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An adaptive inexact Uzawa algorithm based on polynomial-degree-robust a posteriori estimates for the Stokes problem

Martin Čermák[¶] Frédéric Hecht[§] Zuqi Tang[‡] Martin Vohralík[‡]

Abstract

In this paper, we develop an adaptive version of the inexact Uzawa algorithm applied to finite element discretizations of the linear Stokes problem. We base our developments on an equilibrated flux a posteriori error estimate distinguishing the different error components, namely the discretization error component, the inner algebraic solver error component, and the outer Uzawa iteration error component. On each outer Uzawa and inner linear algebraic solver iteration, we prove that our estimate gives a guaranteed upper bound on the total error, as well as a polynomial-degree-robust local efficiency. Our adaptive inexact algorithm stops the outer Uzawa iteration and the inner linear algebraic solver iteration when the Uzawa error component, respectively the algebraic solver error component, do not have a significant influence on the total error. The developed framework covers all standard conforming and conforming stabilized finite element methods. The implementation into the FreeFem++ programming language is invoked and two numerical examples showcase the performance of our adaptive strategy.

Key words: Stokes problem, conforming finite element method, adaptive inexact Uzawa algorithm, a posteriori error estimate, guaranteed reliability, polynomial-degree-robust efficiency, adaptive stopping criterion, interplay between error components

1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be an open bounded polygonal domain with the boundary denoted by $\partial\Omega$. We consider in this paper the steady linear Stokes model problem: find a velocity $\mathbf{u} : \Omega \rightarrow \mathbb{R}^2$ and a pressure $p : \Omega \rightarrow \mathbb{R}$ satisfying:

$$-\Delta \mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega, \quad (1.1a)$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega, \quad (1.1b)$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \partial\Omega, \quad (1.1c)$$

where $\mathbf{f} : \Omega \rightarrow \mathbb{R}^2$ represents the volumetric force. The notations Δ , ∇ , and $\nabla \cdot$ represent respectively the Laplacian, gradient, and divergence operators.

Let $\mathbf{u} = [\mathbf{u}^m]_{1 \leq m \leq 2}$, $\mathbf{v} = [\mathbf{v}^m]_{1 \leq m \leq 2}$ denote vector functions and $\mathbf{u} = [\mathbf{u}^{m,n}]_{1 \leq m, n \leq 2}$, $\mathbf{v} = [\mathbf{v}^{m,n}]_{1 \leq m, n \leq 2}$ denote tensor functions. The respective scalar products are defined as $\mathbf{u} \cdot \mathbf{v} := \sum_{m=1}^2 \mathbf{u}^m \cdot \mathbf{v}^m$ and $\mathbf{u} : \mathbf{v} := \sum_{m=1}^2 \sum_{n=1}^2 \mathbf{u}^{m,n} : \mathbf{v}^{m,n}$. We follow the convention that

$$\nabla \mathbf{u} = \begin{pmatrix} \frac{\partial \mathbf{u}^1}{\partial x} & \frac{\partial \mathbf{u}^1}{\partial y} \\ \frac{\partial \mathbf{u}^2}{\partial x} & \frac{\partial \mathbf{u}^2}{\partial y} \end{pmatrix}, \quad \nabla \cdot \mathbf{v} = \begin{pmatrix} \frac{\partial \mathbf{v}^{1,1}}{\partial x} + \frac{\partial \mathbf{v}^{1,2}}{\partial y} \\ \frac{\partial \mathbf{v}^{2,1}}{\partial x} + \frac{\partial \mathbf{v}^{2,2}}{\partial y} \end{pmatrix},$$

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and

$$\nabla \cdot \mathbf{u} = \frac{\partial \mathbf{u}^1}{\partial x} + \frac{\partial \mathbf{u}^2}{\partial y}.$$

Let $D \subset \mathbb{R}^2$; we denote $(p, q)_D$ the integral $\int_D pq \, d\mathbf{x}$, where $p, q \in L^2(D)$. We use the same symbol $(\cdot, \cdot)_D$ also for the integrals of vector functions with values in \mathbb{R}^2 and of tensor functions with values in $\mathbb{R}^{2 \times 2}$, with the above scalar products. The Lebesgue norm on $L^2(D)$ is denoted by $\|\cdot\|_D$, respectively, and alike for vectors and tensors. When the domain D coincides with Ω , the subscript Ω is dropped. The same symbol $\|\cdot\|$ will also be used for the Euclidean norm of a vector $\mathbf{V} \in \mathbb{R}^N$,

$$\|\mathbf{V}\|^2 := \sum_{m=1}^N (\mathbf{V}_m)^2.$$

Moreover, we denote by $\|\cdot\|_\infty$ the infinity norm. Let us consider the following function spaces:

$$\begin{aligned} \mathbf{V} &:= [H_0^1(\Omega)]^2, \\ Q &:= L_0^2(\Omega) := \{q \in L^2(\Omega); (q, 1) = 0\}. \end{aligned}$$

The weak formulation of problem (1.1) reads:

Definition 1.1 (Weak solution). *Find $(\mathbf{u}, p) \in \mathbf{V} \times Q$ such that*

$$(\nabla \mathbf{u}, \nabla \mathbf{v}) - (\nabla \cdot \mathbf{v}, p) = (\mathbf{f}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V}, \quad (1.3a)$$

$$(\nabla \cdot \mathbf{u}, q) = 0 \quad \forall q \in Q. \quad (1.3b)$$

Problem (1.3) is well-posed (cf. [22]) due to the inf-sup condition (we systematically assume the arguments nonzero)

$$\inf_{q \in Q} \sup_{\mathbf{v} \in \mathbf{V}} \frac{(q, \nabla \cdot \mathbf{v})}{\|\nabla \mathbf{v}\| \|q\|} = \beta, \quad (1.4)$$

where β is a positive constant, only depending on Ω .

Remark 1.2 (Alternative weak formulation). *Alternative equivalent of the variational formulation of Definition 1.1 is, see [35]: find $\mathbf{u} \in \mathbf{V}$ with $\nabla \cdot \mathbf{u} = 0$ such that*

$$(\nabla \mathbf{u}, \nabla \mathbf{v}) = (\mathbf{f}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V} \text{ with } \nabla \cdot \mathbf{v} = 0. \quad (1.5)$$

Let \mathcal{T}_h be a simplicial mesh of Ω , matching in the sense that for two distinct elements of \mathcal{T}_h , their intersection is either an empty set or their common vertex or edge. Let next $\mathbf{V}_h \subset \mathbf{V}$ and $Q_h \subset Q$ be two approximation spaces of piecewise polynomials on \mathcal{T}_h . We consider the discretization of problem (1.3) by the conforming finite element method. The discrete problem reads as:

Definition 1.3 (Finite element method). *Find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ such that*

$$(\nabla \mathbf{u}_h, \nabla \mathbf{v}_h) - (\nabla \cdot \mathbf{v}_h, p_h) = (\mathbf{f}, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \quad (1.6a)$$

$$(\nabla \cdot \mathbf{u}_h, q_h) = 0 \quad \forall q_h \in Q_h. \quad (1.6b)$$

Let $\{\mathbf{v}_h^j\}_{1 \leq j \leq M}$ be a basis of the function space \mathbf{V}_h and let $\{q_h^j\}_{1 \leq j \leq N}$ be a basis of the function space Q_h . The corresponding system of linear algebraic equations is given by: find vectors $\mathbf{U} \in \mathbb{R}^M$ and $\mathbf{P} \in \mathbb{R}^N$ such that

$$\begin{pmatrix} \mathbf{A} & \mathbf{B}^T \\ \mathbf{B} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{U} \\ \mathbf{P} \end{pmatrix} = \begin{pmatrix} \mathbf{F} \\ \mathbf{0} \end{pmatrix}, \quad (1.7)$$

where \mathbf{A} is a symmetric positive definite matrix and \mathbf{B} has full rank.

The Uzawa algorithm, see [5], can be seen as a direct application of the gradient method to the minimization problem of the dual functional of the Stokes problem, see [6] for a general discussion. Applied to problem (1.6), it can be written as, see [19, 30]:

Algorithm 1.4 (Uzawa algorithm).

1: Chose an initial approximation $p_h^0 \in Q_h$ of p , a real constant $\alpha \in (0, 2)$, and a tolerance $\varepsilon > 0$.

2: **For** $k = 0 \dots + \infty$:

a) Compute $\mathbf{u}_h^{k+1} \in \mathbf{V}_h$ such that

$$(\nabla \mathbf{u}_h^{k+1}, \nabla \mathbf{v}_h) = (\nabla \cdot \mathbf{v}_h, p_h^k) + (\mathbf{f}_h, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{V}_h. \quad (1.8)$$

b) Compute $p_h^{k+1} \in Q_h$ such that

$$(p_h^{k+1}, q_h) = (p_h^k, q_h) + \alpha (\nabla \cdot \mathbf{u}_h^{k+1}, q_h) \quad \forall q_h \in Q_h. \quad (1.9)$$

c) If $\|p_h^{k+1} - p_h^k\|_\infty < \varepsilon$, finish the computation.

EndFor

According to [19], the optimal rate of convergence is achieved with the choice of α

$$\alpha = \frac{1}{\lambda_1 + \lambda_2},$$

where λ_1 and λ_2 are respectively the maximal and minimal eigenvalues of $\mathbb{B}\mathbb{A}^{-1}\mathbb{B}^T$, and it is shown that the Uzawa algorithm converges when $0 < \alpha < \frac{2}{\lambda_2}$. The most attractive character of the Uzawa method is its simplicity of implementation and its minimal memory requirements. Unfortunately, its speed of convergence is slow in some cases and therefore it may require too many iterations in order to obtain sufficient accuracy. In this context, inexact Uzawa methods were proposed in [41, 19, 11, 15, 6], see also the references therein. Discussions about the convergence of the different variants of the inexact Uzawa method can be found in [11, 15, 6, 30, 29]. For example, in [19], the inexact Uzawa algorithm is specified as follows:

Algorithm 1.5 (Inexact Uzawa algorithm).

1: Chose an initial approximation $p_h^0 \in Q_h$ of p , an initial approximation $\mathbf{u}_h^0 \in \mathbf{V}_h$ of \mathbf{u} (such that $\nabla \cdot \mathbf{u}_h^0 \neq 0$, i.e., $\|\mathbb{B}\mathbf{U}^0\| \neq 0$), a real constant $\alpha \in (0, 2)$, a tolerance $\varepsilon > 0$, and a constant $\tau > 0$.

2: **For** $k = 0 \dots + \infty$:

a) Consider an iterative algebraic solver for the linear system (1.8). We stop the iterations whenever $\|\mathbf{R}^k\| \leq \tau \|\mathbb{B}\mathbf{U}^k\|$, where $\mathbf{R}^k = [\mathbf{R}_j^k]_{1 \leq j \leq M}$ is the algebraic residual such that

$$(\nabla \mathbf{u}_h^{k+1}, \nabla \mathbf{v}_h^j) = (\nabla \cdot \mathbf{v}_h^j, p_h^k) + (\mathbf{f}_h, \mathbf{v}_h^j) - \mathbf{R}_j^k \quad 1 \leq j \leq M.$$

b) Compute $p_h^{k+1} \in Q_h$ such that

$$(p_h^{k+1}, q_h) = (p_h^k, q_h) + \alpha (\nabla \cdot \mathbf{u}_h^{k+1}, q_h) \quad \forall q_h \in Q_h.$$

c) If $\|p_h^{k+1} - p_h^k\|_\infty < \varepsilon$, finish the computation.

EndFor

In this paper, we are interested in designing an adaptive version of the inexact Uzawa algorithm 1.5, see Algorithm 3.1 below. We constitute it upon the distinction of the different error components arising in the inexact Uzawa algorithm, namely the discretization, inner algebraic solver iteration, and outer Uzawa iteration components. To arrive at this aim, we follow the ideas of [9, 33, 36], see also the survey of [4] and the references therein, but most closely the approach of [26, 20] for the Laplace problems.

There exist various types of a posteriori error estimates for the Stokes problem which aim at controlling the discretization error. The residual-based estimators can be found in [39] for the conforming case and in [40] for the nonconforming case. Reference [31] extends this type of estimates to the Stokes eigenvalue problem.

For stabilized finite element methods, the residual-based a posteriori error estimates have been addressed in [27, 42]. The recovery-based (Zienkiewicz–Zhu) estimator applied to the nonconforming finite element approximation can be found in [14, 38]. The works [39, 7, 2] propose an a posteriori error estimate by solving local residual problems. In [37], guaranteed upper bounds for both energy norms and functional outputs are considered. In [18], guaranteed estimates are derived for nonconforming discretizations, and in [1], the issue of appropriate online selection of the parameter in stabilized methods is addressed. A unified framework for guaranteed a posteriori error estimation for the Stokes problem discretized by different conforming and nonconforming numerical methods is presented in [23]. Finally, the work [3] estimates the total error at each Uzawa step for the exact Uzawa algorithm, and shows that this error in global sense can be controlled by the divergence of the velocity.

We follow here the equilibrated fluxes methodology, see [28, 17, 18, 10, 23, 21] and the references therein, and prove that our error estimate gives a guaranteed upper bound at each moment of the computation. We balance the three arising error components by stopping adaptively the inner algebraic as well as the outer Uzawa solver, so that the unnecessary iterations of both these solvers can be avoided. Following the ideas of [10] and [21], we also prove a polynomial-degree-robust (local) efficiency of our estimate, and this in the presence of algebraic solver and Uzawa iteration errors. All the results are proved for various conforming finite element discretizations, both unstabilized and stabilized.

The outline of the paper is as follows. In Section 2, conforming unstabilized and stabilized finite element methods are respectively introduced. Our adaptive inexact Uzawa method is presented in Section 3. The developments of Section 4 give a general a posteriori error estimate for the adaptive inexact Uzawa method and enable to estimate the error between the exact solution and the approximate solution obtained at each outer Uzawa step and at each inner algebraic solver step. In Section 5, the polynomial-degree-robust efficiency of our estimator is proved, giving a lower bound for the error up to a generic constant only depending on the shape regularity parameter. All these results are derived in an abstract framework. The application of our results necessitates the verification of Assumptions 4.1 and 5.1. We do it in Section 6. Finally in Section 7, two numerical examples are provided to show the behavior of our a posteriori estimates and the computational gains of our adaptive inexact Uzawa method in comparison with the exact Uzawa algorithm 1.4 and the inexact Uzawa algorithm 1.5. Here only a fixed computational mesh is considered; further extensions incorporate mesh adaptivity as well.

2 Conforming finite element methods

We shall give our a posteriori error estimates in a quite general framework. In order to, however, fix ideas, we now give typical practical examples of finite element discretizations of (1.3). Let $\mathbb{P}_l(\mathcal{T}_h)$ denote the piecewise polynomials of order l on \mathcal{T}_h , let h_K stand for the diameter of $K \in \mathcal{T}_h$, and let h_Ω denote the diameter of Ω .

2.1 Unstabilized schemes

We consider the following discretizations (1.6), following [35]:

The Taylor–Hood family: for $l \geq 2$,

$$\mathbf{V}_h := \mathbf{V} \cap [\mathbb{P}_l(\mathcal{T}_h)]^2, \quad (2.1a)$$

$$Q_h := Q \cap [C^0(\overline{\Omega}) \cap \mathbb{P}_{l-1}(\mathcal{T}_h)]. \quad (2.1b)$$

The mini element:

$$\mathbf{V}_h := \mathbf{V} \cap [\mathbb{P}_1^b(\mathcal{T}_h)]^2,$$

$$Q_h := Q \cap [C^0(\overline{\Omega}) \cap \mathbb{P}_1(\mathcal{T}_h)],$$

where $\mathbb{P}_1^b(\mathcal{T}_h)$ stands for $\mathbb{P}_1(\mathcal{T}_h)$ enriched by the cubic bubble functions vanishing on the boundary of each element. We need to set here $l = 3$. The cross-grid \mathbb{P}_1 – \mathbb{P}_1 element, with $l = 1$,

$$\mathbf{V}_h := \mathbf{V} \cap [\mathbb{P}_1(\mathcal{T}_h)]^2,$$

$$Q_h := Q \cap [C^0(\overline{\Omega}) \cap \mathbb{P}_1(\mathcal{T}_H)],$$

where the triangular mesh \mathcal{T}_h was formed from the triangular mesh \mathcal{T}_H by adding the barycenter as a node and cutting the triangle into 3 subtriangles. The \mathbb{P}_1 iso \mathbb{P}_2 - \mathbb{P}_1 element, with $l = 1$,

$$\begin{aligned}\mathbf{V}_h &:= \mathbf{V} \cap [\mathbb{P}_1(\mathcal{T}_h)]^2, \\ Q_h &:= Q \cap [C^0(\bar{\Omega}) \cap \mathbb{P}_1(\mathcal{T}_H)],\end{aligned}$$

where \mathcal{T}_h was formed from \mathcal{T}_H by adding the edge barycenters as nodes and cutting the triangle into 4 subtriangles.

2.2 Stabilized schemes

We will also cover some conforming stabilized methods, where some regularization terms are added to (1.6):

Definition 2.1 (Stabilized finite element method). *Find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ such that*

$$(\nabla \mathbf{u}_h, \nabla \mathbf{v}_h) - (\nabla \cdot \mathbf{v}_h, p_h) + t_h(\mathbf{u}_h, p_h; \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \quad (2.5a)$$

$$(\nabla \cdot \mathbf{u}_h, q_h) + s_h(\mathbf{u}_h, p_h; q_h) = 0 \quad \forall q_h \in Q_h. \quad (2.5b)$$

Let

$$\mathbf{V}_h := \mathbf{V} \cap [\mathbb{P}_l(\mathcal{T}_h)]^2, \quad (2.6a)$$

$$Q_h := Q \cap [C^0(\bar{\Omega}) \cap \mathbb{P}_l(\mathcal{T}_h)] \quad (2.6b)$$

and let $\delta > 0$ be a penalization parameter. Then, for $l = 1$, the Brezzi–Pitkäranta method [13] uses:

$$s_h(\mathbf{u}_h, p_h; q_h) := -\delta \sum_{K \in \mathcal{T}_h} h_K^2 (\nabla p_h, \nabla q_h)_K.$$

For $l \geq 1$, the Hughes–Franca–Balestra method [25] uses:

$$s_h(\mathbf{u}_h, p_h; q_h) := \delta \sum_{K \in \mathcal{T}_h} h_K^2 (\mathbf{f} + \Delta \mathbf{u}_h - \nabla p_h, \nabla q_h)_K.$$

For $l \geq 1$, the Brezzi–Douglas method [12] uses:

$$s_h(\mathbf{u}_h, p_h; q_h) := \delta \sum_{K \in \mathcal{T}_h} h_K^2 \{(\mathbf{f} - \nabla p_h, \nabla q_h)_K + \langle \Delta \mathbf{u}_h \cdot \mathbf{n}_K, q_h \rangle_{\partial K \cap \partial \Omega}\}.$$

In all these methods, $t_h(\mathbf{u}_h, p_h; \mathbf{v}_h) := 0$. Finally, for the choice (2.6) or for (2.1), $l \geq 2$, and a parameter $\rho = -1, 1$, the Hughes–Franca–Balestra method [25] uses:

$$s_h(\mathbf{u}_h, p_h; q_h) := \delta \sum_{K \in \mathcal{T}_h} h_K^2 (\mathbf{f} + \Delta \mathbf{u}_h - \nabla p_h, \nabla q_h)_K,$$

$$t_h(\mathbf{u}_h, p_h; \mathbf{v}_h) := \delta \rho \sum_{K \in \mathcal{T}_h} h_K^2 (\mathbf{f} + \Delta \mathbf{u}_h - \nabla p_h, \Delta \mathbf{v}_h)_K.$$

Henceforth, we only consider (2.5) instead of (1.6), with $s_h(\mathbf{u}_h, p_h; q_h)$ and $t_h(\mathbf{u}_h, p_h; q_h)$ equal to zero for unstabilized methods.

3 Adaptive inexact Uzawa method

In this section, we introduce our adaptive amendment of the classical inexact Uzawa algorithm 1.5, in the spirit of [20]. Let $\eta_{\text{rem}}^{k,i}$, $\eta_{\text{alg}}^{k,i}$, $\eta_{\text{Uza}}^{k,i}$, and $\eta_{\text{disc}}^{k,i}$ be respectively the estimators of the algebraic reminder, the inner algebraic solver error, the outer Uzawa iteration error, and the discretization error on the k -th Uzawa step and i -th algebraic step. We give their precise forms in Section 4 below.

Algorithm 3.1 (Adaptive inexact Uzawa algorithm).

1: Chose an initial approximation p_h^0 of p , an initial approximation \mathbf{u}_h^0 of \mathbf{u} , a real constant $\alpha \in (0, 2)$, a fixed iteration count $\nu_0 > 0$, and real parameters $\gamma_{\text{rem}}, \gamma_{\text{alg}}, \gamma_{\text{Uza}} > 0$, typically of order 10^{-1} .

2: **For** $k = 0 \dots + \infty$:

2.1: Define $\mathbf{u}_h^{k,0} := \mathbf{u}_h^k$ and set $i := \nu_0$. Starting from $\mathbf{u}_h^{k,0}$, perform ν_0 steps of a chosen iterative linear solver for the linear system associated with (2.5a), yielding $\mathbf{u}_h^{k,i} \in \mathbf{V}_h$ and $\mathbf{R}_j^{k,i} \in \mathbb{R}^M$ such that

$$(\nabla \mathbf{u}_h^{k,i}, \nabla \mathbf{v}_h^j) + t_h(\mathbf{u}_h^{k,i}, p_h^k; \mathbf{v}_h^j) = (\nabla \cdot \mathbf{v}_h^j, p_h^k) + (\mathbf{f}, \mathbf{v}_h^j) - \mathbf{R}_j^{k,i} \quad 1 \leq j \leq M. \quad (3.1)$$

2.2: a) Set $\nu := 0$.

b) Perform ν_0 additional steps of the chosen iterative linear solver, starting from $\mathbf{u}_h^{k,i+\nu}$. Set $\nu := \nu + \nu_0$. This yields $\mathbf{u}_h^{k,i+\nu} \in \mathbf{V}_h$ and $\mathbf{R}_j^{k,i+\nu} \in \mathbb{R}^M$ satisfying

$$(\nabla \mathbf{u}_h^{k,i+\nu}, \nabla \mathbf{v}_h^j) + t_h(\mathbf{u}_h^{k,i+\nu}, p_h^k; \mathbf{v}_h^j) = (\nabla \cdot \mathbf{v}_h^j, p_h^k) + (\mathbf{f}, \mathbf{v}_h^j) - \mathbf{R}_j^{k,i+\nu} \quad 1 \leq j \leq M. \quad (3.2)$$

c) Compute the estimators $\eta_{\text{rem}}^{k,i}$ and $\eta_{\text{alg}}^{k,i}$. Check the balancing criterion

$$\eta_{\text{rem}}^{k,i} \leq \gamma_{\text{rem}} \eta_{\text{alg}}^{k,i}. \quad (3.3a)$$

If not satisfied, go back to step 2.2 b).

d) Compute the estimators $\eta_{\text{Uza}}^{k,i}$ and $\eta_{\text{disc}}^{k,i}$ and check the stopping criterion for the inner algebraic solver in the form

$$\eta_{\text{alg}}^{k,i} \leq \gamma_{\text{alg}} \max\{\eta_{\text{disc}}^{k,i}, \eta_{\text{Uza}}^{k,i}\}. \quad (3.3b)$$

If not satisfied, set $i := i + \nu$ and go back to step 2.2 a).

2.3 Check the stopping criterion for the outer Uzawa iteration in the form

$$\eta_{\text{Uza}}^{k,i} \leq \gamma_{\text{Uza}} \eta_{\text{disc}}^{k,i}. \quad (3.3c)$$

If satisfied, finish the computation. If not, set $\mathbf{u}_h^{k+1} := \mathbf{u}_h^{k,i}$ and compute p_h^{k+1} such that

$$(p_h^{k+1}, q_h) = (p_h^k, q_h) + \alpha ((\nabla \cdot \mathbf{u}_h^{k+1}, q_h) + s_h(\mathbf{u}_h^{k+1}, p_h^k; q_h)) \quad \forall q_h \in Q_h.$$

EndFor

Remark 3.2 (Local stopping criteria). Above, the stopping criteria (3.3) are global in the sense that the estimators are summed over all mesh elements. Alternatively, we can also consider the local stopping criteria [26, 20]:

$$\eta_{\text{rem},K}^{k,i} \leq \gamma_{\text{rem},K} \eta_{\text{alg},K}^{k,i}, \quad \forall K \in \mathcal{T}_h, \quad (3.4a)$$

$$\eta_{\text{alg},K}^{k,i} \leq \gamma_{\text{alg},K} \max\{\eta_{\text{disc},K}^{k,i}, \eta_{\text{Uza},K}^{k,i}\}, \quad \forall K \in \mathcal{T}_h, \quad (3.4b)$$

$$\eta_{\text{Uza},K}^{k,i} \leq \gamma_{\text{Uza},K} \eta_{\text{disc},K}^{k,i}, \quad \forall K \in \mathcal{T}_h, \quad (3.4c)$$

where for any $K \in \mathcal{T}_h$, the parameters $\gamma_{\text{rem},K}$, $\gamma_{\text{alg},K}$, and $\gamma_{\text{Uza},K}$ are positive user-given weights, typically of order 10^{-1} .

4 Guaranteed reliability via equilibrated fluxes

Let $\mathbb{H}(\text{div}, \Omega)$ be given by

$$\mathbb{H}(\text{div}, \Omega) = \{\mathbf{v} \in [L^2(\Omega)]^{2 \times 2}; \nabla \cdot \mathbf{v} \in [L^2(\Omega)]^2\}.$$

For $1 \leq m \leq 2$, we denote $\mathbf{e}_m \in \mathbb{R}^2$ the canonical vectors of \mathbb{R}^2 , i.e.,

$$\mathbf{e}_1 := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \mathbf{e}_2 := \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

We also denote by \mathbb{I} the 2-by-2 identity matrix.

Let \mathbf{f}_h be a piecewise polynomial approximation of the source term, verifying

$$(\mathbf{f}_h, \mathbf{e}_m)_K = (\mathbf{f}, \mathbf{e}_m)_K, \quad \forall 1 \leq m \leq 2, \quad \forall K \in \mathcal{T}_h. \quad (4.1)$$

We will use below the Poincaré inequality

$$\|\mathbf{v} - \mathbf{v}_K\|_K \leq \frac{h_K}{\pi} \|\nabla \mathbf{v}\|_K, \quad \forall \mathbf{v} \in [H^1(K)]^2, \quad (4.2)$$

where \mathbf{v}_K denotes the mean value of \mathbf{v} in K , see [34]. We will also employ the Friedrichs inequality stating that

$$\|\mathbf{v}\| \leq h_\Omega \|\nabla \mathbf{v}\|, \quad \forall \mathbf{v} \in \mathbf{V}. \quad (4.3)$$

In order to proceed as generally as possible, we now make:

Assumption 4.1 (Quasi-equilibrated Cauchy stress reconstruction). *For each outer Uzawa iteration $k \geq 1$, each inner algebraic solver iteration $i \geq 1$, and each number $\nu \geq 0$ of additional algebraic solver steps, there exists a Cauchy stress reconstruction $\mathbf{d}_h^{k,i+\nu} \in \mathbb{H}(\text{div}, \Omega)$ and an algebraic remainder function $\mathbf{r}_h^{k,i+\nu} \in [L^2(\Omega)]^2$ such that*

$$-\nabla \cdot \mathbf{d}_h^{k,i+\nu} = \mathbf{f}_h - \mathbf{r}_h^{k,i+\nu}. \quad (4.4)$$

Under Assumption 4.1, we now prove the following theorem, following essentially [3, 18, 20, 23].

Theorem 4.2 (A general a posteriori error estimate for problem (1.1)). *Let (\mathbf{u}, p) be the weak solution given by Definition 1.1 and let $(\mathbf{u}_h^{k,i}, p_h^k) \in \mathbf{V}_h \times Q_h$ be the approximate solution obtained by Algorithm 3.1 on the Uzawa step $k \geq 1$ and the inner algebraic iteration $i \geq 1$. For $\nu \geq 0$ additional inner algebraic solver steps, let Assumption 4.1 be satisfied. For any $K \in \mathcal{T}_h$, define*

the flux estimator:

$$\eta_{\mathbb{F},K}^{k,i} := \|\nabla \mathbf{u}_h^{k,i} - p_h^k \mathbb{I} - \mathbf{d}_h^{k,i+\nu}\|_K, \quad (4.5a)$$

the divergence estimator:

$$\eta_{\mathbb{D},K}^{k,i} := \frac{\|\nabla \cdot \mathbf{u}_h^{k,i}\|_K}{\beta}, \quad (4.5b)$$

the remainder estimator:

$$\eta_{\text{rem},K}^{k,i} := h_\Omega \|\mathbf{r}_h^{k,i+\nu}\|_K, \quad (4.5c)$$

the data oscillation estimator:

$$\eta_{\text{osc},K}^{k,i} := \frac{h_K}{\pi} \|\mathbf{f} - \mathbf{f}_h\|_K. \quad (4.5d)$$

Let the following convention be used

$$\eta_*^{k,i} = \left\{ \sum_{K \in \mathcal{T}_h} (\eta_{*,K}^{k,i})^2 \right\}^{1/2}. \quad (4.6)$$

Then we have

$$\|\nabla(\mathbf{u} - \mathbf{u}_h^{k,i})\| \leq \eta_F^{k,i} + \eta_{\text{osc}}^{k,i} + \eta_{\text{rem}}^{k,i} + \eta_D^{k,i}, \quad (4.7a)$$

$$\beta\|p - p_h^k\| \leq \eta_F^{k,i} + \eta_{\text{osc}}^{k,i} + \eta_{\text{rem}}^{k,i} + \eta_D^{k,i}. \quad (4.7b)$$

Proof. We first bound $\|\nabla(\mathbf{u} - \mathbf{u}_h^{k,i})\|$. Let $\mathbf{s} \in \mathbf{V}$ with $\nabla \cdot \mathbf{s} = 0$ be the solution of

$$(\nabla \mathbf{s}, \nabla \mathbf{v}) = (\nabla \mathbf{u}_h^{k,i}, \nabla \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V} \text{ with } \nabla \cdot \mathbf{v} = 0. \quad (4.8)$$

This problem has a unique solution, cf. Remark 1.2. We have the Pythagorean equality

$$\|\nabla(\mathbf{u} - \mathbf{u}_h^{k,i})\|^2 = \|\nabla(\mathbf{u} - \mathbf{s})\|^2 + \|\nabla(\mathbf{s} - \mathbf{u}_h^{k,i})\|^2, \quad (4.9)$$

which follows from the fact that

$$\begin{aligned} \|\nabla(\mathbf{u} - \mathbf{u}_h^{k,i})\|^2 &= \|\nabla(\mathbf{u} - \mathbf{s} + \mathbf{s} - \mathbf{u}_h^{k,i})\|^2 \\ &= \|\nabla(\mathbf{u} - \mathbf{s})\|^2 + \|\nabla(\mathbf{s} - \mathbf{u}_h^{k,i})\|^2 + 2(\nabla(\mathbf{u} - \mathbf{s}), \nabla(\mathbf{s} - \mathbf{u}_h^{k,i})), \end{aligned}$$

with the last term disappearing thanks the facts that $\mathbf{u} - \mathbf{s} \in \mathbf{V}$ and $\nabla \cdot (\mathbf{u} - \mathbf{s}) = 0$ and to (4.8). We estimate the two terms in (4.9) separately.

Using the expression of the energy norm as a dual norm, we have

$$\|\nabla(\mathbf{u} - \mathbf{s})\| = \sup_{\varphi \in \mathbf{V}; \nabla \cdot \varphi = 0, \|\nabla \varphi\| = 1} (\nabla(\mathbf{u} - \mathbf{s}), \nabla \varphi).$$

Let thus $\varphi \in \mathbf{V}$ with $\nabla \cdot \varphi = 0$ and $\|\nabla \varphi\| = 1$ be fixed. Employing the definitions (1.5) and (4.8), we have

$$(\nabla(\mathbf{u} - \mathbf{s}), \nabla \varphi) = (\mathbf{f}, \varphi) - (\nabla \mathbf{u}_h^{k,i}, \nabla \varphi). \quad (4.10)$$

Next, using that $0 = (p_h^k, \nabla \cdot \varphi) = (p_h^k \mathbb{I}, \nabla \varphi)$, adding and subtracting $(\mathfrak{d}_h^{k,i+\nu}, \nabla \varphi)$, and using the Green theorem, we get

$$(\nabla(\mathbf{u} - \mathbf{s}), \nabla \varphi) = (\mathbf{f} + \nabla \cdot \mathfrak{d}_h^{k,i+\nu}, \varphi) - (\nabla \mathbf{u}_h^{k,i} - p_h^k \mathbb{I} - \mathfrak{d}_h^{k,i+\nu}, \nabla \varphi).$$

Using (4.4), we get

$$(\nabla(\mathbf{u} - \mathbf{s}), \nabla \varphi) = (\mathbf{f} - \mathbf{f}_h, \varphi) + (\mathbf{r}_h^{k,i+\nu}, \varphi) - (\nabla \mathbf{u}_h^{k,i} - p_h^k \mathbb{I} - \mathfrak{d}_h^{k,i+\nu}, \nabla \varphi).$$

We have, for any $K \in \mathcal{T}_h$,

$$(\mathbf{f} - \mathbf{f}_h, \varphi)_K = (\mathbf{f} - \mathbf{f}_h, \varphi - \varphi_K)_K \leq \eta_{\text{osc},K}^{k,i} \|\nabla \varphi\|_K,$$

using (4.2), whereas the estimate

$$(\nabla \mathbf{u}_h^{k,i} - p_h^k \mathbb{I} - \mathfrak{d}_h^{k,i+\nu}, \nabla \varphi)_K \leq \eta_{F,K}^{k,i} \|\nabla \varphi\|_K$$

is immediately obtained by the Cauchy–Schwarz inequality. And with the Friedrichs inequality (4.3), we get

$$(\mathbf{r}_h^{k,i+\nu}, \varphi) \leq h_\Omega \|\mathbf{r}_h^{k,i+\nu}\| \|\nabla \varphi\|.$$

Thus the Cauchy–Schwarz inequality gives

$$\begin{aligned} (\nabla(\mathbf{u} - \mathbf{s}), \nabla \varphi) &\leq \sum_{K \in \mathcal{T}_h} \left\{ (\eta_{F,K}^{k,i} + \eta_{\text{osc},K}^{k,i}) \|\nabla \varphi\|_K \right\} + h_\Omega \|\mathbf{r}_h^{k,i+\nu}\| \|\nabla \varphi\| \\ &\leq \eta_F^{k,i} + \eta_{\text{osc}}^{k,i} + \eta_{\text{rem}}^{k,i}. \end{aligned} \quad (4.11)$$

We now treat the term $\|\nabla(\mathbf{s} - \mathbf{u}_h^{k,i})\|$. We have

$$\|\nabla(\mathbf{s} - \mathbf{u}_h^{k,i})\|^2 = (\nabla(\mathbf{s} - \mathbf{u}_h^{k,i}), \nabla(\mathbf{s} - \mathbf{u}_h^{k,i})).$$

As in Remark 1.2 with respect to Definition 1.1, the following equivalent formulation of (4.8) can be given: find $(\mathbf{s}, w) \in \mathbf{V} \times Q$ such that

$$(\nabla \mathbf{s}, \nabla \mathbf{v}) - (\nabla \cdot \mathbf{v}, w) = (\nabla \mathbf{u}_h^{k,i}, \nabla \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V}, \quad (4.12a)$$

$$(\nabla \cdot \mathbf{s}, q) = 0 \quad \forall q \in Q. \quad (4.12b)$$

Thus we have, as $\mathbf{s} - \mathbf{u}_h^{k,i} \in \mathbf{V}$ can be taken as a test function in (4.12a),

$$(\nabla(\mathbf{s} - \mathbf{u}_h^{k,i}), \nabla(\mathbf{s} - \mathbf{u}_h^{k,i})) = (\nabla \cdot (\mathbf{s} - \mathbf{u}_h^{k,i}), w) = -(\nabla \cdot \mathbf{u}_h^{k,i}, w) \leq \|\nabla \cdot \mathbf{u}_h^{k,i}\| \|w\|.$$

We have also used the fact that $\nabla \cdot \mathbf{s} = 0$ and the Cauchy–Schwarz inequality. To estimate $\|w\|$, we will rely on the inf–sup condition (1.4):

$$\|w\| \leq \frac{1}{\beta} \sup_{\mathbf{v} \in \mathbf{V}} \frac{(w, \nabla \cdot \mathbf{v})}{\|\nabla \mathbf{v}\|} = \frac{1}{\beta} \sup_{\mathbf{v} \in \mathbf{V}} \frac{(\nabla(\mathbf{s} - \mathbf{u}_h^{k,i}), \nabla \mathbf{v})}{\|\nabla \mathbf{v}\|} \leq \frac{1}{\beta} \|\nabla(\mathbf{s} - \mathbf{u}_h^{k,i})\|,$$

where we have employed (4.12a) and the Cauchy–Schwarz inequality. We thus arrive at

$$(\nabla(\mathbf{s} - \mathbf{u}_h^{k,i}), \nabla(\mathbf{s} - \mathbf{u}_h^{k,i})) \leq \frac{\|\nabla \cdot \mathbf{u}_h^{k,i}\|}{\beta} \|\nabla(\mathbf{s} - \mathbf{u}_h^{k,i})\|,$$

whence

$$\|\nabla(\mathbf{s} - \mathbf{u}_h^{k,i})\| \leq \eta_D^{k,i}. \quad (4.13)$$

The term $\|p - p_h^k\|$ is treated through the inf–sup condition (1.4), which in particular gives

$$\|p - p_h^k\| \leq \frac{1}{\beta} \sup_{\varphi \in \mathbf{V}; \|\nabla \varphi\|=1} (p - p_h^k, \nabla \cdot \varphi).$$

Fix $\varphi \in \mathbf{V}$ with $\|\nabla \varphi\| = 1$. The weak solution characterization (1.3a) gives

$$(p, \nabla \cdot \varphi) = (\nabla \mathbf{u}, \nabla \varphi) - (\mathbf{f}, \varphi).$$

Thus using also $(p_h^k, \nabla \cdot \varphi) = (p_h^k \mathbb{I}, \nabla \varphi)$, adding and subtracting $(\mathfrak{d}_h^{k,i+\nu}, \nabla \varphi)$ as well as $(\nabla \mathbf{u}_h^{k,i}, \nabla \varphi)$, and using the Green theorem, we arrive at

$$\begin{aligned} (p - p_h^k, \nabla \cdot \varphi) &= (\nabla(\mathbf{u} - \mathbf{u}_h^{k,i}), \nabla \varphi) - (\nabla \cdot \mathfrak{d}_h^{k,i+\nu} + \mathbf{f}, \varphi) \\ &\quad + (\nabla \mathbf{u}_h^{k,i} - p_h^k \mathbb{I} - \mathfrak{d}_h^{k,i+\nu}, \nabla \varphi). \end{aligned}$$

The two last terms on the above right-hand side could be estimated as in (4.11) and the first one could be bounded by $\|\nabla(\mathbf{u} - \mathbf{u}_h^{k,i})\|$ and consequently by (4.7a). Such a straightforward bound can, however, be substantially improved while proceeding as in [1]. Let $\varphi_C \in \mathbf{V}$ with $\nabla \cdot \varphi_C = 0$ be the solution of

$$(\nabla \varphi_C, \nabla \mathbf{v}) = (\nabla \varphi, \nabla \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V} \text{ with } \nabla \cdot \mathbf{v} = 0.$$

Let $\varphi_{\text{NC}} := \varphi - \varphi_C$, and note that

$$(\nabla \varphi_{\text{NC}}, \nabla \mathbf{v}) = 0 \quad \forall \mathbf{v} \in \mathbf{V} \text{ with } \nabla \cdot \mathbf{v} = 0. \quad (4.14)$$

Then, as in (4.9), we immediately have

$$\|\nabla \varphi\|^2 = \|\nabla \varphi_C\|^2 + \|\nabla \varphi_{\text{NC}}\|^2, \quad (4.15)$$

as $(\nabla \varphi_{\text{NC}}, \nabla \varphi_C) = 0$. Now

$$(p - p_h^k, \nabla \cdot \varphi) = (p - p_h^k, \nabla \cdot \varphi_{\text{NC}})$$

and

$$(p - p_h^k, \nabla \cdot \boldsymbol{\varphi}_{\text{NC}}) = (\nabla(\mathbf{u} - \mathbf{u}_h^{k,i}), \nabla \boldsymbol{\varphi}_{\text{NC}}) - (\nabla \cdot \mathbf{d}_h^{k,i+\nu} + \mathbf{f}, \boldsymbol{\varphi}_{\text{NC}}) \\ + (\nabla \mathbf{u}_h^{k,i} - p_h^k \mathbb{I} - \mathbf{d}_h^{k,i+\nu}, \nabla \boldsymbol{\varphi}_{\text{NC}}).$$

We can estimate

$$-(\nabla \cdot \mathbf{d}_h^{k,i+\nu} + \mathbf{f}, \boldsymbol{\varphi}_{\text{NC}}) + (\nabla \mathbf{u}_h^{k,i} - p_h^k \mathbb{I} - \mathbf{d}_h^{k,i+\nu}, \nabla \boldsymbol{\varphi}_{\text{NC}}) \leq \eta_{\text{F}}^{k,i} + \eta_{\text{osc}}^{k,i} + \eta_{\text{rem}}^{k,i},$$

as in (4.11), since $\|\nabla \boldsymbol{\varphi}_{\text{NC}}\| \leq \|\nabla \boldsymbol{\varphi}\| = 1$ by (4.15). Finally for the term $(\nabla(\mathbf{u} - \mathbf{u}_h^{k,i}), \nabla \boldsymbol{\varphi}_{\text{NC}})$, we have

$$(\nabla(\mathbf{u} - \mathbf{u}_h^{k,i}), \nabla \boldsymbol{\varphi}_{\text{NC}}) = (\nabla(\mathbf{s} - \mathbf{u}_h^{k,i}), \nabla \boldsymbol{\varphi}_{\text{NC}}),$$

as $(\nabla(\mathbf{u} - \mathbf{s}), \nabla \boldsymbol{\varphi}_{\text{NC}}) = 0$ by (4.14) since $\mathbf{u} - \mathbf{s} \in \mathbf{V}$ and $\nabla \cdot (\mathbf{u} - \mathbf{s}) = 0$. Now the Cauchy–Schwarz inequality and (4.13) together with $\|\nabla \boldsymbol{\varphi}_{\text{NC}}\| \leq \|\nabla \boldsymbol{\varphi}\| = 1$ give

$$(\nabla(\mathbf{u} - \mathbf{u}_h^{k,i}), \nabla \boldsymbol{\varphi}_{\text{NC}}) \leq \eta_{\text{D}}^{k,i}.$$

Combining the above results gives (4.7b). \square

Following the idea in [26, Section 7.2] and [20, Section 4], on k -th Uzawa and i -th algebraic solver step, we construct the discretization stress reconstruction $\mathbf{d}_h^{k,i}$. And after ν additional algebraic solver steps, we construct the Cauchy stress reconstruction $\mathbf{d}_h^{k,i+\nu}$ of Assumption 4.1. Define the algebraic error stress reconstruction as

$$\mathbf{a}_h^{k,i} := \mathbf{d}_h^{k,i+\nu} - \mathbf{d}_h^{k,i}. \quad (4.16)$$

Then we have

$$\|\mathbf{a}_h^{k,i}\| \rightarrow 0 \quad \text{as the inner algebraic solver converges,}$$

which is a necessary condition for algebraic error control from [20, Section 3.2]. We then have:

Theorem 4.3 (A posteriori estimate distinguishing the error components). *Let (\mathbf{u}, p) be the weak solution given by Definition 1.1 and let $(\mathbf{u}_h^{k,i}, p_h^k) \in \mathbf{V}_h \times Q_h$ be the approximate solution obtained by Algorithm 3.1 on the Uzawa step $k \geq 1$ and the inner algebraic iteration $i \geq 1$. For $\nu \geq 0$ additional inner algebraic solver steps, let Assumption 4.1 be satisfied. Using the convention (4.6), we then have:*

$$\|\nabla(\mathbf{u} - \mathbf{u}_h^{k,i})\| + \beta \|p - p_h^k\| \leq 2 \left(\eta_{\text{disc}}^{k,i} + \eta_{\text{alg}}^{k,i} + \eta_{\text{Uza}}^{k,i} + \eta_{\text{rem}}^{k,i} + \eta_{\text{osc}}^{k,i} \right), \quad (4.17)$$

with $\eta_{\text{rem}}^{k,i}$ and $\eta_{\text{osc}}^{k,i}$ given respectively by (4.5c) and (4.5d), and with the discretization, inner algebraic solver, and Uzawa estimators respectively given by:

the discretization estimator:

$$\eta_{\text{disc},K}^{k,i} := \|\nabla \mathbf{u}_h^{k,i} - p_h^k \mathbb{I} - \mathbf{d}_h^{k,i}\|_K \\ + \beta^{-1} \|\nabla \cdot \mathbf{u}_h^{k,i} - \delta_h^{k,i}\|_K, \quad (4.18a)$$

the inner algebraic solver estimator:

$$\eta_{\text{alg},K}^{k,i} := \|\mathbf{a}_h^{k,i}\|_K, \quad (4.18b)$$

the Uzawa estimator:

$$\eta_{\text{Uza},K}^{k,i} := \beta^{-1} \|\delta_h^{k,i}\|_K, \quad (4.18c)$$

where $\delta_h^{k,i} \in Q_h$ is given by

$$(\delta_h^{k,i}, q_h) = (\nabla \cdot \mathbf{u}_h^{k,i}, q_h) + s_h(\mathbf{u}_h^{k,i}, p_h^k; q_h), \quad \forall q_h \in Q_h.$$

Proof. The decomposition (4.16) and the triangle inequality yield

$$\|\nabla \mathbf{u}_h^{k,i} - p_h^k \mathbb{I} - \mathbf{d}_h^{k,i+\nu}\|_K \leq \|\nabla \mathbf{u}_h^{k,i} - p_h^k \mathbb{I} - \mathbf{d}_h^{k,i}\|_K + \|\mathbf{a}_h^{k,i}\|_K.$$

Note that it follows from (2.5b) that upon the full convergence of the Uzawa iterations, $\delta_h^{k,i} = 0$. Thus this term represents the error in the Uzawa iteration, and with the triangle inequality, we have

$$\beta^{-1} \|\nabla \cdot \mathbf{u}_h^{k,i}\| \leq \beta^{-1} \|\nabla \cdot \mathbf{u}_h^{k,i} - \delta_h^{k,i}\| + \beta^{-1} \|\delta_h^{k,i}\|,$$

where the first term contributes to the estimator of the discretization error and the second one estimates the Uzawa iteration error. \square

5 Polynomial-degree-robust efficiency

In this section, we prove the polynomial-degree-robust efficiency of the a posteriori error estimate of Theorem 4.3. Define the shape regularity parameter $\kappa_{\mathcal{T}_h}$ by

$$\kappa_{\mathcal{T}_h} := \max_{K \in \mathcal{T}_h} \frac{h_K}{\rho_K},$$

where ρ_K represents the diameter of the largest circle inscribed into the triangle K . We suppose henceforth that there exists a real number $\kappa_{\mathcal{T}} > 0$ which bounds $\kappa_{\mathcal{T}_h}$ for any considered mesh \mathcal{T}_h .

We will need some additional notation. Let \mathcal{V}_h stand for the set of the nodes of \mathcal{T}_h , $\mathcal{V}_h^{\text{int}}$ for the set of the interior nodes of \mathcal{T}_h , and $\mathcal{V}_h^{\text{ext}}$ for the set of the boundary nodes of \mathcal{T}_h . For each $\mathbf{a} \in \mathcal{V}_h$, let $\omega_{\mathbf{a}}$ denote the patch of the vertex \mathbf{a} , i.e., all the elements $K \in \mathcal{T}_h$ which share the vertex \mathbf{a} , and let $h_{\omega_{\mathbf{a}}}$ denote the diameter of the patch $\omega_{\mathbf{a}}$. Let $\mathbb{RTN}_l(\omega_{\mathbf{a}}) \subset \mathbb{H}(\text{div}, \omega_{\mathbf{a}})$ be the Raviart-Thomas-Nédélec space of order $l \geq 0$ on the patch $\omega_{\mathbf{a}}$. Let next $\mathbb{RTN}_l^{\text{N},0}(\omega_{\mathbf{a}})$ be its subspace with zero normal flux through $\partial\omega_{\mathbf{a}}$ for $\mathbf{a} \in \mathcal{V}_h^{\text{int}}$ and with zero normal flux through $\partial\omega_{\mathbf{a}} \setminus \partial\Omega$ for $\mathbf{a} \in \mathcal{V}_h^{\text{ext}}$. $[\mathbb{P}_l^*(\omega_{\mathbf{a}})]^2$ is spanned by piecewise l -th order polynomials on $\omega_{\mathbf{a}}$, with zero mean componentwise on $\omega_{\mathbf{a}}$ when $\mathbf{a} \in \mathcal{V}_h^{\text{int}}$; the mean value condition is not imposed on $\mathbf{a} \in \mathcal{V}_h^{\text{ext}}$. Let $\psi_{\mathbf{a}} \in \mathbb{P}_1(\mathcal{T}_h) \cap C^0(\overline{\Omega})$ stand for the hat basis function associated with the vertex $\mathbf{a} \in \mathcal{V}_h$, i.e., $\psi_{\mathbf{a}}(\mathbf{a}) = 1$ and $\psi_{\mathbf{a}}(\mathbf{a}') = 0$ for $\mathbf{a}' \in \mathcal{V}_h$, $\mathbf{a}' \neq \mathbf{a}$. Let $\boldsymbol{\psi}_{\mathbf{a},m}$, $1 \leq m \leq 2$ be its vector variant:

$$\boldsymbol{\psi}_{\mathbf{a},1} := \begin{pmatrix} \psi_{\mathbf{a}} \\ 0 \end{pmatrix} \text{ and } \boldsymbol{\psi}_{\mathbf{a},2} := \begin{pmatrix} 0 \\ \psi_{\mathbf{a}} \end{pmatrix}.$$

In order to prove polynomial-degree-robust efficiency, we first need to specify the construction of the equilibrated flux. We can do so under the following general hypothesis:

Assumption 5.1. *Let $(\mathbf{u}_h^{k,i}, p_h^k) \in \mathbf{V}_h \times Q_h$ be the approximate solution obtained by Algorithm 3.1 on the Uzawa iteration $k \geq 1$ and the inner algebraic iteration $i \geq 1$. Suppose that for all $\mathbf{a} \in \mathcal{V}_h^{\text{int}}$, there exists $\mathbf{R}_{\mathbf{a}}^{k,i} \in \mathbb{R}^2$ constructed from the algebraic residual vector $\mathbf{R}^{k,i}$ of (3.1) such that*

$$\begin{aligned} (\mathbf{f}, \boldsymbol{\psi}_{\mathbf{a},m})_{\omega_{\mathbf{a}}} - (\nabla \mathbf{u}_h^{k,i}, \nabla \boldsymbol{\psi}_{\mathbf{a},m})_{\omega_{\mathbf{a}}} + (\nabla \cdot \boldsymbol{\psi}_{\mathbf{a},m}, p_h^k)_{\omega_{\mathbf{a}}} - (\mathbf{R}_{\mathbf{a}}^{k,i})_m &= 0, \\ \forall \mathbf{a} \in \mathcal{V}_h^{\text{int}}, 1 \leq m \leq 2. \end{aligned} \quad (5.1)$$

For $i + \nu$ in place of i , $\nu \geq 0$, we have $\mathbf{R}_{\mathbf{a}}^{k,i+\nu} \in \mathbb{R}^2$ and (5.1) corresponds to (3.2).

For simplicity of notation, set $\mathbf{R}_{\mathbf{a}}^{k,i} = \mathbf{0}$ for $\mathbf{a} \in \mathcal{V}_h^{\text{ext}}$. Following the recent work [32], we construct the equilibrated flux by solving the mixed finite element local problems on the patches $\omega_{\mathbf{a}}$:

Definition 5.2 (Discretization stress reconstruction $\mathbf{d}_h^{k,i}$). *For each Uzawa step $k \geq 1$, each inner algebraic iteration $i \geq 1$, and on each patch $\omega_{\mathbf{a}}$, $\mathbf{a} \in \mathcal{V}_h$, find $(\mathbf{d}_{\mathbf{a}}^{k,i}, \mathbf{o}_{\mathbf{a}}^{k,i}) \in \mathbb{RTN}_l^{\text{N},0}(\omega_{\mathbf{a}}) \times [\mathbb{P}_l^*(\omega_{\mathbf{a}})]^2$ such that*

$$(\mathbf{d}_{\mathbf{a}}^{k,i}, \mathbb{v}_h)_{\omega_{\mathbf{a}}} + (\mathbf{o}_{\mathbf{a}}^{k,i}, \nabla \cdot \mathbb{v}_h)_{\omega_{\mathbf{a}}} = ((\nabla \mathbf{u}_h^{k,i} - p_h^k \mathbb{I}) \boldsymbol{\psi}_{\mathbf{a}}, \mathbb{v}_h)_{\omega_{\mathbf{a}}}, \quad \forall \mathbb{v}_h \in \mathbb{RTN}_l^{\text{N},0}(\omega_{\mathbf{a}}), \quad (5.2a)$$

$$\begin{aligned} -((\nabla \cdot \mathbf{d}_{\mathbf{a}}^{k,i})_m, (\mathbf{q}_h)_m)_{\omega_{\mathbf{a}}} &= (\mathbf{f} \cdot \boldsymbol{\psi}_{\mathbf{a},m} - (\nabla \mathbf{u}_h^{k,i} - p_h^k \mathbb{I}) : \nabla \boldsymbol{\psi}_{\mathbf{a},m}, (\mathbf{q}_h)_m)_{\omega_{\mathbf{a}}} \\ &\quad - ((\mathbf{R}_{\mathbf{a}}^{k,i})_m |\omega_{\mathbf{a}}|^{-1}, (\mathbf{q}_h)_m)_{\omega_{\mathbf{a}}}, \\ \forall \mathbf{q}_h &\in [\mathbb{P}_l^*(\omega_{\mathbf{a}})]^2, 1 \leq m \leq 2. \end{aligned} \quad (5.2b)$$

Then we set

$$\mathbf{d}_h^{k,i} := \sum_{\mathbf{a} \in \mathcal{V}_h} \mathbf{d}_{\mathbf{a}}^{k,i}, \quad (5.3)$$

Remark 5.3 (Construction of $\mathbf{d}_h^{k,i+\nu}$). We construct $\mathbf{d}_h^{k,i+\nu}$ via Definition 5.2, with $i + \nu$ in place of i .

Remark 5.4 (Cheaper equilibration). Following [10], one can also carry the reconstruction in $\mathbb{RTN}_{l-1}^{N,0}(\omega_{\mathbf{a}}) \times [\mathbb{P}_{l-1}^*(\omega_{\mathbf{a}})]^2$. Employing the Raviart–Thomas–Nédélec projector, the efficiency of such a simplification has been shown in [20], albeit the polynomial-degree-robustness has not been treated therein.

The problems (5.2) are well-posed thanks to (5.1), which is the corresponding Neumann compatibility condition for $\mathbf{a} \in \mathcal{V}_h^{\text{int}}$. We now follow the work [10] and [20], and extend it to the Stokes setting. For $q \in L^2(\Omega)$, denote by $\Pi_{\mathbb{P}_l(\mathcal{T}_h)} q \in \mathbb{P}_l(\mathcal{T}_h)$ the projection such that:

$$(q - \Pi_{\mathbb{P}_l(\mathcal{T}_h)} q, q_h) = 0 \quad \forall q_h \in \mathbb{P}_l(\mathcal{T}_h).$$

We will use similar notation for $[\mathbb{P}_l(\mathcal{T}_h)]^2$.

Theorem 5.5 (Polynomial-degree-robust efficiency). Let (\mathbf{u}, p) be the weak solution given by Definition 1.1 and let $(\mathbf{u}_h^{k,i}, p_h^k) \in \mathbf{V}_h \times Q_h$ be the approximate solution obtained by Algorithm 3.1 on the Uzawa step $k \geq 1$ and the inner algebraic iteration $i \geq 1$. Let the estimators be constructed following Definition 5.2 and let, for $\nu \geq 0$ additional inner algebraic solver steps, the local stopping criteria (3.4) of Remark 3.2 be satisfied. Suppose that $0 < \gamma_{\text{rem},K}, \gamma_{\text{alg},K}, \gamma_{\text{Uza},K} < 1$. Then, with the constants C_{st} and $C_{\text{PF},\omega_{\mathbf{a}}} > 0$ only depending on the shape regularity parameter $\kappa_{\mathcal{T}}$, see [21] and the references therein, there holds, for all $K \in \mathcal{T}_h$,

$$\begin{aligned} & \eta_{\text{disc},K}^{k,i} + \eta_{\text{alg},K}^{k,i} + \eta_{\text{Uza},K}^{k,i} + \eta_{\text{rem},K}^{k,i} \\ & \leq 4(1 + \gamma_{\text{Uza},K}(1 - \gamma_{\text{Uza},K})^{-1})C_{\text{st}} \sum_{1 \leq m \leq 2} \sum_{\mathbf{a} \in \mathcal{V}_K} \max\{1, \beta^{-1}\} \\ & \quad \times (2C_{\text{PF},\omega_{\mathbf{a}}} h_{\omega_{\mathbf{a}}} \|\nabla \psi_{\mathbf{a},m}\|_{\omega_{\mathbf{a},\infty}} + 1) (\|\nabla(\mathbf{u} - \mathbf{u}_h^{k,i})\|_{\omega_{\mathbf{a}}} + \beta \|p - p_h^k\|_{\omega_{\mathbf{a}}}) \\ & \quad + 4(1 - \gamma_{\text{Uza},K})^{-1} \beta^{-1} \|\nabla(\mathbf{u} - \mathbf{u}_h^{k,i})\|_K + 4(1 + \gamma_{\text{Uza},K}(1 - \gamma_{\text{Uza},K})^{-1})C_{\text{st}} \\ & \quad \times \sum_{1 \leq m \leq 2} \sum_{\mathbf{a} \in \mathcal{V}_K} \left\{ \sum_{K' \in \omega_{\mathbf{a}}} \left(\frac{h_{K'}}{\pi} \|\Pi_{\mathbb{P}_l(\mathcal{T}_h)}(\mathbf{f} \cdot \psi_{\mathbf{a},m}) - \mathbf{f} \cdot \psi_{\mathbf{a},m}\|_{K'} \right)^2 \right\}^{1/2}. \end{aligned} \quad (5.4)$$

Proof. From our local stopping criteria (3.4), we have

$$\eta_{\text{rem},K}^{k,i} + \eta_{\text{alg},K}^{k,i} + \eta_{\text{Uza},K}^{k,i} \leq 3\eta_{\text{disc},K}^{k,i}.$$

Then, by (4.18a), we only need to bound $\|\nabla \mathbf{u}_h^{k,i} - p_h^k \mathbb{I} - \mathbf{d}_h^{k,i}\|_K$ and $\beta^{-1} \|\nabla \cdot \mathbf{u}_h^{k,i} - \delta_h^{k,i}\|_K$.

Using the triangle inequality, the definition (4.18c), $\nabla \cdot \mathbf{u} = 0$, and the local stopping criterion (3.4c), we obtain

$$\begin{aligned} \beta^{-1} \|\nabla \cdot \mathbf{u}_h^{k,i} - \delta_h^{k,i}\|_K & \leq \beta^{-1} \|\nabla \cdot \mathbf{u}_h^{k,i}\|_K + \beta^{-1} \|\delta_h^{k,i}\|_K \\ & \leq \beta^{-1} \|\nabla \cdot (\mathbf{u} - \mathbf{u}_h^{k,i})\|_K + \gamma_{\text{Uza},K} \eta_{\text{disc},K}^{k,i}. \end{aligned}$$

Then the fact $\|\nabla \cdot (\mathbf{u} - \mathbf{u}_h^{k,i})\| \leq \|\nabla(\mathbf{u} - \mathbf{u}_h^{k,i})\|$ and the definition (4.18a) yield

$$\begin{aligned} & (1 - \gamma_{\text{Uza},K}) \beta^{-1} \|\nabla \cdot \mathbf{u}_h^{k,i} - \delta_h^{k,i}\|_K \\ & \leq \beta^{-1} \|\nabla(\mathbf{u} - \mathbf{u}_h^{k,i})\|_K + \gamma_{\text{Uza},K} \|\nabla \mathbf{u}_h^{k,i} - p_h^k \mathbb{I} - \mathbf{d}_h^{k,i}\|_K, \end{aligned}$$

where we also use the assumption $\gamma_{\text{Uza},K} < 1$.

To bound $\|\nabla \mathbf{u}_h^{k,i} - p_h^k \mathbb{I} - \mathbf{d}_h^{k,i}\|_K$, we use the same arguments as in [21] and proceed in three steps.

Step 1: Let $\mathbf{a} \in \mathcal{V}_K$ and consider the patch $\omega_{\mathbf{a}}$ of the vertex \mathbf{a} . Define the function space $H_*^1(\omega_{\mathbf{a}})$

$$\begin{aligned} H_*^1(\omega_{\mathbf{a}}) & = \{v \in H^1(\omega_{\mathbf{a}}), (v, 1)_{\omega_{\mathbf{a}}} = 0\}, & \mathbf{a} \in \mathcal{V}_h^{\text{int}}, \\ H_*^1(\omega_{\mathbf{a}}) & = \{v \in H^1(\omega_{\mathbf{a}}), v = 0 \text{ on } \partial\omega_{\mathbf{a}} \cap \partial\Omega\}, & \mathbf{a} \in \mathcal{V}_h^{\text{ext}}. \end{aligned}$$

Let $1 \leq m \leq 2$ and define $r_{\mathbf{a},m} \in H_*^1(\omega_{\mathbf{a}})$ such that for $v \in H_*^1(\omega_{\mathbf{a}})$,

$$\begin{aligned} (\nabla r_{\mathbf{a},m}, \nabla v)_{\omega_{\mathbf{a}}} &= -((\nabla \mathbf{u}_h^{k,i} - p_h^k \mathbb{I}) \boldsymbol{\psi}_{\mathbf{a},m}, \nabla v)_{\omega_{\mathbf{a}}} \\ &\quad + (\mathbf{f} \cdot \boldsymbol{\psi}_{\mathbf{a},m} - (\nabla \mathbf{u}_h^{k,i} - p_h^k \mathbb{I}) : \nabla \boldsymbol{\psi}_{\mathbf{a},m} - (\mathbf{R}_{\mathbf{a}}^{k,i})_m |\omega_{\mathbf{a}}|^{-1}, v)_{\omega_{\mathbf{a}}}. \end{aligned} \quad (5.5)$$

We now prove that

$$\begin{aligned} \|\nabla r_{\mathbf{a},m}\|_{\omega_{\mathbf{a}}} &\leq \max\{1, \beta^{-1}\} (2C_{\text{PF},\omega_{\mathbf{a}}} h_{\omega_{\mathbf{a}}} \|\nabla \boldsymbol{\psi}_{\mathbf{a},m}\|_{\omega_{\mathbf{a},\infty}} + 1) \\ &\quad \times (\|\nabla(\mathbf{u} - \mathbf{u}_h^{k,i})\|_{\omega_{\mathbf{a}}} + \beta \|p - p_h^k\|_{\omega_{\mathbf{a}}}). \end{aligned} \quad (5.6)$$

We have

$$\|\nabla r_{\mathbf{a},m}\|_{\omega_{\mathbf{a}}} = \sup_{v \in H_*^1(\omega_{\mathbf{a}}), \|\nabla v\|_{\omega_{\mathbf{a}}}=1} (\nabla r_{\mathbf{a},m}, \nabla v)_{\omega_{\mathbf{a}}}.$$

Fix $v \in H_*^1(\omega_{\mathbf{a}})$ with $\|\nabla v\|_{\omega_{\mathbf{a}}} = 1$; then (5.5) gives

$$\begin{aligned} &(\nabla r_{\mathbf{a},m}, \nabla v)_{\omega_{\mathbf{a}}} \\ &= -((\nabla \mathbf{u}_h^{k,i} - p_h^k \mathbb{I}) \boldsymbol{\psi}_{\mathbf{a},m}, \nabla v)_{\omega_{\mathbf{a}}} \\ &\quad + (\mathbf{f} \cdot \boldsymbol{\psi}_{\mathbf{a},m} - (\nabla \mathbf{u}_h^{k,i} - p_h^k \mathbb{I}) : \nabla \boldsymbol{\psi}_{\mathbf{a},m} - (\mathbf{R}_{\mathbf{a}}^{k,i})_m |\omega_{\mathbf{a}}|^{-1}, v)_{\omega_{\mathbf{a}}} \\ &= (\mathbf{f} \cdot \boldsymbol{\psi}_{\mathbf{a},m}, v)_{\omega_{\mathbf{a}}} - (\nabla \mathbf{u}_h^{k,i} - p_h^k \mathbb{I}, \nabla(\boldsymbol{\psi}_{\mathbf{a},m} v))_{\omega_{\mathbf{a}}} - ((\mathbf{R}_{\mathbf{a}}^{k,i})_m |\omega_{\mathbf{a}}|^{-1}, v)_{\omega_{\mathbf{a}}} \\ &= ((\nabla \mathbf{u} - p \mathbb{I}), \nabla(\boldsymbol{\psi}_{\mathbf{a},m} v))_{\omega_{\mathbf{a}}} - (\nabla \mathbf{u}_h^{k,i} - p_h^k \mathbb{I}, \nabla(\boldsymbol{\psi}_{\mathbf{a},m} v))_{\omega_{\mathbf{a}}} \\ &\quad - ((\mathbf{R}_{\mathbf{a}}^{k,i})_m |\omega_{\mathbf{a}}|^{-1}, v)_{\omega_{\mathbf{a}}} \\ &\leq (\|\nabla(\mathbf{u} - \mathbf{u}_h^{k,i})\|_{\omega_{\mathbf{a}}} + \|p - p_h^k\|_{\omega_{\mathbf{a}}}) \|\nabla(\boldsymbol{\psi}_{\mathbf{a},m} v)\|_{\omega_{\mathbf{a}}} \\ &\quad + \|(\mathbf{R}_{\mathbf{a}}^{k,i})_m |\omega_{\mathbf{a}}|^{-1}\|_{\omega_{\mathbf{a}}} \|v\|_{\omega_{\mathbf{a}}} \\ &\leq \max\{1, \beta^{-1}\} (\|\nabla(\mathbf{u} - \mathbf{u}_h^{k,i})\|_{\omega_{\mathbf{a}}} + \beta \|p - p_h^k\|_{\omega_{\mathbf{a}}}) \|\nabla(\boldsymbol{\psi}_{\mathbf{a},m} v)\|_{\omega_{\mathbf{a}}} \\ &\quad + |(\mathbf{R}_{\mathbf{a}}^{k,i})_m| |\omega_{\mathbf{a}}|^{-1/2} \|v\|_{\omega_{\mathbf{a}}}, \end{aligned} \quad (5.7)$$

where we used the fact that $(\mathbf{f}, \boldsymbol{\psi}_{\mathbf{a},m} v)_{\omega_{\mathbf{a}}} = (\mathbf{f}, \boldsymbol{\psi}_{\mathbf{a},m} v)_{\Omega}$ as $\boldsymbol{\psi}_{\mathbf{a},m} v = 0$ on the boundary of $\omega_{\mathbf{a}}$. Moreover, employing $\|\nabla v\|_{\omega_{\mathbf{a}}} = 1$ and $\|\boldsymbol{\psi}_{\mathbf{a},m}\|_{\omega_{\mathbf{a},\infty}} = 1$, we have

$$\begin{aligned} \|\nabla(\boldsymbol{\psi}_{\mathbf{a},m} v)\|_{\omega_{\mathbf{a}}} &= \|\nabla \boldsymbol{\psi}_{\mathbf{a},m} v + \boldsymbol{\psi}_{\mathbf{a},m} \nabla v\|_{\omega_{\mathbf{a}}} \\ &\leq \|\nabla \boldsymbol{\psi}_{\mathbf{a},m}\|_{\omega_{\mathbf{a},\infty}} \|v\|_{\omega_{\mathbf{a}}} + \|\boldsymbol{\psi}_{\mathbf{a},m}\|_{\omega_{\mathbf{a},\infty}} \|\nabla v\|_{\omega_{\mathbf{a}}} \\ &\leq C_{\text{PF},\omega_{\mathbf{a}}} h_{\omega_{\mathbf{a}}} \|\nabla \boldsymbol{\psi}_{\mathbf{a},m}\|_{\omega_{\mathbf{a},\infty}} + 1, \end{aligned} \quad (5.8)$$

where we use the Poincaré inequality (4.2) on the patch $\omega_{\mathbf{a}}$ for $\mathbf{a} \in \mathcal{V}_h^{\text{int}}$, and the Friedrichs inequality (4.3)

$$\|v\|_{\omega_{\mathbf{a}}} \leq h_{\omega_{\mathbf{a}}} \|\nabla v\|_{\omega_{\mathbf{a}}}, \quad \forall v \in H_*^1(\omega_{\mathbf{a}}),$$

for $\mathbf{a} \in \mathcal{V}_h^{\text{ext}}$. For any $\mathbf{a} \in \mathcal{V}_h^{\text{int}}$, with (5.1), we have

$$\begin{aligned} (\mathbf{R}_{\mathbf{a}}^{k,i})_m &= (\mathbf{f}, \boldsymbol{\psi}_{\mathbf{a},m})_{\omega_{\mathbf{a}}} - (\nabla \mathbf{u}_h^{k,i}, \nabla \boldsymbol{\psi}_{\mathbf{a},m})_{\omega_{\mathbf{a}}} + (\nabla \cdot \boldsymbol{\psi}_{\mathbf{a},m}, p_h^k)_{\omega_{\mathbf{a}}} \\ &= (\nabla \mathbf{u}, \nabla \boldsymbol{\psi}_{\mathbf{a},m})_{\omega_{\mathbf{a}}} - (\nabla \cdot \boldsymbol{\psi}_{\mathbf{a},m}, p)_{\omega_{\mathbf{a}}} - (\nabla \mathbf{u}_h^{k,i}, \nabla \boldsymbol{\psi}_{\mathbf{a},m})_{\omega_{\mathbf{a}}} \\ &\quad + (\nabla \cdot \boldsymbol{\psi}_{\mathbf{a},m}, p_h^k)_{\omega_{\mathbf{a}}} \\ &\leq (\|\nabla(\mathbf{u} - \mathbf{u}_h^{k,i})\|_{\omega_{\mathbf{a}}} + \|p - p_h^k\|_{\omega_{\mathbf{a}}}) \|\nabla \boldsymbol{\psi}_{\mathbf{a},m}\|_{\omega_{\mathbf{a}}} \\ &\leq \max\{1, \beta^{-1}\} (\|\nabla(\mathbf{u} - \mathbf{u}_h^{k,i})\|_{\omega_{\mathbf{a}}} + \beta \|p - p_h^k\|_{\omega_{\mathbf{a}}}) \|\nabla \boldsymbol{\psi}_{\mathbf{a},m}\|_{\omega_{\mathbf{a},\infty}} |\omega_{\mathbf{a}}|^{1/2}. \end{aligned} \quad (5.9)$$

Employing again $\|v\|_{\omega_{\mathbf{a}}} \leq C_{\text{PF},\omega_{\mathbf{a}}} h_{\omega_{\mathbf{a}}} \|\nabla v\|_{\omega_{\mathbf{a}}}$, (5.7)–(5.9) yield (5.6).

Step 2: Still for $\mathbf{a} \in \mathcal{V}_K$, let $\tilde{r}_{\mathbf{a},m} \in H_*^1(\omega_{\mathbf{a}})$, $1 \leq m \leq 2$, be defined by

$$\begin{aligned} &(\nabla \tilde{r}_{\mathbf{a},m}, \nabla v)_{\omega_{\mathbf{a}}} \\ &= -((\nabla \mathbf{u}_h^{k,i} - p_h^k \mathbb{I}) \boldsymbol{\psi}_{\mathbf{a},m}, \nabla v)_{\omega_{\mathbf{a}}} + (\Pi_{\mathbb{P}_l(\mathcal{T}_h)}(\mathbf{f} \cdot \boldsymbol{\psi}_{\mathbf{a},m}) \\ &\quad - (\nabla \mathbf{u}_h^{k,i} - p_h^k \mathbb{I}) : \nabla \boldsymbol{\psi}_{\mathbf{a},m} - (\mathbf{R}_{\mathbf{a}}^{k,i})_m |\omega_{\mathbf{a}}|^{-1}, (\mathbf{q}_h)_m)_{\omega_{\mathbf{a}}}. \end{aligned}$$

Using that this problem, in contrast to (5.5), has piecewise polynomial data and that the equilibration of Definition 5.2 is done with the corresponding degree l and componentwise, we can use [21, Corollary 3.16] (relying on [16, 10]) to infer that, for $1 \leq m \leq 2$,

$$\|((\nabla \mathbf{u}_h^{k,i} - p_h^k \mathbb{I})\psi_{\mathbf{a}} - \mathfrak{d}_{\mathbf{a}}^{k,i})_m\|_{\omega_{\mathbf{a}}} \leq C_{\text{st}} \|\nabla \tilde{r}_{\mathbf{a},m}\|_{\omega_{\mathbf{a}}},$$

with the constant C_{st} only depending on the shape regularity parameter $\kappa_{\mathcal{T}}$. Finally, using

$$\begin{aligned} & \sup_{v \in H_*^1(\omega_{\mathbf{a}}); \|\nabla v\|_{\omega_{\mathbf{a}}} = 1} (\Pi_{\mathbb{P}_l(\mathcal{T}_h)}(\mathbf{f} \cdot \boldsymbol{\psi}_{\mathbf{a},m}) - \mathbf{f} \cdot \boldsymbol{\psi}_{\mathbf{a},m}, v)_{\omega_{\mathbf{a}}} \\ &= \sup_{v \in H_*^1(\omega_{\mathbf{a}}); \|\nabla v\|_{\omega_{\mathbf{a}}} = 1} \left\{ \sum_{K' \in \omega_{\mathbf{a}}} (\Pi_{\mathbb{P}_l(\mathcal{T}_h)}(\mathbf{f} \cdot \boldsymbol{\psi}_{\mathbf{a},m}) - \mathbf{f} \cdot \boldsymbol{\psi}_{\mathbf{a},m}, v - v_K)_{K'} \right\} \\ &\leq \left\{ \sum_{K' \in \omega_{\mathbf{a}}} \left(\frac{h_{K'}}{\pi} \|\Pi_{\mathbb{P}_l(\mathcal{T}_h)}(\mathbf{f} \cdot \boldsymbol{\psi}_{\mathbf{a},m}) - \mathbf{f} \cdot \boldsymbol{\psi}_{\mathbf{a},m}\|_{K'} \right)^2 \right\}^{1/2} \end{aligned}$$

leads to

$$\begin{aligned} & \|((\nabla \mathbf{u}_h^{k,i} - p_h^k \mathbb{I})\psi_{\mathbf{a}} - \mathfrak{d}_{\mathbf{a}}^{k,i})_m\|_{\omega_{\mathbf{a}}} \\ &\leq C_{\text{st}} \|\nabla r_{\mathbf{a},m}\|_{\omega_{\mathbf{a}}} + C_{\text{st}} \left\{ \sum_{K' \in \omega_{\mathbf{a}}} \left(\frac{h_{K'}}{\pi} \|\Pi_{\mathbb{P}_l(\mathcal{T}_h)}(\mathbf{f} \cdot \boldsymbol{\psi}_{\mathbf{a},m}) - \mathbf{f} \cdot \boldsymbol{\psi}_{\mathbf{a},m}\|_{K'} \right)^2 \right\}^{1/2}. \end{aligned} \quad (5.10)$$

Step 3: Employing (5.3) and the fact $\sum_{\mathbf{a} \in \mathcal{V}_K} \psi_{\mathbf{a}}|_K = 1$, we get

$$\begin{aligned} \|(\nabla \mathbf{u}_h^{k,i} - p_h^k \mathbb{I} - \mathfrak{d}_h^{k,i})_m\|_K &= \left\| \sum_{\mathbf{a} \in \mathcal{V}_K} ((\nabla \mathbf{u}_h^{k,i} - p_h^k \mathbb{I})\psi_{\mathbf{a}} - \mathfrak{d}_{\mathbf{a}}^{k,i})_m \right\|_K \\ &\leq \sum_{\mathbf{a} \in \mathcal{V}_K} \|((\nabla \mathbf{u}_h^{k,i} - p_h^k \mathbb{I})\psi_{\mathbf{a}} - \mathfrak{d}_{\mathbf{a}}^{k,i})_m\|_K \\ &\leq \sum_{\mathbf{a} \in \mathcal{V}_K} \|((\nabla \mathbf{u}_h^{k,i} - p_h^k \mathbb{I})\psi_{\mathbf{a}} - \mathfrak{d}_{\mathbf{a}}^{k,i})_m\|_{\omega_{\mathbf{a}}}. \end{aligned} \quad (5.11)$$

Combining (5.6), (5.10), and (5.11) yields the remaining assertion. \square

Remark 5.6 (Data oscillation). *As shown in [21, Remark 3.6 and Remark 3.18], if the function \mathbf{f} is elementwise smooth enough, the data oscillation term in (5.4) superconverges.*

Remark 5.7 (Global efficiency). *Using the global stopping criteria (3.3) in place of the local ones (3.4), global efficiency follows.*

6 Applications

In Section 4, we need Assumption 4.1 to derive the guaranteed reliability of our estimate, and in Section 5 we need Assumption 5.1 to derive the polynomial-degree-robust efficiency. We now show how for the different conforming finite element discretizations of Section 2, we can verify these two assumptions. In fact, Assumption 4.1 is a consequence of Assumption 5.1:

Theorem 6.1 (Verification of Assumption 4.1). *On Uzawa step $k \geq 1$, algebraic iteration $i \geq 1$, and for $\nu \geq 0$ additional algebraic iterations, let Assumption 5.1 hold, and construct $\mathfrak{d}_h^{k,i+\nu}$ via Definition 5.2, with $i + \nu$ in place of i . Set*

$$\begin{aligned} \mathbf{r}_h^{k,i+\nu}|_K &:= \sum_{\mathbf{a} \in \mathcal{V}_K} |\omega_{\mathbf{a}}|^{-1} \mathbf{R}_{\mathbf{a}}^{k,i+\nu}, \\ \mathbf{f}_h &:= \Pi_{[\mathbb{P}_l(\mathcal{T}_h)]^2} \mathbf{f}. \end{aligned} \quad (6.1)$$

Then Assumption 4.1 holds.

Proof. All $\mathbf{d}_\mathbf{a}^{k,i+\nu}$ of (5.2) belong to $\mathbb{H}(\operatorname{div}, \Omega)$ (more precisely to $\mathbb{RTN}_l(\Omega)$), so their sum $\mathbf{d}_h^{k,i+\nu}$ also belongs to $\mathbb{H}(\operatorname{div}, \Omega)$ (to $\mathbb{RTN}_l(\Omega)$). Fix now an arbitrary $K \in \mathcal{T}_h$ and $\mathbf{q}_h \in [\mathbb{P}_l(K)]^2$, only supported on the element K . For any $\mathbf{a} \in \mathcal{V}_K$, take this test function in (5.2b) (this is possible thanks to the Neumann compatibility condition (5.1)). Then, for $1 \leq m \leq 2$, we have

$$\begin{aligned} & - ((\nabla \cdot \mathbf{d}_h^{k,i+\nu})_m, (\mathbf{q}_h)_m)_K \\ = & - \sum_{\mathbf{a} \in \mathcal{V}_K} ((\nabla \cdot \mathbf{d}_\mathbf{a}^{k,i+\nu})_m, (\mathbf{q}_h)_m)_K \\ = & \sum_{\mathbf{a} \in \mathcal{V}_K} (\mathbf{f} \cdot \boldsymbol{\psi}_{\mathbf{a},m} - (\nabla \mathbf{u}_h^{k,i+\nu} - p_h^k \mathbb{I}) : \nabla \boldsymbol{\psi}_{\mathbf{a},m}, (\mathbf{q}_h)_m)_K \\ & - (|\omega_\mathbf{a}|^{-1} (\mathbf{R}_\mathbf{a}^{k,i+\nu})_m, (\mathbf{q}_h)_m)_K \\ = & ((\mathbf{f}_h - \mathbf{r}_h^{k,i+\nu})_m, (\mathbf{q}_h)_m)_K, \end{aligned}$$

employing the partition of unity

$$\sum_{\mathbf{a} \in \mathcal{V}_K} \boldsymbol{\psi}_{\mathbf{a},m}|_K = 1 \quad \text{and} \quad \sum_{\mathbf{a} \in \mathcal{V}_K} \nabla \boldsymbol{\psi}_{\mathbf{a},m}|_K = 0$$

and the definitions (6.1). □

We are thus left to check Assumption 5.1:

Theorem 6.2 (Verification of Assumption 5.1). *On Uzawa step $k \geq 1$, algebraic iteration $i \geq 1$, and for $\mathbf{a} \in \mathcal{V}_h^{\text{int}}$ and $1 \leq m \leq 2$, define from (3.1)*

$$(\mathbf{R}_\mathbf{a}^{k,i})_m := \mathbf{R}_j^{k,i},$$

for the basis function \mathbf{v}_h^j of \mathbf{V}_h given by $\boldsymbol{\psi}_{\mathbf{a},m}$. Then Assumption 5.1 holds.

Proof. The claim is straightforward for all unstabilized schemes of Section 2.1, as well as for those stabilized schemes of Section 2.2 where $t_h(\mathbf{u}_h, p_h; \mathbf{v}_h) = 0$ (remark we are only working with the first block equation of (2.5a)). The crucial observation for the Hughes–Franca–Balestra stabilized method is that $t_h(\mathbf{u}_h, p_h; \mathbf{v}_h)$ only contains the elementwise Laplacian of the test function, which is zero for $\boldsymbol{\psi}_{\mathbf{a},m}$. □

7 Numerical examples

In this section, two numerical examples with known analytic solution will be described in detail. The first one has a smooth solution, whereas the second one has a solution with a singularity in a reentrant corner of the domain. We consider the three different types of Uzawa approaches presented in this paper, the exact Uzawa method of Algorithm 1.4, the inexact Uzawa method of Algorithm 1.5, and our adaptive inexact Uzawa method of Algorithm 3.1. For the exact Uzawa method, we use the CG method to solve the linear system; for the regular case, we do not use a preconditioner. We stop the CG iterations whenever the relative algebraic residual gets below 10^{-8} and set $\epsilon = 10^{-8}$ for the stopping criteria on the Uzawa iteration in Algorithm 1.4. For the inexact Uzawa method, we take the same stopping criterion for the Uzawa iterations, and we set $\tau = 1$ for the inner stopping criterion. Finally, for our adaptive Uzawa method, the parameters for the stopping criteria are set as:

$$\gamma_{\text{rem}} := 1, \gamma_{\text{Uza}} := 0.5, \gamma_{\text{alg}} := 0.5.$$

The choice of the number of additional steps is taken as $\nu_0 := 5$.

Recall that (\mathbf{u}, p) is the exact solution, (\mathbf{u}_h, p_h) is the solution obtained by the finite element discretization with exact linear system solve, $(\mathbf{u}_h^{k,i}, p_h^k)$ is the solution obtained on the k -th step of Uzawa iteration and

the i -th step of linear algebraic solver, and $(\mathbf{u}_h^{k,\infty}, p_h^k)$ is the solution obtained on the k -th step of Uzawa iteration with an exact algebraic solve ($\mathbf{R}^{k,i} = \mathbf{0}$ in (3.1)). The different errors components are given as:

$$\begin{aligned} \text{total error} &:= \|\nabla(\mathbf{u} - \mathbf{u}_h^{k,i})\| + \beta\|p - p_h^k\|, \\ \text{discretization error} &:= \|\nabla(\mathbf{u} - \mathbf{u}_h)\| + \beta\|p - p_h\|, \\ \text{Uzawa (outer) error} &:= \|\nabla(\mathbf{u}_h - \mathbf{u}_h^{k,\infty})\| + \beta\|p_h - p_h^k\|, \\ \text{algebraic (inner) error} &:= \|\nabla(\mathbf{u}_h^{k,\infty} - \mathbf{u}_h^{k,i})\|. \end{aligned}$$

In the following, we employ the Taylor–Hood conforming finite element discretization of order $l = 2$ (there is no stabilization) and use the cheaper reconstruction of Remark 5.4 of order 1. We set the Uzawa parameter $\alpha := 1$. The implementation has been done with the FreeFem++ programming language [24].

7.1 Smooth solution

We consider here the computational domain as $\Omega = (0, 1)^2$, with homogeneous Dirichlet boundary conditions. The exact solution is given by, cf. [23] with the value $\alpha = 1$,

$$\mathbf{u} = \begin{pmatrix} 2x^2y(x-1)^2(y-1)^2 + x^2y^2(2y-2)(x-1)^2 \\ -2xy^2(x-1)^2(y-1)^2 - x^2y^2(2x-2)(y-1)^2 \end{pmatrix}$$

and

$$p = x + y - 1.$$

The volumetric force \mathbf{f} is chosen accordingly. We consider eight levels of uniform mesh refinement. The inf–sup constant β is approximately equal to 0.44 here, see [23] for a discussion.

Figure 1 presents the total error and the various estimators as a function of the number of mesh elements. It can be seen that the three approaches yield almost indistinguishable value for the total error; in this smooth case, the convergence order is $O(h^2)$. For the exact and inexact Uzawa method, the estimators $\eta_{\text{Uza}}^{k,i}$ take the values around 10^{-8} , and the $\eta_{\text{alg}}^{k,i}$ estimators are much smaller in the exact case than in the inexact one. For the adaptive inexact method, we terminate the computation with much larger $\eta_{\text{Uza}}^{k,i}$ and $\eta_{\text{alg}}^{k,i}$ estimators, just sufficient not to influence the total error. For all the three methods, the total estimators converge with the same speed as the total error, as proven in Theorem 5.5 (cf. also Remark 5.7).

Figure 2 focuses on the finest mesh, and displays the dependence of the total error and of the various estimators on the Uzawa iteration. The total error and the total estimator decrease rapidly for the first 30–40 Uzawa iterations and then only very slowly, since therefrom the influence of the Uzawa iteration error gets negligible. This is precisely the point where our adaptive inexact Uzawa method stops, leading to an important economy of the Uzawa iterations used.

In Figure 3 we compare the overall performance of the three approaches. We present the number of Uzawa iterations as a function of the refinement level, the number of algebraic solver iterations as a function of Uzawa iteration on the last refined mesh, and the total number of algebraic solver iterations as a function of the refinement level. Concerning the total number of Uzawa iterations, the exact and inexact methods show no significant difference. The adaptive inexact method reduces this count by a factor bigger than two. Gratifyingly, the inexact and the adaptive inexact methods need almost the same number of CG iterations on each Uzawa step. In combination, the adaptive inexact method needs the smallest number of total algebraic solver iterations; on the eighth refined mesh, the three approaches need respectively 31499, 11162, and 6295 total algebraic solver iterations.

The total effectivity indices for the three approaches are presented in Figure 4. There are computed as the ratio between the total estimator and the total error, i.e.,

$$\text{total effectivity index} := \frac{2(\eta_{\text{F}}^{k,i} + \eta_{\text{rem}}^{k,i} + \eta_{\text{D}}^{k,i} + \eta_{\text{osc}}^{k,i})}{\text{total error}},$$

see (4.7). This index for the adaptive inexact Uzawa algorithm appears to be lower than for the two other considered algorithms; setting, however, the parameters γ_{rem} , γ_{Uza} , and γ_{alg} small enough, we obtain almost the same effectivity indices.

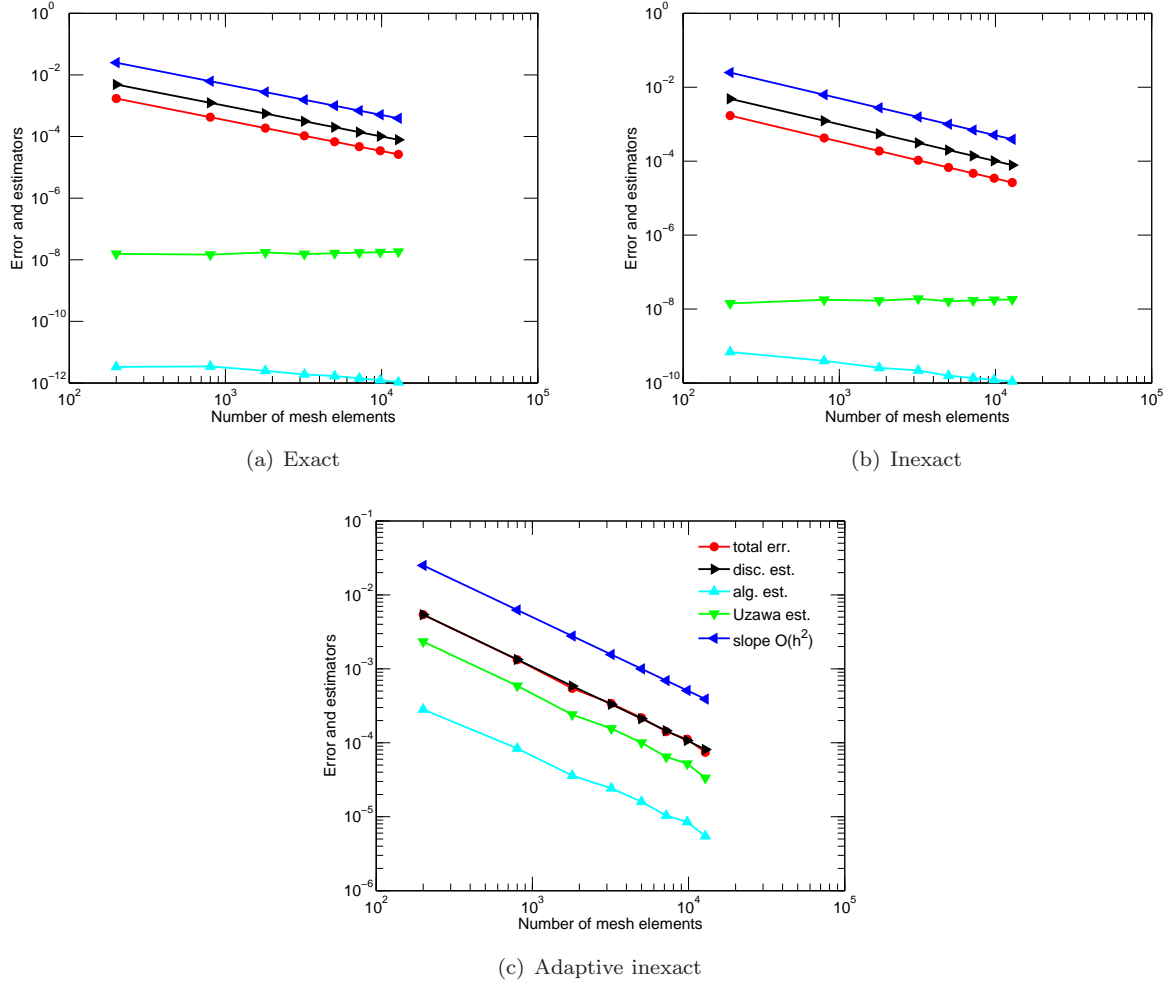


Figure 1: Error and estimators on uniformly refined meshes

Similarly as for the total effectivity index, we also define the effectivity indices for each error component in the adaptive inexact method:

$$\text{discretization effectivity index} := \frac{\eta_{\text{disc}}^{k,i}}{\text{discretization error}},$$

$$\text{Uzawa effectivity index} := \frac{\eta_{\text{Uza}}^{k,i}}{\text{Uzawa error}},$$

$$\text{algebraic effectivity index} := \frac{\eta_{\text{alg}}^{k,i}}{\text{algebraic error}}.$$

We can see in Figure 5 that all of them take values nicely close to the optimal value of one; note that the total estimate can indeed be smaller than the discretization estimate, compare (4.7) with (4.17).

Finally, in Figure 6, we display the spatial distribution of the different error components (left) and of the corresponding estimators (right), on the eighth refined mesh. We can observe a nice match.

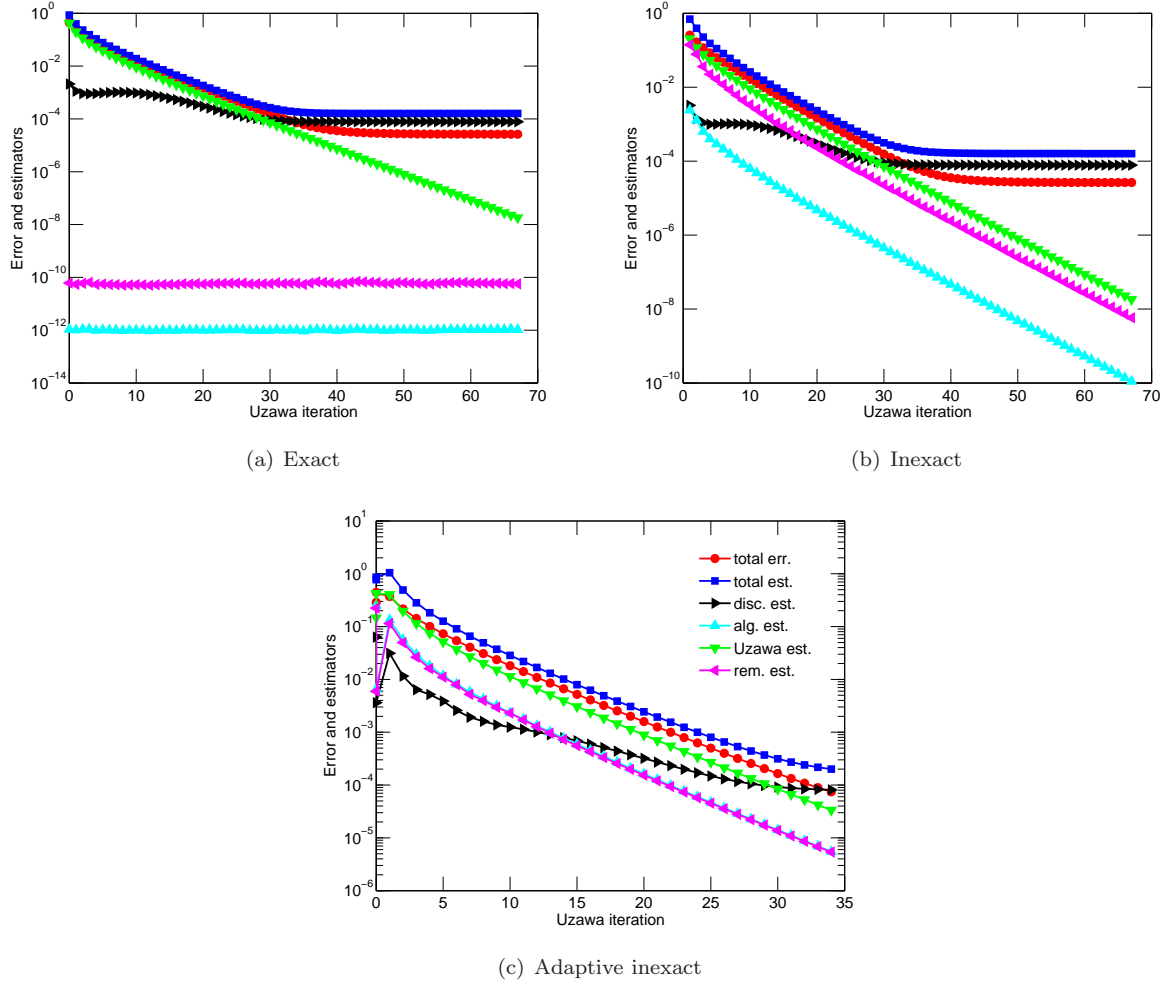


Figure 2: Error and estimators as a function of Uzawa iterations

7.2 Singular solution

We consider here the L -shaped domain $\Omega = (-1, 1)^2 \setminus (0, 1) \times (-1, 0)$, with an inhomogeneous Dirichlet boundary condition prescribed by the analytic solution, given in given in polar coordinates by, cf. [8],

$$\mathbf{u}(r, \varphi) = r^\kappa \begin{bmatrix} \cos(\varphi)\psi'(\varphi) + (1 + \kappa)\sin(\varphi)\psi(\varphi) \\ \sin(\varphi)\psi'(\varphi) - (1 + \kappa)\cos(\varphi)\psi(\varphi) \end{bmatrix}$$

and

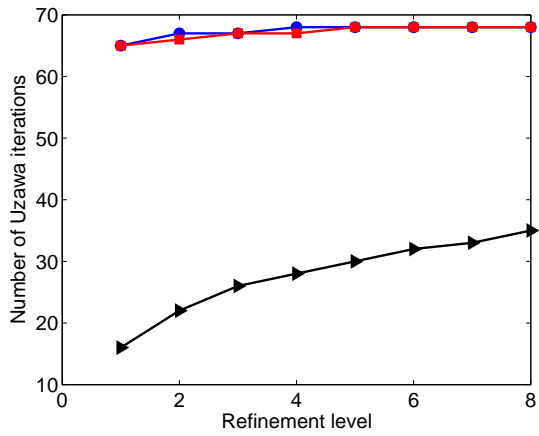
$$p(r, \varphi) = -r^{\kappa-1} \frac{(1 + \kappa)^2 \psi'(\varphi) + \psi'''(\varphi)}{1 - \kappa},$$

with the function $\psi(\varphi)$ defined as

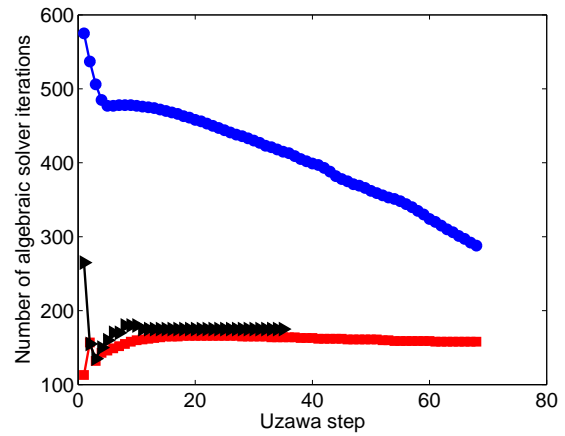
$$\psi(\varphi) = \frac{\sin((1 + \kappa)\varphi) \cos(\kappa\omega)}{1 + \kappa} - \cos((1 + \kappa)\varphi) + \frac{\sin((\kappa - 1)\varphi) \cos(\kappa\omega)}{1 - \kappa} - \cos((\kappa - 1)\varphi).$$

We consider $\kappa = 856399/1572864 \approx 0.54$ and $\omega = 3\pi/2$. There is the typical corner singularity in the re-entrant corner $(0, 0)$. The volumetric force is here $\mathbf{f} = \mathbf{0}$ and the inf-sup constant β is used with value 0.3, see [18] and the references therein.

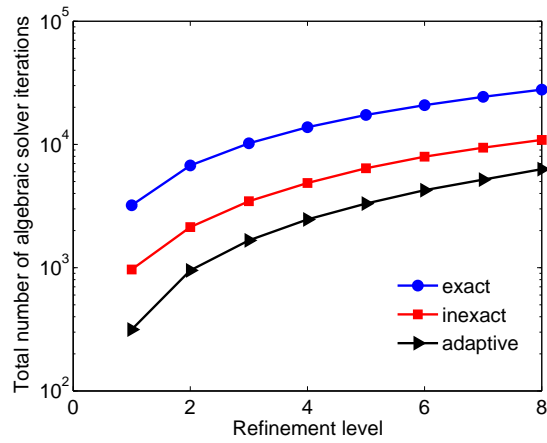
Similar to the preceding example, we consider eight levels of uniform mesh refinement. The estimators



(a) Number of Uzawa iterations per refinement level



(b) Number of linear solver iterations per Uzawa step on the eighth-level mesh



(c) Total number of linear solver iterations per refinement level

Figure 3: Number of linear solver and Uzawa iterations

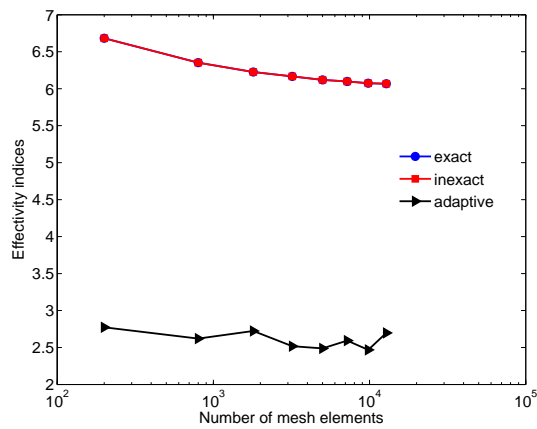


Figure 4: Effectivity indices

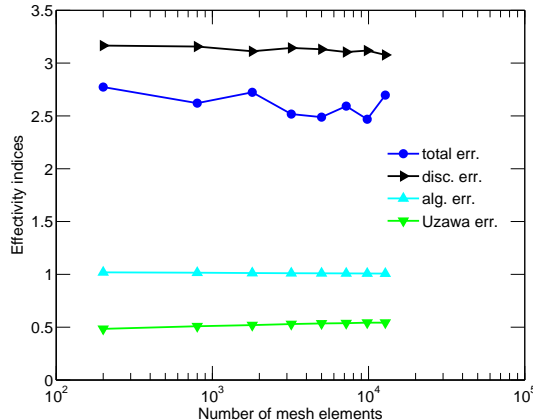


Figure 5: Effectivity indices for each error component in the adaptive inexact case

and total error as a function of the number of mesh elements are shown in Figure 7. It can be seen that the three approaches yield almost indistinguishable values for the total error for this test case as well, while $\eta_{\text{Uza}}^{k,i}$ and $\eta_{\text{alg}}^{k,i}$ differ significantly between them. The convergence order here is $O(h^\kappa)$.

In Figure 8, we show the dependence of the total error and of the estimators in (4.7) and (4.17) on the Uzawa iterations, for the last refined mesh. This time, we find our adaptive inexact Uzawa method needs considerably fewer Uzawa iterations than the two other methods, namely less than 10% iterations of the exact method.

The overall performance of the three approaches is compared in Figure 9. Concerning the total number Uzawa iterations, the exact and inexact methods are almost the same, whereas the adaptive inexact method systematically needs less than 10% of this count. For each Uzawa iteration, the adaptive inexact method takes a little more iterations than the inexact one; the exact method is much more expensive in this respect. Comparing the total number of algebraic solver iterations on the eighth refined mesh, the three approaches need respectively 78530, 14454, and 1959 iterations.

The effectivity indices for three approaches are next presented in Figure 10, and the effectivity indices for each error component in the adaptive inexact method are given in Figure 11. Similar conclusions as in Section 7.1 can be drawn; in particular, the singularity has here no influence.

Finally, in Figure 12, we display the distribution of the error components and of the corresponding estimators on the eighth refined mesh. We again observe a nice match, except for the Uzawa error, presumably influenced here by the strong singularity.

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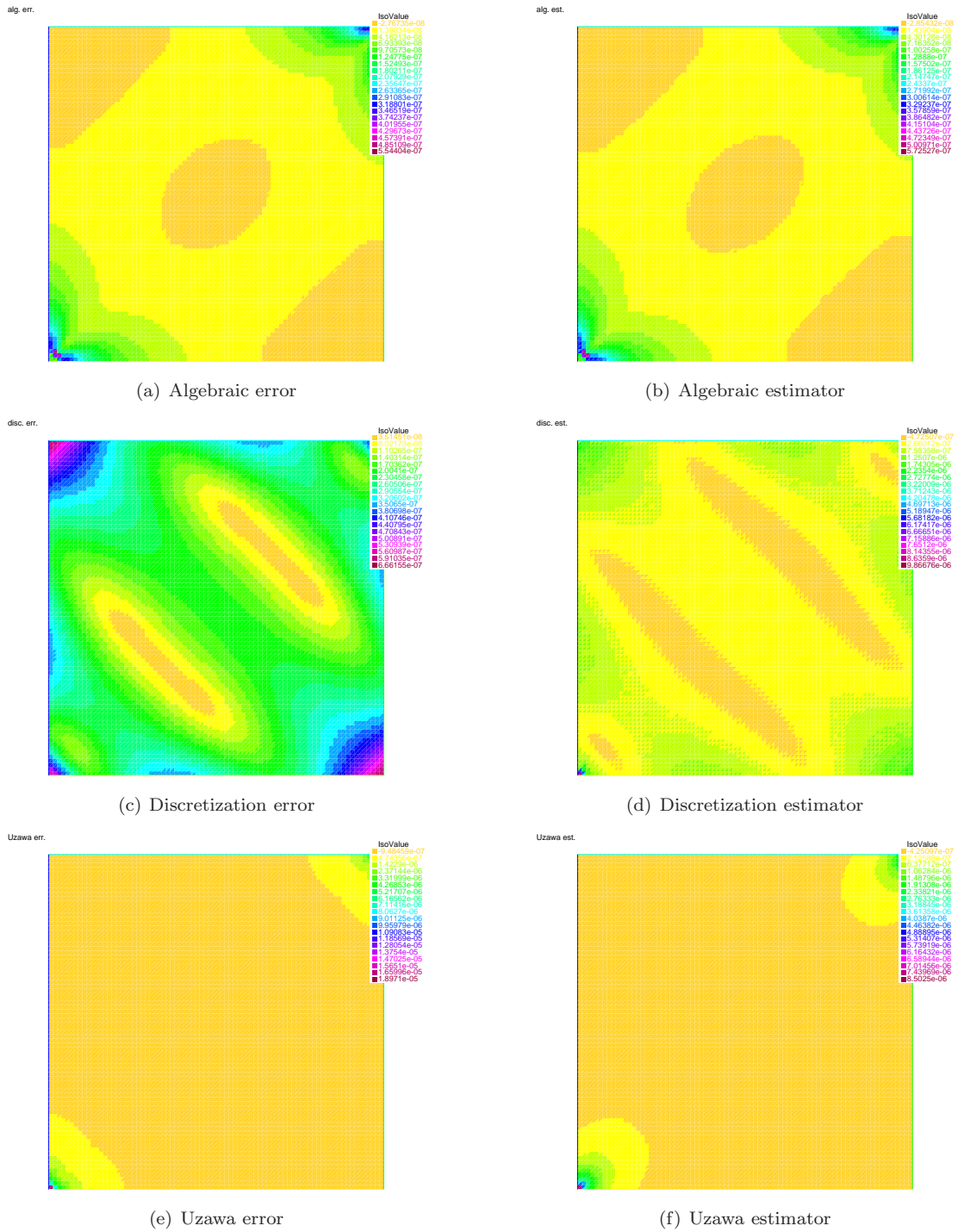


Figure 6: Spatial distributions of the different error components and of the corresponding estimates

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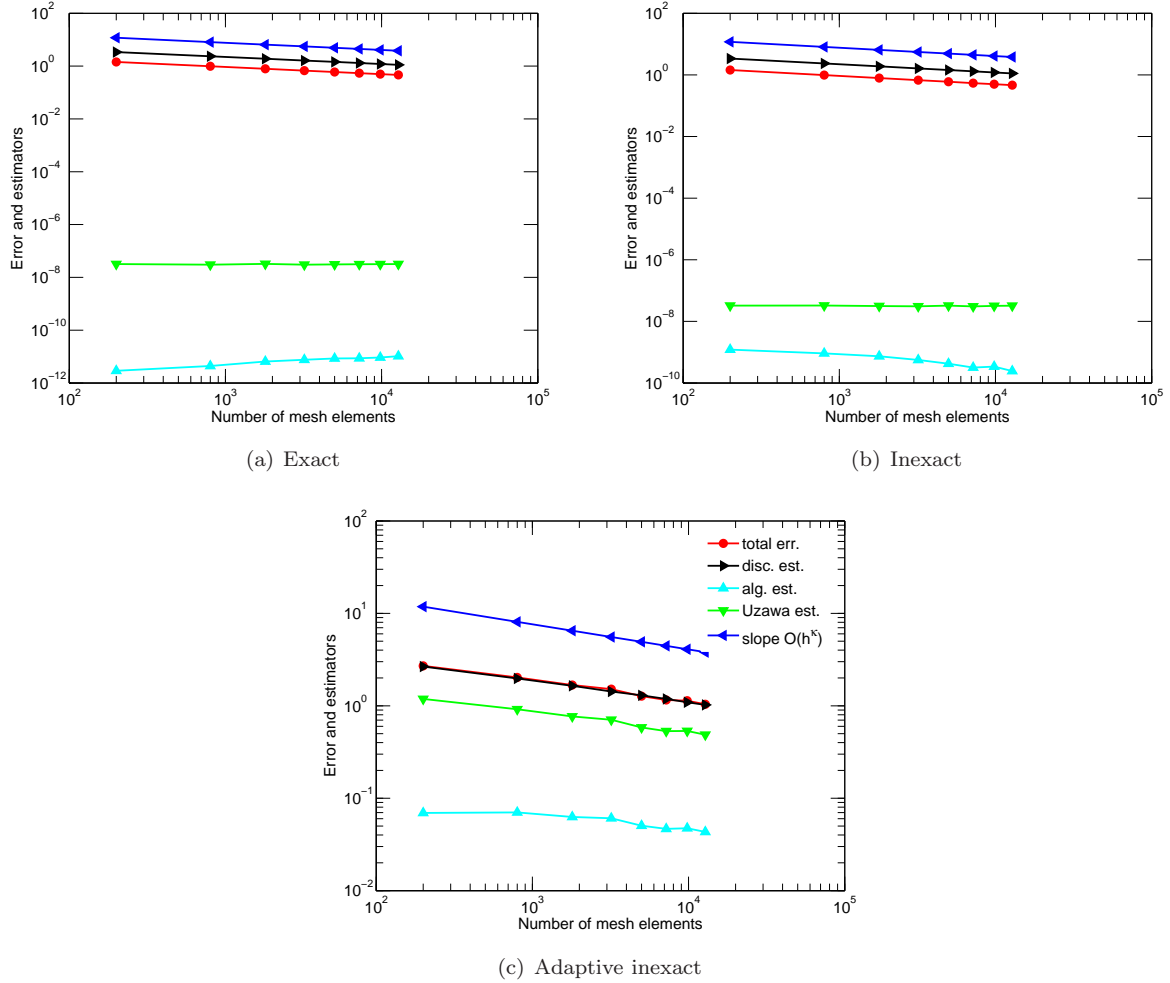


Figure 7: Error and estimators on uniformly refined meshes

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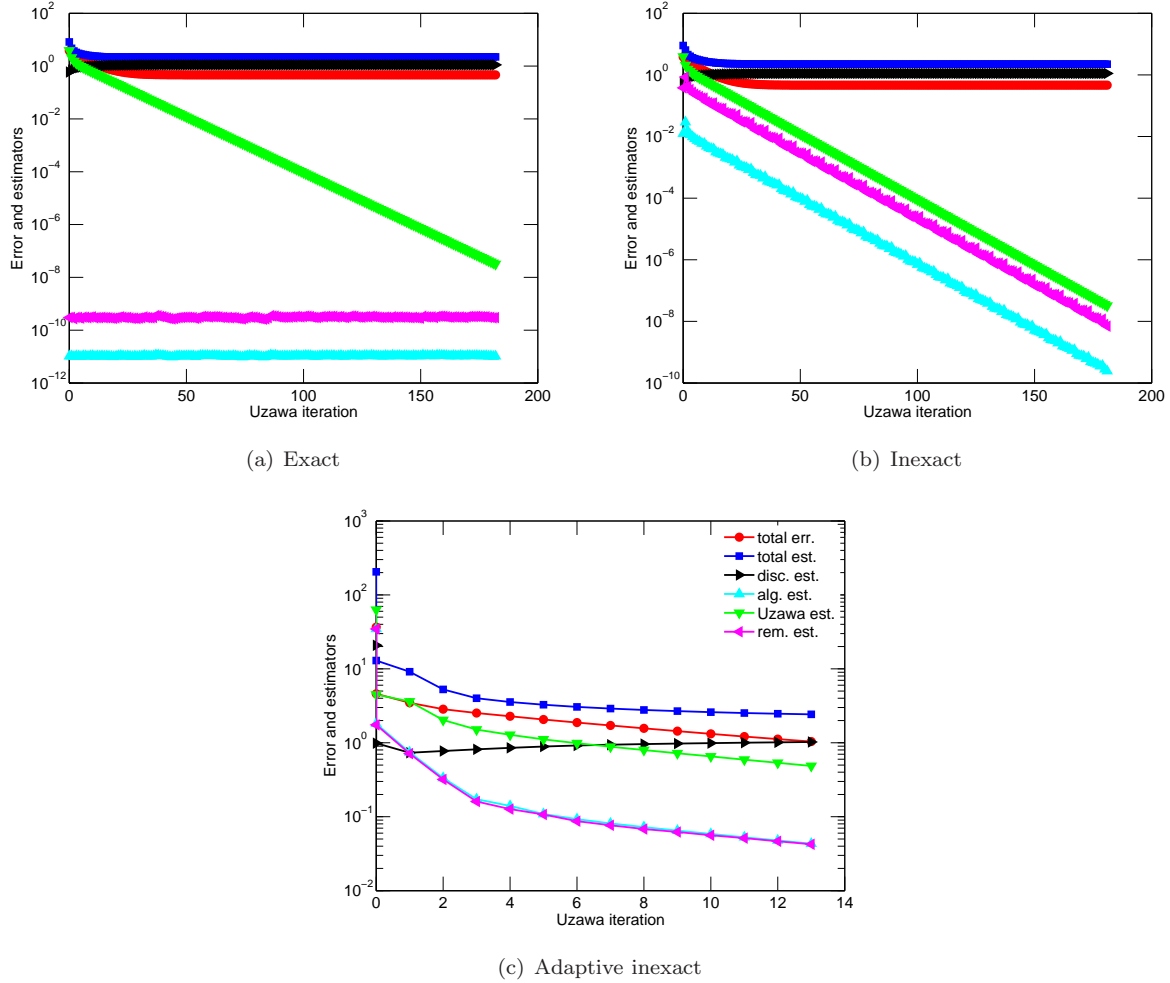
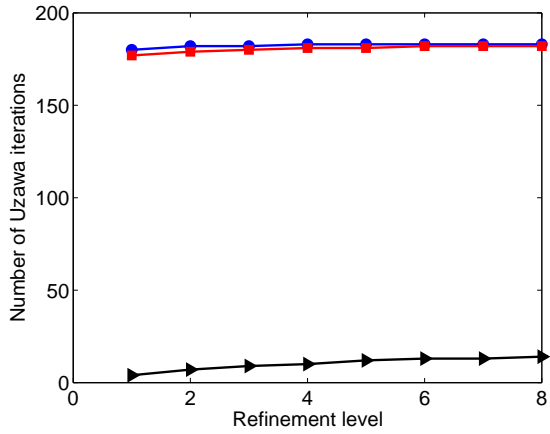
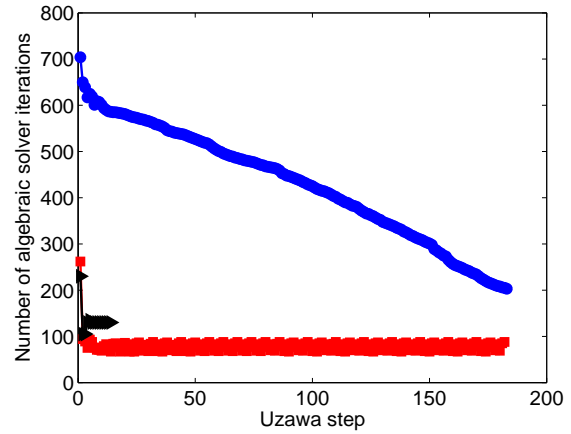


Figure 8: Error and estimators as a function of Uzawa iterations

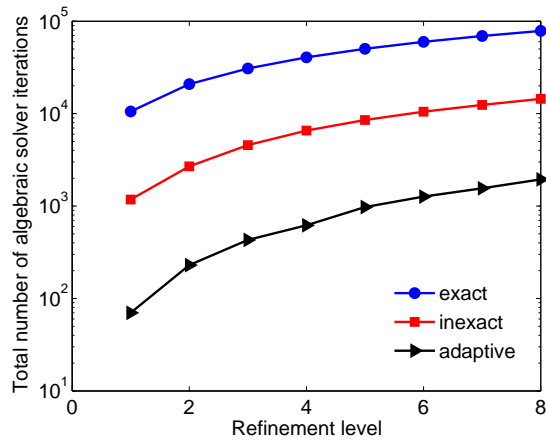
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(a) Number of Uzawa iterations per refinement level



(b) Number of linear solver iterations per Uzawa step on the eighth level mesh



(c) Total number of linear solver iterations per refinement level

Figure 9: Number of linear solver and Uzawa iterations

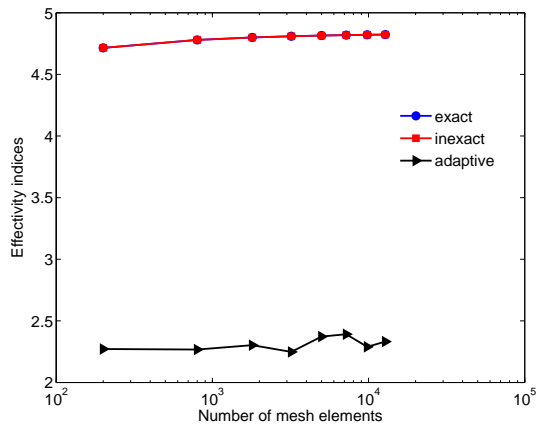


Figure 10: Effectivity indices

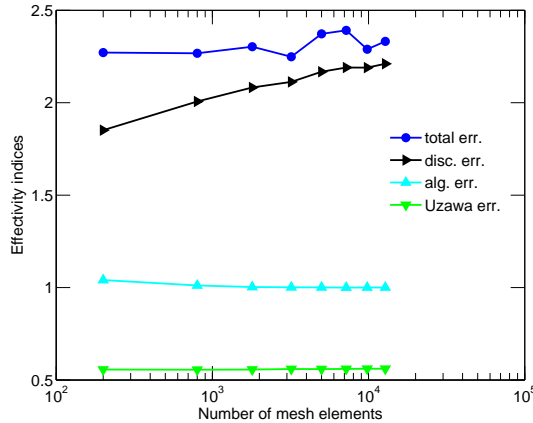


Figure 11: Effectivity indices for each error component in the adaptive inexact case

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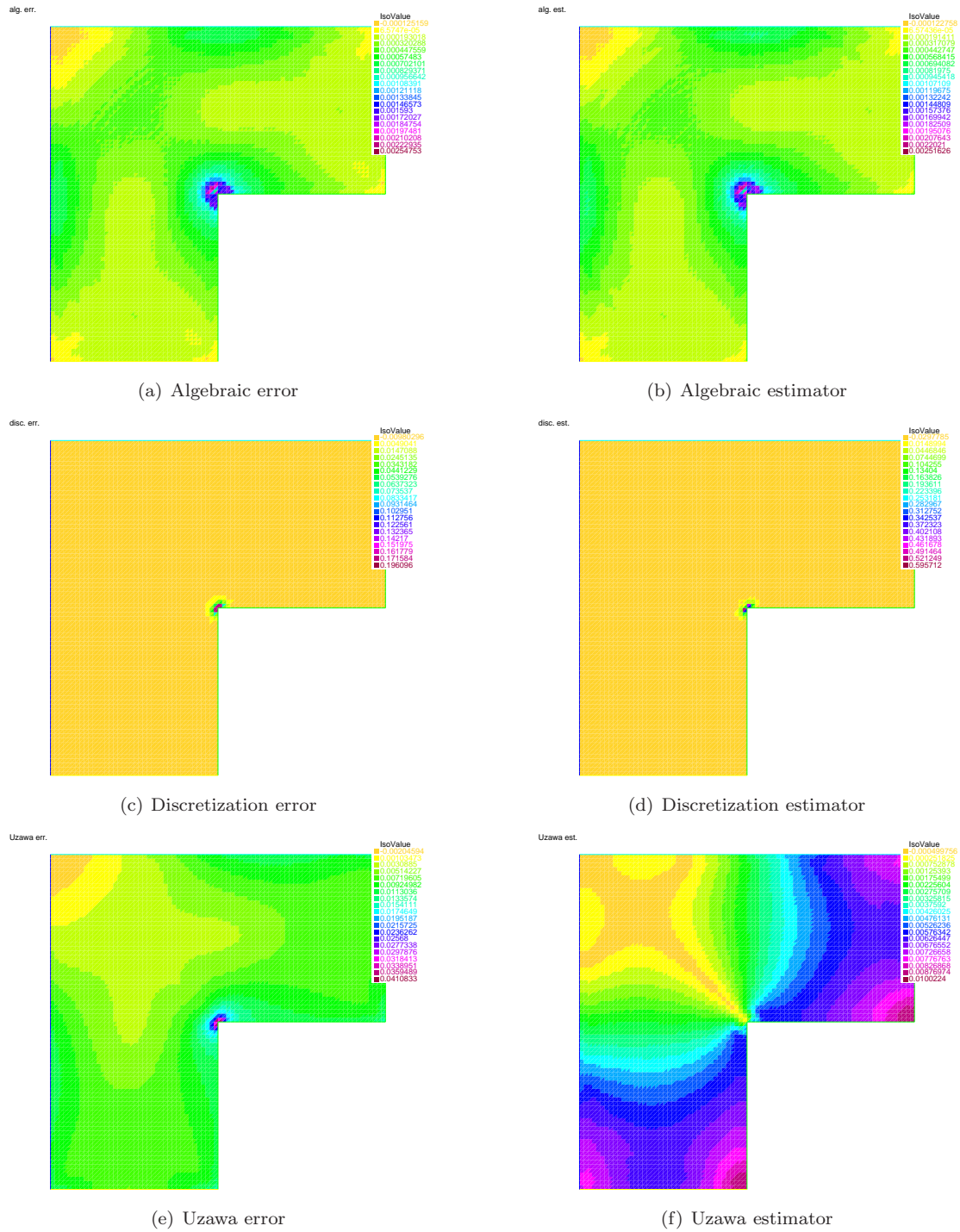


Figure 12: Spatial distributions of the different error components and of the corresponding estimates

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