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### ► To cite this version:

Konstantin Avrachenkov, Giovanni Neglia, Vikas Vikram Singh. Network Formation Games with Teams. [Research Report] RR-8657, Inria. 2014, pp.15. hal-01097318v1

**HAL Id: hal-01097318**

**<https://inria.hal.science/hal-01097318v1>**

Submitted on 19 Dec 2014 (v1), last revised 23 Mar 2015 (v2)

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**RESEARCH  
REPORT**

**N° 8657**

December 2014

Project-Teams Maestro





## Network Formation Games with Teams

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Project-Teams Maestro

Research Report n° 8657 — December 2014 — 15 pages

**Abstract:** Network formation games have been proposed as a tool to explain the topological characteristics of existing networks. They assume that each node is an autonomous decision-maker, ignoring that in many cases different nodes are under the control of the same authority (e.g. an Autonomous System) and then they operate as a team. In this paper we introduce the concept of network formation games for teams of nodes and show how very different network structures can arise also for some simple games studied in the literature. Beside extending the usual definition of pairwise stable networks to this new setting, we define a more general concept of stability toward deviations from a specific set  $\mathcal{C}$  of teams' coalitions ( $\mathcal{C}$ -stability). We study then a trembling-hand dynamics, where at each time a coalition of teams can create or sever links in order to reduce its cost, but it can also take wrong decisions with some small probability. We show that this stochastic dynamics selects in the long run  $\mathcal{C}$ -stable networks or closed cycles as the error probability vanishes.

**Key-words:** Network Formation Games, Teams, Network stability, AS Network

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## Les Jeux de Formation du Réseau avec des Équipes

**Résumé :** Les jeux de formation du réseau ont été proposés comme un outil théorique pour expliquer les topologies des réseaux existants. Il est généralement supposé que chaque noeud est un agent autonome. Ce point de vue ne tient pas compte du fait que dans des nombreux cas les noeuds peuvent être sous le contrôle du même autorité (par exemple, sous le contrôle du même Internet Autonomous System). Donc, ils opèrent comme une équipe. Dans cet article, nous introduisons les jeux de la formation du réseau avec des équipes de noeuds et montrent comment des structures topologiques très différentes peuvent survenir. En outre, nous étendons le concept de la stabilité du réseau par paires au concept plus général de  $\mathcal{C}$ -stabilité, où  $\mathcal{C}$  est un ensemble spécifique des coalitions des équipes. Nous étudions ensuite une dynamique de “main tremblante”, quand à chaque instant une coalition d’équipes peut créer ou rompre des liens pour réduire son coût. Cependant, il peut également prendre de mauvaises décisions avec quelques petits probabilité. Nous montrons que cette dynamique sélectionne dans le long terme des réseaux  $\mathcal{C}$ -stables ou cycles fermés quand la probabilité d’erreur s’annule.

**Mots-clés :** Les Jeux de Formation du Réseau, Équipes, Stabilité du Réseau, Réseau AS

## 1 Introduction

Network formation games are nowadays a consolidated branch of game theory (see for example the recent books [6], [5], [10], [9]). They study which networks' structures arise when nodes are selfish rational players, who can sever or create some links in order to increase the utility they perceive from the network. In particular, it is usually assumed that each node can unilaterally sever a link to one of its neighbors, while the creation of a new link requires the approval of both the participating nodes. This idea has led to the concept of pairwise-stable networks, i.e. networks for which every existent link is beneficial to both the connected nodes and every inexistent link is not beneficial to at least one of the two nodes it would connect. Different dynamics for links' creation/destruction have been studied. Specific network formation games have been proposed to explain the topological characteristics of existing networks, including the Autonomous Systems' (ASs') network taking for example into account both costs for routing traffic and for a lack of end-to-end connectivity [3], [1], [14]. They also have been used to investigate the distributed formation of overlay topologies on top of the Internet [7], or routing topologies among relay stations in future cellular networks [15], or among nodes in multi-hop wireless networks to prevent eavesdropping [16].

Network formation games consider that each node is an autonomous decision-maker, ignoring that in many cases different nodes are under the control of the same authority (e.g. an AS) and then they constitute a *team*. We observe that by introducing teams in a simple network formation game proposed in [4], gives different stable network structures. This motivates us to study network formation games with teams and define new stability notions for this new class of games. To the best of our knowledge this paper is the first to introduce teams in network formation games and aims to open a new interesting research direction. Although, the concept of teams in game theory has been studied before in different contexts. For instance, there is a recent work on team formation games by Boncinelli and Pin [2], where each agent can be the part of different teams that are involved in specific projects.

Our first contribution is then to define network formation games for teams of nodes and define team-pairwise stability analogous to pairwise stability in standard network formation games. The second contribution of our paper is to provide a more general concept of stability toward the deviations from a specific set  $\mathcal{C}$  of teams' coalitions (we talk then about  $\mathcal{C}$ -stability). This idea can capture the fact that some groups of teams are more likely to cooperate than others. For example, ASs located in far-away regions may have no incentive to connect directly. Apart from above static stability notions we also consider the dynamic formation of networks in the presence of trembling (mutations). We then discuss the dynamic stability notion called as stochastic stability for networks, i.e., which networks are selected by the dynamics as trembling vanishes. So, as a final contribution we define a general trembling-hand dynamics for which we prove that only  $\mathcal{C}$ -stable networks or closed cycles are selected with non-zero probability on the long term when the trembling vanishes. Jackson and Watts [12] are the first ones to consider trembling-hand dynamics in standard network formation games for the case of pairwise interaction between nodes.

The paper is organized as follows. We start providing the basic definitions for network formation games with teams in Section 2. Then, in Section 3 we provide a simple network formation game proposed in [4] showing that different network structures arise in the presence of teams. We move then to introduce the general concept of  $\mathcal{C}$ -stability and our trembling-hand dynamics in Section 4. We show some numerical results for specific games in Section 5. Finally, we conclude our paper in Section 6.

## 2 Network Formation Games with Teams: basic definitions

In this section we define network formation games for teams and we start extending some usual concepts for network formation games to our setting. In particular we define the notion of team-pairwise stability to characterize meaningful equilibria.

Let  $N = \{1, 2, \dots, n\}$  be a finite set of nodes. Let  $\mathcal{P} = \{T_1, T_2, \dots, T_m\}$ ,  $m \leq n$  be a partition of  $N$ . Undirected edges can be formed both between nodes belonging to the same set  $T_i$  (called internal links) as well as belonging to different sets  $T_i$  and  $T_j$  (called external links). The edge between nodes  $k$  and  $l$  is denoted by  $kl$ . We say that there is a link between  $T_i$  and  $T_j$  if there is at least one link  $kl$  with  $k \in T_i$

and  $l \in T_j$ . The collection of edges defines a network  $g$ .

The cost of each set  $T_i$  ( $c(T_i, g)$ ) depends in general on the structure of the whole network  $g$ . In usual network formation games, each node  $k$  has its own cost  $c_k(g)$ . In network formation games with teams we consider that nodes belonging to each set  $T_i$  share the same cost  $c(T_i, g)$ . As a consequence, they will form or sever links only if this is beneficial for the whole set  $T_i$ , i.e. if the cost of  $T_i$  is reduced. For this reason we refer to each  $T_i$   $i = 1, 2, \dots, m$  as a team. In particular, nodes in a team will agree to create an internal link  $kh$  with  $k, h \in T_i$ , if  $c(T_i, g + kh) < c(T_i, g)$ , and will agree to sever an internal link  $kh$  if  $c(T_i, g - kh) < c(T_i, g)$ . This leads to following definition of internal stability:

**Definition 2.1** (Internal stability). *A network  $g$  is internally stable in team  $T_i$  if no further link can be created or severed within the nodes from  $T_i$ , i.e., if  $g'$  is a network which is obtained from  $g$  via the addition of new links within the nodes of  $T_i$  or destruction of existing links within the nodes of  $T_i$  then  $c(T_i, g') > c(T_i, g)$ . A network is internally stable if it is internally stable in all the teams.*

The destruction of an external link is unilateral: any of the two nodes can sever the link if this is beneficial for its team. On the contrary, the creation of an external link requires the agreement of both the teams involved, as it is often considered in literature. The link will then be created only if it does not increase the cost of any of the two teams and it decreases the cost of at least one of them. These link formation rules leads to the following notion of equilibria for network formation games with teams:

**Definition 2.2** (Team-pairwise stability). *A network  $g$  is said to be team-pairwise stable if*

1.  $g$  is internally stable, and
2. for all pair  $(T_i, T_j)$  if  $kl \in g$ ,  $k \in T_i$  and  $l \in T_j$  then  $c(T_i, g) \leq c(T_i, g - kl)$  and  $c(T_j, g) \leq c(T_j, g - kl)$ , and
3. for all pair  $(T_i, T_j)$  if  $kl \notin g$ ,  $k \in T_i$  and  $l \in T_j$  and if  $c(T_i, g + kl) < c(T_i, g)$  then  $c(T_j, g + kl) > c(T_j, g)$ .

### 3 Network Creation Game with Teams: A Motivating Example

We introduce teams in a network creation game considered by Corbo and Parkes [4] and we observe that some different stable network structure arises even for very simple cases. This shows the usefulness of new stability notions defined in Section 2 and it is worth studying this new class of network formation games. In this section we use the cost function given in [4]. But, in Section 2 and in general in the whole paper we consider the general cost function unless mentioned otherwise.

Fabrikant et al. [8] introduced a network creation game where each node can add as well as sever links unilaterally. While in network creation game by Corbo and Parkes [4] the link creation is bilateral and link destruction is unilateral. We consider network creation game by Corbo and Parkes because we are interested in pairwise interaction among teams. First we give the description of the bilateral network creation game given in [4]. Let  $N = \{1, 2, \dots, n\}$  be a finite set of players (nodes). The action set of player  $i$  is  $A_i = 2^{N - \{i\}}$ , i.e., each player chooses a subset of players with whom he wants to create a link. For each  $i \in N$ , the action set  $A_i$  can be represented as  $A_i = \{(a_{ij})_{j \neq i} | a_{ij} \in \{0, 1\}\}$ . Player  $i$  seeks contact with player  $j$  if  $a_{ij} = 1$ . Player  $i$  and player  $j$  are connected with a link if  $a_{ij} = a_{ji} = 1$ . Let  $a_i = (a_{ij})_{j \neq i}$  denotes an action of player  $i$  and  $|a_i| = \sum_{j \neq i} a_{ij}$  which is the number of connections player  $i$  establishes with other players. Players simultaneously announce the list of the players with whom they wish to be connected. Each action profile  $a = (a_i)_{i=1}^n$  gives an undirected network  $g(a)$ . The shortest distance between nodes  $i$  and  $j$  in a network  $g(a)$  is defined as the minimal number of hops along a path connecting them and it is denoted by  $d_{g(a)}(i, j)$ . At action profile  $a$  the cost incurred by player  $i$  is

$$c_i(a) = \alpha \cdot |a_i| + \sum_{j \in N; j \neq i} d_{g(a)}(i, j), \quad (1)$$

The above cost function implies a sunk cost for player  $i$  for provisioning for links that may not actually form (since the link formation is bilateral). It is clear that the sunk cost is absent at equilibrium. At

network  $g$  if  $ij \in g$  then both the nodes  $i$  and  $j$  incur cost  $\alpha$  for creating this link, i.e., the total cost of creating the link  $ij$  is  $2\alpha$ . The cost of a network  $g$  is the sum of the cost of all its nodes. A network  $g$  is said to be efficient if the cost of  $g$  is minimum among all the networks. Corbo and Parkes showed that for  $\alpha < 1$  the complete network is unique efficient and pairwise stable network and for  $\alpha > 1$  the star network is only efficient network which is also pairwise stable but not unique.

We consider the network creation game defined above. We assume that the nodes are divided into disjoint sets  $\{T_i\}_{i=1}^m$  and the cost of a link between two nodes from a same set is  $\alpha$  and the cost of a link between two nodes from different sets  $T_i$  and  $T_j$  is  $2\alpha$ , i.e., there is a cost heterogeneity among links. Let the nodes from each set form a team. So, each team incurs cost  $\alpha$  for forming internal links and also incurs cost  $\alpha$  for forming external links. For a given network  $g$  the cost of a team  $T_i$  is defined as

$$c(T_i, g) = \alpha(E(T_i) + \sum_{\substack{T_j \in \mathcal{P} \\ j \neq i}} E(T_i T_j)) + \sum_{k \in T_i} \sum_{\substack{j \in N \\ j \neq k}} d_g(k, j), \quad (2)$$

where  $E(T_i)$  is the total number of internal links of  $T_i$  and  $E(T_i T_j)$  is the total number of external links between  $T_i$  and  $T_j$ . The cost of a network  $g$  is defined as the sum of the cost of all the teams.

For illustration purpose, let us first consider a simple case when  $1 < \alpha < 2$  and see which network is a team-pairwise stable and whether that network is also pairwise stable if there are no teams.

**Proposition 3.1.** *Consider a bilateral network creation game with teams and  $1 < \alpha < 2$ , then a network  $g$  is internally stable if all the nodes from each team form a complete network internally.*

*Proof.* We prove this result by contradiction. Let  $g$  be an internally stable network. Assume that there exist a team  $T_i$  whose nodes do not form a complete network internally, i.e., there exist two nodes  $k$  and  $l$  such that they do not have a link between them. Let  $g' = g + kl$ , the cost of team  $T_i$  at  $g'$  is increased by  $\alpha$  due to the creation of link  $kl$  but the cost is reduced by at least  $2 > \alpha$  units due to the distance between  $k$  and  $l$ , so  $c(T_i, g') < c(T_i, g)$ . This implies that  $g$  is not internally stable in  $T_i$  which gives a contradiction. Hence, at internally stable network the nodes of each team form a complete network internally.  $\square$

Similarly to the star network in standard network formation games, we define a star network at team level which we call as a team-star. A network is a team-star if each team has at least two nodes and all the nodes of each team form a complete network internally and there is a central team with a node called central node such that all the nodes of other teams are connected to the central node.

**Proposition 3.2.** *A team-star network is a team-pairwise stable in a bilateral network creation game with teams but it is not a pairwise stable if each node is an independent rational player.*

*Proof.* Let  $g$  be a team-star network, then by definition  $g$  is internally stable. From the definition of  $g$  it is clear that the distance between any two nodes of  $g$  is at most 2. Now consider a pair of non-central teams  $(T_i, T_j)$  then both teams have no incentive to create a link between them because it will cost each team  $\alpha > 1$  but their cost due to distance can be reduced only by 1. Now consider a pair of teams  $(T_i, T_j)$  such that  $T_i$  is the central team then consider a link  $kl$  between a central node  $k$  of  $T_i$  and a node  $l$  of  $T_j$  then this link cannot be deleted by both the teams because the cost of both the teams get reduced only by  $\alpha$  but the cost due to distance is increased by at least  $2 > \alpha$ . No further links between a node of  $T_i$  other than central node  $k$  and a node of  $T_j$  can be added because it will increase the cost by  $\alpha > 1$  but the cost due to distance is reduced only by 1. So, team-star is a team-pairwise stable.

Assume that there are no teams, i.e., the nodes belonging to each set  $T_i$ ,  $i = 1, 2, \dots, m$  is an independent player. Then the network  $g$  is not a pairwise stable because the central node  $k$  from the central set  $T_i$  will form maximum one link from each set  $T_j$ ,  $j \neq i$ .  $\square$

**Remark 3.3.** *Introducing teams in a standard network formation game leads to a team-pairwise stable network which is not a pairwise stable network even for very simple cases.*

Now there is an interesting question: which network is pairwise stable when nodes belonging to sets  $\{T_i\}_{i=1}^m$  do not form a team. To get an idea how a pairwise stable network looks like, we consider



an instance of bilateral network creation game with 9 nodes that are divided into 3 sets, i.e.,  $N = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$  and  $T_1 = \{1, 2, 3\}$ ,  $T_2 = \{4, 5, 6\}$ ,  $T_3 = \{7, 8, 9\}$ . The team-star network given in Figure 1 is a team-pairwise stable but not pairwise stable from Proposition 3.2. A pairwise stable network in this case when nodes from  $T_1$ ,  $T_2$  and  $T_3$  does not form a team is given in Figure 2.

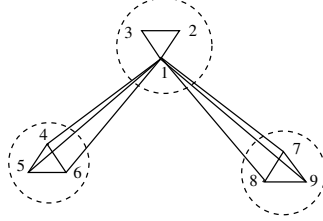


Figure 1: Team-star: A team-pairwise stable network

The cost of the team-star network from Figure 1 is  $c(\text{team-star}) = 21\alpha + 114$  and the cost of network  $g$  from Figure 2 is  $c(g) = 27\alpha + 108$ . It is easy to check that for  $\alpha > 1$ ,  $c(\text{team-star}) < c(g)$ . For these simple cases of network creation game the introducing teams give us a team-pairwise stable network which is not a pairwise stable and it is more efficient than a pairwise stable network. This shows that in general introducing teams in standard network formation games may give team-pairwise stable networks which are completely different from pairwise stable networks and more efficient.

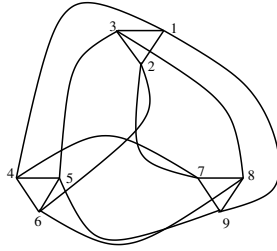


Figure 2: Pairwise stable network

## 4 Stability of Networks Against the Coalition of Teams

Unlike in team-pairwise stability where at a time only one link between two teams can be changed we consider a general situation where two or more teams form a coalition and they can make all possible changes in a given network by creating new links or severing existing links in such a way that a resulting network is at most as costly, for all the teams that are the part of the coalition, as in the current network and there exists at least one team whose cost is strictly less. We define the stability concept of networks against the deviation of teams' coalitions. In some situations, it may not be possible to form all types of teams' coalitions, e.g. some ASs may not form coalition due to geographical constraints, so, let  $\mathcal{C} \subset 2^{\mathcal{P}}$  denotes the set of all possible teams' coalitions. Under this constraint on the coalition formation there will be a restriction on the set of all possible networks, e.g., if there exists a pair of teams such that both of them cannot be the part of a coalition together, then there cannot exist a link between them. Let  $G(\mathcal{C})$  be the set of all possible networks. The stability notions against the deviation of coalitions in the standard network formation games has been considered in literature before (see [13], [11]).

**Definition 4.1.** A network  $g'$  is obtainable from  $g$  via deviation by a coalition  $S \in \mathcal{C}$  as denoted by  $g \rightarrow_S g'$ , if

1.  $kl \in g'$  and  $kl \notin g$  such that  $k \in T_i$  and  $l \in T_j$ , then  $\{T_i, T_j\} \subset S$ .
2.  $kl \in g$  and  $kl \notin g'$  such that  $k \in T_i$  and  $l \in T_j$  then  $\{T_i, T_j\} \cap S \neq \phi$ .

The condition (i) of above definition requires that new links can be added only between the teams which are the part of a coalition  $S$ . The condition (ii) requires that at least one team of any deleted link has to be the part of  $S$ . We denote  $G(S, g)$  as a set of all networks which are obtainable from  $g$  via deviation by  $S$ , i.e.,  $G(S, g) = \{g' \in G(\mathcal{C}) | g \rightarrow_S g'\}$ .

**Definition 4.2.** A deviation by a coalition  $S$  from a network  $g$  to a network  $g'$  is said to be improving if

1.  $g \rightarrow_S g'$ ,
2.  $c(T, g') \leq c(T, g)$ ,  $\forall T \in S$  (with at least one strict inequality).

We denote  $G_f(S, g)$  as a set of all networks  $g'$  which are obtainable from  $g$  by an improving deviation of  $S$ , i.e.,

$$G_f(S, g) = \{g' \in G(S, g) | c(T, g') \leq c(T, g), \forall T \in S, \\ c(T', g') < c(T', g) \text{ for some } T' \in S\}.$$

It is clear that  $g \notin G_f(S, g)$ . We denote  $G_{nf}(S) = G(S, g) \setminus G_f(S, g)$  is the set of networks  $g'$  which are obtainable from  $g$  by taking the wrong decision and it is always nonempty because  $g \in G_{nf}(S, g)$  for all  $S \in \mathcal{C}$ .

**Definition 4.3.** A network  $g \in G(\mathcal{C})$  is  $\mathcal{C}$ -stable if it is not possible for any coalition  $S \in \mathcal{C}$  to make an improving deviation from a network  $g$  to some other network  $g'$ .

A  $\mathcal{C}$ -stable network need not always exist and in that case there exist some set of networks lies on a closed cycle and all the networks in a closed cycle can be reached from each other via an improving path. We next give the definitions of an improving path and closed cycle.

**Definition 4.4** (Improving Path). An improving path from  $g$  to  $g'$  is a sequence of networks and coalitions  $g_1, S_1, g_2, \dots, g_{n-1}, S_{n-1}, g_n$  such that  $g_1 = g$ ,  $g_n = g'$  and  $g_{k+1} \in G_f(S_k, g_k)$  for all  $k = 1, 2, \dots, n-1$ .

**Definition 4.5** (Closed Cycles). A set of networks  $Q$  form a closed cycle if for any  $g, g' \in Q$  there exists an improving path connecting  $g$  and  $g'$  and no network in  $Q$  lies on an improving path leading to a network that is not in  $Q$ .

**Theorem 4.6.** There exists at least one  $\mathcal{C}$ -stable network or a closed cycle of networks.

*Proof.* The proof follows in similar lines to the proof given in [12] for pairwise stable network.  $\square$

Now, we consider a situation where networks are formed dynamically over time. Since, at each time teams are allowed to form a coalition and they can make all possible changes in a given network, so we consider the network formation rules by Jackson and van den Nouweland [11] as given below:

- Link addition is bilateral, i.e., forming a link between the teams  $T_i$  and  $T_j$  requires the consent of both teams.
- Link destruction is unilateral, i.e., severing a link between the teams  $T_i$  and  $T_j$  requires the consent of team  $T_i$  or team  $T_j$ .
- At a time more than one link can be created and severed by the teams.

Jackson and Watts [12] considered dynamic formation of networks in standard network formation games without teams, where at each time only a pair of players are selected and only a link between them can be changed.

#### 4.1 A Stochastic Dynamic Process

We consider a situation where at each time  $t = 1, 2, \dots$  a group of teams form a coalition and discuss whether it is possible to make some changes in a current network such that at resulting network the cost of each team in the coalition is at most as much as it is in the current network and at least one team has strictly less cost. The selected coalition makes a move to such an improving network from a current network. If there are more than one improving networks for selected coalition then the probability of selection of each improving network is positive. That is the above dynamics is stochastic. If there are no improving networks that can be reached by a selected coalition then at next stage the network will be same. The same thing repeats at next stage and this continues for infinite periods. We assume that the coalition formation is random and at a time only one coalition can be formed. We assume that at each time a selected coalition makes some error in calculation whether making a move is at most as costly as before for all the teams in the coalition with at least one team has strictly less cost. This happens with small probability that depends on the size of the coalition. As a consequence at next stage there will be a network with small probability at which all the teams in the coalition are not better off. Here we study the evolution and persistence of network in the long run when the probability of error tends to zero. The above situation can be modeled as a Markov chain over the set of networks  $G(\mathcal{C})$ . We first consider the case when coalition of teams does not make error over time and in that case we have unperturbed Markov chain. Next, we consider the case where at each time selected coalition makes error with small probability. So, in this case there will be a perturbed Markov chain.

We first assume that coalition does not make mistakes, i.e., at each time  $t = 1, 2, \dots$ , a coalition  $S_t \in \mathcal{C}$  is randomly selected with probability  $p_{S_t} > 0$  who can make all the possible improving deviations from the current network  $g_t$ . As a consequence at time  $t + 1$  the network is  $g_{t+1} \in G_f(S_t, g_t)$  with probability  $p_f(g_{t+1}|S_t, g_t)$ . If it is not possible to make an improving deviation by a coalition  $S_t$  then  $g_{t+1} = g_t$ . The transition law  $P^0$  of the Markov chain over finite set  $G(\mathcal{C})$  is defined as:

$$P^0(g'|g) = \sum_{S \in \mathcal{C}; G_f(S, g) \neq \emptyset} p_S p_f(g'|S, g) 1_{G_f(S, g)}(g') + \sum_{S \in \mathcal{C}; G_f(S, g) = \emptyset} p_S 1_{\{g'=g\}}(g'), \quad (3)$$

where  $p_f(\cdot|S, g)$  is a probability measure over  $G_f(S, g)$  and  $1_B(\cdot)$  is an indicator function on a given set  $B$ . It is clear that a  $\mathcal{C}$ -stable network is an absorbing state and a closed cycle is a recurrent class of  $P^0$  that contains more than one network.

Now, we assume that at each time  $t$ , there is an error made by a coalition  $S_t$  with small probability. This adds mutations as well as another level of stochasticity in the dynamics. The chance of making error by a coalition increases with its size because the number of possible network choices for coalition increases with its size. We assume that coalition  $S_t$  makes error with  $|S_t|\varepsilon$  probability for sufficiently small  $0 < \varepsilon < \frac{1}{|\mathcal{P}|}$ ;  $|B|$  denotes the cardinality of a given set  $B$ . So, at time  $t + 1$  network is  $g_{t+1} \in G_f(S_t, g_t)$  with  $(1 - |S_t|\varepsilon)p_f(g_{t+1}|S_t, g_t)$  probability and  $g_{t+1} \in G_{nf}(S_t, g_t)$  with  $|S_t|\varepsilon p_{nf}(g_{t+1}|S_t, g_t)$  probability. In the situation where there is no improving move by coalition  $S_t$ , the network  $g_{t+1} = g_t$  with probability  $1 - |S_t|\varepsilon$  and  $g_{t+1} \in G_{nf}(S_t, g_t) \setminus \{g_t\}$  with probability  $|S_t|\varepsilon p_{nf \setminus g_t}(g_{t+1}|S_t, g_t)$ . If  $G_{nf}(S_t, g_t) = \{g_t\}$ , which arises when  $S_t$  has single team with only one node which is isolated in the network  $g_t$ , then  $g_{t+1} = g_t$ . The transition law  $P^\varepsilon$  of the perturbed Markov chain of  $P^0$  is defined as below:

$$\begin{aligned} P^\varepsilon(g'|g) = & \sum_{S \in \mathcal{C}; G_f(S, g) \neq \emptyset} p_S ((1 - |S|\varepsilon)p_f(g'|S, g) 1_{G_f(S, g)}(g') + |S|\varepsilon p_{nf}(g'|S, g) 1_{G_{nf}(S, g)}(g')) \\ & + \sum_{S \in \mathcal{C}; G_f(S, g) = \emptyset, |G_{nf}(S, g)| > 1} p_S ((1 - |S|\varepsilon) 1_{\{g'=g\}}(g') + |S|\varepsilon p_{nf \setminus g}(g'|S, g) 1_{G_{nf}(S, g) \cap \{g' \neq g\}}(g')) \\ & + \sum_{S \in \mathcal{C}; G_f(S, g) = \emptyset, |G_{nf}(S, g)| = 1} p_S 1_{\{g'=g\}} \end{aligned} \quad (4)$$

for all  $g, g' \in G(\mathcal{C})$ , where  $p_{nf}(\cdot|S, g)$  is a probability measure over set  $G_{nf}(S, g)$  and  $p_{nf \setminus g}(\cdot|S, g)$  is a probability measure over set  $G_{nf}(S, g) \setminus \{g\}$ .

The Markov chain  $P^\varepsilon$  is irreducible because given nonzero mutations it is possible to reach all the networks starting from any network in finite steps. It is also aperiodic because with positive probability Markov chain makes self transition. Hence, there exists a unique stationary distribution  $\mu^\varepsilon$  of

Markov chain  $P^\varepsilon$ . We are interested in the networks to which stationary distribution  $\mu^\varepsilon$  assigns positive probability as  $\varepsilon \rightarrow 0$ . This leads to the definition of stochastic stability:

**Definition 4.7.** *A network  $g$  is stochastically stable relative to process  $P^\varepsilon$  if  $\lim_{\varepsilon \rightarrow 0} \mu_g^\varepsilon > 0$ .*

If  $P^\varepsilon(g'|g) > 0$  then we define one step resistance from network  $g$  to  $g'$ , which is denoted by  $r(g, g')$ , as minimum number of mutations which are required for Markov chain to make a transition from  $g$  to  $g'$  in one step. From (4) it is clear that the transition from a network  $g$  to  $g'$  has the probability of order  $\varepsilon$  if  $g' \notin G_f(S, g)$  for all  $S$  and thus has resistance 1 and is of order 1 otherwise, so has resistance 0, i.e.,  $r(g, g') \in \{0, 1\}$  for all  $g, g' \in G(\mathcal{C})$ . A zero resistance between two nodes corresponds to transition with positive probability under  $P^0$ . Since  $P^\varepsilon$  is an irreducible Markov chain then there must exist at least one directed path between any two recurrent classes  $G_i$  and  $G_j$  of  $P^0$  which starts from  $G_i$  and ends at  $G_j$ . The resistance of any path is defined as the sum of the resistances of the corresponding edges. The resistance of an edge in a directed path represents one step resistance from starting network to end network of an edge. The resistance of a path which is minimum among all paths from  $G_i$  to  $G_j$  is called as resistance from  $G_i$  to  $G_j$  and it is denoted by  $r_{ij}$ . The resistance from any network  $g_i \in G_i$  to any network  $g_j \in G_j$  is  $r_{ij}$  because inside  $G_i$  and  $G_j$  networks are connected with a path of zero resistance.

Now we recall the concept known as stochastic potential of a recurrent class  $G_i$  of  $P^0$  from [17]. Construct a graph  $\mathcal{G}$  where total number of nodes are the number of recurrent classes of  $P^0$  (one network from each recurrent class) and a directed edge from  $g_i$  to  $g_j$  is weighted by  $r_{ij}$ . Take a node  $g_i \in \mathcal{G}$  and consider all the spanning trees such that from every node  $g_j \in \mathcal{G}$ ,  $g_j \neq g_i$ , there is a unique path directed from  $g_j$  to  $g_i$ . Such spanning trees are called as  $g_i$ -trees. The resistance of a  $g_i$ -tree is the sum of the resistances of its edges. The stochastic potential of  $g_i$  is the resistance of a  $g_i$ -tree having minimum resistance among all  $g_i$ -trees. The stochastic potential of each node in  $G_i$  is same [17], which is a stochastic potential of  $G_i$ .

**Theorem 4.8.** *A stochastically stable network is either a  $\mathcal{C}$ -stable network or a closed cycle having minimum stochastic potential.*

*Proof.* We know that the Markov chain  $P^\varepsilon$  is aperiodic and irreducible. From (3) and (4) it is easy to see that

$$\lim_{\varepsilon \rightarrow 0} P^\varepsilon(g'|g) = P^0(g'|g), \forall g, g' \in G(\mathcal{C}).$$

From (4) it is clear that, if  $P^\varepsilon(g'|g) > 0$  for some  $\varepsilon \in (0, \varepsilon_0]$ , then we have

$$0 < \varepsilon^{-r(g, g')} P^\varepsilon(g'|g) < \infty.$$

So, Markov chain  $P^\varepsilon$  is a regular perturbed process because it satisfies all three conditions given in [17]. Hence it follows from Theorem 4 of Young [17] that as  $\varepsilon \rightarrow 0$ ,  $\mu^\varepsilon$  converges to a stationary distribution  $\mu^0$  of  $P^0$  and a network  $g$  is stochastically stable, i.e.,  $\mu_g^0 > 0$  if and only if  $g$  is contained in a recurrent class of  $P^0$  having minimum stochastic potential. We know that the recurrent classes of Markov chain  $P^0$  are  $\mathcal{C}$ -stable networks or closed cycles. So, a stochastically stable network is either a  $\mathcal{C}$ -stable network or a closed cycle having minimum stochastic potential.  $\square$

#### 4.1.1 Strong stability

We consider the case where there are no restrictions on the formation of teams' coalitions, i.e.,  $\mathcal{C} = 2^P$ . In this case  $\mathcal{C}$ -stability is called as strong stability which is analogous to strong stability in standard network formation games [11].

**Corollary 4.9.** *All the strongly stable networks and closed cycles are stochastically stable.*

*Proof.* Since, we have no restrictions on the coalition formation, then all types of networks are possible to form among teams. Now, the recurrent classes of  $P^0$  are the strongly stable networks or the closed cycles. From Theorem 4.8 a stochastically stable network is either a strongly stable network or a closed cycle having minimum stochastic potential. The resistance between any two distinct classes  $G_i$  and  $G_j$  is always 1 because under  $P^\varepsilon$  it is always possible to reach one network from another network by at

most one mutation when all types of teams' coalitions are allowed. Then the resistance of each  $g_i$ -tree is  $J - 1$ , where  $J$  is the total number of recurrent classes of  $P^0$ . Hence, the stochastic potential of each recurrent class of  $P^0$  is  $J - 1$ . So, from Theorem 4.8 all the strongly stable networks and closed cycles are stochastically stable.  $\square$

## 5 Simulation Results

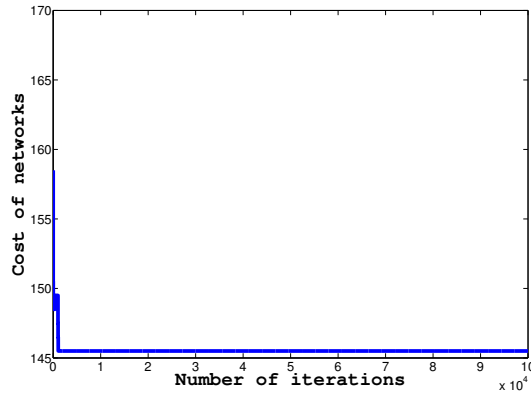
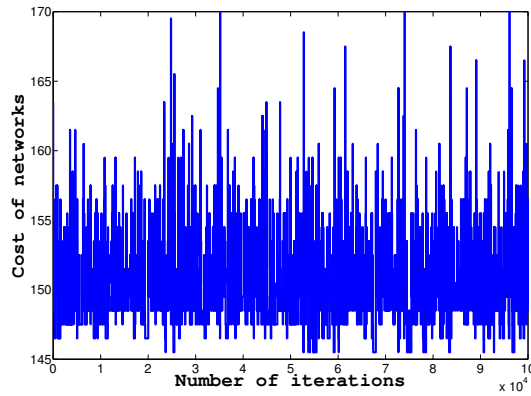
In the previous section we have characterized which networks are selected on the long run by the usual trembling-hand network dynamics, when the error probability  $\varepsilon$  converges to 0. Studying the dynamics itself for a finite  $\varepsilon$  is a harder problem, so we resort to simulations to investigate i) which networks appear more frequently during the dynamics and ii) which *quasi-stable* networks arise and how fast. By quasi-stable networks we mean networks that appear for a long period of time before a random error let them disappear. We cannot be sure that such networks are indeed stable with respect to the set of coalitions we consider in the specific experiment (in general only a direct inspection of all the possible deviations could reach such conclusion). Nevertheless, specially for larger values of  $\varepsilon$  quasi-stable networks can appear over time durations comparable with those of stable networks, so that practically speaking, they are as important as stable networks. The dynamics clearly depends on the value of  $\varepsilon$ , but also on the specific set  $\mathcal{C}$  of coalitions we consider: we are going to explore the effect of both.

As case study, we consider a simple instance of the bilateral network creation game with teams defined in Section 3. Let the set of nodes be  $N = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ . Nodes are split in three teams,  $T_1 = \{1, 2, 3\}$ ,  $T_2 = \{4, 5, 6\}$  and  $T_3 = \{7, 8, 9\}$ . The cost of link creation is  $\alpha = 1.5$ . For any experiment the initial network topology is selected uniformly at random among all the possible networks. Moreover, for any set  $\mathcal{C}$  considered in the experiments, every coalition in  $\mathcal{C}$  is equally likely to be selected at any iteration, i.e., for all  $t$ ,  $p_{S_t} = \frac{1}{|\mathcal{C}|}$  for every  $S_t \in \mathcal{C}$ . Also the probability measures over the set  $G_f(S_t, g_t)$  and  $G_{nf}(S_t, g_t)$  are uniform, i.e., any network  $g \in G_f(S_t, g_t)$  is equally likely to be selected as an improving deviation at any iteration and any network  $g \in G_{nf}(S_t, g_t)$  is equally likely to be selected when coalition  $S_t$  makes error at any iteration. We know that in this game at internally stable network all the teams form a complete network internally. So, in the dynamics we update the networks in such a way that the internal stability of networks is preserved.

### The case of all coalitions

Here we consider the case where all the 7 coalitions are allowed to form. We start considering a very small error probability  $\varepsilon = 0.00001$ . Figure 3 shows the time-evolution of the cost of the network selected by the dynamics. After less than 1000 iterations, the cost reaches a constant value equal to 145.5, that is the cost of the team-star network for this specific value of  $\alpha$ . The curve then suggests that the network dynamics has selected (very fast) a quasi-stable network and that this network may be a team-star. An inspection of the topologies of the networks selected show that this is actually the case: the dynamics reaches a specific team-star network and then nothing changes until the end of the simulation. Note that we do not know if the team-star is stable with respect to the set of all the possible coalitions, but this experiment (and many others we carried on) suggest that this is the case, because we never observed an improved deviation.

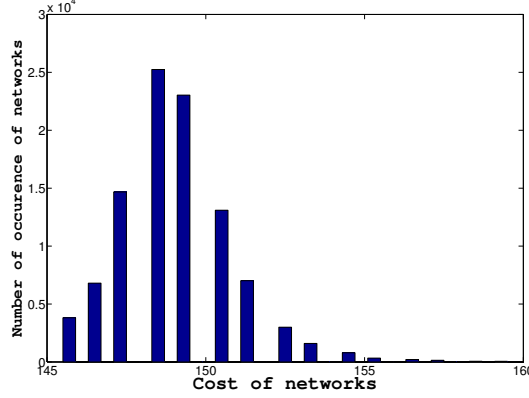
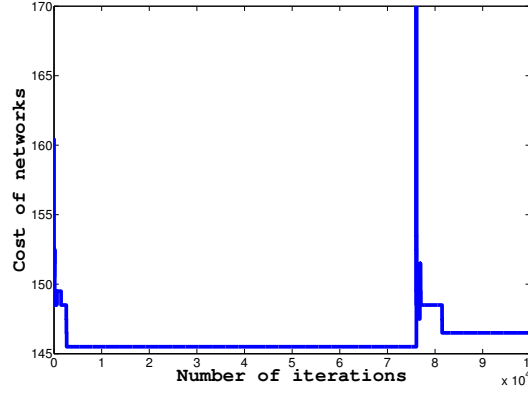
Next, we consider the case of much higher error probability  $\varepsilon = 0.01$ . We can see from Figure 4 that the evolution of network cost is now very noisy, this is clear because the dynamics jumps to a random network on average every 100 iterations, then the network cost usually decreases because of improving deviations selected by the coalitions until the next error occurs that leads the dynamics to a new random network whose cost is in general higher. Even a stable network would not survive for a long time because of the high error probability. Figure 5 shows the empirical distribution of network costs observed during the simulation. The network dynamics visits now more often networks whose cost is higher than the cost of the team-star. We checked some of these networks which appear more frequently and none of them was stable. In particular, for all of them, the team-star was an improving deviation for the grand coalition. One could then wonder why these networks appear more often than the team-star during the dynamics. The reason behind is that such networks appear in much more "forms" than team-star. In fact there are

Figure 3: Sample path of network dynamics for  $\varepsilon = 0.00001$ Figure 4: Sample path of network dynamics for  $\varepsilon = 0.01$ 

only 9 team-star networks (in each of them a different node is the center of the star). For example some of the networks appearing more frequently are team-stars with a few more links. There are many different ways to place such extra links in order to have a network with the same cost. Said in other words, the classes of isomorphic networks have different sizes, and frequency with which a representant of the class appears during the network dynamics depends on its stability versus deviations but also simply on the size of the class.

### Without the grand coalition

Here we consider the case where the grand coalition cannot form, but only the coalitions of size up to 2 can be formed. We first consider  $\varepsilon = 0.00001$ . In Figure 6 we observe first an evolution similar to that in Figure 3 with the network dynamics selecting very fast a team-star, but in this case an error after the 70000-th iteration produces a restart. Interestingly, after the restart, the cost reaches a constant value equal to 146.5. In this case the constant cost hides in reality a continuously changing topology. In fact, there are multiple networks with cost 146.5 which can be obtained one from the other by improving deviations from a coalition of size 2 and this is actually what is happening in this simulation. After the second random start, the dynamics keeps selecting different networks but all with the same cost. These networks are not even quasi-stable according to our definition, but from the point of view of the global cost, the situation does not change. As a final remark, we checked that also for these networks the team-star is an improving deviation, but it can only be enforced by the grand coalition, which is not

Figure 5: Empirical cost distribution for  $\varepsilon = 0.01$ Figure 6: Sample path of network dynamics for  $\varepsilon = 0.00001$ 

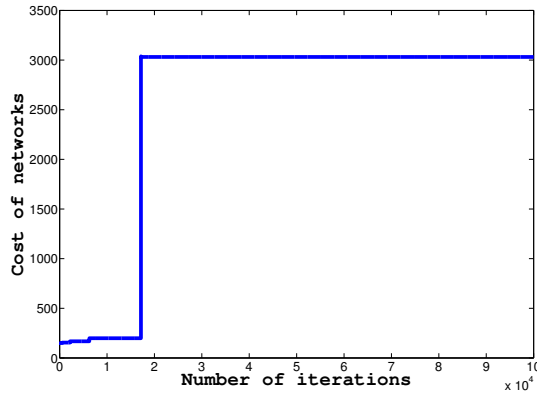
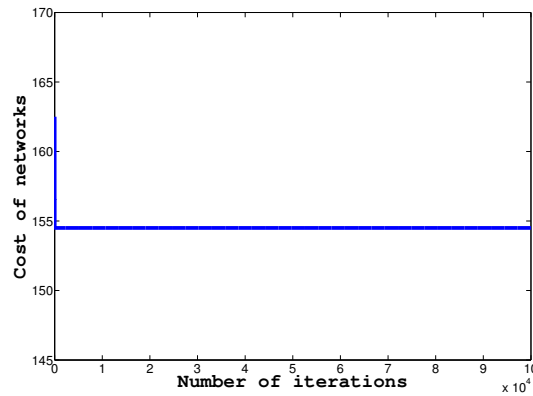
considered in such a case.

For the larger error probability  $\varepsilon = 0.01$  the dynamics appear to be similar to those when all the coalitions can be selected.

### Only single team coalitions

We now move to the case where only coalitions with a single team are considered. Note that the creation of an external link requires a coalition of the two teams, that is excluded here. Hence the only possible change in this case is due to team severing links to other teams if it is beneficial for them or because of a random error. As soon as one of the team will be disconnected from the other, there will be no more improving deviation, and then the network will evolve because of random errors until all the teams are disconnected. This is the absorbing state for the network dynamics. We first consider an error probability  $\varepsilon = 0.001$ . From Figure 7 it appears that the situation where one team is disconnected from the other two teams is reached after less than 20000 iterations (the cost would be infinite according to (2), but in the simulations the cost of a disconnected network is 3000). For small  $\varepsilon$  like  $\varepsilon = 0.00001$ , the situation may be very different.

From Figure 8 we see that the dynamics is quite stable and a network with cost 154.5 (quite costly in comparison to what observed in the previous cases) survives for long time. Inspection of the network topology shows that this network is indeed stable to deviations from single teams. This network will then survive until an error will not lead to one team accidentally severing a link.

Figure 7: Sample path of network dynamics for  $\varepsilon = 0.001$ Figure 8: Sample path of network dynamics for  $\varepsilon = 0.00001$ 

## 6 Conclusions

To the best of our knowledge this is the first paper where the concept of teams is introduced in network formation games. We extend the concept of pairwise stable networks to this new class of games, but also define a new concept of network stability toward deviations from a specific set  $\mathcal{C}$  of teams' coalitions. We show that  $\mathcal{C}$ -stable networks (and closed network cycles) are selected by a trembling-hand dynamics when the error probability vanishes. Finally, we resort to simulations to study the evolution of networks in a specific game by using our stochastic dynamic. We think that the idea of teams in network formation games may capture many practical phenomena such as connectivity pattern in the Internet Autonomous systems and it opens new research directions.

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ISSN 0249-6399