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QC-MDPC-McEliece: A public-key code-based encryption scheme based on quasi-cyclic moderate density parity check codes

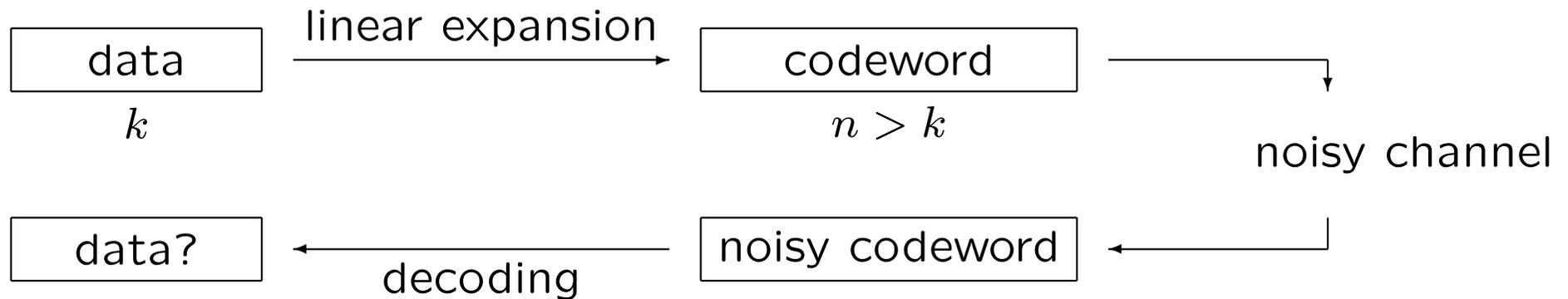
Workshop “Post-Quantum Cryptography: Recent Results and Trends”
Fukuoka, Japan, November 3-4, 2014

Nicolas Sendrier

(joint work with R. Misoczki, J.-P. Tillich, and P. Barreto)

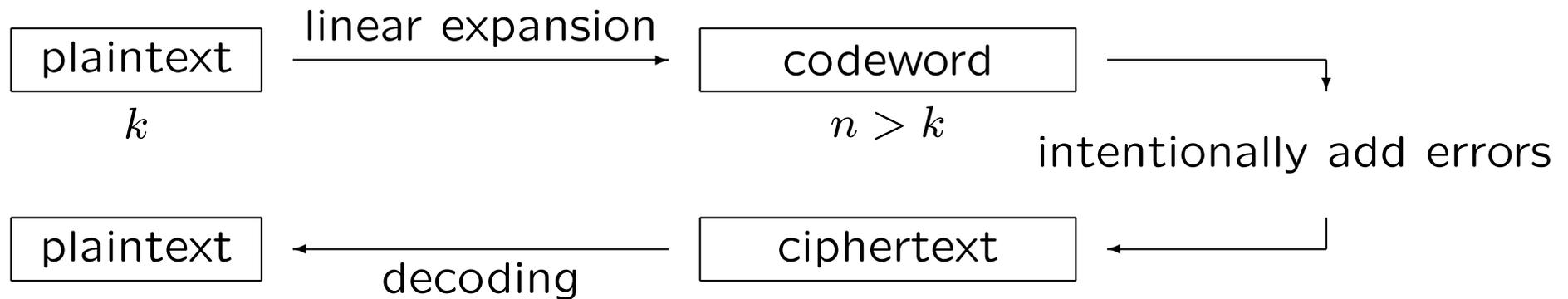


Error Correcting Codes for Public-Key Encryption



- If a random linear expansion is used, no one can decode efficiently
- If a “good” error correcting code is used for the expansion, anyone who knows the structure has access to a fast decoder

Error Correcting Codes for Public-Key Encryption



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- If a “good” error correcting code is used for the expansion, anyone who knows the structure has access to a fast decoder

Assuming that the knowledge of the linear expansion does not reveal the code structure:

- The linear expansion is public and anyone can encrypt
- The decoder is known to the legitimate user who can decrypt
- For anyone else, the public linear expansion looks random

McEliece Public-key Encryption Scheme – Overview

\mathcal{F} a family of t -error correcting binary linear $[n, k]$ code

Key generation:

$\mathcal{C} \in \mathcal{F} \rightarrow \left\{ \begin{array}{l} \text{Public Key: } G \in \{0, 1\}^{k \times n}, \text{ a generator matrix} \\ \text{Secret Key: } \Phi : \{0, 1\}^n \rightarrow \mathcal{C}, \text{ a } t\text{-bounded decoder} \end{array} \right.$

Encryption: $\left[\begin{array}{l} E_G : \{0, 1\}^k \rightarrow \{0, 1\}^n \\ x \mapsto xG + e \end{array} \right]$ with e random of weight t

Decryption: $\left[\begin{array}{l} D_\Phi : \{0, 1\}^n \rightarrow \{0, 1\}^k \\ y \mapsto \Phi(y)G^* \end{array} \right]$ where $GG^* = 1$

[McEliece, 1978] \mathcal{F} is a family of binary Goppa codes

$n = 1024, k = 524, t = 50$

Hardness of Decoding

[Berlekamp, McEliece, & van Tilborg, 78]

Syndrome Decoding

NP-complete

Instance: $H \in \{0, 1\}^{(n-k) \times n}$, $s \in \{0, 1\}^{n-k}$, w integer

Question: Is there $e \in \{0, 1\}^n$ such that $\text{wt}(e) \leq w$ and $eH^T = s$?

[Alekhovich, 03]

Conjectured difficult on average for $w = n^\varepsilon$ and any $\varepsilon > 0$

Best known decoder for w errors in an $[n, k]$ code has complexity

$$W_{\text{SD}}(n, k, w) = 2^{(c+o(1))w \log_2 \frac{n}{n-k}}$$

[Prange, 62] Information Set Decoding, $c = 1$

...

[Becker & Joux & May & Meurer, 12] $c \approx 0.9$ when $w = O(n)$

[Bernstein, 09] quantum $c = 1/2$

Security Reduction

For given parameters n , k , and t

$\mathcal{K} = \{0, 1\}^{k \times n}$ the “apparent” key space

$\mathcal{G} \subset \mathcal{K}$ the set of all public keys

Theorem

If there exists an efficient *adversary* against McEliece then

- either there exists an efficient *distinguisher* for \mathcal{G} versus \mathcal{K}
- or there exists an efficient *generic decoder* for t errors in $[n, k]$ codes

In other words, if we assume that

1. \mathcal{G} is pseudorandom
2. decoding is hard on average

then McEliece’s scheme (with public keys in \mathcal{G}) is secure “on average”

+ a semantically secure conversion \rightarrow any desirable security level

More on Semantic Security

Because the scheme is malleable (replay attack [Berson, 97], reaction attack [Kobara & Imai, 00]) a semantically secure conversion is **mandatory**

First semantically secure conversion: [Kobara & Imai, 01]

With a semantic security layer the public key can be in **systematic form** [Biswas & S.,08]

$$G = \begin{array}{|c|c|} \hline 1 & \\ \hline & \backslash \\ \hline & 1 \\ \hline \end{array}$$

→ smaller key size, easier encryption

Quasi-Cyclic instances of McEliece's Scheme (1/2)

(similar to NTRU, Ring LWE, ideal lattices)

The public key is formed of circulant blocks, for instance:

$$G = \begin{array}{|c|c|} \hline \begin{array}{c} 1 \\ \diagdown \\ 1 \end{array} & \begin{array}{c} \boxed{g} \\ \circlearrowright \end{array} \\ \hline \end{array}$$

$$G = \begin{array}{|c|c|c|c|c|} \hline \begin{array}{c} 1 \\ \diagdown \\ 1 \end{array} & & \begin{array}{c} \boxed{g_{0,0}} \\ \circlearrowright \end{array} & \begin{array}{c} \boxed{g_{0,1}} \\ \circlearrowright \end{array} & \begin{array}{c} \boxed{g_{0,2}} \\ \circlearrowright \end{array} \\ \hline & \begin{array}{c} 1 \\ \diagdown \\ 1 \end{array} & \begin{array}{c} \boxed{g_{1,0}} \\ \circlearrowright \end{array} & \begin{array}{c} \boxed{g_{1,1}} \\ \circlearrowright \end{array} & \begin{array}{c} \boxed{g_{1,2}} \\ \circlearrowright \end{array} \\ \hline \end{array}$$

Advantage: much smaller key size

Difficulty: hide the code structure (*i.e.* the secret decoder)

Quasi-Cyclic instances of McEliece's Scheme (2/2)

- Goppa (or alternant) codes, initiated by [Gaborit, 05]

Too much algebraic structure, some attempts have failed, to be used with care

- “Disguised” LDPC (Low Density Parity Check) codes [Baldi & Chiaraluce, 07]

Less structure but still no convincing security reduction

- MDPC (Moderate Density Parity Check) codes [Misoczki & Tillich & S. & Barreto, 13]

Even less structure, a security reduction

[Misoczki & Barreto, 09]

Also possible with dyadic blocks instead of circulant blocks

MDPC McEliece

QC-MDPC-McEliece Scheme (1/2)

Parameters: n, k, w, t

(for instance $n = 9601, k = 4801, w = 90, t = 84$)

Key generation: (rate $1/2, n = 2p, k = p$)

Pick a (sparse) vector $(h_0, h_1) \in \{0, 1\}^p \times \{0, 1\}^p$ of weight w

$$H_{\text{secret}} = \begin{array}{|c|c|} \hline \boxed{h_0} & \boxed{h_1} \\ \hline \circlearrowleft & \circlearrowleft \\ \hline \end{array}$$

with $h_0(x)$ invertible in $\mathbf{F}_2[x]/(x^p - 1)$

(circulant binary $p \times p$ matrices are isomorphic to $\mathbf{F}_2[x]/(x^p - 1)$)

Publish $h(x) = h_1(x)h_0^{-1}(x) \pmod{x^p - 1}$ or $g(x) = \overline{h(x)/x}$

$$H = \begin{array}{|c|c|} \hline \mathbf{1} & \boxed{h} \\ \hline \backslash & \circlearrowleft \\ \hline & \mathbf{1} \\ \hline \end{array} \quad \text{or} \quad G = \begin{array}{|c|c|} \hline \boxed{g} & \mathbf{1} \\ \hline \circlearrowleft & \backslash \\ \hline & \mathbf{1} \\ \hline \end{array}$$

H a parity check matrix, G a generator matrix

QC-MDPC-McEliece Scheme (2/2)

Encryption: (rate $1/2$, $n = 2p$, $k = p$)

$$\begin{aligned} \mathbf{F}_2[x]/(x^p - 1) &\rightarrow \mathbf{F}_2[x]/(x^p - 1) \times \mathbf{F}_2[x]/(x^p - 1) \\ m(x) &\mapsto (m(x)g(x) + e_0(x), m(x) + e_1(x)) \end{aligned}$$

The error $e(x) = (e_0(x), e_1(x))$ has weight t

Decryption:

Iterative decoding (as for LDPC codes) which only requires the sparse parity check matrix. For instance the “bit flipping” algorithm

Parameters are chosen such that the decoder fails to correct t errors with negligible probability

Each iteration has a cost proportional to $w \cdot (n - k)$, the number of iterations is small (3 to 5 in practice)

QC-MDPC-McEliece Security Reduction

$$H = \left[\begin{array}{c|c} 1 & \boxed{h} \\ \hline & \text{⌚} \\ & 1 \end{array} \right] \text{ with } h(x) = \frac{h_1(x)}{h_0(x)} \pmod{x^p - 1}$$

Secure under two assumptions

1. Pseudorandomness of the public key

Hard to decide whether there exists a sparse vector in the code spanned by H (the dual of the MDPC code)

2. Hardness of generic decoding of QC codes

Hard to decode in the code of parity check matrix H (for an arbitrary value of h)

Sparse Polynomial Problems

The security reduction and the attacks can be stated in terms of polynomials

1. Key Security

Given $h(x)$, find non-zero $(h_0(x), h_1(x))$ such that

$$\begin{cases} h_0(x) + h(x)h_1(x) = 0 \pmod{x^p - 1} \\ \text{wt}(h_0) + \text{wt}(h_1) \leq w \end{cases}$$

or simply decide the existence of a solution \rightarrow distinguisher

2. Message Security

Given $h(x)$ and $S(x)$, find $e_0(x)$ and $e_1(x)$ such that

$$\begin{cases} e_0(x) + h(x)e_1(x) = S(x) \pmod{x^p - 1} \\ \text{wt}(e_0) + \text{wt}(e_1) \leq t \end{cases}$$

In both cases, best known solutions use generic decoding algorithms

Practical Security – Best Known Attacks

Let $W_{SD}(n, k, t)$ denote the cost for the generic decoding of t errors in a binary $[n, k]$ code

We consider a QC-MDPC-McEliece instance with parameters n, k, w, t and circulant blocks of size p .

1. **Key Attack:** find a word of weight w in a quasi-cyclic binary $[n, n - k]$ code

$$W_K(n, k, w) \geq \frac{W_{SD}(n, n - k, w)}{n - k}$$

(there are $n - k$ words of weight w)

2. **Message Attack:** decode t errors in a quasi-cyclic binary $[n, k]$ code

$$W_M(n, k, t, p) \geq \frac{W_{SD}(n, k, t)}{\sqrt{p}}$$

(Decoding One Out of Many [S., 11] \rightarrow factor \sqrt{p})

Parameter Selection

Choose a code rate k/n and a security exponent S (for instance 80 or 128). Then increase the block size until the following succeeds:

- find w the smallest integer such that $W_K(n, k, w) \geq 2^S$
- find t the error correcting capability of the corresponding MDPC code
- check that $W_M(n, k, t, p) \geq 2^S$

80 bits of security

$$n = 9602$$

$$k = 4801$$

$$p = 4801$$

$$w = 90$$

$$t = 84$$

128 bits of security

$$n = 19714$$

$$k = 9857$$

$$p = 9857$$

$$w = 142$$

$$t = 134$$

Scalability

A binary $[n, k]$ code with $n - k$ parity equations of weight w will correct t errors with an LDPC-like decoding algorithm as long as $t \cdot w \lesssim n$

For LDPC codes, we have essentially $w = O(1)$. For MDPC codes we have $w = O(\sqrt{n})$ and thus $t = O(\sqrt{n})$.

The optimal trade-off between the key size (K) and the security (S) is obtained for codes of rate $1/2$ and

$$K \approx cS^2 \text{ with } c < 1$$

For Goppa code, the optimal code rate is ≈ 0.8 and

$$K \approx c(S \log_2 S)^2 \text{ with } c \approx 2$$

Conclusion

QC-MDPC-McEliece is a promising variant which enjoys

- a reasonable key size
- good security arguments (very little structure)
- secure against quantum computers
- easy implementation (including lightweight implementation)
[Heyse & von Maurich & Güneysu, 13]

Thank you for your attention

Bit-Flipping Decoding

Parameter: a threshold T

input: $y \in \{0, 1\}^n$, $H \in \{0, 1\}^{(n-k) \times n}$

Repeat

 Compute the syndrome Hy^T

 for $j = 1, \dots, n$

 if more than T parity equations involving j are violated then
 flip y_j

$$Hy^T = \begin{pmatrix} s_1 \\ \vdots \\ s_{n-k} \end{pmatrix}, \text{ if } s_i \neq 0 \text{ the } i\text{-th parity equation is violated}$$

If H is sparse enough and y close to the code of parity check matrix H then the algorithm finds the closest codeword after a few iterations