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## Application of optimal transport to data assimilation

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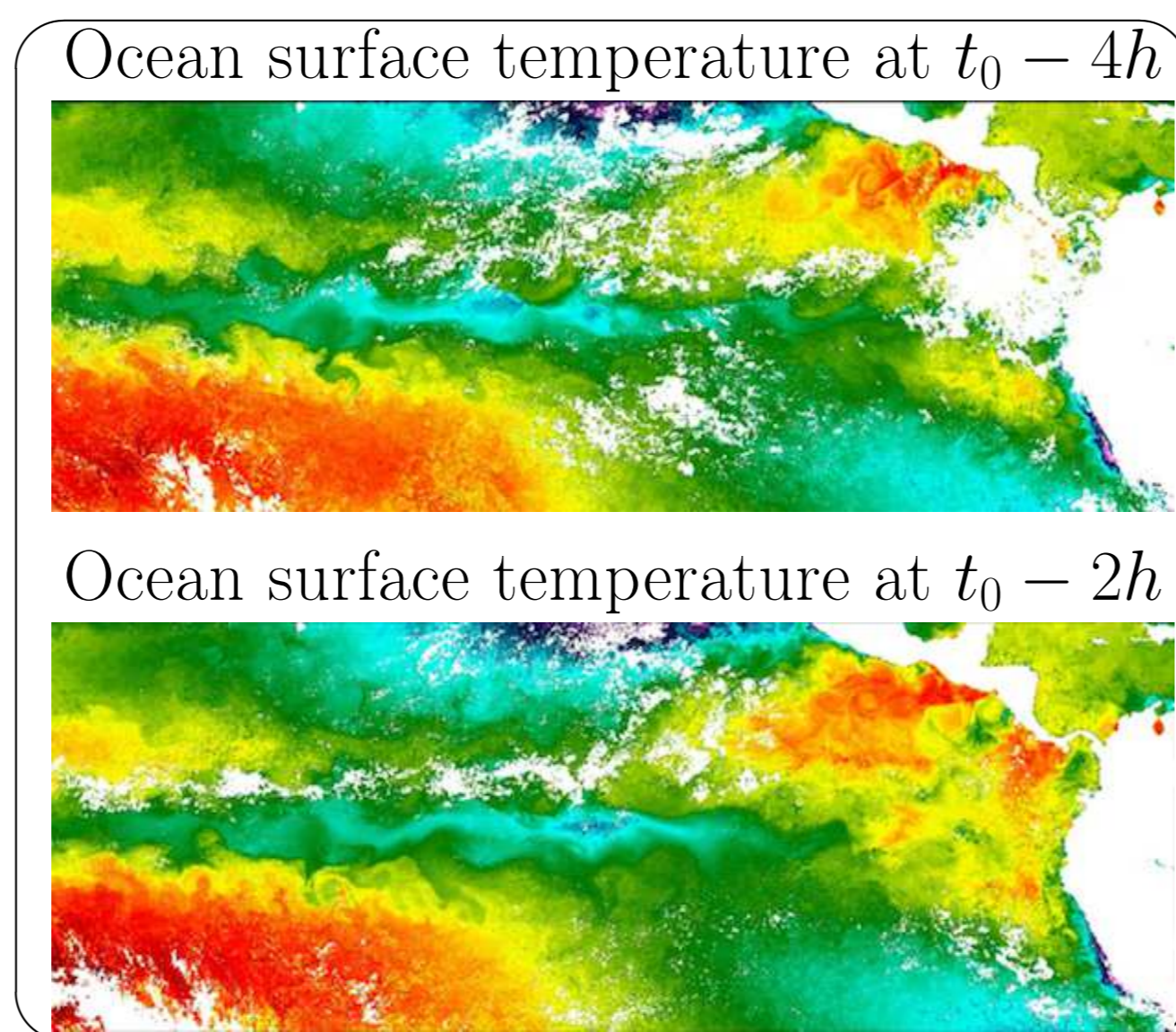
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## EXAMPLE OF OCEANOGRAPHIC DATA ASSIMILATION

Given a state  $\mathbf{x}$  whose evolution is simulated through a model  $\mathcal{M}$ , **data assimilation** is an inverse problem aiming at reconstructing an initial condition  $\mathbf{x}_0$  of  $\mathbf{x}$  given only several partial observations  $(\mathbf{y}_i^o)_i$  of  $\mathbf{x}$ . For example, the state  $\mathbf{x}(t, x, y, z)$  of the ocean gathers :



- Temperature  $T(t, x, y, z)$
- Velocities  $\mathbf{u}(t, x, y, z)$
- Salinity  $S(t, x, y, z)$
- Water height  $h(t, x, y)$

The question is then: What is the complete state of the ocean at  $t = 0$ ,  $\mathbf{x}_0(x, y, z)$ ?

In variational data assimilation this is done by minimizing a cost function  $\mathcal{J}$  defined as

$$\mathcal{J}(\mathbf{x}_0) = \frac{1}{2} \sum_i d(\mathcal{H}_i(\mathbf{x}_0), \mathbf{y}_i^o)^2 + \frac{\gamma}{2} d(\mathbf{x}_0, \mathbf{x}_0^b)^2, \quad (1)$$

$$\approx d(\mathbf{x}_0, \mathbf{x}_0^t)^2$$

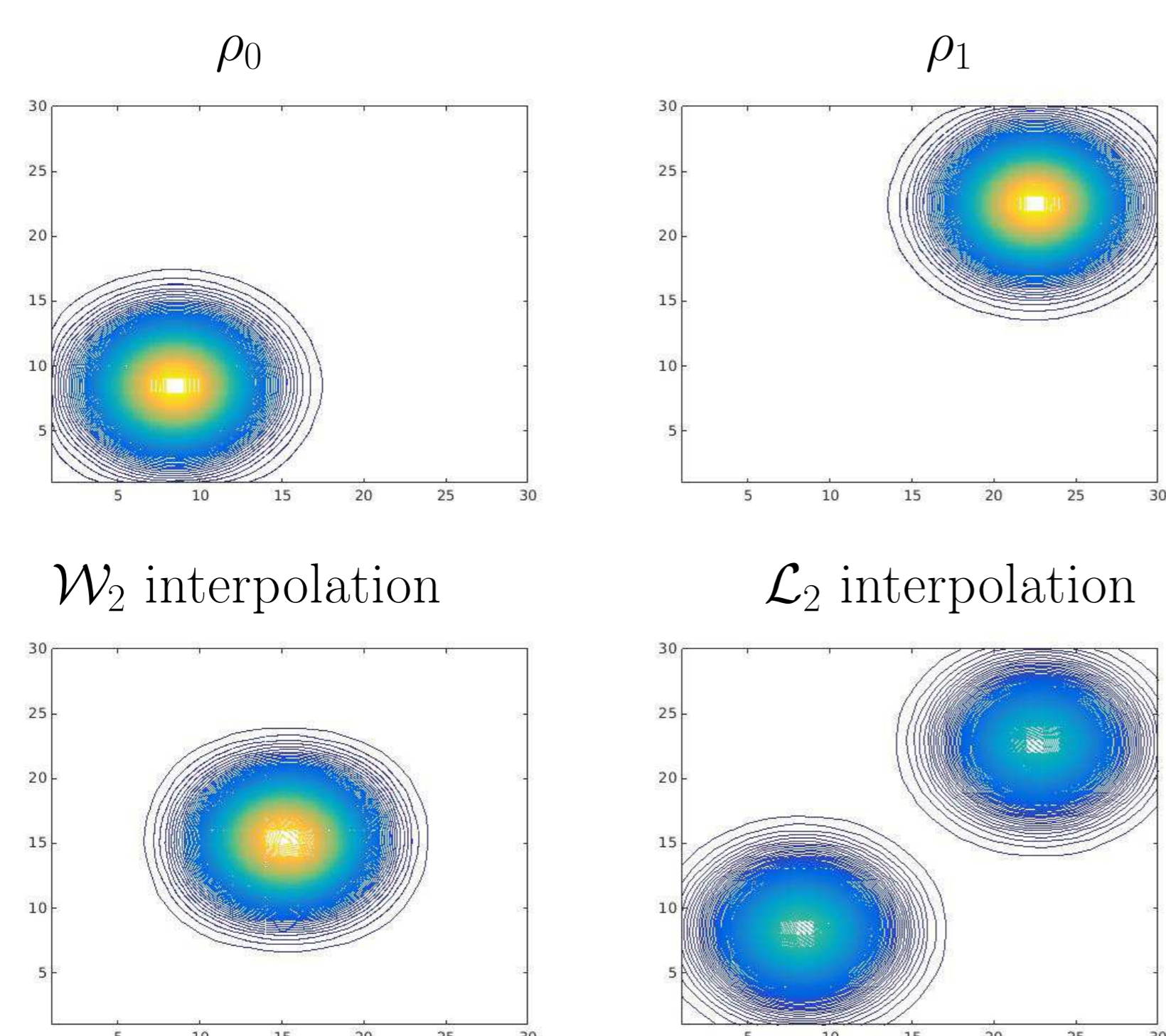
It is common for the distance  $d$  to be a weighted  $\mathcal{L}_2$  distance. Yet, with this  $\mathcal{L}_2$  distance, variational data assimilation cannot well cope with displacement errors. By choosing distances  $d$  that take into account the data more in its entirety, we hope get a more realistic variational data assimilation. That is why we are interested in the Wasserstein distance  $d = \mathcal{W}_2$ .

## ADVANTAGES OF OPTIMAL TRANSPORTATION IN CASE OF DISPLACEMENT ERROR

Knowing two densities  $\rho_0(x)$  and  $\rho_1(x)$ , the **Wasserstein distance**  $\mathcal{W}_2(\rho_0, \rho_1)$  is defined as

$$\mathcal{W}_2^2(\rho_0, \rho_1) := \inf_{\substack{(\rho(t, x), v(t, x)) \\ \partial_t \rho + \text{div}(\rho v) = 0 \\ \rho(0, x) = \rho_0(x), \rho(1, x) = \rho_1(x)}} \int \int_{[0,1] \times \Omega} \rho(t, x) |v(t, x)|^2 dt dx.$$

Example of interpolation using Wasserstein and  $\mathcal{L}_2$  distances. (Interpolation in the sense of minimizing  $d(\rho, \rho_0)^2 + d(\rho, \rho_1)^2$ ).



For Wasserstein distance to be defined, one needs  $\rho_0 \geq 0$ ,  $\rho_1 \geq 0$  and  $\int_{\Omega} \rho_0 = \int_{\Omega} \rho_1$ .

## GRADIENT DESCENT WITH THE WASSERSTEIN DISTANCE

Assuming  $\mathbf{x}_0$ ,  $\mathcal{H}_i(\mathbf{x}_0)$  and  $\mathbf{y}_i^o$  are probability measures, the cost function  $\mathcal{J}_W$  is

$$\mathcal{J}_W(\mathbf{x}_0) = \frac{1}{2} \sum_i \mathcal{W}_2^2(\mathcal{H}_i(\mathbf{x}_0), \mathbf{y}_i^o)$$

The gradient descent algorithm for minimizing  $\mathcal{J}_W$  consists in using the following iterative algorithm

$$\mathbf{x}_0^{n+1} = \mathbf{x}_0^n - \alpha \text{grad} \mathcal{J}_W(\mathbf{x}_0^n)$$

so that  $\mathcal{J}_W(\mathbf{x}_0^{n+1}) < \mathcal{J}_W(\mathbf{x}_0^n)$ . To get the gradient, the **differential** must be computed and also an **inner product**. Indeed,  $\text{grad} \mathcal{J}(\mathbf{x}_0)$  is such that for all  $\eta$ ,  $(\eta, \text{grad} \mathcal{J}(\mathbf{x}_0)) = D\mathcal{J}[\mathbf{x}_0].\eta$ .

### The differential:

Let's differentiate  $\mathcal{J}_W$  by computing  $\mathcal{J}_W(\mathbf{x}_0 + \epsilon \eta)$  :

- $\mathcal{H}_i(\mathbf{x}_0 + \epsilon \eta) = \mathcal{H}_i(\mathbf{x}_0) + \epsilon L[t_i, \mathbf{x}_0].\eta$  with  $L$  the tangent model.
- $\frac{1}{2} \mathcal{W}_2^2(\mathbf{y} + \epsilon \mu, \mathbf{y}') = \epsilon \langle \mu, \psi \rangle_2$  with  $\psi$  the **Kantorovich potential** of optimal transportation between  $\mathbf{y}$  and  $\mathbf{y}'$ .

Then,

$$D\mathcal{J}_W[\mathbf{x}_0].\eta = \sum_i \left\langle L[t_i, \mathbf{x}_0].\eta, \psi_i \right\rangle_2$$

with  $\psi_i$  the Kantorovich potential of the optimal transportation between  $\mathcal{H}_i(\mathbf{x}_0)$  and  $\mathbf{y}_i^o$ .

### The inner product:

For convergence reason, instead of using the  $\mathcal{L}_2$  inner product, we use the  $\mathcal{W}_2$  inner product defined in  $\mathbf{x}_0$  by

$$(\eta, \eta') = \int_{\Omega} \mathbf{x}_0 \nabla \Phi \cdot \nabla \Phi',$$

with  $\Phi, \Phi'$  s.t.  $\eta = -\text{div}(\mathbf{x}_0 \nabla \Phi)$ ,  $\eta' = -\text{div}(\mathbf{x}_0 \nabla \Phi')$

Finally, the gradient of  $\mathcal{J}_W$  w.r.t.  $\mathcal{W}_2$  inner product is, ( $L^*$  is the **adjoint model**),

$$\text{grad} \mathcal{J}_W(\mathbf{x}_0) = -\text{div} \left( \mathbf{x}_0 \nabla \left( \sum_i L^*[t_i, \mathbf{x}_0].\psi_i \right) \right).$$

## ASSIMILATION OF NON-PROBABILITY MEASURE VARIABLES

In the case where  $\mathcal{H}_i(\mathbf{x}_0)$  and  $\mathbf{y}_0$  are probability measures, but *not*  $\mathbf{x}_0$ , it is not possible to write the background term  $\mathcal{J}^b$  with  $\mathcal{W}_2$ . For example, for the Shallow-water system

$$\begin{cases} \partial_t h + \text{div}(hu) = 0 \\ \partial_t u + (u \cdot \nabla)u + g \nabla h = \mu \Delta u \end{cases} \quad (2)$$

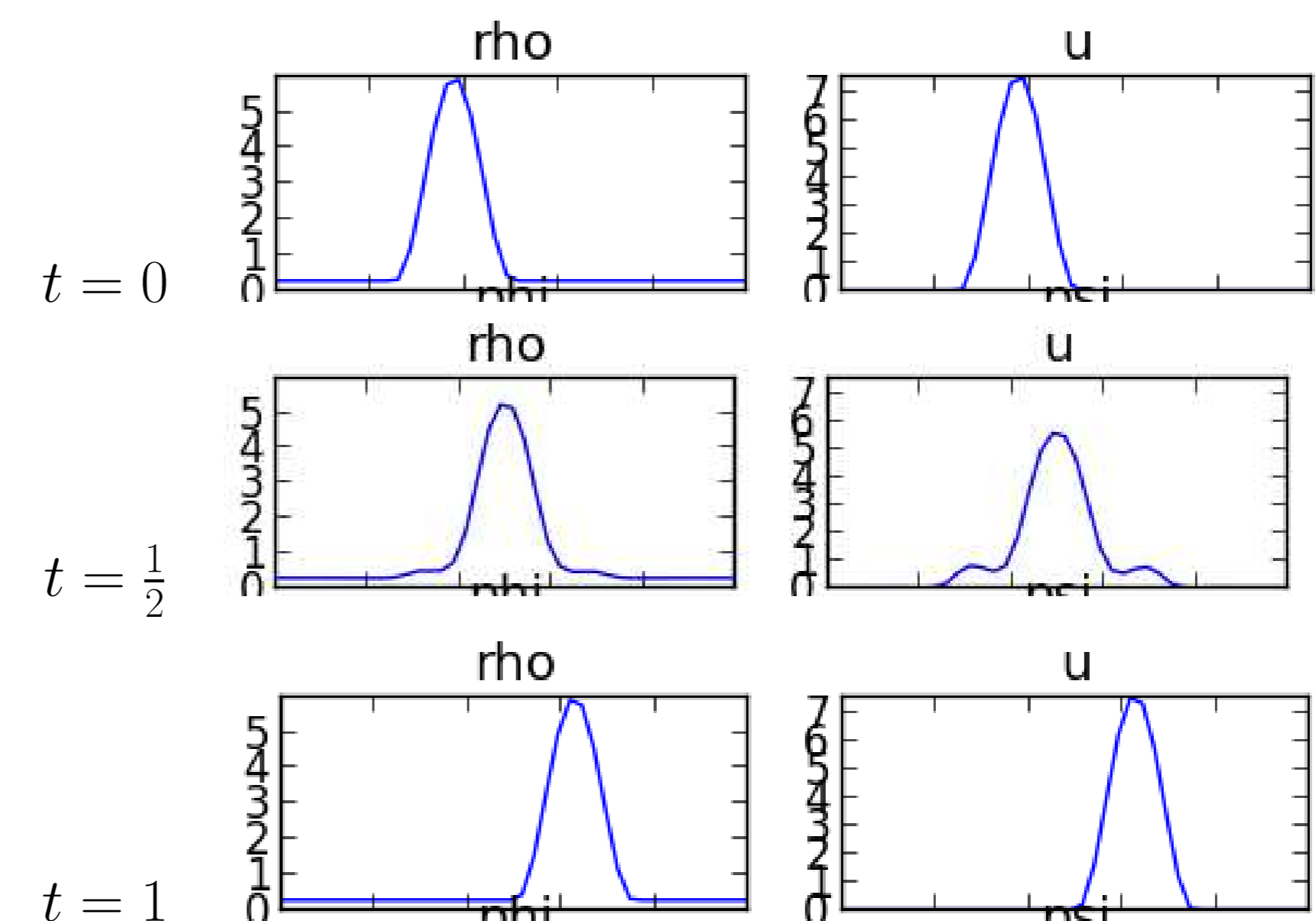
where  $(h_0, u_0)$  are to be assimilated using observations of  $h$  only. The cost function writes

$$\mathcal{J}(h_0, u_0) = \frac{1}{2} \sum_i \mathcal{W}_2^2(\mathcal{H}_i(h_0, u_0), h_i^o) + \gamma \mathcal{J}^b(h_0, u_0).$$

As it is impossible to write  $\mathcal{W}_2^2(u_0, u_0^b)$ , we rather use the following background term

$$\mathcal{J}^b(h_0, u_0) = \inf_{\substack{\partial_t \rho + \text{div}(\rho v) = 0 \\ \partial_t u + \text{div}(\rho w) = 0 \\ \rho(0, x) = h_0, \rho(1, x) = h_0^b \\ u(0, x) = u_0, u(1, x) = u_0^b}} \int \int_{[0,1] \times \Omega} (|v|^2 + |w|^2) \rho dt dx, \quad (3)$$

Using this, the interpolation of two shifted  $(h_0, u_0)$  and  $(h_1, u_1)$  is in-between:



The inner product to be chosen for computing the gradient will be

$$\left( \begin{pmatrix} \eta \\ v \end{pmatrix}, \begin{pmatrix} \eta' \\ v' \end{pmatrix} \right) = \int_{\Omega} h_0 \nabla \Phi \cdot \nabla \Phi' + \int_{\Omega} h_0 \nabla \Psi \cdot \nabla \Psi'$$

with  $\Phi, \Phi', \Psi, \Psi'$  such that

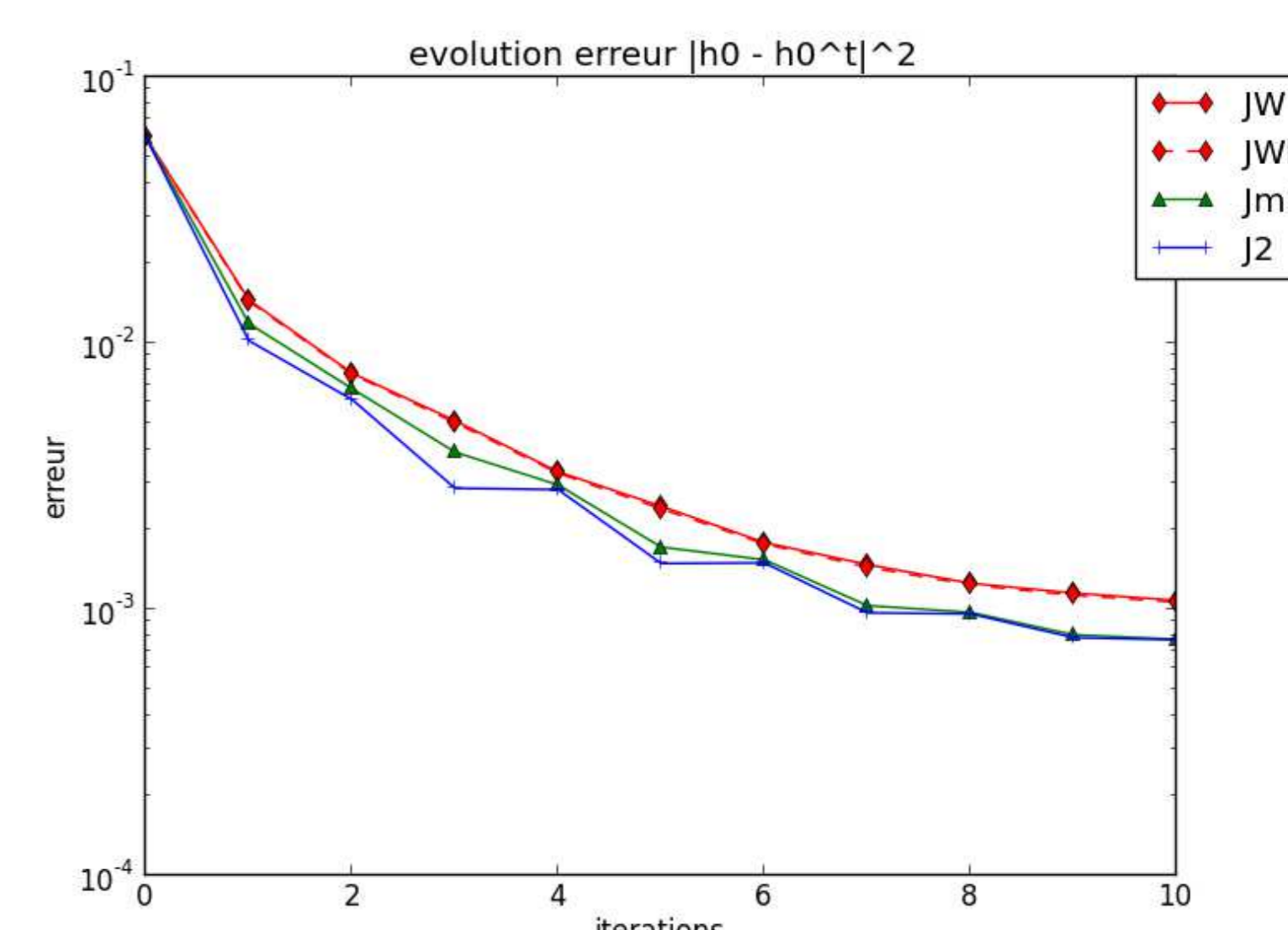
$$\begin{aligned} \eta &= -\text{div}(h_0 \nabla \Phi), & \eta' &= -\text{div}(h_0 \nabla \Phi') \\ v &= -\text{div}(h_0 \nabla \Psi), & v' &= -\text{div}(h_0 \nabla \Psi'). \end{aligned}$$

## DIFFICULTIES OF USING THE WASSERSTEIN DISTANCE

- The Wasserstein distance is only defined for probability measures.
- When  $\rho_0, \rho_1 \approx 1$ , the  $\mathcal{W}_2$  interpolation looks like  $\mathcal{L}_2$  interpolation...
- When  $\mathcal{J}(h_0^n) \rightarrow \min_{h_0} \mathcal{J}(h_0)$ , one only has  $h_0^n \rightarrow \arg \min_{h_0} \mathcal{J}(h_0)$ .
- The computing time is larger for  $\mathcal{W}_2$  than for  $\mathcal{L}_2$  [Peyré, Papadakis, Oudet, 2013].

## RESULTS AND PROPECTS

With some tests on the assimilation of (2), we compare the error  $\|h_0 - h_0^b\|_2^2$  by using  $d = \mathcal{L}_2$  and  $d = \mathcal{W}_2$  in the cost function. The behaviors seem correct.



The background term (3) is still to be implemented.