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# DORT method for the detection of small moving disks in a fluid

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## Abstract

We are interested in determining the positions and the velocities of moving rigid solids in a bounded cavity filled with a perfect fluid. We assume that the solids are small disks and that they move slowly. Using an integral formulation, we first derive the asymptotic expansion of the DtN map of the problem as the diameters of the disks tend to zero. Then, combining a suitable choice of exponential type data and the DORT technique (which is usually used in inverse scattering for the detection of point-like scatterers), we propose a reconstruction method for the unknown positions and velocities.

## 1 Introduction

The geometric inverse problem that we consider in this paper is to determine the positions and the velocities of rigid solids moving in a perfect fluid. More precisely, let  $\Omega$  be an open and simply connected bounded domain of  $\mathbb{R}^2$  with smooth boundary  $\Gamma := \partial\Omega$ . The domain  $\Omega$  is supposed to be filled with a perfect fluid and it contains  $M$  rigid solids  $D_m$ ,  $m = 1, \dots, M$ , where  $D_m \subset \Omega$  is a closed disk with boundary  $\gamma_m$ . The domain occupied by the fluid is denoted by  $\mathcal{F} := \Omega \setminus \left(\sum_{m=1}^M D_m\right)$ . We also introduce the unit normal  $n$  to  $\partial\mathcal{F}$  directed towards the exterior of the fluid and  $\tau$  the unit tangent vector to  $\partial\mathcal{F}$  such that  $\tau = n^\perp$  (throughout the paper, we set  $x^\perp := (-x_2, x_1)$  for all  $x = (x_1, x_2) \in \mathbb{R}^2$ ). We assume that every solid moves with a velocity  $V_m \in \mathbb{R}^2$ . At every given time, we can consider the Eulerian velocity field  $\mathbf{U}(x)$ ,  $x \in \mathcal{F}$ , of the fluid. We assume that the flow is irrotational

$$\text{rot}(\mathbf{U}) = 0 \text{ in } \mathcal{F}, \quad (1.1a)$$

and circulation free, so that for every rectifiable Jordan curve  $\gamma \subset \mathcal{F}$ , there holds

$$\int_{\gamma} \mathbf{U}(s) \cdot \tau(s) ds = 0. \quad (1.1b)$$

The slip boundary conditions on  $\partial\mathcal{F}$  read

$$\mathbf{U} \cdot n = V_m \cdot n \quad \text{on } \gamma_m, \quad m = 1, \dots, M \quad (1.2a)$$

$$\mathbf{U} \cdot n = 0 \quad \text{on } \Gamma. \quad (1.2b)$$

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We shall address the inverse problem of determining the positions and the velocities of the solids by using actuators and sensors located on the outer boundary  $\Gamma$ . We assume that we can prescribe the normal velocity of the fluid on  $\Gamma$ , leading to the modification of equation (1.2b) into:

$$\mathbf{U} \cdot n = F \quad \text{on } \Gamma \quad (1.3)$$

where  $F$  is given. Regarding the available output, we assume that we can measure the tangential velocity  $\mathbf{U} \cdot \tau$  on  $\Gamma$ . According to conditions (1.1a) and (1.1b), we classically introduce the stream function  $\psi : \mathcal{F} \rightarrow \mathbb{R}$  such that  $\mathbf{U} = -\nabla^\perp \psi$  in  $\mathcal{F}$ . Then, equations (1.1a) and (1.2) read

$$-\Delta \psi = 0 \quad \text{in } \mathcal{F} \quad (1.4a)$$

$$\psi = V_m^\perp \cdot x + c_m \quad \text{on } \gamma_m, \quad m = 1, \dots, M \quad (1.4b)$$

$$\psi = f \quad \text{on } \Gamma, \quad (1.4c)$$

where  $\partial_\tau f = F$  on  $\Gamma$  and the constants  $c_m \in \mathbb{R}$ ,  $k = 1, \dots, M$ , are such that

$$\int_{\gamma_m} \partial_n \psi(s) \, ds = 0. \quad (1.4d)$$

With these settings, the measurement reads  $\mathbf{U} \cdot \tau = -\partial_n \psi$ . The detection problem under consideration is then to recover the centers, the diameters and the velocities of the solids from the DtN map  $\Lambda : f \in H^{1/2}(\Gamma) \mapsto \partial_n \psi \in H^{-1/2}(\Gamma)$ .

Although such geometric inverse problems of detecting moving solids in a fluid appear in many applications, the associated literature is quite limited, as most contributions deal with the case of motionless solids (i.e. obstacles). In particular, the case of a fluid described by the Stokes equations has been investigated using optimization methods by Caubet *et al.* in [12, 5, 11, 10] and more recently by Bourgeois *et al.* in [6] using the quasi-reversibility method combined to a level set method. The case of small obstacles has been studied in Caubet and Dambrine [9]. Regarding moving obstacles, Conca *et al.* show in [15, 14] that the position and the velocity for a single disk moving in a perfect fluid can be recovered from one measurement of the velocity on part of the boundary. Linear stability estimates are also provided. In Conca *et al.* [16], the authors consider a moving rigid solid immersed in a potential fluid and provide examples of detectable (ellipses for instance) and undetectable shapes. Conca *et al.* obtained in [17] an identifiability result in the case of a rigid solid immersed in a viscous fluid.

In this work, we restrict the analysis to the case where the solids are small disks (see Figure 1) of typical size  $\varepsilon$ , so that rotation plays no role. For such configurations, we show an identifiability result and we provide a reconstruction method. More precisely, we denote each closed disk  $D_m^\varepsilon$ ,  $m = 1, \dots, M$  and we denote by  $\gamma_m^\varepsilon$  its boundary. We assume that  $D_m^\varepsilon$  is centered at  $r_m$  and is of radius  $\varepsilon R_m$ , where the parameter  $\varepsilon$  is meant to tend to 0. Denoting by  $\mathcal{F}^\varepsilon = \Omega \setminus (\cup_{m=1}^M D_m^\varepsilon)$  the domain occupied by the fluid and letting  $\psi^\varepsilon$  be the corresponding solution of (1.4), our inverse problem can be formulated as follows: *Knowing the DtN map*

$$\Lambda^\varepsilon : f \in H^{1/2}(\Gamma) \mapsto \partial_n \psi^\varepsilon \in H^{-1/2}(\Gamma) \quad (1.5)$$

*is it possible, and if so how, to recover the number  $M$  of disks, their positions  $r_m$ ,  $m = 1, \dots, M$ , their (rescaled) radii  $R_m$  and their velocities  $V_m$ ?*

We answer this question in several steps. In Section 2, we first derive the asymptotic expansion of the DtN operator  $\Lambda^\varepsilon$  as  $\varepsilon \rightarrow 0^+$  (Theorem 1) using a boundary integral formulation of the forward

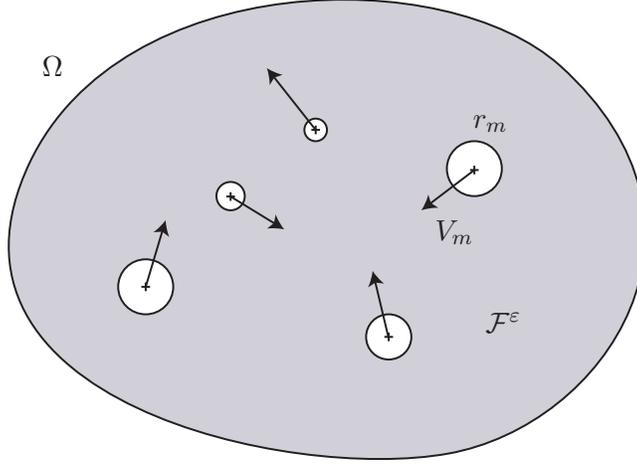


Figure 1: The domain  $\Omega$  containing  $M$  small rigid disks  $D_m^\varepsilon$ ,  $m = 1, \dots, M$  and filled with a perfect fluid occupying the domain  $\mathcal{F}^\varepsilon = \Omega \setminus (\cup_{m=1}^M D_m^\varepsilon)$ .

problem. In Section 3, we combine this expansion with the DORT method<sup>1</sup> to recover the number of disks and their positions, provided they are distant enough. Initially introduced by Fink and Prada [18], this method has been justified mathematically and used in the framework of wave systems for the detection of distant point-like scatterers in acoustics [21, 24, 7] and electromagnetics [4]. Once the positions have been determined, the velocities and rescaled radii can be easily recovered using suitable data. Finally, we collect in Section 4 some numerical examples to illustrate the efficiency of the proposed reconstruction method.

## 2 Asymptotic expansion for small disks

The following result provides the asymptotic expansion of the DtN map  $\Lambda^\varepsilon$  as  $\varepsilon \rightarrow 0^+$ . Although the literature dealing with small inhomogeneities or inclusions is very rich (see for instance the review paper [2] and the book [3] by Ammari and Kang, or the article by Friedman and Vogelius [19]), and as far we know, the needed asymptotics is not available in the framework of our problem.

**Theorem 1** *For every  $f \in H^{1/2}(\Gamma)$ , we denote by  $U^f \in H^1(\Omega)$  the solution of the boundary value problem*

$$\begin{cases} -\Delta U^f = 0 & \text{in } \Omega \\ U^f = f & \text{on } \Gamma \end{cases} \quad (2.1)$$

*Let  $\Lambda^0 \in \mathcal{L}(H^{1/2}(\Gamma), H^{-1/2}(\Gamma))$  the DtN map  $\Lambda^0 : f \in H^{1/2}(\Gamma) \mapsto \partial_n U^f \in H^{-1/2}(\Gamma)$ . Then, the DtN map  $\Lambda^\varepsilon \in \mathcal{L}(H^{1/2}(\Gamma), H^{-1/2}(\Gamma))$  defined by (1.5) admits the following expansion as  $\varepsilon \rightarrow 0^+$ :*

$$\Lambda^\varepsilon = \Lambda^0 + \varepsilon^2 \Lambda^2 + O(\varepsilon^3),$$

*where for every  $f, g \in H^{1/2}(\Gamma)$ :*

$$\langle \Lambda^2 f, g \rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)} = 2\pi \sum_{m=1}^M R_m^2 \nabla U^f(r_m) \cdot \nabla U^g(r_m) - 2\pi \sum_{m=1}^M R_m^2 \nabla U^g(r_m) \cdot V_m^\perp.$$

<sup>1</sup>DORT is the French acronym for Decomposition of the Time-Reversal Operator.

The rest of this section is devoted to the proof of this result. The first ingredient is a superposition result, usually referred to as Kirchhoff principle in the context of fluid dynamics. By linearity, the stream function  $\psi^\varepsilon$  can be decomposed into:

$$\psi^\varepsilon = u^\varepsilon + \sum_{m=1}^M V_m^\perp \cdot \begin{pmatrix} v_m^\varepsilon \\ w_m^\varepsilon \end{pmatrix}, \quad (2.2)$$

where  $u^\varepsilon$  satisfies

$$\begin{aligned} -\Delta u^\varepsilon &= 0 && \text{in } \mathcal{F}^\varepsilon \\ u^\varepsilon &= a_m^\varepsilon && \text{on } \gamma_m^\varepsilon, \quad (m = 1, \dots, M) \\ u^\varepsilon &= f && \text{on } \Gamma, \end{aligned} \quad (2.3)$$

and  $v_m^\varepsilon$  and  $w_m^\varepsilon$  solve for every  $m = 1, \dots, M$  the following boundary value problems

$$\left| \begin{array}{ll} -\Delta v_m^\varepsilon = 0 & \text{in } \mathcal{F}^\varepsilon \\ v_m^\varepsilon = x_1 + b_{m,m}^\varepsilon & \text{on } \gamma_m^\varepsilon \\ v_m^\varepsilon = b_{m,k}^\varepsilon & \text{on } \gamma_k^\varepsilon, \quad k \neq m \\ v_m^\varepsilon = 0 & \text{on } \Gamma, \end{array} \right| \quad \left| \begin{array}{ll} -\Delta w_m^\varepsilon = 0 & \text{in } \mathcal{F}^\varepsilon \\ w_m^\varepsilon = x_2 + c_{m,m}^\varepsilon & \text{on } \gamma_m^\varepsilon \\ w_m^\varepsilon = c_{m,k}^\varepsilon & \text{on } \gamma_k^\varepsilon, \quad k \neq m \\ w_m^\varepsilon = 0 & \text{on } \Gamma, \end{array} \right.$$

where the constants  $a_m^\varepsilon$ ,  $b_{m,k}^\varepsilon$  and  $c_{m,k}^\varepsilon$  are such that

$$\int_{\gamma_m^\varepsilon} \partial_n u^\varepsilon(s) \, ds = \int_{\gamma_m^\varepsilon} \partial_n v^\varepsilon(s) \, ds = \int_{\gamma_m^\varepsilon} \partial_n w^\varepsilon(s) \, ds = 0, \quad m = 1, \dots, M. \quad (2.4)$$

**Remark 1** *With a slight abuse of notation, the boundary integrals above should be understood in the sense of duality between the spaces  $H^{-1/2}$  and  $H^{1/2}$ , e.g.*

$$\int_{\gamma_m^\varepsilon} \partial_n u^\varepsilon(s) \, ds = \langle \partial_n u^\varepsilon, 1 \rangle_{H^{-1/2}(\gamma_m^\varepsilon), H^{1/2}(\gamma_m^\varepsilon)}.$$

The DtN operator is then given by the relation

$$\Lambda^\varepsilon f = \partial_n u^\varepsilon + \sum_{m=1}^M V_m^\perp \cdot \begin{pmatrix} \partial_n v_m^\varepsilon \\ \partial_n w_m^\varepsilon \end{pmatrix}.$$

We first detail in §2.1 the derivation of the asymptotics of  $u^\varepsilon$ , based on a boundary integral formulation of problem (2.3)-(2.4). Then, and since the proofs are quite similar, we only sketch in §2.2 the proofs of the asymptotics of  $v_m^\varepsilon$  and  $w_m^\varepsilon$ . For the sake of clarity and the reader's convenience, we preferred to give a constructive proof of these expansions.

## 2.1 Asymptotic expansion for the function $u^\varepsilon$

The next result collects some well-known properties of the single layer potential that will be used in the sequel (see e.g. Mc Lean [23], Rjasanow and Steinbach [25] or Steinbach [26, Chapter 6]).

**Proposition 1** For  $x \in \mathbb{R}^2 \setminus \{0\}$ , let  $G(x) := -\frac{1}{2\pi} \log|x|$  denote the Green function of the operator  $-\Delta$  in  $\mathbb{R}^2$ . Let  $\gamma$  denote the smooth boundary of a bounded (possibly multiply connected) domain of  $\mathbb{R}^2$ . We introduce the single layer potential associated to a given density  $p$ :

$$(\mathcal{S}_\gamma p)(x) := \int_\gamma G(x-y) p(y) d\sigma_y, \quad x \in \mathbb{R}^2 \setminus \gamma. \quad (2.5)$$

Then, the following assertions hold true:

1.  $\mathcal{S}_\gamma$  defines a continuous operator from  $H^{-1/2}(\gamma)$  onto  $H^1(\mathbb{R}^2)$ .
2. The trace of  $\mathcal{S}_\gamma p$  on  $\gamma$  is given by the boundary integral operator

$$(\mathcal{S}_\gamma p)(x) := \int_\gamma G(x-y) p(y) d\sigma_y, \quad x \in \gamma. \quad (2.6)$$

Moreover,  $\mathcal{S}_\gamma$  defines an isomorphism from  $H^{-1/2}(\gamma)$  to  $H^{1/2}(\gamma)$  and the quantity  $\|\cdot\|_{-\frac{1}{2}} := \langle \cdot, \mathcal{S}_\gamma \cdot \rangle_{H^{-1/2}(\gamma), H^{1/2}(\gamma)}^{1/2}$  defines on  $H^{-1/2}(\gamma)$  a norm equivalent to the classical norm.

3. The normal derivative of  $\mathcal{S}_\gamma p$  on  $\gamma$  is given by the so-called jump condition

$$p = \left[ \frac{\partial(\mathcal{S}_\gamma p)}{\partial n} \right] := \frac{\partial(\mathcal{S}_\gamma p)}{\partial n} \Big|_- - \frac{\partial(\mathcal{S}_\gamma p)}{\partial n} \Big|_+ \quad (2.7)$$

in which  $n$  denotes the unit outgoing normal to  $\gamma$  and the  $+$  and  $-$  signs refer respectively to the normal derivatives coming from the exterior or the interior of  $\gamma$ .

4. Assume that  $\gamma$  has only one connected component. Then, there exists a unique density  $\psi^{\text{eq}} \in H^{-1/2}(\gamma)$ , called the equilibrium density of  $\gamma$ , such that  $\mathcal{S}_\gamma \psi^{\text{eq}}$  is constant on  $\gamma$  and satisfies the normalization condition  $\int_\gamma \psi^{\text{eq}}(y) d\sigma_y = 1$ .

We seek  $u^\varepsilon$  as a single layer potential:

$$u^\varepsilon(x) = \sum_{m=1}^M \mathcal{S}_{\gamma_m^\varepsilon} p_m^\varepsilon(x) + \mathcal{S}_\Gamma q^\varepsilon(x), \quad x \in \mathcal{F}^\varepsilon, \quad (2.8)$$

where  $p_m^\varepsilon$  and  $q^\varepsilon$  are densities in  $H^{-1/2}(\Gamma)$  and  $H^{-1/2}(\gamma_m^\varepsilon)$  to be determined.

**Remark 2** According to (2.7),  $u^\varepsilon$  satisfies for all  $m = 1, \dots, M$ :

$$\int_{\gamma_m^\varepsilon} p_m^\varepsilon(y) d\sigma_y = - \int_{\gamma_m^\varepsilon} \partial_n u^\varepsilon(y) d\sigma_y = 0,$$

since by Green formula  $\int_{\gamma_m^\varepsilon} \frac{\partial}{\partial n} (\mathcal{S}_{\gamma_m^\varepsilon} p_m^\varepsilon) \Big|_- = 0$ .

Taking into account Remark 2, system (2.3)-(2.4) satisfied by  $u^\varepsilon$  is equivalent to the following system of  $(M + 1)$  coupled integral equations on  $\gamma_1^\varepsilon, \dots, \gamma_M^\varepsilon$  and  $\Gamma$ :

$$\mathcal{S}_{\gamma_\ell^\varepsilon} p_\ell^\varepsilon + \sum_{\substack{m=1 \\ m \neq \ell}}^M (\mathcal{S}_{\gamma_m^\varepsilon} p_m^\varepsilon)|_{\gamma_\ell^\varepsilon} + (\mathcal{S}_\Gamma q^\varepsilon)|_{\gamma_\ell^\varepsilon} = a_\ell^\varepsilon, \quad \text{on } \gamma_\ell^\varepsilon, \ell = 1, \dots, M, \quad (2.9a)$$

$$\sum_{m=1}^M (\mathcal{S}_{\gamma_m^\varepsilon} p_m^\varepsilon)|_\Gamma + \mathcal{S}_\Gamma q^\varepsilon = f, \quad \text{on } \Gamma, \quad (2.9b)$$

$$\int_{\gamma_\ell^\varepsilon} p_\ell^\varepsilon(y) d\sigma_y = 0, \quad \ell = 1, \dots, M. \quad (2.9c)$$

Defining the curve  $\gamma_\ell = r_\ell + \varepsilon^{-1}(\gamma_\ell^\varepsilon - r_\ell)$  and setting  $r_{\ell m} = r_\ell - r_m$ , a simple rescaling leads us to rewrite (2.9a)-(2.9b) in the equivalent form:

$$\begin{aligned} G(\varepsilon) \int_{\gamma_\ell} \varepsilon p_\ell^\varepsilon(r_\ell + \varepsilon(y - r_\ell)) d\sigma_y + \int_{\gamma_\ell} G(x - y) \varepsilon p_\ell^\varepsilon(r_\ell + \varepsilon(y - r_\ell)) d\sigma_y \\ + \sum_{\substack{m=1 \\ m \neq \ell}}^M \int_{\gamma_m} G(r_{\ell m} + \varepsilon(x - y - r_{\ell m})) \varepsilon p_m^\varepsilon(r_m + \varepsilon(y - r_m)) d\sigma_y \\ + \int_\Gamma G(r_\ell - y + \varepsilon(x - r_\ell)) q^\varepsilon(y) d\sigma_y = a_\ell^\varepsilon, \quad x \in \gamma_\ell, \end{aligned} \quad (2.10a)$$

$$\begin{aligned} \sum_{m=1}^M \int_{\gamma_m} G(x - r_m - \varepsilon(y - r_m)) \varepsilon p_m^\varepsilon(r_m + \varepsilon(y - r_m)) d\sigma_y \\ + \int_\Gamma G(x - y) q^\varepsilon(y) d\sigma_y = f(x), \quad x \in \Gamma. \end{aligned} \quad (2.10b)$$

The circulation-free condition (2.9c), once the moving disks are rescaled, becomes:

$$\int_{\gamma_m} \varepsilon p_m^\varepsilon(r_m + \varepsilon(y - r_m)) d\sigma_y = 0 \quad (m = 1, \dots, M). \quad (2.11)$$

For every  $\varepsilon > 0$ , we introduce the following a priori decomposition of the densities:

$$\varepsilon p_\ell^\varepsilon(r_\ell + \varepsilon(y - r_\ell)) = p_{\ell,0}(y) + \varepsilon p_{\ell,1}(y) + \varepsilon^2 \mathcal{P}_\ell^\varepsilon(y), \quad y \in \gamma_\ell, \quad (2.12a)$$

$$q^\varepsilon(y) = q_0(y) + \varepsilon q_1(y) + \varepsilon^2 q_2(y) + \varepsilon^3 \mathcal{Q}^\varepsilon(y), \quad y \in \Gamma. \quad (2.12b)$$

Likewise, we write the constants  $a_\ell^\varepsilon$  as:

$$a_\ell^\varepsilon = a_{\ell,0} + \varepsilon a_{\ell,1} + \varepsilon^2 \mathcal{A}_\ell^\varepsilon \quad (2.13)$$

where all of the terms arising in the right hand sides of (2.12) and (2.13) have to be determined. Classically, this can be achieved by plugging the above a priori expansions in the system of integral

equations (2.10) and identifying the terms of same order in the expansion. More precisely, equations (2.10) read

$$\begin{aligned}
G(\varepsilon) & \int_{\gamma_\ell} [p_{\ell,0}(y) + \varepsilon p_{\ell,1}(y) + \varepsilon^2 \mathcal{P}_\ell^\varepsilon(y)] d\sigma_y \\
& + \int_{\gamma_\ell} G(x-y) [p_{\ell,0}(y) + \varepsilon p_{\ell,1}(y) + \varepsilon^2 \mathcal{P}_\ell^\varepsilon(y)] d\sigma_y \\
& + \sum_{\substack{m=1 \\ m \neq \ell}}^M \int_{\gamma_m} G(r_{\ell m} + \varepsilon(x-y-r_{\ell m})) [p_{m,0}(y) + \varepsilon p_{m,1}(y) + \varepsilon^2 \mathcal{P}_m^\varepsilon(y)] d\sigma_y \\
& + \int_{\Gamma} G(r_\ell - y + \varepsilon(x-r_\ell)) [q_0(y) + \varepsilon q_1(y) + \varepsilon^2 q_2(y) + \varepsilon^3 \mathcal{Q}^\varepsilon(y)] d\sigma_y \\
& = a_{\ell,0} + \varepsilon a_{\ell,1} + \varepsilon^2 \mathcal{A}_\ell^\varepsilon, \quad x \in \gamma_\ell, \ell = 1, \dots, M \quad (2.14a)
\end{aligned}$$

$$\begin{aligned}
& \sum_{m=1}^M \int_{\gamma_m} G(x-r_m - \varepsilon(y-r_m)) [p_{m,0}(y) + \varepsilon p_{m,1}(y) + \varepsilon^2 \mathcal{P}_m^\varepsilon(y)] d\sigma_y \\
& + \int_{\Gamma} G(x-y) [q_0(y) + \varepsilon q_1(y) + \varepsilon^2 q_2(y) + \varepsilon^3 \mathcal{Q}^\varepsilon(y)] d\sigma_y = f(x), \quad x \in \Gamma, \quad (2.14b)
\end{aligned}$$

while the circulation free condition (2.11) implies the following conditions on the different terms of the asymptotic expansion (2.12) of  $p_\ell^\varepsilon$ :

$$\int_{\gamma_\ell} p_{\ell,0}(y) d\sigma_y = \int_{\gamma_\ell} p_{\ell,1}(y) d\sigma_y = \int_{\gamma_\ell} \mathcal{P}_\ell^\varepsilon(y) d\sigma_y = 0, \quad \ell = 1, \dots, M. \quad (2.15)$$

For every  $X, Y \in \mathbb{R}^2$ ,  $X \neq 0$ , we have:

$$G(X + \varepsilon Y) = G(X) + \varepsilon \eta_1^\varepsilon(X, Y) = G(X) + \varepsilon \nabla G(X) \cdot Y + \varepsilon^2 \eta_2^\varepsilon(X, Y)$$

where

$$\begin{aligned}
\eta_1^\varepsilon(X, Y) & = \int_0^1 \nabla G(X + \varepsilon s Y) \cdot Y ds, \\
\eta_2^\varepsilon(X, Y) & = \int_0^1 (1-s) \langle D^2 G(X + \varepsilon s Y), Y, Y \rangle ds.
\end{aligned}$$

Using the above Green's function asymptotics in (2.14) and identifying the terms of order 0 in  $\varepsilon$  yields

$$\mathcal{S}_{\gamma_\ell} p_{\ell,0}(x) + \mathcal{S}_\Gamma q_0(r_\ell) = a_{\ell,0}, \quad x \in \gamma_\ell, \ell = 1, \dots, M \quad (2.16a)$$

$$\mathcal{S}_\Gamma q_0(x) = f(x), \quad x \in \Gamma. \quad (2.16b)$$

The last equation uniquely determines  $q_0 = \mathcal{S}_\Gamma^{-1} f \in H^{-1/2}(\Gamma)$ . In particular, recalling the notation  $U^f$  introduced in (2.1), we have

$$\mathcal{S}_\Gamma q_0(x) = U^f(x), \quad \forall x \in \Omega.$$

Choosing  $p_{\ell,0} = 0$  and  $a_{\ell,0} = U^f(r_\ell)$  for every  $\ell = 1, \dots, M$ , the first equation in (2.16) is then fulfilled.

We move now to identify the terms of order 1 in  $\varepsilon$  of System (2.10), which yields:

$$\mathbf{S}_{\gamma_\ell} p_{\ell,1}(x) + \mathbf{S}_\Gamma q_1(r_\ell) + \nabla U^f(r_\ell) \cdot (x - r_\ell) = a_{\ell,1}, \quad x \in \gamma_\ell, \ell = 1, \dots, M \quad (2.17a)$$

$$\mathbf{S}_\Gamma q_1(x) = 0, \quad x \in \Gamma. \quad (2.17b)$$

From the last equation,  $q_1 = 0$ . Taking the scalar product of the first equation of (2.17) with the equilibrium density  $\psi_\ell^{\text{eq}}$  of  $\gamma_\ell$  (see Proposition 1), we get

$$\nabla U^f(r_\ell) \cdot \left( \int_{\gamma_\ell} (x - r_\ell) \psi_\ell^{\text{eq}}(x) d\sigma_x \right) = a_{\ell,1}.$$

But since the moment of order 1 of the equilibrium density of a circle coincides with its center (this follows in particular from relation (5.4) in Lemma 3, with  $V = 0$  and  $c = (2\pi R_\ell)^{-1}$ ), the last integral is zero and thus  $a_{\ell,1} = 0$ . According to the first equation in (2.17), the density  $p_{\ell,1} \in H^{-1/2}(\gamma_\ell)$  is thus uniquely determined by the equation

$$\mathbf{S}_{\gamma_\ell} p_{\ell,1}(x) = \nabla U^f(r_\ell) \cdot (r_\ell - x), \quad x \in \gamma_\ell, \ell = 1, \dots, M. \quad (2.18)$$

Let us consider now the terms of order 2 in (2.10b) (and this will be enough to derive the expected asymptotic expansion of  $\Lambda^\varepsilon$ ). We easily obtain that

$$\mathbf{S}_\Gamma q_2(x) = \sum_{m=1}^M \nabla G(x - r_m) \cdot \left( \int_{\gamma_m} (y - r_m) p_{m,1}(y) d\sigma_y \right), \quad x \in \Gamma. \quad (2.19)$$

Since  $\gamma_m$  is a circle, the integrals in the right hand side can be explicitly computed by applying Lemma 3 of the Appendix 5.2, with  $V = -\nabla U^f(r_\ell)$  and  $c = \nabla U^f(r_\ell) \cdot r_\ell$ . More precisely, it follows easily from identity (5.4) that for every  $m \in \{1, \dots, M\}$ :

$$\int_{\gamma_m} (y - r_\ell) p_{m,1}(y) d\sigma_y = -2\pi R_m^2 \nabla U^f(r_\ell). \quad (2.20)$$

Consequently, (2.19) shows that  $q_2$  solves the boundary integral equation

$$\mathbf{S}_\Gamma q_2(x) = - \sum_{m=1}^M 2\pi R_m^2 \nabla U^f(r_m) \cdot \nabla G(x - r_m), \quad x \in \Gamma. \quad (2.21)$$

Let us focus now on the remainders  $\mathcal{P}_\ell^\varepsilon \in H^{-1/2}(\gamma_\ell)$ ,  $\mathcal{Q}^\varepsilon \in H^{-1/2}(\Gamma)$  and  $\mathcal{A}^\varepsilon \in \mathbb{R}$  appearing in (2.12) and (2.13), and prove that they are bounded. This will provide a justification of the formal a priori expansion of the densities  $p_\ell^\varepsilon$ ,  $q^\varepsilon$  and the constants  $a_\ell^\varepsilon$ . We will need the following general result, proved in the Appendix.

**Lemma 1** *Let  $\Gamma_1$  and  $\Gamma_2$  be two rectifiable Jordan curves contained in two open, bounded and non intersecting sets  $\mathcal{V}_1$  and  $\mathcal{V}_2$ . Let  $K : (x, y) \in \mathcal{V}_1 \times \mathcal{V}_2 \mapsto K(x, y) \in \mathbb{R}$  be a function of class  $C^\infty$ . Then there exists a constant  $C > 0$  such that:*

$$\left| \int_{\Gamma_1} \int_{\Gamma_2} K(x, y) p(x) q(y) d\sigma_y d\sigma_x \right| \leq C \|p\|_{H^{-1/2}(\Gamma_1)} \|q\|_{H^{-1/2}(\Gamma_2)},$$

for every  $p \in H^{-1/2}(\Gamma_1)$  and  $q \in H^{-1/2}(\Gamma_2)$ . Moreover,  $C$  depends only on  $\mathcal{V}_1$ ,  $\mathcal{V}_2$  and  $\|K\|_{H^2(\mathcal{V}_1 \times \mathcal{V}_2)}$ .

We are now in position to prove the following result.

**Lemma 2** *The remainders  $\mathcal{P}_\ell^\varepsilon \in H^{-1/2}(\gamma_\ell)$ ,  $\mathcal{Q}^\varepsilon \in H^{-1/2}(\Gamma)$  and  $\mathcal{A}^\varepsilon \in \mathbb{R}$  are uniformly bounded in  $\varepsilon$ .*

*Proof.* On  $\Gamma$ , we have:

$$S_\Gamma \mathcal{Q}^\varepsilon(x) = - \sum_{m=1}^M \int_{\gamma_m} \eta_1^\varepsilon(x - r_m, r_m - y) \mathcal{P}_m^\varepsilon(y) d\sigma_y + \int_{\gamma_m} \eta_2^\varepsilon(x - r_m, r_m - y) p_{m,1}(y) d\sigma_y.$$

Multiplying by  $\mathcal{Q}^\varepsilon$ , we get (recall that  $\|\mathcal{Q}^\varepsilon\|_{-\frac{1}{2}}^2 = \langle \mathcal{Q}^\varepsilon, S_\Gamma \mathcal{Q}^\varepsilon \rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)}$ ):

$$\begin{aligned} \|\mathcal{Q}^\varepsilon\|_{-\frac{1}{2}}^2 &= - \sum_{m=1}^M \int_\Gamma \int_{\gamma_m} \eta_1^\varepsilon(x - r_m, r_m - y) \mathcal{P}_m^\varepsilon(y) \mathcal{Q}^\varepsilon(x) d\sigma_y d\sigma_x \\ &\quad + \int_\Gamma \int_{\gamma_m} \eta_2^\varepsilon(x - r_m, r_m - y) p_{m,1}(y) \mathcal{Q}^\varepsilon(x) d\sigma_y d\sigma_x. \end{aligned}$$

Lemma 1 implies the existence of a constant  $C > 0$  independent of  $\varepsilon > 0$  such that

$$\|\mathcal{Q}^\varepsilon\|_{-\frac{1}{2}} \leq C \left( 1 + \sum_{m=1}^M \|\mathcal{P}_m^\varepsilon\|_{-\frac{1}{2}} \right). \quad (2.22)$$

On  $\gamma_\ell$ , we have:

$$\begin{aligned} S_{\gamma_\ell} \mathcal{P}_\ell^\varepsilon(x) &+ \sum_{\substack{m=1 \\ m \neq \ell}}^M \int_{\gamma_m} \eta_1^\varepsilon(r_{\ell m}, x - y - r_{\ell m}) p_{m,1}(y) d\sigma_y \\ &+ \varepsilon \int_{\gamma_m} \eta_1^\varepsilon(r_{\ell m}, x - y - r_{\ell m}) \mathcal{P}_m^\varepsilon(y) d\sigma_y + \int_\Gamma \eta_2^\varepsilon(r_\ell - y, x - r_\ell) q_0(y) d\sigma_y \\ &+ \int_\Gamma G(r_\ell - y + \varepsilon(x - r_\ell)) q_2(y) d\sigma_y + \varepsilon \int_\Gamma G(r_\ell - y + \varepsilon(x - r_\ell)) \mathcal{Q}^\varepsilon(y) d\sigma_y = \mathcal{A}_\ell^\varepsilon. \end{aligned} \quad (2.23)$$

Multiplying by the equilibrium density  $\psi_\ell^{\text{eq}}$  of  $\gamma_\ell$ , integrating on  $\gamma_\ell$  and applying Lemma 1, we get, since  $\int_\gamma \mathcal{P}_\ell^\varepsilon(y) d\sigma_y = 0$ :

$$|\mathcal{A}_\ell^\varepsilon| \leq C \left( 1 + \varepsilon \sum_{\substack{m=1 \\ m \neq \ell}}^M \|\mathcal{P}_m^\varepsilon\|_{-\frac{1}{2}} + \varepsilon \|\mathcal{Q}^\varepsilon\|_{-\frac{1}{2}} \right), \quad (2.24)$$

where the constant  $C > 0$  does not depend on  $\varepsilon$ . Multiplying now (2.23) by  $\mathcal{P}_\ell^\varepsilon$ , integrating on  $\gamma_\ell$  and using once again Lemma 1, we get:

$$\|\mathcal{P}_\ell^\varepsilon\|_{-\frac{1}{2}} \leq C \left( 1 + \varepsilon \sum_{\substack{m=1 \\ m \neq \ell}}^M \|\mathcal{P}_m^\varepsilon\|_{-\frac{1}{2}} + \varepsilon \|\mathcal{Q}^\varepsilon\|_{-\frac{1}{2}} \right). \quad (2.25)$$

Combining the estimates (2.22) and (2.25), we immediately obtain that

$$\|\mathcal{P}_\ell^\varepsilon\|_{-\frac{1}{2}} \leq C \left( 1 + \varepsilon \sum_{\substack{m=1 \\ m \neq \ell}}^M \|\mathcal{P}_m^\varepsilon\|_{-\frac{1}{2}} \right),$$

which clearly implies that  $\|\mathcal{P}_\ell^\varepsilon\|_{-\frac{1}{2}}$  is uniformly bounded, for every  $\ell = 1, \dots, M$ . The conclusion of the Lemma follows then from (2.22), (2.24).  $\square$

Summing up, we have proved the following result:

**Proposition 2** *With the above notation, we have for every  $\ell = 1, \dots, M$ :*

$$\varepsilon p_\ell^\varepsilon(r_\ell + \varepsilon(y - r_\ell)) = \varepsilon p_{\ell,1}(y) + \varepsilon^2 \mathcal{P}_\ell^\varepsilon(y), \quad y \in \gamma_\ell, \quad (2.26a)$$

$$q^\varepsilon(y) = q_0(y) + \varepsilon^2 q_2(y) + \varepsilon^3 \mathcal{Q}^\varepsilon(y), \quad y \in \Gamma, \quad (2.26b)$$

$$a_\ell^\varepsilon = a_{\ell,0} + \varepsilon^2 \mathcal{A}_\ell^\varepsilon, \quad (2.26c)$$

where

$$\begin{aligned} a_{\ell,0} &= U^f(r_\ell) & p_{\ell,1} &= \mathcal{S}_{\gamma_\ell}^{-1} \left( \nabla U^f(r_\ell) \cdot (r_\ell - \cdot) \right) \\ q_0 &= \mathcal{S}_\Gamma^{-1} f & q_2 &= -\mathcal{S}_\Gamma^{-1} \left( \sum_{m=1}^M 2\pi R_m^2 \nabla U^f(r_m) \cdot \nabla G(\cdot - r_m) \right) \end{aligned}$$

and where  $\|\mathcal{P}_\ell^\varepsilon\|_{H^{-1/2}(\gamma_\ell)}$ ,  $\|\mathcal{Q}^\varepsilon\|_{H^{-1/2}(\Gamma)}$  and  $\mathcal{A}_\ell^\varepsilon$  are uniformly bounded in  $\varepsilon$ .

We are now able to derive the asymptotics of  $\partial_n u^\varepsilon$  on the outer boundary  $\Gamma$ . To achieve this, we first compute the normal derivative on  $\Gamma$  of  $u^\varepsilon$  from formula (2.8) and get

$$\partial_n u^\varepsilon(x) = \frac{1}{2} q^\varepsilon(x) + \int_\Gamma \nabla G(x - y) \cdot n(x) q^\varepsilon(y) d\sigma_y + \sum_{\ell=1}^M \int_{\gamma_\ell^\varepsilon} \nabla G(x - y) \cdot n(x) p_\ell^\varepsilon(y) d\sigma_y.$$

A change of variables in the integrals on  $\gamma_\ell^\varepsilon$  yields

$$\begin{aligned} \partial_n u^\varepsilon(x) &= \frac{1}{2} q^\varepsilon(x) + \int_\Gamma \nabla G(x - y) \cdot n(x) q^\varepsilon(y) d\sigma_y \\ &\quad + \sum_{\ell=1}^M \int_{\gamma_\ell} \nabla G(x - r_\ell - \varepsilon(y - r_\ell)) \cdot n(x) \varepsilon p_\ell^\varepsilon(r_\ell + \varepsilon(y - r_\ell)) d\sigma_y, \quad x \in \Gamma. \end{aligned} \quad (2.27)$$

Using the expansions (2.26), we get the following asymptotic expansion for  $\partial_n u^\varepsilon$ :

$$\partial_n u^\varepsilon = \partial_n U^f + \varepsilon^2 \partial_n u_2 + \varepsilon^3 \partial_n \mathcal{U}^\varepsilon \quad (2.28)$$

where

$$\begin{aligned} u_2(x) &= \mathcal{S}_\Gamma q_2(x) - \sum_{\ell=1}^M \nabla G(x - r_\ell) \cdot \int_{\gamma_\ell} (y - r_\ell) p_{\ell,1}(y) d\sigma_y \\ \mathcal{U}^\varepsilon(x) &:= \mathcal{S}_\Gamma \mathcal{Q}^\varepsilon(x) + \sum_{\ell=1}^M \int_{\gamma_\ell} \eta_1^\varepsilon(x - r_\ell, r_\ell - y) \mathcal{P}_\ell^\varepsilon(y) d\sigma_y \\ &\quad + \sum_{\ell=1}^M \int_{\gamma_\ell} \eta_2^\varepsilon(x - r_\ell, r_\ell - y) p_{\ell,1}(y) d\sigma_y. \end{aligned}$$

Thanks to (2.20), we can rewrite  $u_2(x)$  as:

$$u_2(x) = \mathcal{S}_\Gamma q_2(x) + \sum_{\ell=1}^M 2\pi R_\ell^2 \nabla G(x - r_\ell) \cdot \nabla U^f(r_\ell).$$

Therefore, we have :  $\Delta u_2 = -\sum_{\ell=1}^M 2\pi R_\ell^2 \nabla U^f(r_\ell) \cdot \nabla \delta_{r_\ell}$  in  $\Omega$ . On the other hand, equation (2.21) shows that  $u_2 = 0$  on  $\Gamma$ . We note that for every  $g \in H^{1/2}(\Gamma)$ , since  $\Delta U^g = 0$  in  $\Omega$  and since  $u_2 = 0$  on  $\Gamma$ , Green's formula yields (a rigorous justification of the formal derivation below is classical, and follows from the introduction of small balls around the singularities  $r_m$  and letting their diameters go to 0):

$$\langle \partial_n u_2, g \rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)} = \int_\Omega \Delta u_2(x) U^g(x) dx = \sum_{m=1}^M 2\pi R_m^2 \nabla U^f(r_m) \cdot \nabla U^g(r_m).$$

Plugging the last relation in (2.28), and noting that  $U^\varepsilon$  is uniformly (in  $\varepsilon$ ) bounded in  $H^1(\mathcal{V})$  where  $\mathcal{V}$  is any open bounded set containing  $\Gamma$  and far enough from the  $r_m$  ( $m = 1, \dots, M$ ) (and thus its normal trace on  $\Gamma$  is uniformly bounded in  $H^{-1/2}(\Gamma)$ ), we have proved the following result:

**Theorem 2** *Let  $u^\varepsilon \in H^1(\mathcal{F}^\varepsilon)$  be the solution of (2.3)-(2.4). Then, with the notation of Theorem 1, we have for every  $g \in H^{1/2}(\Gamma)$ :*

$$\langle \partial_n u^\varepsilon, g \rangle = \langle \partial_n U^f, g \rangle + \varepsilon^2 \sum_{m=1}^M 2\pi R_m^2 \nabla U^f(r_m) \cdot \nabla U^g(r_m) + O(\varepsilon^3).$$

## 2.2 Asymptotic expansion for the functions $v_m^\varepsilon$ and $w_m^\varepsilon$

Let us deal with the function  $v_1^\varepsilon$ , as the other cases ( $v_m^\varepsilon$ ,  $m \geq 2$  and  $w_m^\varepsilon$ ,  $m = 1, \dots, M$ ) can be treated in a similar way. For the sake of simplicity, we drop the subscript referring to this disk's number in the proof and we denote merely this function by  $v^\varepsilon$ . Similarly, we rename  $D^\varepsilon$  the disk  $D_1^\varepsilon$ , and  $\gamma^\varepsilon$  its boundary, while the other disks are renamed  $D_m^\varepsilon$ ,  $m = 1, \dots, N := M - 1$ , with boundaries  $\gamma_m^\varepsilon$ . With this notation, the function  $v^\varepsilon$  solves:

$$-\Delta v^\varepsilon = 0 \quad \text{in } \mathcal{F}^\varepsilon \tag{2.29a}$$

$$v^\varepsilon = x_1 + b^\varepsilon \quad \text{on } \gamma^\varepsilon \tag{2.29b}$$

$$v^\varepsilon = b_m^\varepsilon \quad \text{on } \gamma_m^\varepsilon, m = 1, \dots, N \tag{2.29c}$$

$$v^\varepsilon = 0 \quad \text{on } \Gamma, \tag{2.29d}$$

where the constants  $b^\varepsilon$  and  $b_m^\varepsilon$ ,  $m = 1, \dots, N$ , are such that

$$\int_{\gamma^\varepsilon} \partial_n v^\varepsilon d\sigma_y = 0, \quad \int_{\gamma_m^\varepsilon} \partial_n v^\varepsilon d\sigma_y = 0. \tag{2.29e}$$

We seek the solution as a single layer potential:

$$v^\varepsilon(x) = \mathcal{S}_{\gamma^\varepsilon} p^\varepsilon(x) + \sum_{m=1}^N \mathcal{S}_{\gamma_m^\varepsilon} p_m^\varepsilon(x) + \mathcal{S}_\Gamma q^\varepsilon(x), \quad x \in \mathcal{F}^\varepsilon,$$

where the densities  $p^\varepsilon \in H^{-1/2}(\gamma^\varepsilon)$ ,  $p_m^\varepsilon \in H^{-1/2}(\gamma_m^\varepsilon)$  and  $q^\varepsilon \in H^{-1/2}(\Gamma)$  are to be determined. Problem (2.29) is equivalent to the following system of integral equations:

$$\begin{aligned} \mathcal{S}_{\gamma^\varepsilon} p^\varepsilon(x) + \sum_{m=1}^N (\mathcal{S}_{\gamma_m^\varepsilon} p_m^\varepsilon)|_{\gamma^\varepsilon}(x) + (\mathcal{S}_\Gamma q^\varepsilon)|_{\gamma^\varepsilon}(x) &= x_1 + b^\varepsilon, & x \in \gamma^\varepsilon, \\ (\mathcal{S}_{\gamma^\varepsilon} p^\varepsilon)|_{\gamma_\ell^\varepsilon}(x) + \mathcal{S}_{\gamma_\ell^\varepsilon} p_\ell^\varepsilon(x) + \sum_{\substack{m=1 \\ m \neq \ell}}^N (\mathcal{S}_{\gamma_m^\varepsilon} p_m^\varepsilon)|_{\gamma_\ell^\varepsilon}(x) + (\mathcal{S}_\Gamma q^\varepsilon)|_{\gamma_\ell^\varepsilon}(x) &= b_\ell^\varepsilon, & x \in \gamma_\ell^\varepsilon, \\ (\mathcal{S}_{\gamma^\varepsilon} p^\varepsilon)|_\Gamma(x) + \sum_{m=1}^N (\mathcal{S}_{\gamma_m^\varepsilon} p_m^\varepsilon)|_\Gamma(x) + \mathcal{S}_\Gamma q^\varepsilon(x) &= 0, & x \in \Gamma, \end{aligned}$$

with the constraints (see Remark 2):

$$\int_{\gamma^\varepsilon} p^\varepsilon(y) d\sigma_y = \int_{\gamma_m^\varepsilon} p_m^\varepsilon(y) d\sigma_y = 0, \quad m = 1, \dots, N.$$

Rescaling the above equations, we obtain with obvious notation:

$$\begin{aligned} \int_\gamma G(x-y) \varepsilon p^\varepsilon(r + \varepsilon(y-r)) d\sigma_y \\ + \sum_{m=1}^N \int_{\gamma_m} G(r-r_m + \varepsilon(x-r-y+r_m)) \varepsilon p_m^\varepsilon(r_m + \varepsilon(y-r_m)) d\sigma_y \\ + \int_\Gamma G(r + \varepsilon(x-r) - y) q^\varepsilon(y) d\sigma_y = \varepsilon x_1 + b^\varepsilon, \quad x \in \gamma, \end{aligned} \quad (2.30a)$$

$$\begin{aligned} \int_\gamma G(r_\ell - r + \varepsilon(x-r_\ell-y+r)) \varepsilon p^\varepsilon(r + \varepsilon(y-r)) d\sigma_y + \int_{\gamma_\ell} G(x-y) \varepsilon p_\ell^\varepsilon(r_\ell + \varepsilon(y-r_\ell)) d\sigma_y \\ + \sum_{\substack{m=1 \\ m \neq \ell}}^N \int_\gamma G(r_\ell - r_m + \varepsilon(x-r_\ell-y+r_m)) \varepsilon p_m^\varepsilon(r_m + \varepsilon(y-r_m)) d\sigma_y \\ + \int_\Gamma G(r_\ell + \varepsilon(x-r_\ell) - y) q^\varepsilon(y) d\sigma_y = b_\ell^\varepsilon, \quad x \in \gamma_\ell, \end{aligned} \quad (2.30b)$$

$$\begin{aligned} \int_\gamma G(x-r - \varepsilon(y-r)) \varepsilon p^\varepsilon(r + \varepsilon(y-r)) d\sigma_y \\ + \sum_{m=1}^N \int_{\gamma_m} G(x-r_m - \varepsilon(y-r_m)) \varepsilon p_m^\varepsilon(r_m + \varepsilon(y-r_m)) d\sigma_y \\ + \int_\Gamma G(x-y) q^\varepsilon(y) d\sigma_y = 0, \quad x \in \Gamma, \end{aligned} \quad (2.30c)$$

with the constraints:

$$\int_\gamma p_\varepsilon(r + \varepsilon(y-r)) d\sigma_y = \int_{\gamma_m} p_m^\varepsilon(r + \varepsilon(y-r)) d\sigma_y = 0, \quad m = 1, \dots, N. \quad (2.30d)$$

For all  $\ell = 1, \dots, N$ , we introduce the formal asymptotic expansions of the quantities:

$$\varepsilon p^\varepsilon(r + \varepsilon(y - r)) = p_0(y) + \varepsilon p_1(y) + \varepsilon^2 \mathcal{P}^\varepsilon(y), \quad y \in \gamma, \quad (2.31a)$$

$$\varepsilon p_\ell^\varepsilon(r + \varepsilon(y - r)) = p_{\ell,0}(y) + \varepsilon p_{\ell,1}(y) + \varepsilon^2 \mathcal{P}_\ell^\varepsilon(y), \quad y \in \gamma_\ell, \quad (2.31b)$$

$$q^\varepsilon(y) = q_0(y) + \varepsilon q_1(y) + \varepsilon^2 q_2(y) + \varepsilon^3 \mathcal{Q}^\varepsilon(y), \quad y \in \Gamma, \quad (2.31c)$$

$$b^\varepsilon = b_0 + \varepsilon b_1 + \varepsilon^2 \mathcal{B}^\varepsilon, \quad (2.31d)$$

$$b_\ell^\varepsilon = b_{\ell,0} + \varepsilon b_{\ell,1} + \varepsilon^2 \mathcal{B}_\ell^\varepsilon. \quad (2.31e)$$

Using these expressions in System (2.30), the order 0 (in  $\varepsilon$ ) reads:

$$\begin{aligned} \mathcal{S}_\gamma p_0(x) + \mathcal{S}_\Gamma q_0(r) &= b_0, & x \in \gamma, \\ \mathcal{S}_{\gamma_\ell} p_{\ell,0}(x) + \mathcal{S}_\Gamma q_0(r_\ell) &= b_{\ell,0}, & x \in \gamma_\ell, \ell = 1, \dots, N, \\ \mathcal{S}_\Gamma q_0(x) &= 0, & x \in \Gamma, \end{aligned}$$

which, thanks to (2.30d), leads to  $q_0 = p_0 = p_{\ell,0} = 0$  and  $b_0 = b_{\ell,0} = 0$ . Identifying the terms of order 1 in  $\varepsilon$  of System (2.30) leads to:

$$\begin{aligned} \mathcal{S}_\gamma p_1(x) + \mathcal{S}_\Gamma q_1(r) &= x_1 + b_1 & x \in \gamma, \\ \mathcal{S}_{\gamma_\ell} p_{\ell,1}(x) + \mathcal{S}_\Gamma q_1(r_\ell) &= b_{\ell,1} & x \in \gamma_\ell, \ell = 1, \dots, N, \\ \mathcal{S}_\Gamma q_1(x) &= 0 & x \in \Gamma. \end{aligned}$$

Multiplying the first equation by the equilibrium density  $\psi_\gamma^{\text{eq}}$  of  $\gamma$  and integrating in  $x$  over  $\gamma$ , we find that

$$b_1 = - \int_\gamma \psi_\gamma^{\text{eq}}(x) x_1 d\sigma_x = -r \cdot e_1,$$

since the first momentum of the equilibrium density of a circle is its center. From the other equations we deduce that  $q_1 = 0$ ,  $p_{\ell,1} = 0$  and  $b_{\ell,1} = 0$ . Finally,  $p_1$  solves

$$\mathcal{S}_\gamma p_1(x) = x_1 - r \cdot e_1, \quad x \in \gamma.$$

Lemma 3 shows that in the local coordinates of  $\gamma$ :  $p_1(x) = -(r \cdot e_1)/R + 2 \cos \theta$ . Finally, the identification of the second order terms in  $\varepsilon$  in equation (2.30c) yields:

$$\nabla G(x - r) \cdot \left( \int_\gamma (r - y) p_1(y) d\sigma_y \right) + \mathcal{S}_\Gamma q_2(x) = 0, \quad x \in \Gamma.$$

Applying identity (5.4) of Lemma 3 with  $V = e_1$ , we get that  $\int_\gamma (r - y) p_1(y) d\sigma_y = -2\pi R^2 e_1$ , and therefore,  $q_2$  is the unique solution of the integral equation

$$\mathcal{S}_\Gamma q_2(x) = 2\pi R^2 \nabla G(x - r) \cdot e_1, \quad x \in \Gamma.$$

The remainders in (2.31) can be shown to be bounded using the same arguments as in section 2.1. Summing up, we have proved the following result:

**Proposition 3** *With the above notation, we have for every  $\ell = 1, \dots, N$ :*

$$\varepsilon p^\varepsilon(r + \varepsilon(y - r)) = \varepsilon p_1(y) + \varepsilon^2 \mathcal{P}^\varepsilon(y), \quad y \in \gamma, \quad (2.32a)$$

$$\varepsilon p_\ell^\varepsilon(r + \varepsilon(y - r)) = \varepsilon^2 \mathcal{P}_\ell^\varepsilon(y), \quad y \in \gamma_\ell, \quad (2.32b)$$

$$q^\varepsilon(y) = \varepsilon^2 q_2(y) + \varepsilon^3 \mathcal{Q}^\varepsilon(y), \quad y \in \Gamma \quad (2.32c)$$

$$b^\varepsilon = \varepsilon b_1 + \varepsilon^2 \mathcal{B}^\varepsilon, \quad (2.32d)$$

$$b_\ell^\varepsilon = \varepsilon^2 \mathcal{B}_\ell^\varepsilon. \quad (2.32e)$$

where

$$b_1 = -r \cdot e_1 \quad p_1 = \mathcal{S}_\gamma^{-1}((\cdot - r) \cdot e_1) \quad q_2 = \mathcal{S}_\Gamma^{-1}(2\pi R^2 \nabla G(\cdot - r) \cdot e_1)$$

and where the quantities  $\|\mathcal{P}^\varepsilon\|_{H^{-1/2}(\gamma)}$ ,  $\|\mathcal{P}_\ell^\varepsilon\|_{H^{-1/2}(\gamma_\ell)}$ ,  $\|\mathcal{Q}^\varepsilon\|_{H^{-1/2}(\Gamma)}$ ,  $\mathcal{B}^\varepsilon$  and  $\mathcal{B}_\ell^\varepsilon$  are uniformly bounded in  $\varepsilon$ .

Finally, to obtain the asymptotic expansion of the normal derivative of  $v^\varepsilon$  on  $\Gamma$ , we use a formula similar to (2.27), namely:

$$\begin{aligned} \partial_n v^\varepsilon(x) &= \frac{1}{2} q^\varepsilon(x) + \int_\Gamma \nabla G(x - y) \cdot n(x) q^\varepsilon(y) d\sigma_y \\ &\quad + \int_\gamma \nabla G(x - r - \varepsilon(y - r)) \cdot n(x) \varepsilon p^\varepsilon(r + \varepsilon(y - r)) d\sigma_y \\ &\quad + \sum_{m=1}^N \int_{\gamma_m} \nabla G(x - r_m - \varepsilon(y - r_m)) \cdot n(x) \varepsilon p_m^\varepsilon(r_m + \varepsilon(y - r_m)) d\sigma_y, \quad x \in \Gamma, \end{aligned}$$

to get, on  $\Gamma$ :

$$\partial_n v^\varepsilon = \varepsilon^2 \partial_n v_2 + \varepsilon^3 \partial_n \mathcal{V}^\varepsilon$$

where  $v_2(x) = \mathcal{S}_\Gamma q_2(x) - 2\pi R^2 \nabla G(x - r) \cdot e_1$  and  $\partial_n \mathcal{V}^\varepsilon$  is a uniformly (in  $\varepsilon$ ) bounded function in  $H^{-1/2}(\Gamma)$ . As  $\langle \partial_n v_2, g \rangle = 2\pi R^2 \nabla U^g(r) \cdot e_1$  for every  $g \in H^{1/2}(\Gamma)$ , we have proved the following result:

**Theorem 3** *Denote by  $v^\varepsilon \in H^1(\mathcal{F}^\varepsilon)$  the solution of (2.29). Then, for every  $g \in H^{1/2}(\Gamma)$ , we have*

$$\langle \partial_n v^\varepsilon, g \rangle = -\varepsilon^2 (2\pi R^2 \nabla U^g(r) \cdot e_1) + O(\varepsilon^3).$$

According to the splitting (2.2), Theorem 1 follows now from Theorems 2 and 3 and the proof is now complete.

### 3 Target identification using exterior sources

Our goal is to determine the number  $M$ , the positions  $r_m$ , the rescaled radii  $R_m$  and the velocities  $V_m$  of the small moving disks from the knowledge of:

$$a(f, g) := \sum_{m=1}^M R_m^2 \nabla U^f(r_m) \cdot \nabla U^g(r_m) - \sum_{m=1}^M R_m^2 \nabla U^g(r_m) \cdot V_m^\perp, \quad (3.1)$$

for every  $f, g \in H^{1/2}(\Gamma)$ . Among all the unknowns, the positions are the hardest one to determine. To achieve this, we first notice that we can easily isolate the first sum in the right hand side of (3.1) by setting:

$$a_0(f, g) := a(f, g) - a(0, g) = \sum_{m=1}^M R_m^2 \nabla u^f(r_m) \cdot \nabla u^g(r_m), \quad (3.2)$$

and hence get a measurement in which the unknown velocities  $V_m$  do not appear anymore (actually,  $a_0(f, g)$  coincides with  $a(f, g)$  in case the targets are motionless). The next step is to relate the Laplace type inverse problem under consideration to a Helmholtz type inverse problem. We use for this purpose a suitably chosen family of excitations  $f$  and test functions  $g$  of exponential type, following an idea due to Calderón [8]. Next, we make use of the so-called DORT Method to recover the unknown positions  $r_m$ . Based on time reversal techniques, this method has been successively used for the detection of well separated point like scatterers from far field measurements, for many wave type systems in acoustics [21, 24, 7] and electromagnetics [4]. The underlying idea of this method is that the eigenfunctions of some finite rank integral operator (the so-called time-reversal operator, that can be computed from the measurements) generate waves that selectively focus on each target. Of course, one can also use other reconstruction methods which are classically used for the detection of point-like targets in scattering theory, like the MUSIC algorithm [13, 22, 20].

### 3.1 Motionless targets

Here, we exclusively use the quantity  $a_0(f, g)$  defined by (3.2), which corresponds to the zero velocity case. Following an idea introduced by Calderón [8], let us choose for a every given  $\eta \in \mathbb{R}^2$  the excitation data  $f$  and the test functions  $g$  defined by

$$f(x) = e^{i(\eta+i\eta^\perp) \cdot x}, \quad g(x) = e^{i(\eta-i\eta^\perp) \cdot x}, \quad x \in \Gamma. \quad (3.3)$$

Noting that these expressions define in fact harmonic functions on  $\mathbb{R}^2$ , we have  $U^f(x) = f(x)$  and  $U^g(x) = g(x)$  and hence

$$a_0(f, g) = - \sum_{m=1}^M 2|\eta|^2 R_m^2 e^{2i\eta \cdot r_m}.$$

Choosing  $k > 0$  and  $\eta = \frac{1}{2}k(\alpha - \beta)$  where  $\alpha, \beta$  are two given vectors of the unit sphere  $\mathbf{S}$  of  $\mathbb{R}^2$  (so that  $|\eta|^2 = \frac{1}{2}k^2(1 - \alpha \cdot \beta)$ ), we can compute from the measurements

$$A(\alpha, \beta) := -\frac{1}{k^2} a_0(f, g) = \sum_{m=1}^M R_m^2 (1 - \alpha \cdot \beta) e^{ik(\alpha - \beta) \cdot r_m}.$$

Define the selfadjoint finite rank (and thus compact) integral operator  $\mathcal{A} \in \mathcal{L}(\mathbf{S})$  associated with the degenerate kernel  $A(\alpha, \beta)$ :

$$(\mathcal{A}\varphi)(\beta) := \int_{\mathbf{S}} A(\alpha, \beta) \varphi(\alpha) d\alpha.$$

In order to determine its spectrum and its eigenfunctions, we introduce the following functions of  $L^2(\mathbf{S})$  for every  $m = 1, \dots, M$ :

$$\varphi_{m,0}(\alpha) = \frac{1}{\sqrt{2\pi}} e^{-ik\alpha \cdot r_m} \quad \varphi_{m,1}(\alpha) = \frac{1}{\sqrt{\pi}} \alpha_1 e^{-ik\alpha \cdot r_m} \quad \varphi_{m,2}(\alpha) = \frac{1}{\sqrt{\pi}} \alpha_2 e^{-ik\alpha \cdot r_m}.$$

Then, setting  $\lambda_{m,0} = 2\pi R_m^2$  and  $\lambda_{m,\ell} = -\pi R_m^2$  for  $\ell = 1, 2$ , we can write  $A(\alpha, \beta)$  in the more compact form  $A(\alpha, \beta) = \sum_{m=1}^M \sum_{\ell=1}^3 \lambda_{m,\ell} \overline{\varphi_{m,\ell}(\alpha)} \varphi_{m,\ell}(\beta)$ , which yields

$$\mathcal{A}\varphi = \sum_{m=1}^M \sum_{\ell=1}^3 \lambda_{m,\ell} (\varphi, \varphi_{m,\ell})_{L^2(\mathbf{S})} \varphi_{m,\ell}.$$

Since  $\mathcal{A}$  is at most of rank  $3M$ , it has  $3M$  non zero eigenvalues (and  $M$  positive eigenvalues). More precisely, we can easily check that for every  $m = 1, \dots, M$ , the three functions  $\varphi_{m,0}$ ,  $\varphi_{m,1}$  and  $\varphi_{m,2}$  are orthonormal in  $L^2(\mathbf{S})$ . Moreover

$$(\varphi_{m,\ell}, \varphi_{n,\ell'})_{L^2(\mathbf{S})} = \int_{\mathbf{S}} \psi_{\ell,\ell'}(\alpha) e^{ik\alpha \cdot (r_n - r_m)} d\alpha, \quad 1 \leq m \neq n \leq M,$$

where  $\psi_{\ell,\ell'}(\alpha)$ , for  $\ell, \ell' \in \{0, 1, 2\}$ , are given in Table 1 (with  $\alpha = (\alpha_1, \alpha_2)$ ).

$(\ell, \ell')$	$\ell' = 0$	$\ell' = 1$	$\ell' = 2$
$\ell = 0$	1	$\alpha_1$	$\alpha_2$
$\ell = 1$	$\alpha_1$	$(\alpha_1)^2$	$\alpha_1 \alpha_2$
$\ell = 2$	$\alpha_2$	$\alpha_1 \alpha_2$	$(\alpha_2)^2$

Table 1: The values of  $\psi_{\ell,\ell'}(\alpha)$ .

One can easily check that (see for instance [1, p. 360])

$$(\varphi_{m,0}, \varphi_{n,1})_{L^2(\mathbf{S})} = 2 \int_0^\pi \cos \theta e^{ik|r_m - r_n| \cos \theta} d\theta = -2i\pi J_1(k|r_m - r_n|),$$

where  $J_n$  denotes the Bessel function of the first kind of order  $n$ , while

$$(\varphi_{m,0}, \varphi_{n,2})_{L^2(\mathbf{S})} = (\varphi_{m,1}, \varphi_{n,2})_{L^2(\mathbf{S})} = 0.$$

This shows that for  $1 \leq m \neq n \leq M$ , we have (see [1, p. 364])

$$(\varphi_{m,\ell}, \varphi_{n,\ell'})_{L^2(\mathbf{S})} = \mathcal{O}\left(\frac{1}{\sqrt{k|r_m - r_n|}}\right), \quad k \rightarrow +\infty.$$

Summing up, this proves that

$$\mathcal{A}\varphi_{m,\ell} = \lambda_{m,\ell} \varphi_{m,\ell} + \mathcal{O}\left(\frac{1}{\sqrt{k r_{\min}}}\right), \quad k \rightarrow +\infty,$$

where  $r_{\min} := \min_{m \neq n} |r_m - r_n|$  is the minimal distance between the solids. In other words, for every  $m = 1, \dots, M$ ,  $\varphi_{m,0}$ ,  $\varphi_{m,1}$  and  $\varphi_{m,2}$  constitute approximate eigenfunctions of  $\mathcal{A}$  as  $k \rightarrow +\infty$  (for high frequency and distant disks), the corresponding eigenvalues being respectively  $2\pi R_m^2$  (simple eigenvalue) and  $-\pi R_m^2$  (with multiplicity two). These (approximate) eigenfunctions can be used to recover the positions of the disks. Indeed, by associating to each eigenfunction  $\varphi_{m,\ell}$  the corresponding Herglotz wave, i.e. the superposition of plane waves defined by

$$u_{m,\ell}(x) := \int_{\mathbf{S}} \varphi_{m,\ell}(\alpha) e^{ik\alpha \cdot x} d\alpha, \quad x \in \mathbb{R}^2, \quad (3.4)$$

we see that

$$u_{m,\ell}(x) = \mathcal{O}\left(\frac{1}{\sqrt{k|x-r_m|}}\right), \quad k \rightarrow +\infty, \quad x \neq r_m.$$

Hence, the function  $u_{m,\ell}$  generates a wave that selectively focuses on the target  $m$  as  $k \rightarrow \infty$  and allows us to recover its location. We note that for the eigenfunction associated with the largest eigenvalue (in modulus) of  $\mathcal{A}$ , the focusing profile is described by the function

$$u_{m,0}(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{S}} e^{ik\alpha \cdot (x-r_m)} d\alpha = 2\pi J_0(k|x-r_m|).$$

In particular,  $u_{m,0}$  reaches its maximum exactly at the point  $x = r_m$ .

### 3.2 Moving targets

Now that we have determined the positions  $r_m$  ( $m = 1, \dots, M$ ) of the solids, it remains to compute their rescaled radii  $R_m$  and their velocities  $V_m$ . To this purpose, we introduce the complex polynomials:

$$P_m(z) = (z - r_1)^2 (z - r_2)^2 \dots (z - r_{m-1})^2 (z - r_{m+1})^2 (z - r_M)^2, \quad m = 1, \dots, M,$$

and the following harmonic functions (we identify  $x$  and  $r_j$  with their complex representations):

$$g_m^1(x) := \Re(P_m(x)/P'_m(r_m)), \quad g_m^2(x) := \Im(P_m(x)/P'_m(r_m)).$$

These functions, which are well defined since  $P'_m(r_m) \neq 0$ , enjoy the properties ( $\delta_{m,\ell}$  denotes here Kronecker's symbol):

$$Ug_m^j = g_m^j, \quad \nabla Ug_m^1(r_\ell) = \begin{pmatrix} \delta_{m,\ell} \\ 0 \end{pmatrix}, \quad \nabla Ug_m^2(r_\ell) = \begin{pmatrix} 0 \\ \delta_{m,\ell} \end{pmatrix},$$

for every  $m, \ell = 1, \dots, M$  and  $j = 1, 2$ . Substituting these last two relations in (3.1) and (3.2), the expressions of the radii and the velocities follow then straightforwardly, as for every  $m = 1, \dots, M$ :

$$R_m = \sqrt{a_0(g_m^1, g_m^1)}, \quad V_m \cdot e_1 = -\frac{a(0, g_m^2)}{R_m^2}, \quad V_m \cdot e_2 = \frac{a(0, g_m^1)}{R_m^2}.$$

## 4 Numerical tests

In this section, we present some numerical results to illustrate our reconstruction method. The solutions of the boundary value problems involved, in particular for the data generation, are computed using a `Matlab` boundary integral equation solver<sup>2</sup>. As a typical configuration, we consider three small disks located in a (smoothened) rectangular domain  $\Omega$ . The radii of the three small disks are  $(m+1)\varepsilon$ , where  $\varepsilon = 10^{-3}$  and  $m = 1, 2, 3$ . As a preliminary step, we first generate numerically the data that will be used for the reconstruction. Given a wavenumber  $k$  and a uniform discretization of  $[0, 2\pi]$  with mesh size  $h = 1/(N+1)$ , we compute the matrix  $\mathbb{A} = (A_{i,j})_{1 \leq i, j \leq N}$  corresponding to the discretization of the kernel  $A(\alpha, \beta)$ . More precisely, we have  $A_{i,j} = -1/k^2 (\Lambda^\varepsilon f_{i,j}, g_{i,j})$ , where  $f_{i,j}, g_{i,j}$  are the functions obtained in (3.3) for  $\eta = \frac{1}{2}k(\alpha_i - \beta_j)$ , where  $\alpha_i$  and  $\beta_j$  belong to the chosen

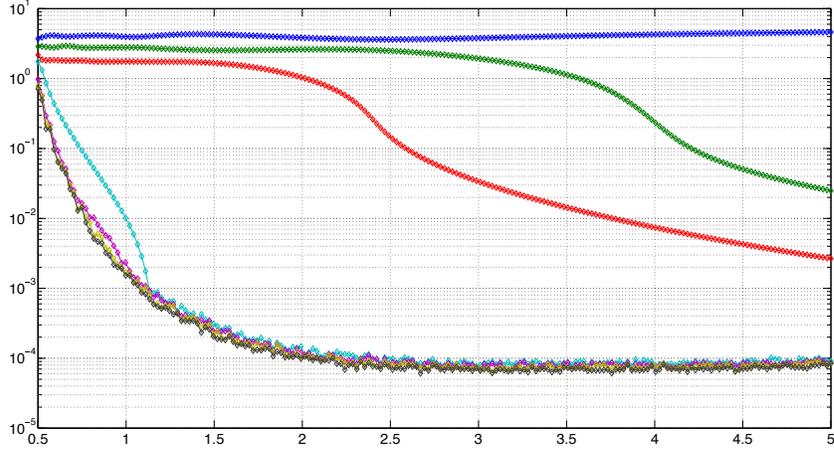


Figure 2: Dependence of the positive eigenvalues  $\lambda_{m,0}$  (in log scale) on the source's wavelength  $\lambda = 2\pi/k$ .

discretization of  $[0, 2\pi]$ . This requires to solve the forward problem (1.4) for  $N^2$  different right-hand sides  $f_{i,j}$ ,  $1 \leq i, j \leq N$ .

Once these data available, we can apply our DORT reconstruction procedure. We compute the eigenvalues and eigenfunctions of the discretized integral operator  $\mathcal{A}_h$  associated to the kernel  $\mathbb{A}$ . As expected from the theoretical analysis of Section 3, it turns out that this integral operator has indeed three significant eigenvalues for a wide range of values of  $k$ . We show on Fig. 2 the dependence of the positive eigenvalues with respect to the wavenumber  $\lambda = 2\pi/k$  of the oscillating source. According to this figure, we can recover the number of disks as soon as  $k$  is larger than  $\pi$  (i.e. for  $\lambda \geq 2$ ). Fixing such a value of  $k$ , the centers of the disks can be obtained as the points where the Herglotz waves (3.4) generated by the eigenfunctions associated to the largest eigenvalue reach their maximum (in modulus), as shown on Fig. 3. Finally, let us emphasize that although our reconstruction method is theoretically justified only for small disks ( $\varepsilon \rightarrow 0$ ), it turns out to be still efficient numerically for “extended” disks.

## 5 Appendix

### 5.1 Appendix A : Proof of Lemma 1

Define the maps:

$$\begin{aligned} \mathcal{K} : p \in H^{-1/2}(\Gamma_1) &\longmapsto \int_{\Gamma_1} K(\cdot, y)p(y) d\sigma_y \in H^1(\mathcal{V}_2), \\ \mathcal{L} : u \in H^1(\mathcal{V}_2) &\longmapsto \int_{\mathcal{V}_2} \nabla_x K(x, \cdot) \cdot \nabla u(x) + K(x, \cdot)u(x) dx \in H^1(\mathcal{V}_1) \end{aligned}$$

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<sup>2</sup>For more information about the solver, see <http://www.iecn.u-nancy.fr/~munier/IES/>

and  $\mathcal{L}^\dagger := \gamma_{\Gamma_1} \circ \mathcal{L}$ , where  $\gamma_{\Gamma_1}$  is the trace operator from  $H^1(\mathcal{V}_1)$  into  $H^{1/2}(\Gamma_1)$ . Let us verify that  $\mathcal{L}$  is continuous. For smooth functions  $u$ , we have:

$$\begin{aligned} \|\mathcal{L}u\|_{H^1(\mathcal{V}_1)}^2 &= \int_{\mathcal{V}_1} \left( \int_{\mathcal{V}_2} \nabla_x K(x, y) \cdot \nabla u(x) + K(x, y)u(x) \, dx \right)^2 dy + \\ &\quad \int_{\mathcal{V}_1} \left| \int_{\mathcal{V}_2} D_{xy}^2 K(x, y) \nabla u(x) + \nabla_y K(x, y)u(x) \, dx \right|^2 dy, \end{aligned}$$

where  $D_{xy}^2 K := (\partial_{x_i y_j}^2 K)_{1 \leq i, j \leq 2}$ . Applying Jensen then Cauchy-Schwarz inequality, we get:

$$\|\mathcal{L}u\|_{H^1(\mathcal{V}_1)} \leq C \|K\|_{H^2(\mathcal{V}_1 \times \mathcal{V}_2)} \|u\|_{H^1(\mathcal{V}_2)},$$

where the constant  $C$  depends on  $\mathcal{V}_1$  and  $\mathcal{V}_2$  only. By density, this estimate remains true for every  $u \in H^1(\mathcal{V}_2)$ . Assuming now that  $p$  and  $u$  are smooth, one can easily verify that:

$$(\mathcal{K}p, u)_{H^1(\mathcal{V}_2)} = \langle p, \mathcal{L}^\dagger u \rangle_{H^{-1/2}(\Gamma_1) \times H^{1/2}(\Gamma_1)}.$$

We deduce from the continuity of the trace operator  $\gamma_{\Gamma_1} : H^1(\mathcal{V}_1) \rightarrow H^{1/2}(\Gamma_1)$ , that:

$$|(\mathcal{K}p, u)_{H^1(\mathcal{V}_2)}| \leq C \|K\|_{H^2(\mathcal{V}_1 \times \mathcal{V}_2)} \|p\|_{H^{-1/2}(\Gamma_1)} \|u\|_{H^1(\mathcal{V}_2)},$$

where the constant  $C > 0$  does not depend on  $p$ ,  $u$  and  $K$ . This last inequality ensures the continuity of  $\mathcal{K}$ . We get the conclusion of the lemma by writing that:

$$\int_{\Gamma_2} \int_{\Gamma_1} K(x, y) p(y) q(x) \, d\sigma_y d\sigma_x = \int_{\Gamma_2} (\gamma_{\Gamma_2} \circ \mathcal{K}p)(x) q(x) \, d\sigma_x,$$

where  $\gamma_{\Gamma_2} : H^1(\mathcal{V}_2) \rightarrow H^{1/2}(\Gamma_2)$  is the continuous trace operator.

## 5.2 Appendix B : Single layer integral equation for a circle

**Lemma 3** *Denote by  $\gamma$  the circle of radius  $R$  centered at the origin and let be given  $c \in \mathbb{R}$  and  $V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \in \mathbb{R}^2$ . Then, the solution of the integral equation*

$$\mathcal{S}_\gamma p(x) = c + V \cdot x, \quad \forall x \in \gamma \tag{5.1}$$

*is given in polar coordinates by*

$$p(x) = \frac{c}{R} + 2(V_1 \cos \theta + V_2 \sin \theta), \quad \forall x = \begin{pmatrix} R \cos \theta \\ R \sin \theta \end{pmatrix} \in \gamma. \tag{5.2}$$

*The corresponding simple layer potential is*

$$(\mathcal{S}_\gamma p)(x) = \begin{cases} c + V \cdot x & \text{for } |x| < R \\ c \log \left( \frac{R}{|x|} \right) + \left( \frac{R}{|x|} \right)^2 V \cdot x & \text{for } |x| > R. \end{cases} \tag{5.3}$$

*Furthermore, we have*

$$\int_\gamma xp(x) \, d\sigma_x = 2\pi R^2 V. \tag{5.4}$$

*Proof.* One can easily check that the function defined by (5.3) satisfies

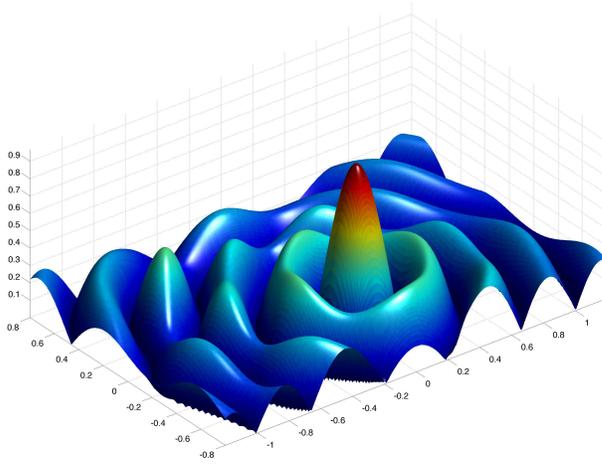
$$\begin{aligned} -\Delta\varphi &= 0 && \text{in } \mathbb{R}^2 \setminus \gamma \\ \varphi &= c + V \cdot x && \text{on } \gamma \\ [\varphi] &= 0 && \text{on } \gamma \\ \varphi(x) &= \mathcal{O}(\log|x|) && \text{as } |x| \rightarrow +\infty. \end{aligned}$$

This implies that  $\varphi = \mathcal{S}_\gamma p$  where  $p$  solves (5.1). Equation (5.2) follows then immediately from the formula  $p = \left[ \frac{\partial\varphi}{\partial n} \right]$  (see Proposition 1). Finally, using (5.2), equation (5.4) follows from a straightforward computation.  $\square$

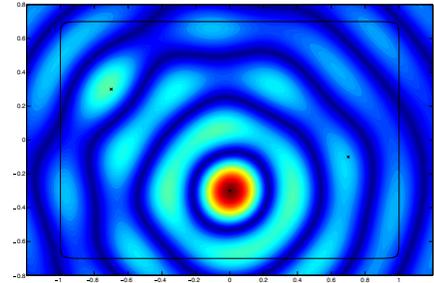
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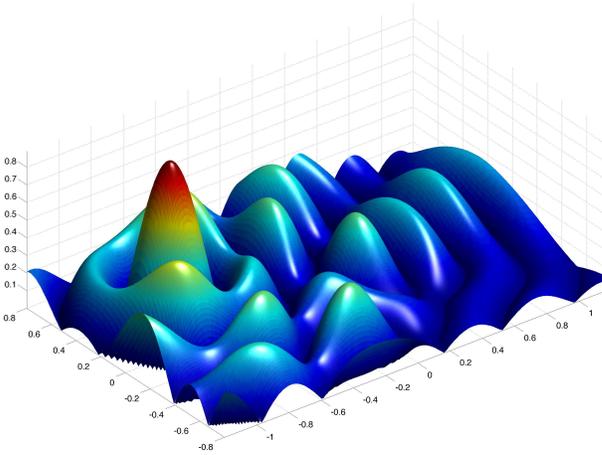
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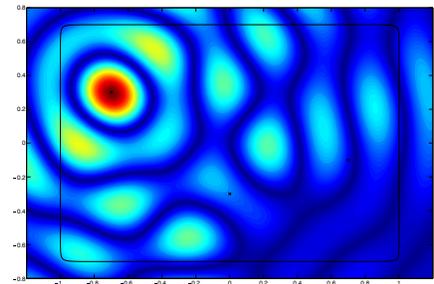
(a)



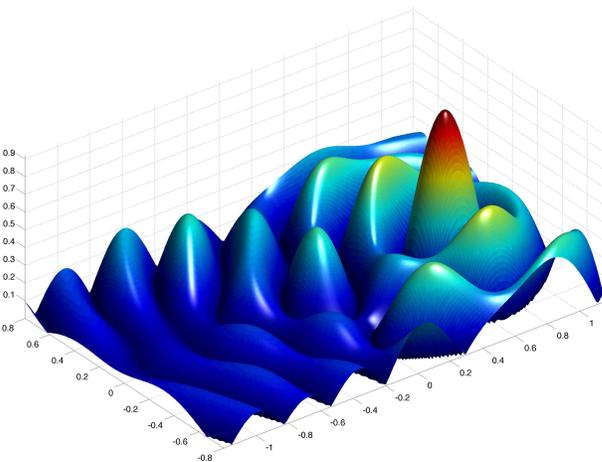
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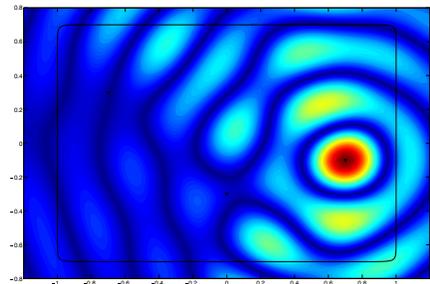
(c)



(d)



(e)



(f)

Figure 3: 3D (left) and 2D (right) visualization of the focusing properties of the eigenfunctions corresponding to the three largest positive eigenvalues for  $k = 10.5$ . The focusing property leads to an accurate recovery of the centers of the disks.