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# **QC-MDPC-McEliece: A public-key code-based encryption scheme based on quasi-cyclic moderate density parity check codes**

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# QC-MDPC-McEliece: A public-key code-based encryption scheme based on quasi-cyclic moderate density parity check codes

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Workshop “Post-Quantum Cryptography: Recent Results and Trends”  
Fukuoka, Japan, November 3-4, 2014

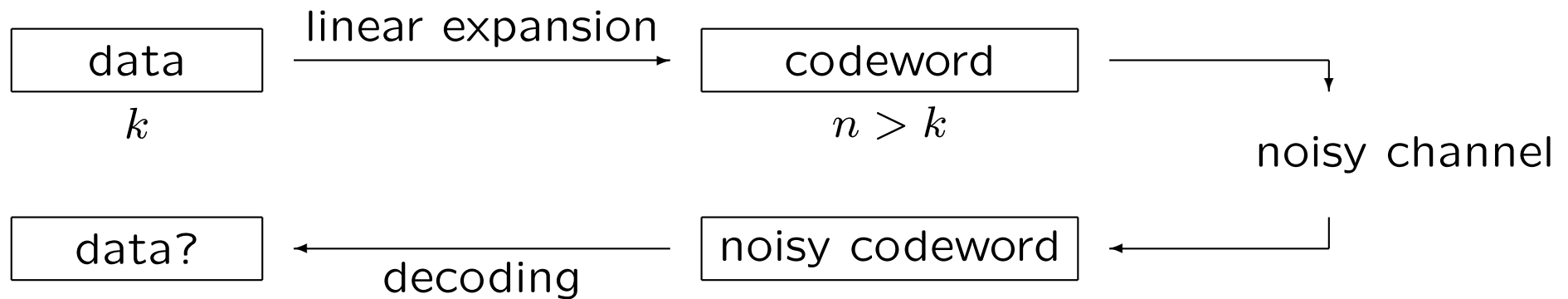
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Nicolas Sendrier

(joint work with R. Misoczki, J.-P. Tillich, and P. Barreto)

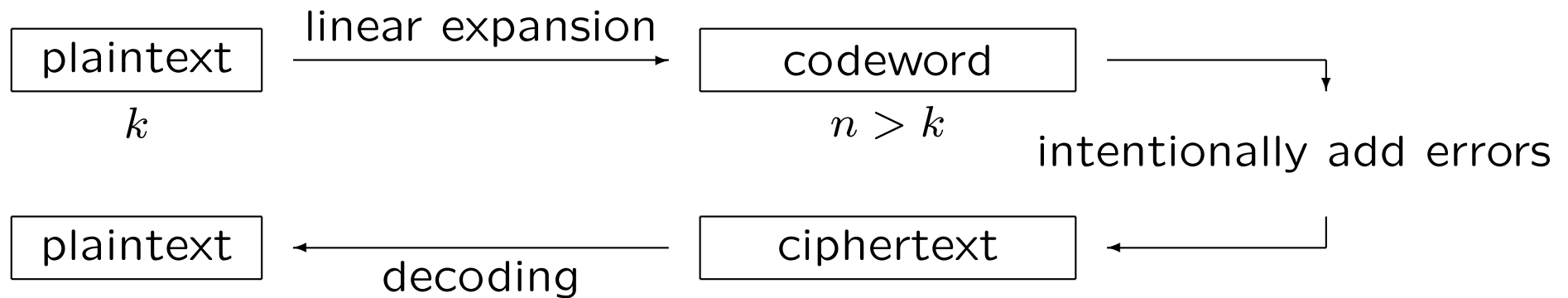


## Error Correcting Codes for Public-Key Encryption



- If a random linear expansion is used, no one can decode efficiently
- If a “good” error correcting code is used for the expansion, anyone who knows the structure has access to a fast decoder

## Error Correcting Codes for Public-Key Encryption



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- If a “good” error correcting code is used for the expansion, anyone who knows the structure has access to a fast decoder

Assuming that the knowledge of the linear expansion does not reveal the code structure:

- The linear expansion is public and anyone can encrypt
- The decoder is known to the legitimate user who can decrypt
- For anyone else, the public linear expansion looks random

## McEliece Public-key Encryption Scheme – Overview

$\mathcal{F}$  a family of  $t$ -error correcting binary linear  $[n, k]$  code

### Key generation:

$\mathcal{C} \in \mathcal{F} \rightarrow \begin{cases} \text{Public Key: } G \in \{0, 1\}^{k \times n}, \text{ a generator matrix} \\ \text{Secret Key: } \Phi : \{0, 1\}^n \rightarrow \mathcal{C}, \text{ a } t\text{-bounded decoder} \end{cases}$

**Encryption:**  $\left[ \begin{array}{l} E_G : \{0, 1\}^k \rightarrow \{0, 1\}^n \\ x \mapsto xG + e \end{array} \right]$  with  $e$  random of weight  $t$

**Decryption:**  $\left[ \begin{array}{l} D_\Phi : \{0, 1\}^n \rightarrow \{0, 1\}^k \\ y \mapsto \Phi(y)G^* \end{array} \right]$  where  $GG^* = 1$

[McEliece, 1978]  $\mathcal{F}$  is a family of binary Goppa codes

$n = 1024, k = 524, t = 50$

## Hardness of Decoding

[Berlekamp, McEliece, & van Tilborg, 78]

### Syndrome Decoding

NP-complete

*Instance:*  $H \in \{0, 1\}^{(n-k) \times n}$ ,  $s \in \{0, 1\}^{n-k}$ ,  $w$  integer

*Question:* Is there  $e \in \{0, 1\}^n$  such that  $\text{wt}(e) \leq w$  and  $eH^T = s$ ?

[Alekhovich, 03]

Conjectured difficult on average for  $w = n^\varepsilon$  and any  $\varepsilon > 0$

Best known decoder for  $w$  errors in an  $[n, k]$  code has complexity

$$W_{\text{SD}}(n, k, w) = 2^{(c+o(1))w \log_2 \frac{n}{n-k}}$$

[Prange, 62] Information Set Decoding,  $c = 1$

...

[Becker & Joux & May & Meurer, 12]  $c \approx 0.9$  when  $w = O(n)$

[Bernstein, 09] quantum  $c = 1/2$

## Security Reduction

For given parameters  $n$ ,  $k$ , and  $t$

$\mathcal{K} = \{0, 1\}^{k \times n}$  the “apparent” key space

$\mathcal{G} \subset \mathcal{K}$  the set of all public keys

### Theorem

If there exists an efficient *adversary* against McEliece then

- either there exists an efficient *distinguisher* for  $\mathcal{G}$  versus  $\mathcal{K}$
- or there exists an efficient *generic decoder* for  $t$  errors in  $[n, k]$  codes

In other words, if we assume that

1.  $\mathcal{G}$  is pseudorandom
2. decoding is hard on average

then McEliece’s scheme (with public keys in  $\mathcal{G}$ ) is secure “on average”

+ a semantically secure conversion  $\rightarrow$  any desirable security level

## More on Semantic Security

Because the scheme is malleable (replay attack [Berson, 97], reaction attack [Kobara & Imai, 00]) a semantically secure conversion is **mandatory**

First semantically secure conversion: [Kobara & Imai, 01]

With a semantic security layer the public key can be in **systematic form** [Biswas & S.,08]

$$G = \begin{array}{|c|c|} \hline 1 & \\ \hline & \backslash \\ \hline & 1 \\ \hline \end{array}$$

→ smaller key size, easier encryption

## Quasi-Cyclic instances of McEliece's Scheme (1/2)

(similar to NTRU, Ring LWE, ideal lattices)

The public key is formed of circulant blocks, for instance:

$$G = \begin{array}{|c|c|} \hline \begin{array}{c} 1 \\ \diagdown \\ 1 \end{array} & \begin{array}{c} \boxed{g} \\ \circlearrowleft \end{array} \\ \hline \end{array}$$

$$G = \begin{array}{|c|c|c|c|c|} \hline \begin{array}{c} 1 \\ \diagdown \\ 1 \end{array} & & \begin{array}{c} \boxed{g_{0,0}} \\ \circlearrowleft \end{array} & \begin{array}{c} \boxed{g_{0,1}} \\ \circlearrowleft \end{array} & \begin{array}{c} \boxed{g_{0,2}} \\ \circlearrowleft \end{array} \\ \hline & \begin{array}{c} 1 \\ \diagdown \\ 1 \end{array} & \begin{array}{c} \boxed{g_{1,0}} \\ \circlearrowleft \end{array} & \begin{array}{c} \boxed{g_{1,1}} \\ \circlearrowleft \end{array} & \begin{array}{c} \boxed{g_{1,2}} \\ \circlearrowleft \end{array} \\ \hline \end{array}$$

Advantage: much smaller key size

Difficulty: hide the code structure (*i.e.* the secret decoder)

## Quasi-Cyclic instances of McEliece's Scheme (2/2)

- Goppa (or alternant) codes, initiated by [Gaborit, 05]

Too much algebraic structure, some attempts have failed, to be used with care

- “Disguised” LDPC (Low Density Parity Check) codes [Baldi & Chiaraluce, 07]

Less structure but still no convincing security reduction

- MDPC (Moderate Density Parity Check) codes [Misoczki & Tillich & S. & Barreto, 13]

Even less structure, a security reduction

[Misoczki & Barreto, 09]

Also possible with dyadic blocks instead of circulant blocks

MDPC McEliece

## QC-MDPC-McEliece Scheme (1/2)

Parameters:  $n, k, w, t$

(for instance  $n = 9601, k = 4801, w = 90, t = 84$ )

**Key generation:** (rate  $1/2, n = 2p, k = p$ )

Pick a (sparse) vector  $(h_0, h_1) \in \{0, 1\}^p \times \{0, 1\}^p$  of weight  $w$

$$H_{\text{secret}} = \begin{array}{|c|c|} \hline \boxed{h_0} & \boxed{h_1} \\ \hline \circlearrowright & \circlearrowright \\ \hline \end{array}$$

with  $h_0(x)$  invertible in  $\mathbf{F}_2[x]/(x^p - 1)$

(circulant binary  $p \times p$  matrices are isomorphic to  $\mathbf{F}_2[x]/(x^p - 1)$ )

Publish  $h(x) = h_1(x)h_0^{-1}(x) \pmod{x^p - 1}$  or  $g(x) = \overline{h(x)/x}$

$$H = \begin{array}{|c|c|} \hline \mathbf{1} & \boxed{h} \\ \hline \backslash & \circlearrowright \\ \hline & \mathbf{1} \\ \hline \end{array} \quad \text{or} \quad G = \begin{array}{|c|c|} \hline \boxed{g} & \mathbf{1} \\ \hline \circlearrowright & \backslash \\ \hline & \mathbf{1} \\ \hline \end{array}$$

$H$  a parity check matrix,  $G$  a generator matrix

## QC-MDPC-McEliece Scheme (2/2)

**Encryption:** (rate  $1/2$ ,  $n = 2p$ ,  $k = p$ )

$$\begin{aligned}\mathbf{F}_2[x]/(x^p - 1) &\rightarrow \mathbf{F}_2[x]/(x^p - 1) \times \mathbf{F}_2[x]/(x^p - 1) \\ m(x) &\mapsto (m(x)g(x) + e_0(x), m(x) + e_1(x))\end{aligned}$$

The error  $e(x) = (e_0(x), e_1(x))$  has weight  $t$

**Decryption:**

Iterative decoding (as for LDPC codes) which only requires the sparse parity check matrix. For instance the “bit flipping” algorithm

Parameters are chosen such that the decoder fails to correct  $t$  errors with negligible probability

Each iteration has a cost proportional to  $w \cdot (n - k)$ , the number of iterations is small (3 to 5 in practice)

## QC-MDPC-McEliece Security Reduction

$$H = \left[ \begin{array}{c|c} 1 & \boxed{h} \\ \hline & \text{⌚} \\ & 1 \end{array} \right] \text{ with } h(x) = \frac{h_1(x)}{h_0(x)} \pmod{x^p - 1}$$

Secure under two assumptions

1. Pseudorandomness of the public key

Hard to decide whether there exists a sparse vector in the code spanned by  $H$  (the dual of the MDPC code)

2. Hardness of generic decoding of QC codes

Hard to decode in the code of parity check matrix  $H$  (for an arbitrary value of  $h$ )

# Sparse Polynomial Problems

The security reduction and the attacks can be stated in terms of polynomials

## 1. Key Security

Given  $h(x)$ , find non-zero  $(h_0(x), h_1(x))$  such that

$$\begin{cases} h_0(x) + h(x)h_1(x) = 0 \pmod{x^p - 1} \\ \text{wt}(h_0) + \text{wt}(h_1) \leq w \end{cases}$$

or simply decide the existence of a solution  $\rightarrow$  distinguisher

## 2. Message Security

Given  $h(x)$  and  $S(x)$ , find  $e_0(x)$  and  $e_1(x)$  such that

$$\begin{cases} e_0(x) + h(x)e_1(x) = S(x) \pmod{x^p - 1} \\ \text{wt}(e_0) + \text{wt}(e_1) \leq t \end{cases}$$

In both cases, best known solutions use generic decoding algorithms

## Practical Security – Best Known Attacks

Let  $W_{SD}(n, k, t)$  denote the cost for the generic decoding of  $t$  errors in a binary  $[n, k]$  code

We consider a QC-MDPC-McEliece instance with parameters  $n, k, w, t$  and circulant blocks of size  $p$ .

1. **Key Attack:** find a word of weight  $w$  in a quasi-cyclic binary  $[n, n - k]$  code

$$W_K(n, k, w) \geq \frac{W_{SD}(n, n - k, w)}{n - k}$$

(there are  $n - k$  words of weight  $w$ )

2. **Message Attack:** decode  $t$  errors in a quasi-cyclic binary  $[n, k]$  code

$$W_M(n, k, t, p) \geq \frac{W_{SD}(n, k, t)}{\sqrt{p}}$$

(Decoding One Out of Many [S., 11]  $\rightarrow$  factor  $\sqrt{p}$ )

## Parameter Selection

Choose a code rate  $k/n$  and a security exponent  $S$  (for instance 80 or 128). Then increase the block size until the following succeeds:

- find  $w$  the smallest integer such that  $W_K(n, k, w) \geq 2^S$
- find  $t$  the error correcting capability of the corresponding MDPC code
- check that  $W_M(n, k, t, p) \geq 2^S$

80 bits of security

$$n = 9602$$

$$k = 4801$$

$$p = 4801$$

$$w = 90$$

$$t = 84$$

128 bits of security

$$n = 19714$$

$$k = 9857$$

$$p = 9857$$

$$w = 142$$

$$t = 134$$

## Scalability

A binary  $[n, k]$  code with  $n - k$  parity equations of weight  $w$  will correct  $t$  errors with an LDPC-like decoding algorithm as long as  $t \cdot w \lesssim n$

For LDPC codes, we have essentially  $w = O(1)$ . For MDPC codes we have  $w = O(\sqrt{n})$  and thus  $t = O(\sqrt{n})$ .

The optimal trade-off between the key size ( $K$ ) and the security ( $S$ ) is obtained for codes of rate  $1/2$  and

$$K \approx cS^2 \text{ with } c < 1$$

For Goppa code, the optimal code rate is  $\approx 0.8$  and

$$K \approx c(S \log_2 S)^2 \text{ with } c \approx 2$$

## Conclusion

QC-MDPC-McEliece is a promising variant which enjoys

- a reasonable key size
- good security arguments (very little structure)
- secure against quantum computers
- easy implementation (including lightweight implementation)  
[Heyse & von Maurich & Güneysu, 13]

Thank you for your attention

## Bit-Flipping Decoding

Parameter: a threshold  $T$

input:  $y \in \{0, 1\}^n$ ,  $H \in \{0, 1\}^{(n-k) \times n}$

Repeat

    Compute the syndrome  $Hy^T$

    for  $j = 1, \dots, n$

        if more than  $T$  parity equations involving  $j$  are violated then  
            flip  $y_j$

$$Hy^T = \begin{pmatrix} s_1 \\ \vdots \\ s_{n-k} \end{pmatrix}, \text{ if } s_i \neq 0 \text{ the } i\text{-th parity equation is violated}$$

If  $H$  is sparse enough and  $y$  close to the code of parity check matrix  $H$  then the algorithm finds the closest codeword after a few iterations