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Enhanced Recovery Conditions for OMP/OLS by Exploiting both Coherence and Decay

Cédric Herzet¹, Charles Soussen².

¹INRIA Centre Rennes - Bretagne Atlantique, Rennes, France.

²Centre de Recherche en Automatique de Nancy, Université de Lorraine, Nancy, France.

1 Introduction

In this paper, we focus on two popular instances of greedy algorithms, namely orthogonal matching pursuit (OMP) [1] and orthogonal least squares¹ (OLS) [5, 6].

The suboptimal nature of OMP and OLS has led many researchers to study conditions under which these procedures succeed in recovering the true sparse vector. This question has been widely addressed for OMP in the recent years, including worst-case uniform [7, 8] and probabilistic analyses [9]. Although OLS has been known in the literature for a few decades (under different names [10]), exact recovery analysis for OLS have only appeared very recently, see [11, 12, 13].

Most of the existing works deal with *uniform* guarantees: these conditions ensure the success of OMP/OLS for a given sparsity level (or a given support) *irrespective* of the magnitude of the non-zero coefficients. In contrast with these works, we derive new guarantees of success accounting for the decay of the non-zero elements of the sparse vector. Our conditions are expressed in terms of the mutual coherence of the dictionary μ and encompass, as particular cases, some well-known results of the literature. The proofs of the results are reported in our technical report [14].

2 Context and Main Results

Let us assume that $\mathbf{y} \in \mathbb{R}^m$ is a (noisy) linear combination of k columns of $\mathbf{A} \in \mathbb{R}^{m \times n}$ indexed by \mathcal{Q}^* , that is

$$\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{w} \quad \text{with} \quad \begin{cases} x_i \neq 0 \Leftrightarrow i \in \mathcal{Q}^* \\ \text{Card}\{\mathcal{Q}^*\} = k \end{cases} \quad (1)$$

where $\mathbf{w} \in \mathbb{R}^m$ denotes some noise vector and $\text{Card}\{\cdot\}$ stands for the cardinality operator. We assume that the columns \mathbf{a}_i of the dictionary are normalized: $\|\mathbf{a}_i\|_2 = 1 \forall i$.

Let us first consider the noiseless case ($\mathbf{w} = \mathbf{0}$). Theorem 1 provides sufficient conditions of success for OMP/OLS accounting for the possible decay of non-zero coefficients in \mathbf{x} . In our statement, we assume without loss of generality that

$$\mathcal{Q}^* = \{1, 2, \dots, k\}, \quad (2)$$

and

$$|x_1| \geq |x_2| \geq \dots \geq |x_k| > 0. \quad (3)$$

Theorem 1. *If*

$$\mu < \frac{1}{k}, \quad (4)$$

¹The OLS algorithm is also known as forward selection [2], Order Recursive Matching Pursuit (ORMP) [3] and Optimized Orthogonal Matching Pursuit (OOMP) [4] in the literature.

and

$$|x_i| > \frac{2\mu(k-i)}{1-i\mu} |x_{i+1}| \quad \forall i \in \{1, \dots, k\}, \quad (5)$$

then Oxx selects atoms in \mathcal{Q}^* from noiseless data during the first k iterations.

In our technical report [14], we show that this kind of result can also be extended to the characterization of the success of OMP/OLS in the noisy setting ($\|\mathbf{w}\|_2 \leq \epsilon$) or the partial recovery of the support by the latter algorithms. In particular, we show that a well-known result by Donoho *et al.* [15, Th. 5.1] can be further relaxed as

Theorem 2. *If*

$$\mu < \frac{1}{2k-1}, \quad (6)$$

and

$$|x_i| > \frac{2\epsilon}{1-(2k-i)\mu} \quad \forall i \in \{1, \dots, k\}, \quad (7)$$

then Oxx selects atoms in \mathcal{Q}^* during the first k iterations.

More specifically, we see that the condition by Donoho *et al.* in [15, Th. 5.1] is sufficient for (7) to be satisfied and is therefore stronger than the conditions mentioned in Theorem 2. On the other hand, the two types of conditions become equivalent as soon as the nonzero elements of \mathbf{x} have the same magnitude.

References

- [1] Y. C. Pati, R. Rezaifar, and P. S. Krishnaprasad, "Orthogonal matching pursuit: recursive function approximation with applications to wavelet decomposition," in *Proc. 27th Ann. Asilomar Conf. Signals, Systems, and Computers*, 1993.
- [2] Alan Miller, *Subset Selection in Regression, Second Edition*, Chapman and Hall/CRC, 2 edition, Apr. 2002.
- [3] S. F. Cotter, J. Adler, B. D. Rao, and K. Kreutz-Delgado, "Forward sequential algorithms for best basis selection," *IEE Proc. Vision, Image and Signal Processing*, vol. 146, no. 5, pp. 235–244, Oct. 1999.
- [4] L. Rebollo-Neira and D. Lowe, "Optimized orthogonal matching pursuit approach," *IEEE Signal Processing Letters*, vol. 9, no. 4, pp. 137–140, Apr. 2002.
- [5] S. Chen, S. A. Billings, and W. Luo, "Orthogonal least squares methods and their application to non-linear system identification," *International Journal of Control*, vol. 50, no. 5, pp. 1873–1896, Nov. 1989.
- [6] B. K. Natarajan, "Sparse approximate solutions to linear systems," *SIAM J. Comput.*, vol. 24, pp. 227–234, Apr. 1995.

- [7] J. A. Tropp, “Greed is good: algorithmic results for sparse approximation,” *IEEE Trans. Inf. Theory*, vol. 50, no. 10, pp. 2231–2242, Oct. 2004.
- [8] M. A. Davenport and M. B. Wakin, “Analysis of orthogonal matching pursuit using the restricted isometry property,” *IEEE Trans. Inf. Theory*, vol. 56, no. 9, pp. 4395–4401, 2010.
- [9] J. A. Tropp and A. C. Gilbert, “Signal recovery from random measurements via orthogonal matching pursuit,” *IEEE Trans. Inf. Theory*, vol. 53, no. 12, pp. 4655–4666, Dec. 2007.
- [10] T. Blumensath and M. E. Davies, “On the difference between orthogonal matching pursuit and orthogonal least squares,” Tech. Rep., University of Edinburgh, Mar. 2007.
- [11] S. Foucart, “Stability and robustness of weak orthogonal matching pursuits,” in *Recent advances in harmonic analysis and applications*, D. Bilyk, L. De Carli, A. Petukhov, A. M. Stokolos, and B. D. Wick, Eds. 2013, vol. 25, pp. 395–405, Springer proceedings in Mathematics & Statistics.
- [12] C. Soussen, R. Gribonval, J. Idier, and C. Herzet, “Joint k -step analysis of orthogonal matching pursuit and orthogonal least squares,” *IEEE Trans. Inf. Theory*, vol. 59, no. 5, pp. 3158–3174, May 2013.
- [13] C. Herzet, C. Soussen, J. Idier, and R. Gribonval, “Exact recovery conditions for sparse representations with partial support information,” *IEEE Trans. Inf. Theory*, vol. 59, no. 11, pp. 7509–7524, Nov. 2013.
- [14] Cédric Herzet, Angélique Drémeau, and Charles Soussen, “Relaxed recovery conditions for OMP/OLS by exploiting both coherence and decay,” Tech. Rep., available at <http://arxiv.org/abs/1401.7533>, Aug. 2014.
- [15] D. L. Donoho, M. Elad, and V. N. Temlyakov, “Stable recovery of sparse overcomplete representations in the presence of noise,” *IEEE Trans. Inf. Theory*, vol. 52, no. 1, pp. 6–18, Jan. 2006.