *With the above notation, we have for every $\ell = 1, \dots, M$:*

$$(2.27a) \quad \varepsilon p_\ell^\varepsilon(r_\ell + \varepsilon(y - r_\ell)) = \varepsilon p_{\ell,1}(y) + \varepsilon^2 \mathcal{P}_\ell^\varepsilon(y), \quad y \in \gamma_\ell,$$

$$(2.27b) \quad q^\varepsilon(y) = q_0(y) + \varepsilon^2 q_2(y) + \varepsilon^3 \mathcal{Q}^\varepsilon(y), \quad y \in \Gamma,$$

$$(2.27c) \quad a_\ell^\varepsilon = a_{\ell,0} + \varepsilon^2 \mathcal{A}_\ell^\varepsilon,$$

where

$$\begin{aligned} a_{\ell,0} &= U^f(r_\ell), & p_{\ell,1} &= \frac{2}{R_\ell} \nabla U^f(r_\ell) \cdot (r_\ell - x), \\ q_0 &= S_\Gamma^{-1} f, & q_2 &= -S_\Gamma^{-1} \left(\sum_{m=1}^M 2\pi R_m^2 \nabla U^f(r_m) \cdot \nabla G(\cdot - r_m) \right), \end{aligned}$$

and where $\|\mathcal{P}_\ell^\varepsilon\|_{H^{-1/2}(\gamma_\ell)}$, $\|\mathcal{Q}^\varepsilon\|_{H^{-1/2}(\Gamma)}$ and $\mathcal{A}_\ell^\varepsilon$ are uniformly bounded in ε .

Now, we can compute the contribution $[u^\varepsilon, U^g]_{\gamma_\ell^\varepsilon}$ to the expansion of Λ^ε (see (2.2) and (2.6)). On γ_ℓ^ε , we have on the one hand $u_\ell^\varepsilon = a_\ell^\varepsilon$ and, on the other hand, $p_\ell^\varepsilon = [\partial_n u^\varepsilon] = \partial_n u^\varepsilon$ (the first equality follows from the jump relation (2.9) and the second one from the fact that the continuous harmonic extension of u^ε inside D_ℓ^ε is the constant function a_ℓ^ε). Using the asymptotic expansions of Proposition 2.5, simple computations lead to the following result.

THEOREM 2.6. *Let $u^\varepsilon \in H^1(\mathcal{F}^\varepsilon)$ be the solution of (2.4)-(2.5). Using the notation (2.3), we have*

$$[u^\varepsilon, U^g]_{\gamma_\ell^\varepsilon} = \varepsilon^2 (2\pi R_\ell^2 \nabla U^f(r_\ell) \cdot \nabla U^g(r_\ell)) + O(\varepsilon^3).$$

2.2. Asymptotic expansion for the functions v_m^ε and w_m^ε . Let us deal with the function v_1^ε , as the other cases (v_m^ε , $m \geq 2$ and w_m^ε , $m = 1, \dots, M$) can be treated in a similar way. For the sake of simplicity, we drop the subscript referring to this disk's number in the proof and we denote merely this function by v^ε . Similarly, we rename D^ε the disk D_1^ε , and γ^ε its boundary, while the other disks are renamed D_m^ε ,

$m = 1, \dots, N := M - 1$, with boundaries γ_m^ε . With this notation, v^ε solves:

$$\begin{aligned} (2.28a) \quad & -\Delta v^\varepsilon = 0 && \text{in } \mathcal{F}^\varepsilon \\ (2.28b) \quad & v^\varepsilon = x_1 + b^\varepsilon && \text{on } \gamma^\varepsilon \\ (2.28c) \quad & v^\varepsilon = b_m^\varepsilon && \text{on } \gamma_m^\varepsilon, m = 1, \dots, N \\ (2.28d) \quad & v^\varepsilon = 0 && \text{on } \Gamma, \end{aligned}$$

where the constants b^ε and b_m^ε , $m = 1, \dots, N$, are such that

$$(2.28e) \quad \int_{\gamma^\varepsilon} \partial_n v^\varepsilon \, d\sigma_y = 0, \quad \int_{\gamma_m^\varepsilon} \partial_n v^\varepsilon \, d\sigma_y = 0.$$

We seek v^ε as a single layer potential: $v^\varepsilon(x) = \mathcal{S}_{\gamma^\varepsilon} p^\varepsilon(x) + \sum_{m=1}^N \mathcal{S}_{\gamma_m^\varepsilon} p_m^\varepsilon(x) + \mathcal{S}_\Gamma q^\varepsilon(x)$, for $x \in \mathcal{F}^\varepsilon$, where $p^\varepsilon \in H^{-1/2}(\gamma^\varepsilon)$, $p_m^\varepsilon \in H^{-1/2}(\gamma_m^\varepsilon)$ and $q^\varepsilon \in H^{-1/2}(\Gamma)$ are to be determined. Problem (2.28) is equivalent to the system of integral equations:

$$\begin{aligned} & \mathcal{S}_{\gamma^\varepsilon} p^\varepsilon(x) + \sum_{m=1}^N (\mathcal{S}_{\gamma_m^\varepsilon} p_m^\varepsilon)_{|\gamma^\varepsilon}(x) + (\mathcal{S}_\Gamma q^\varepsilon)_{|\gamma^\varepsilon}(x) = x_1 + b^\varepsilon, && x \in \gamma^\varepsilon, \\ (\mathcal{S}_{\gamma^\varepsilon} p^\varepsilon)_{|\gamma_\ell^\varepsilon}(x) + \mathcal{S}_{\gamma_\ell^\varepsilon} p_\ell^\varepsilon(x) + \sum_{\substack{m=1 \\ m \neq \ell}}^N (\mathcal{S}_{\gamma_m^\varepsilon} p_m^\varepsilon)_{|\gamma_\ell^\varepsilon}(x) + (\mathcal{S}_\Gamma q^\varepsilon)_{|\gamma_\ell^\varepsilon}(x) &= b_\ell^\varepsilon, && x \in \gamma_\ell^\varepsilon, \\ (\mathcal{S}_{\gamma^\varepsilon} p^\varepsilon)_{|\Gamma}(x) + \sum_{m=1}^N (\mathcal{S}_{\gamma_m^\varepsilon} p_m^\varepsilon)_{|\Gamma}(x) + \mathcal{S}_\Gamma q^\varepsilon(x) &= 0, && x \in \Gamma, \end{aligned}$$

with the constraints (see Remark 2): $\int_{\gamma^\varepsilon} p^\varepsilon(y) \, d\sigma_y = \int_{\gamma_m^\varepsilon} p_m^\varepsilon(y) \, d\sigma_y = 0$ for all $m = 1, \dots, N$. Rescaling the above equations, we obtain with obvious notation:

$$(2.29a) \quad \begin{aligned} & \int_{\gamma} G(x-y) \varepsilon p^\varepsilon(r + \varepsilon(y-r)) \, d\sigma_y \\ & + \sum_{m=1}^N \int_{\gamma_m} G(r-r_m + \varepsilon(x-r-y+r_m)) \varepsilon p_m^\varepsilon(r_m + \varepsilon(y-r_m)) \, d\sigma_y \\ & + \int_{\Gamma} G(r + \varepsilon(x-r) - y) q^\varepsilon(y) \, d\sigma_y = \varepsilon x_1 + b^\varepsilon, \quad x \in \gamma, \end{aligned}$$

$$(2.29b) \quad \begin{aligned} & \int_{\gamma} G(r_\ell - r + \varepsilon(x-r_\ell - y + r)) \varepsilon p^\varepsilon(r + \varepsilon(y-r)) \, d\sigma_y + \int_{\gamma_\ell} G(x-y) \varepsilon p_\ell^\varepsilon(r_\ell + \varepsilon(y-r_\ell)) \, d\sigma_y \\ & + \sum_{\substack{m=1 \\ m \neq \ell}}^N \int_{\gamma} G(r_\ell - r_m + \varepsilon(x-r_\ell - y + r_m)) \varepsilon p_m^\varepsilon(r_m + \varepsilon(y-r_m)) \, d\sigma_y \\ & + \int_{\Gamma} G(r_\ell + \varepsilon(x-r_\ell) - y) q^\varepsilon(y) \, d\sigma_y = b_\ell^\varepsilon, \quad x \in \gamma_\ell, \end{aligned}$$

$$\begin{aligned}
(2.29c) \quad & \int_{\gamma} G(x-r-\varepsilon(y-r))\varepsilon p^\varepsilon(r+\varepsilon(y-r)) \, d\sigma_y \\
& + \sum_{m=1}^N \int_{\gamma_m} G(x-r_m-\varepsilon(y-r_m))\varepsilon p^\varepsilon(r_m+\varepsilon(y-r_m)) \, d\sigma_y \\
& + \int_{\Gamma} G(x-y)q^\varepsilon(y) \, d\sigma_y = 0, \quad x \in \Gamma,
\end{aligned}$$

with the constraints:

$$(2.29d) \quad \int_{\gamma} p_\varepsilon(r+\varepsilon(y-r)) \, d\sigma_y = \int_{\gamma_m} p_\varepsilon^m(r+\varepsilon(y-r)) \, d\sigma_y = 0, \quad m = 1, \dots, N.$$

For all $\ell = 1, \dots, N$, we introduce the formal asymptotic expansions of the quantities:

$$(2.30a) \quad \varepsilon p^\varepsilon(r+\varepsilon(y-r)) = p_0(y) + \varepsilon p_1(y) + \varepsilon^2 \mathcal{P}^\varepsilon(y), \quad y \in \gamma,$$

$$(2.30b) \quad \varepsilon p_\ell^\varepsilon(r+\varepsilon(y-r)) = p_{\ell,0}(y) + \varepsilon p_{\ell,1}(y) + \varepsilon^2 \mathcal{P}_\ell^\varepsilon(y), \quad y \in \gamma_\ell,$$

$$(2.30c) \quad q^\varepsilon(y) = q_0(y) + \varepsilon q_1(y) + \varepsilon^2 q_2(y) + \varepsilon^3 \mathcal{Q}^\varepsilon(y), \quad y \in \Gamma,$$

$$(2.30d) \quad b^\varepsilon = b_0 + \varepsilon b_1 + \varepsilon^2 \mathcal{B}^\varepsilon,$$

$$(2.30e) \quad b_\ell^\varepsilon = b_{\ell,0} + \varepsilon b_{\ell,1} + \varepsilon^2 \mathcal{B}_\ell^\varepsilon.$$

Using these expressions in System (2.29), the order 0 (in ε) reads:

$$\begin{aligned}
\mathcal{S}_\gamma p_0(x) + \mathcal{S}_\Gamma q_0(r) &= b_0, & x \in \gamma, \\
\mathcal{S}_{\gamma_\ell} p_{\ell,0}(x) + \mathcal{S}_\Gamma q_0(r_\ell) &= b_{\ell,0}, & x \in \gamma_\ell, \ell = 1, \dots, N, \\
\mathcal{S}_\Gamma q_0(x) &= 0, & x \in \Gamma,
\end{aligned}$$

which, thanks to (2.29d), leads to $q_0 = p_0 = p_{\ell,0} = 0$ and $b_0 = b_{\ell,0} = 0$. Identifying the terms of order 1 in ε of System (2.29) leads to:

$$\begin{aligned}
\mathcal{S}_\gamma p_1(x) + \mathcal{S}_\Gamma q_1(r) &= x_1 + b_1 & x \in \gamma, \\
\mathcal{S}_{\gamma_\ell} p_{\ell,1}(x) + \mathcal{S}_\Gamma q_1(r_\ell) &= b_{\ell,1} & x \in \gamma_\ell, \ell = 1, \dots, N, \\
\mathcal{S}_\Gamma q_1(x) &= 0 & x \in \Gamma.
\end{aligned}$$

Multiplying the first equation by the ψ_γ^{eq} of γ and integrating over γ , we find that $b_1 = -\int_{\gamma} \psi_\gamma^{\text{eq}}(x) x_1 \, d\sigma_x = -r \cdot e_1$, since the first momentum of the equilibrium density of a circle is its center. From the other equations we deduce that $q_1 = 0$, $p_{\ell,1} = 0$ and $b_{\ell,1} = 0$. Finally, $\mathcal{S}_\gamma p_1(x) = (x-r) \cdot e_1$ for $x \in \gamma$ and according to Lemma B.1, we have: $p_1(x) = (2/R)(x-r) \cdot e_1$. Finally, the identification of the second order terms in ε in equation (2.29c) yields:

$$\nabla G(x-r) \cdot \left(\int_{\gamma} (r-y)p_1(y) \, d\sigma_y \right) + \mathcal{S}_\Gamma q_2(x) = 0, \quad x \in \Gamma.$$

Identity (B.4) of Lemma B.1 shows that $\int_{\gamma} (r-y)p_1(y) \, d\sigma_y = -2\pi R^2 e_1$, and therefore, q_2 is the unique solution of the integral equation

$$\mathcal{S}_\Gamma q_2(x) = 2\pi R^2 \nabla G(x-r) \cdot e_1, \quad x \in \Gamma.$$

The remainders in (2.30) can be shown to be bounded using the same arguments as in section 2.1. Summing up, we have proved the following result:

PROPOSITION 2.7. *With the above notation, we have for every $\ell = 1, \dots, N$:*

$$\begin{aligned} (2.31a) \quad & \varepsilon p^\varepsilon(r + \varepsilon(y - r)) = \varepsilon p_1(y) + \varepsilon^2 \mathcal{P}^\varepsilon(y), & y \in \gamma, \\ (2.31b) \quad & \varepsilon p_\ell^\varepsilon(r + \varepsilon(y - r)) = \varepsilon^2 \mathcal{P}_\ell^\varepsilon(y), & y \in \gamma_\ell, \\ (2.31c) \quad & q^\varepsilon(y) = \varepsilon^2 q_2(y) + \varepsilon^3 \mathcal{Q}^\varepsilon(y), & y \in \Gamma \\ (2.31d) \quad & b^\varepsilon = \varepsilon b_1 + \varepsilon^2 \mathcal{B}^\varepsilon, \\ (2.31e) \quad & b_\ell^\varepsilon = \varepsilon^2 \mathcal{B}_\ell^\varepsilon. \end{aligned}$$

where

$$b_1 = -r \cdot e_1, \quad p_1(x) = \frac{2(x - r) \cdot e_1}{R}, \quad q_2 = \mathbb{S}_\Gamma^{-1} (2\pi R^2 \nabla G(\cdot - r) \cdot e_1),$$

and where the quantities $\|\mathcal{P}^\varepsilon\|_{H^{-1/2}(\gamma)}$, $\|\mathcal{P}_\ell^\varepsilon\|_{H^{-1/2}(\gamma_\ell)}$, $\|\mathcal{Q}^\varepsilon\|_{H^{-1/2}(\Gamma)}$, \mathcal{B}^ε and $\mathcal{B}_\ell^\varepsilon$ are uniformly bounded in ε .

Using these expansions and similar formulae for w_m^ε , one can easily check that the contributions $[v_m^\varepsilon, U^g]_{\gamma_\ell^\varepsilon}$ and $[w_m^\varepsilon, U^g]_{\gamma_\ell^\varepsilon}$ to the expansion of Λ^ε (see (2.2) and (2.6)) read as follows.

THEOREM 2.8. *Let $\ell \in \{1, \dots, M\}$ and let \mathcal{I}_ℓ be defined in (2.3). For all $m \in \{1, \dots, M\}$, we have*

$$V_m^\perp \cdot \begin{pmatrix} [v_m^\varepsilon, U^g]_{\gamma_\ell^\varepsilon} \\ [w_m^\varepsilon, U^g]_{\gamma_\ell^\varepsilon} \end{pmatrix} = \begin{cases} O(\varepsilon^3), & \text{if } m \neq \ell, \\ \varepsilon^2 (2\pi R_\ell^2 \nabla U^g(r_\ell) \cdot V_\ell^\perp) + O(\varepsilon^3), & \text{if } m = \ell. \end{cases}$$

According to (2.2) and to the splitting (2.6), Theorem 2.1 follows then immediately from Theorem 2.6 and Theorem 2.8.

3. Target identification using exterior sources. Our goal is to determine the number M , the positions r_m , the rescaled radii R_m and the velocities V_m of the small moving disks from the knowledge of the bilinear form defined for all $f, g \in H^{1/2}(\Gamma)$ by:

$$(3.1) \quad a(f, g) := \sum_{m=1}^M R_m^2 \nabla U^f(r_m) \cdot \nabla U^g(r_m) + \sum_{m=1}^M R_m^2 \nabla U^g(r_m) \cdot V_m^\perp,$$

Let us start by determining the positions which are the hardest parameters to reconstruct. We first note that we can isolate the first sum in the right hand side of (3.1):

$$(3.2) \quad a_0(f, g) := a(f, g) - a(0, g) = \sum_{m=1}^M R_m^2 \nabla u^f(r_m) \cdot \nabla u^g(r_m),$$

and hence get a measurement in which the unknown velocities V_m do not appear anymore. Actually, $a_0(f, g)$ coincides to the case of motionless targets. Of course, this does not mean that we have to realize two experiments (one with moving targets and the other one with still ones), but simply that the available data (the full DtN map) can be used to obtain the new data $a_0(\cdot, \cdot)$. The next step is to relate these Laplace type data to a Helmholtz type inverse problem. To do so, following an idea

due to Calderón [8], we use a suitably chosen family of exponential type excitations f and test functions g . Next, we make use of the so-called DORT Method to recover the unknown positions r_m . Based on time reversal techniques, this method has been successively used for the detection of well separated point like scatterers from far field measurements, for many wave type systems in acoustics [21, 26, 7] and electromagnetics [4]. The underlying idea of this method is that the eigenfunctions of some finite rank integral operator (the so-called time-reversal operator, that can be computed from the measurements) generate waves that selectively focus on each target. For the elliptic Laplace problem considered here, let us emphasize that time reversal is not performed experimentally, but only used numerically in the reconstruction algorithm through the use of suitably chosen excitations and test functions. Of course, one can also use other reconstruction methods which are classically used for the detection of point-like targets in scattering theory, like the MUSIC algorithm [13, 23, 20].

3.1. Motionless targets. Here, we exclusively use the quantity $a_0(f, g)$ defined by (3.2), which corresponds to the zero velocity case. Following an idea introduced by Calderón [8], let us choose for a every given $\eta \in \mathbb{R}^2$ the excitation data f and the test functions g defined by

$$(3.3) \quad f(x) = e^{i(\eta+i\eta^\perp)\cdot x}, \quad g(x) = e^{i(\eta-i\eta^\perp)\cdot x}, \quad x \in \Gamma.$$

Noting that these expressions define in fact harmonic functions on \mathbb{R}^2 , we have $U^f(x) = f(x)$ and $U^g(x) = g(x)$ and hence

$$a_0(f, g) = - \sum_{m=1}^M 2|\eta|^2 R_m^2 e^{2i\eta\cdot r_m}.$$

Choosing $k > 0$ and $\eta = \frac{1}{2}k(\alpha - \beta)$ where α, β are two given vectors of the unit sphere \mathbf{S} of \mathbb{R}^2 (so that $|\eta|^2 = \frac{1}{2}k^2(1 - \alpha \cdot \beta)$), we can compute from the measurements

$$A(\alpha, \beta) := -\frac{1}{k^2} a_0(f, g) = \sum_{m=1}^M R_m^2 (1 - \alpha \cdot \beta) e^{ik(\alpha-\beta)\cdot r_m}.$$

Define the selfadjoint finite rank integral operator $\mathcal{A} \in \mathcal{L}(\mathbf{S})$ associated with the degenerate kernel $A(\alpha, \beta)$:

$$(\mathcal{A}\varphi)(\beta) := \int_{\mathbf{S}} A(\alpha, \beta)\varphi(\alpha) d\alpha.$$

In order to determine its spectrum and its eigenfunctions, we introduce the following functions of $L^2(\mathbf{S})$ for every $m = 1, \dots, M$:

$$\varphi_{m,0}(\alpha) = \frac{1}{\sqrt{2\pi}} e^{-ik\alpha\cdot r_m} \quad \varphi_{m,1}(\alpha) = \frac{1}{\sqrt{\pi}} \alpha_1 e^{-ik\alpha\cdot r_m} \quad \varphi_{m,2}(\alpha) = \frac{1}{\sqrt{\pi}} \alpha_2 e^{-ik\alpha\cdot r_m}.$$

Then, setting $\lambda_{m,0} = 2\pi R_m^2$ and $\lambda_{m,\ell} = -\pi R_m^2$ for $\ell = 1, 2$, we can write $A(\alpha, \beta)$ in the more compact form $A(\alpha, \beta) = \sum_{m=1}^M \sum_{\ell=1}^3 \lambda_{m,\ell} \overline{\varphi_{m,\ell}(\alpha)} \varphi_{m,\ell}(\beta)$, which yields

$$\mathcal{A}\varphi = \sum_{m=1}^M \sum_{\ell=1}^3 \lambda_{m,\ell} (\varphi, \varphi_{m,\ell})_{L^2(\mathbf{S})} \varphi_{m,\ell}.$$

\mathcal{A} being at most of rank $3M$, it has at most $3M$ non zero eigenvalues. More precisely, for every $m = 1, \dots, M$, $\varphi_{m,0}$, $\varphi_{m,1}$ and $\varphi_{m,2}$ are orthonormal in $L^2(\mathbf{S})$ and

$$(\varphi_{m,\ell}, \varphi_{n,\ell'})_{L^2(\mathbf{S})} = \int_{\mathbf{S}} \psi_{\ell,\ell'}(\alpha) e^{ik\alpha \cdot (r_n - r_m)} d\alpha, \quad 1 \leq m \neq n \leq M,$$

where $\psi_{\ell,\ell'}(\alpha)$, for $\ell, \ell' \in \{0, 1, 2\}$, are given in Table 1 (with $\alpha = (\alpha_1, \alpha_2)$).

(ℓ, ℓ')	$\ell' = 0$	$\ell' = 1$	$\ell' = 2$
$\ell = 0$	1	α_1	α_2
$\ell = 1$	α_1	$(\alpha_1)^2$	$\alpha_1 \alpha_2$
$\ell = 2$	α_2	$\alpha_1 \alpha_2$	$(\alpha_2)^2$

TABLE 1
The values of $\psi_{\ell,\ell'}(\alpha)$.

One can easily check that (see for instance [1, p. 360])

$$(\varphi_{m,0}, \varphi_{n,1})_{L^2(\mathbf{S})} = 2 \int_0^\pi \cos \theta e^{ik|r_m - r_n| \cos \theta} d\theta = -2i\pi J_1(k|r_m - r_n|),$$

where J_n denotes the Bessel function of the first kind of order n , while

$$(\varphi_{m,0}, \varphi_{n,2})_{L^2(\mathbf{S})} = (\varphi_{m,1}, \varphi_{n,2})_{L^2(\mathbf{S})} = 0.$$

This shows that for $1 \leq m \neq n \leq M$, we have (see [1, p. 364])

$$(\varphi_{m,\ell}, \varphi_{n,\ell'})_{L^2(\mathbf{S})} = \mathcal{O}\left((k|r_m - r_n|)^{-1/2}\right), \quad k \rightarrow +\infty.$$

Summing up, this proves that

$$\mathcal{A}\varphi_{m,\ell} = \lambda_{m,\ell} \varphi_{m,\ell} + \mathcal{O}\left((kr_{\min})^{-1/2}\right), \quad k \rightarrow +\infty,$$

where $r_{\min} := \min_{m \neq n} |r_m - r_n|$ is the minimal distance between the solids. In other words, for every $m = 1, \dots, M$, $\varphi_{m,0}$, $\varphi_{m,1}$ and $\varphi_{m,2}$ constitute approximate eigenfunctions of \mathcal{A} as $k \rightarrow +\infty$ (for high frequency and distant disks), the corresponding eigenvalues being respectively $2\pi R_m^2$ (simple eigenvalue) and $-\pi R_m^2$ (with multiplicity two). These approximate eigenfunctions can be used to recover the positions of the disks by constructing the corresponding Herglotz wave

$$(3.4) \quad u_{m,\ell}(x) := \int_{\mathbf{S}} \varphi_{m,\ell}(\alpha) e^{ik\alpha \cdot x} d\alpha, \quad x \in \mathbb{R}^2,$$

we see that

$$u_{m,\ell}(x) = \mathcal{O}\left((k|x - r_m|)^{-1/2}\right), \quad k \rightarrow +\infty, \quad x \neq r_m.$$

Hence, $u_{m,\ell}$ generates a wave that selectively focuses on the target m as $k \rightarrow \infty$ and allows to recover its location. In particular, for $\ell = 0$, we have

$$u_{m,0}(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{S}} e^{ik\alpha \cdot (x - r_m)} d\alpha = 2\pi J_0(k|x - r_m|),$$

and thus, $u_{m,0}$ reaches its maximum exactly at the point $x = r_m$.

3.2. Moving targets. Now that we have determined the positions r_m ($m = 1, \dots, M$) of the solids, it remains to compute their rescaled radii R_m and their velocities V_m . To this purpose, we introduce the complex polynomials:

$$P_m(z) = \prod_{1 \leq \ell \neq m \leq M} (z - r_\ell)^2$$

and the following harmonic functions (we identify x and r_j with their complex representations):

$$g_m^1(x) := \Re(P_m(x)/P'_m(r_m)), \quad g_m^2(x) := \Im(P_m(x)/P'_m(r_m)).$$

These functions, which are well defined since $P'_m(r_m) \neq 0$, enjoy the properties:

$$U^{g_m^j} = g_m^j, \quad \nabla U^{g_m^1}(r_\ell) = \begin{pmatrix} \delta_{m,\ell} \\ 0 \end{pmatrix}, \quad \nabla U^{g_m^2}(r_\ell) = \begin{pmatrix} 0 \\ \delta_{m,\ell} \end{pmatrix},$$

for every $m, \ell = 1, \dots, M$ and $j = 1, 2$ ($\delta_{m,\ell}$ denotes Kronecker's symbol). Substituting these last two relations in (3.1) and (3.2), the expressions of the radii and the velocities follow then easily, as for every $m = 1, \dots, M$:

$$(3.5) \quad R_m = \sqrt{a_0(g_m^1, g_m^1)}, \quad V_m \cdot e_1 = \frac{a(0, g_m^2)}{R_m^2}, \quad V_m \cdot e_2 = -\frac{a(0, g_m^1)}{R_m^2}.$$

4. Numerical tests. In this section, we present some numerical results to illustrate our reconstruction method. The solutions of the boundary value problems involved, in particular for the data generation, are computed using a `Matlab` boundary integral equation solver². As a typical configuration, we consider three small disks located in a (smoothened) rectangular domain Ω . The three small disks are of radii $(m+1)\varepsilon$, where $\varepsilon = 10^{-3}$ and $m = 1, 2, 3$. They are located respectively at the points $(0.7, -0.1)$, $(-0.7, 0.3)$ and $(0, -0.3)$ and their velocities are $(0, 1)$, $(1, 1)$ and $(1, 0)$. As a preliminary step, we first generate numerically the data that will be used for the reconstruction. Given a wavenumber k and a uniform discretization of $[0, 2\pi]$ with mesh size $h = 1/(N+1)$, we compute the matrix $\mathbb{A} = (A_{i,j})_{1 \leq i, j \leq N}$ corresponding to the discretization of the kernel $A(\alpha, \beta)$. More precisely, we have $A_{i,j} = -1/k^2(\Lambda^\varepsilon f_{i,j}, g_{i,j})$, where $f_{i,j}, g_{i,j}$ are the functions obtained in (3.3) for $\eta = k(\alpha_i - \beta_j)/2$, where α_i and β_j belong to the chosen discretization of $[0, 2\pi]$. This requires to solve the forward problem (1.4) for N^2 different right-hand sides $f_{i,j}$, $1 \leq i, j \leq N$. The data are generated using $N = 40$ angles and then perturbed them artificially by adding 10% of noise.

Our DORT based reconstruction procedure is applied to these noisy data. We compute the eigenvalues and eigenfunctions of the discretized integral operator \mathcal{A}_h associated to the kernel \mathbb{A} . As expected from the theoretical analysis of Section 3, it turns out that this integral operator has indeed three significant eigenvalues for a wide range of values of k .

We show on Fig. 2 the dependence of the positive eigenvalues with respect to the wavelength $\lambda = 2\pi/k$ of the oscillating source (without noise). According to this figure, we can recover the number of disks as soon as k is chosen in the shaded region on Fig. 2, roughly corresponding to a constant number of significant eigenvalues. The

²For more information, see: <http://iecl.univ-lorraine.fr/~Alexandre.Munnier/IES/>

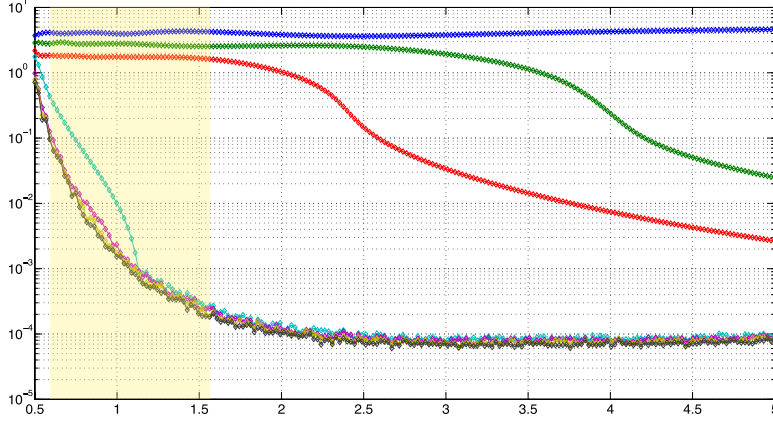


FIG. 2. Dependence of the positive eigenvalues $\lambda_{m,0}$ (in log scale) on the source's wavelength $\lambda = 2\pi/k$. The shaded region corresponds to the region of admissible wavenumbers leading to identifiability of the number of targets.

wavenumber is now chosen to be $k = 10$ which yields accurate location identification. The centers of the disks can be obtained as the points where the Herglotz waves (3.4) generated by the eigenfunctions associated to the largest eigenvalue reach their maximum (in modulus), as shown on Fig. 4. We obtain the following estimated positions $(0.0084, -0.2955)$, $(-0.7075, 0.3099)$ and $(0.7123, -0.1017)$, estimated radii 1.9299, 2.9140, 4.0851 and estimated velocities $(-0.0143, 1.0322)$, $(1.0129, 1.0143)$ and $(0.9569, -0.0594)$. Finally, let us emphasize that although our reconstruction method is theoretically justified only for small disks ($\varepsilon \rightarrow 0$), it turns out to be still efficient numerically for “extended” disks as long as the centers are concerned.

Appendix A. Proof of Lemma 2.3.

Define the maps:

$$\begin{aligned} \mathcal{K} : p \in H^{-1/2}(\Gamma_1) &\longmapsto \int_{\Gamma_1} K(\cdot, y)p(y) \, d\sigma_y \in H^1(\mathcal{V}_2), \\ \mathcal{L} : u \in H^1(\mathcal{V}_2) &\longmapsto \int_{\mathcal{V}_2} \nabla_x K(x, \cdot) \cdot \nabla u(x) + K(x, \cdot)u(x) \, dx \in H^1(\mathcal{V}_1) \end{aligned}$$

and $\mathcal{L}^\dagger := \gamma_{\Gamma_1} \circ \mathcal{L}$, where γ_{Γ_1} is the trace operator from $H^1(\mathcal{V}_1)$ into $H^{1/2}(\Gamma_1)$. Let us verify that \mathcal{L} is continuous. For smooth functions u , we have:

$$\begin{aligned} \|\mathcal{L}u\|_{H^1(\mathcal{V}_1)}^2 &= \int_{\mathcal{V}_1} \left(\int_{\mathcal{V}_2} \nabla_x K(x, y) \cdot \nabla u(x) + K(x, y)u(x) \, dx \right)^2 dy + \\ &\quad \int_{\mathcal{V}_1} \left| \int_{\mathcal{V}_2} D_{xy}^2 K(x, y) \nabla u(x) + \nabla_y K(x, y)u(x) \, dx \right|^2 dy, \end{aligned}$$

where $D_{xy}^2 K := (\partial_{x_i y_j}^2 K)_{1 \leq i, j \leq 2}$. Applying Jensen then Cauchy-Schwarz inequality, we get:

$$\|\mathcal{L}u\|_{H^1(\mathcal{V}_1)} \leq C \|K\|_{H^2(\mathcal{V}_1 \times \mathcal{V}_2)} \|u\|_{H^1(\mathcal{V}_2)},$$

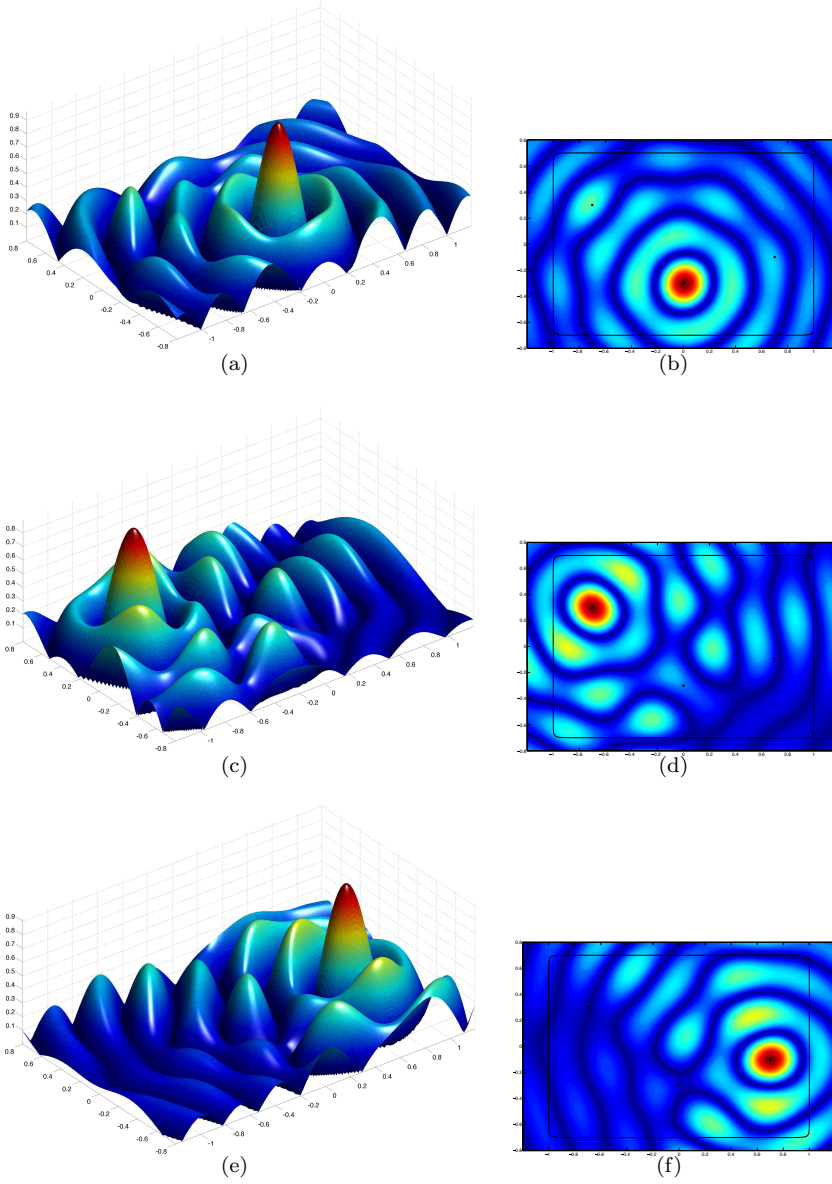


FIG. 3. 3D (left) and 2D (right) visualization of the focusing properties of the eigenfunctions corresponding to the three largest positive eigenvalues for $k = 10.5$. The focusing property leads to an accurate recovery of the centers of the disks.

where C depends only on \mathcal{V}_1 and \mathcal{V}_2 . By density, this estimate remains true for every $u \in H^1(\mathcal{V}_2)$. For smooth p and u , one can easily verify that:

$$(\mathcal{K}p, u)_{H^1(\mathcal{V}_2)} = \langle p, \mathcal{L}^\dagger u \rangle_{H^{-1/2}(\Gamma_1) \times H^{1/2}(\Gamma_1)}.$$

We deduce from the continuity of the trace operator $\gamma_{\Gamma_1} : H^1(\mathcal{V}_1) \rightarrow H^{1/2}(\Gamma_1)$, that:

$$|(\mathcal{K}p, u)_{H^1(\mathcal{V}_2)}| \leq C \|K\|_{H^2(\mathcal{V}_1 \times \mathcal{V}_2)} \|p\|_{H^{-1/2}(\Gamma_1)} \|u\|_{H^1(\mathcal{V}_2)},$$

where $C > 0$ does not depend on p , u and K . This inequality ensures the continuity of \mathcal{K} . We get the conclusion of the lemma by writing that:

$$\int_{\Gamma_2} \int_{\Gamma_1} K(x, y) p(y) q(x) d\sigma_y d\sigma_x = \int_{\Gamma_2} (\gamma_{\Gamma_2} \circ \mathcal{K}p)(x) q(x) d\sigma_x,$$

where $\gamma_{\Gamma_2} : H^1(\mathcal{V}_2) \rightarrow H^{1/2}(\Gamma_2)$ is the continuous trace operator.

Appendix B. Single layer integral equation for a circle.

LEMMA B.1. *Denote by γ the circle of radius R centered at the origin and let be given $c \in \mathbb{R}$ and $V \in \mathbb{R}^2$. Then, the solution of the integral equation*

$$(B.1) \quad \mathcal{S}_\gamma p(x) = c + V \cdot x, \quad \forall x \in \gamma$$

is given by

$$(B.2) \quad p(x) = \frac{c}{R} + \frac{2}{R} V \cdot x, \quad \forall x \in \gamma.$$

The corresponding simple layer potential is

$$(B.3) \quad (\mathcal{S}_\gamma p)(x) = \begin{cases} c + V \cdot x & \text{for } |x| < R \\ c \log\left(\frac{eR}{|x|}\right) + \left(\frac{R}{|x|}\right)^2 V \cdot x & \text{for } |x| > R. \end{cases}$$

Furthermore, we have

$$(B.4) \quad \int_\gamma xp(x) d\sigma_x = 2\pi R^2 V.$$

Proof. One can easily check that the function defined by (B.3) satisfies

$$\begin{aligned} -\Delta\varphi &= 0 && \text{in } \mathbb{R}^2 \setminus \gamma \\ \varphi &= c + V \cdot x && \text{on } \gamma \\ [\varphi] &= 0 && \text{on } \gamma \\ \varphi(x) &= \mathcal{O}(\log|x|) && \text{as } |x| \rightarrow +\infty. \end{aligned}$$

Hence $\varphi = \mathcal{S}_\gamma p$ where p solves (B.1). (B.2) follows then from $p = \left[\frac{\partial\varphi}{\partial n}\right]$ (see Proposition 2.2). Using (B.2), (B.4) follows from a straightforward computation. \square

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