

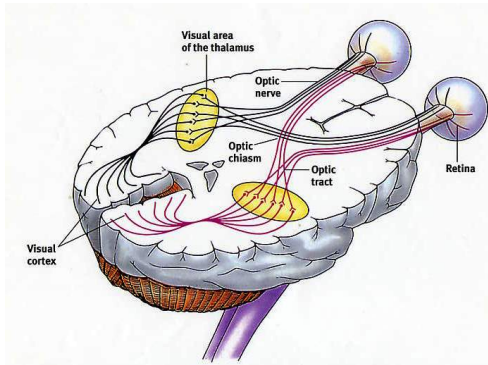
From retina to statistical physics

Bruno Cessac

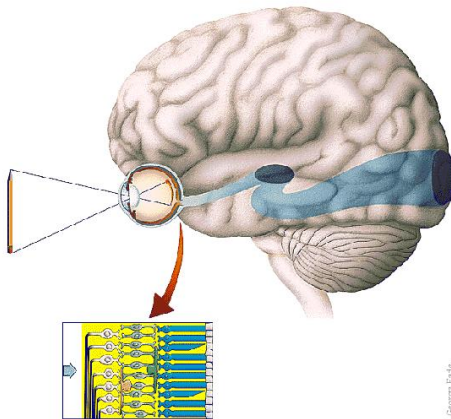
NeuroMathComp Team, INRIA Sophia Antipolis, France.

4ème journée de la physique niçoise, 20-06-14

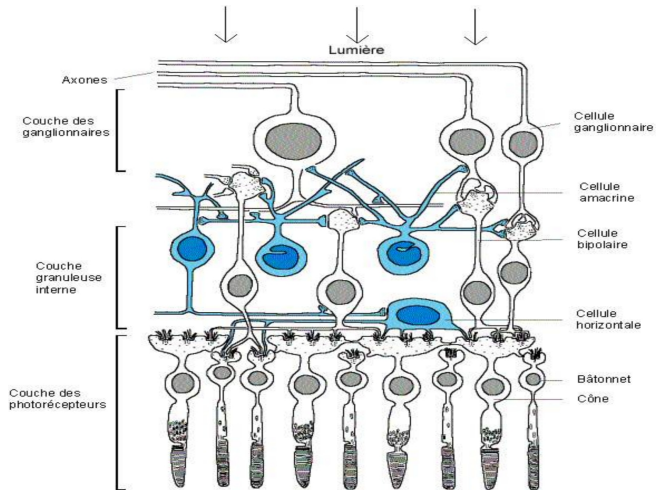
Visual system



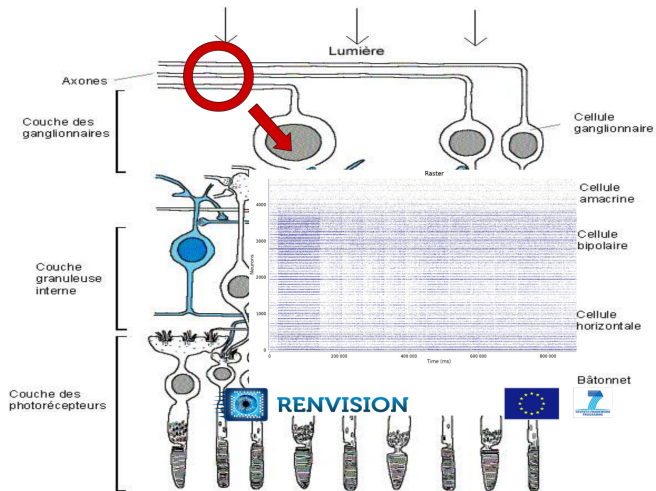
Visual system



Visual system



Visual system



Multi Electrodes Array

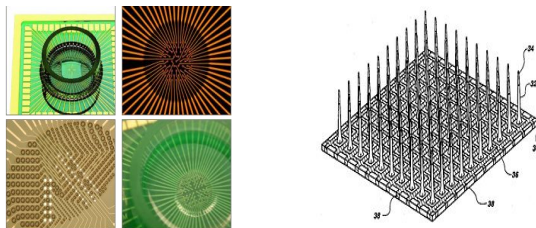
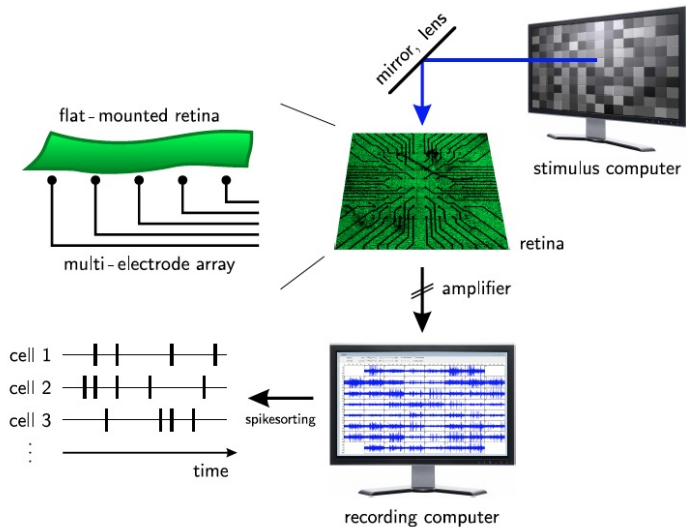
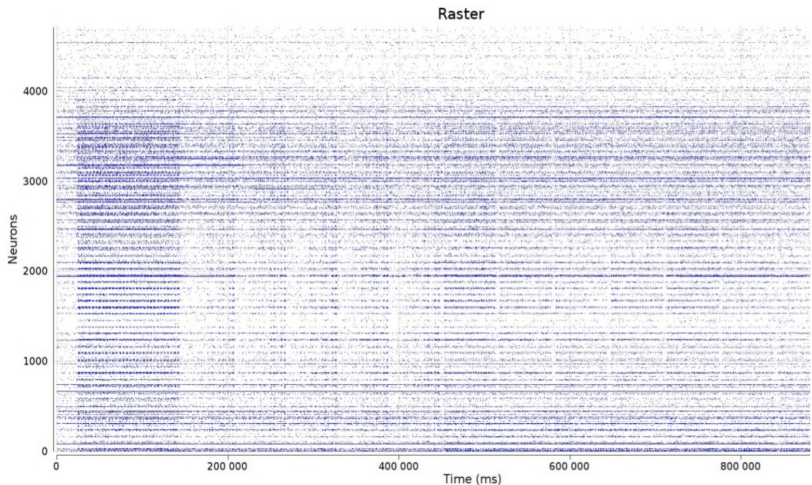


Figure: Multi-Electrodes Array.

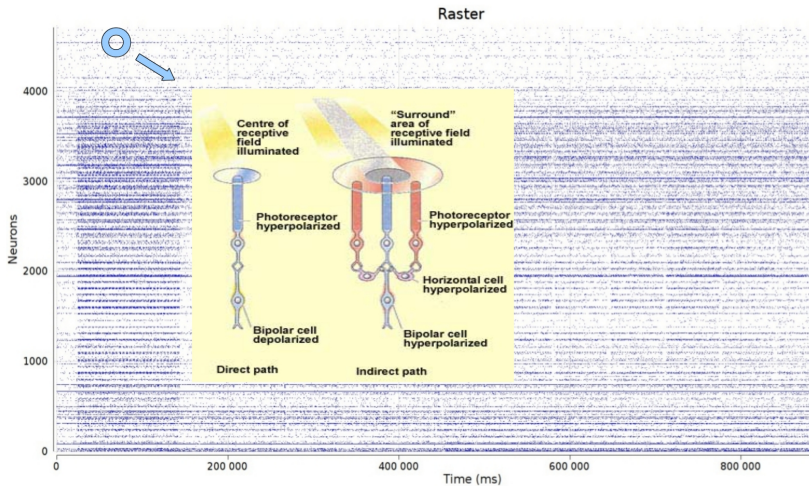
Multi Electrodes Array



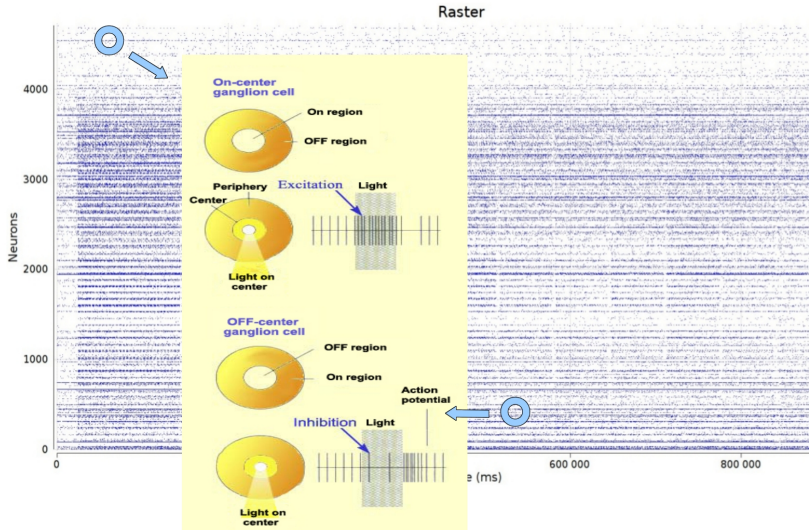
Encoding a visual scene



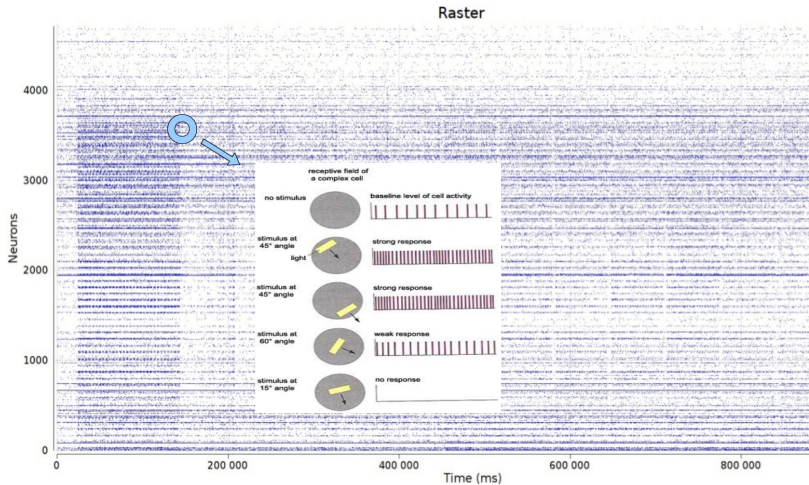
Encoding a visual scene



Encoding a visual scene



Encoding a visual scene



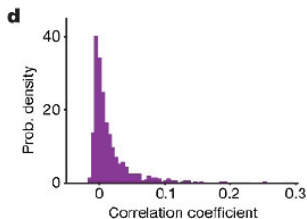
Do Ganglion cells act as **independent** encoders ?

Do Ganglion cells act as **independent** encoders ?

Or do their dynamical (spatio-temporal) **correlations** play a role in encoding a visual scene (*population coding*) ?

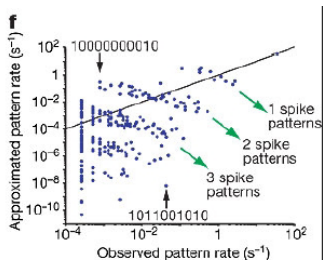
Let us measure (instantaneous pairwise) correlations

E. Schneidman, M.J. Berry, R. Segev, and W. Bialek. "Weak pairwise correlations imply strongly correlated network states in a neural population". Nature, 440(7087):1007-1012, 2006.



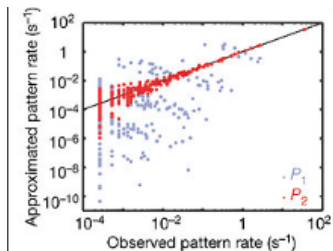
Let us measure (instantaneous pairwise) correlations

E. Schneidman, M.J. Berry, R. Segev, and W. Bialek. "Weak pairwise correlations imply strongly correlated network states in a neural population". Nature, 440(7087):1007-1012, 2006.



Let us measure (instantaneous pairwise) correlations

E. Schneidman, M.J. Berry, R. Segev, and W. Bialek. "Weak pairwise correlations imply strongly correlated network states in a neural population". Nature, 440(7087):1007-1012, 2006.



Constructing a statistical model handling measured correlations

- Assume **stationarity**.
- Measure empirical correlations.
- Select **the** probability distribution which maximizes the entropy and reproduces these correlations.

Spike events

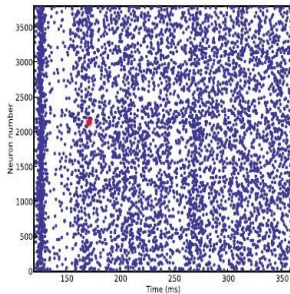


Figure: Spike state.

Spike state

$$\omega_k(n) \in \{0, 1\}$$

Spike events

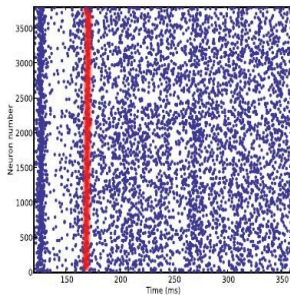


Figure: Spike pattern.

Spike state

$$\omega_k(n) \in \{0, 1\}$$

Spike pattern

$$\omega(n) = (\omega_k(n))_{k=1}^N$$

Spike events

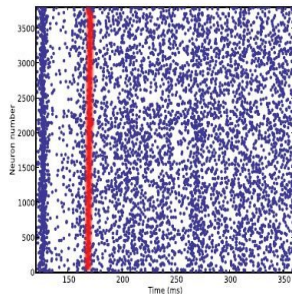


Figure: Spike pattern.

Spike state

$$\omega_k(n) \in \{0, 1\}$$

Spike pattern

$$\omega(n) = (\omega_k(n))_{k=1}^N \quad \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Spike events

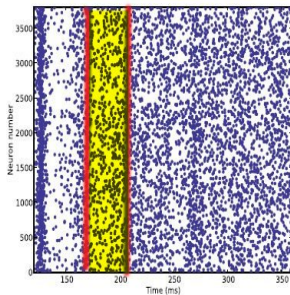


Figure: Spike block.

Spike state

$$\omega_k(n) \in \{0, 1\}$$

Spike pattern

$$\omega(n) = (\omega_k(n))_{k=1}^N$$

Spike block

$$\omega_m^n = \{\omega(m)\omega(m+1)\dots\omega(n)\}$$

Spike events

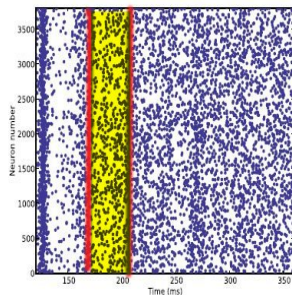


Figure: Spike block.

Spike state

$$\omega_k(n) \in \{0, 1\}$$

Spike pattern

$$\omega(n) = (\omega_k(n))_{k=1}^N$$

Spike block

$$\omega_m^n = \{\omega(m)\omega(m+1)\dots\omega(n)\}$$
$$\begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

Spike events

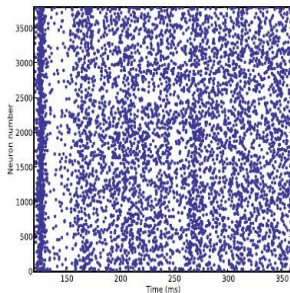


Figure: Raster plot/Spike train.

Spike state

$$\omega_k(n) \in \{0, 1\}$$

Spike pattern

$$\omega(n) = (\omega_k(n))_{k=1}^N$$

Spike block

$$\omega_m^n = \{\omega(m)\omega(m+1)\dots\omega(n)\}$$

Raster plot

$$\omega \stackrel{\text{def}}{=} \omega_0^T$$

Constructing a statistical model handling measured correlations

- Let $\pi_\omega^{(T)}$ be the **empirical measure**:

$$\pi_\omega^{(T)} [f] = \frac{1}{T} \sum_{t=1}^T f \circ \sigma^t(\omega)$$

e.g. $\pi_\omega^{(T)} [\omega_i] = \frac{1}{T} \sum_{t=1}^T \omega_i(t)$: firing rate;

$$\pi_\omega^{(T)} [\omega_i \omega_j] = \frac{1}{T} \sum_{t=1}^T \omega_i(t) \omega_j(t).$$

- Find **the** (stationary) probability distribution μ that maximizes **statistical entropy** under the constraints:

$$\pi_\omega^{(T)} [\omega_i] = \mu(\omega_i); \pi_\omega^{(T)} [\omega_i \omega_j] = \mu(\omega_i \omega_j)$$

Constructing a statistical model handling measured correlations

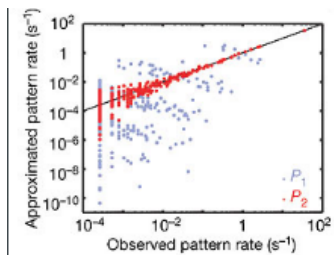
There is a **unique** probability distribution which satisfies these conditions.

This is the **Gibbs distribution** with potential:

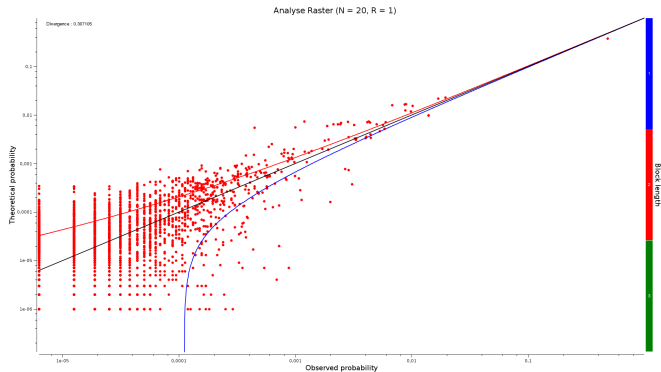
$$H(\omega(0)) = \sum_{i=1}^N h_i \omega_i(0) + \sum_{i,j=1}^N J_{ij} \omega_i(0) \omega_j(0)$$

Ising model

End of the story ?



End of the story ?



The Ising potential:

$$H(\omega(0)) = \sum_{i=1}^N h_i \omega_i(0) + \sum_{i,j=1}^N J_{ij} \omega_i(0) \omega_j(0)$$

does not consider **time correlations** between neurons.
It is therefore bad at predicting **spatio-temporal patterns** !

Which correlations ?

Spikes correlations seem to play a role in spike coding.

Which correlations ?

Spikes correlations seem to play a role in spike coding.

Although this statement depends on several assumption that could bias statistics

Stationarity;

Binning;

Stimulus dependence ?

Which correlations ?

Spikes correlations seem to play a role in spike coding.

Although this statement depends on several assumption that could bias statistics

Stationarity;

Binning;

Stimulus dependence ?

Modulo these remarks, Maximum entropy seems to be a relevant setting to study the role of spatio-temporal spike correlations in retina coding.

OK. So let us consider spatio-temporal constraints.

OK. So let us consider spatio-temporal constraints.

Easy !

$$H(\omega_0^D) = \sum_{i=1}^N h_i \omega_i(0) + \sum_{i,j=1}^N J_{ij}^{(0)} \omega_i(0) \omega_j(0)$$

OK. So let us consider spatio-temporal constraints.

Easy !

$$H(\omega_0^D) = \sum_{i=1}^N h_i \omega_i(0) + \sum_{i,j=1}^N J_{ij}^{(0)} \omega_i(0) \omega_j(0) \\ + \sum_{i,j=1}^N J_{ij}^{(1)} \omega_i(0) \omega_j(1)$$

OK. So let us consider spatio-temporal constraints.

Easy !

$$\begin{aligned} H(\omega_0^D) &= \sum_{i=1}^N h_i \omega_i(0) + \sum_{i,j=1}^N J_{ij}^{(0)} \omega_i(0) \omega_j(0) \\ &\quad + \sum_{i,j=1}^N J_{ij}^{(1)} \omega_i(0) \omega_j(1) \\ &\quad + \sum_{i,j,k=1}^N J_{ijk}^{(2)} \omega_i(0) \omega_j(1) \omega_k(2) \end{aligned}$$

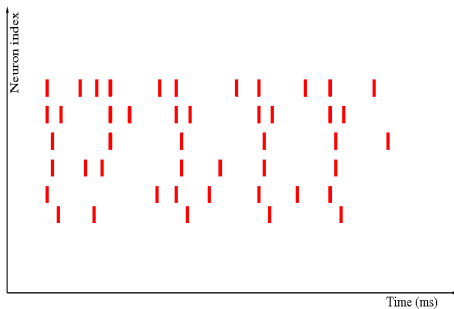
OK. So let us consider spatio-temporal constraints.

Easy ! **Euh... In fact not so easy.**

$$\begin{aligned} H(\omega_0^D) &= \sum_{i=1}^N h_i \omega_i(0) + \sum_{i,j=1}^N J_{ij}^{(0)} \omega_i(0) \omega_j(0) \\ &+ \sum_{i,j=1}^N J_{ij}^{(1)} \omega_i(0) \omega_j(1) \\ &+ \sum_{i,j,k=1}^N J_{ijk}^{(2)} \omega_i(0) \omega_j(1) \omega_k(2) \\ &+ \text{?????} \end{aligned}$$

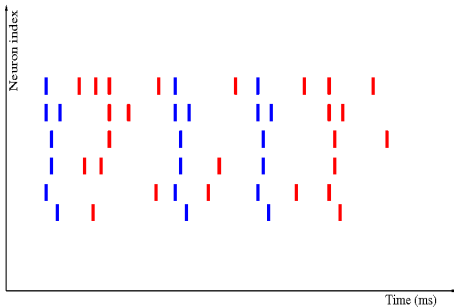
Two "small" problems.

Handling temporality and memory.



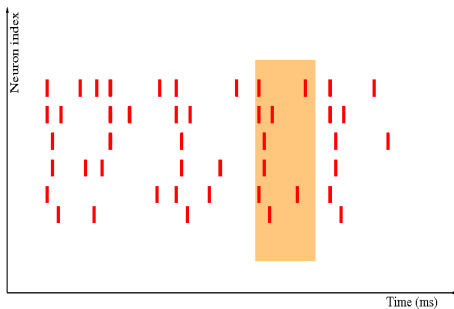
Two "small" problems.

Handling temporality and memory.



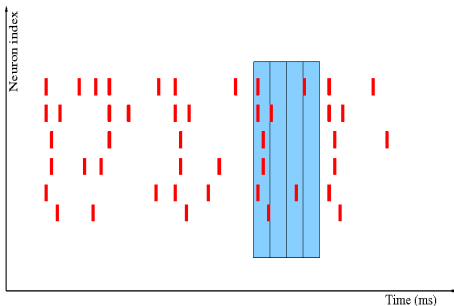
Two "small" problems.

Handling temporality and memory.



Two "small" problems.

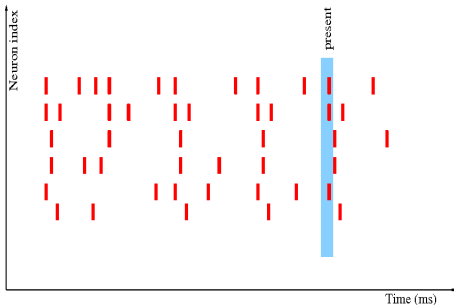
Handling temporality and memory.



Ising model considers successive times as **independent**

Two "small" problems.

Handling temporality and memory.

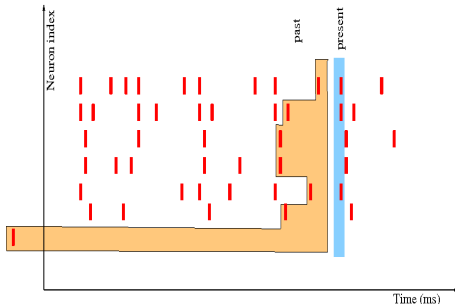


Probability of characteristic spatio-temporal patterns

The probability of a spike pattern
depends on the **network history** (transition probabilities).

Two "small" problems.

Handling temporality and memory.

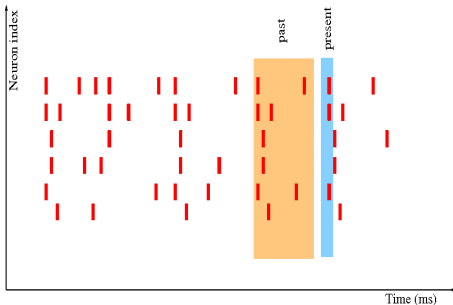


Probability of characteristic spatio-temporal patterns

The probability of a spike pattern
depends on the **network history** (transition probabilities).

Two "small" problems.

Handling temporality and memory.

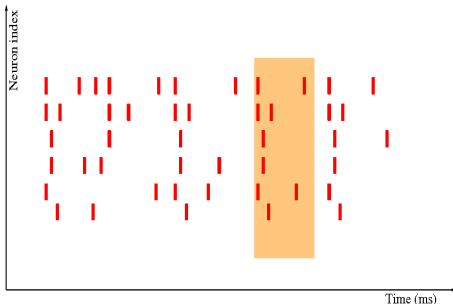


Probability of characteristic spatio-temporal patterns

The probability of a spike pattern
depends on the **network history** (transition probabilities).

Two "small" problems.

Handling temporality and memory.

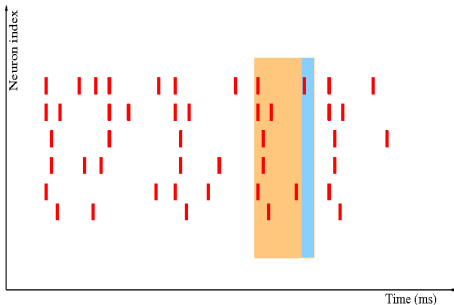


Probability of characteristic spatio-temporal patterns

The probability of a spike pattern
depends on the **network history** (transition probabilities).

Two "small" problems.

Handling temporality and memory.

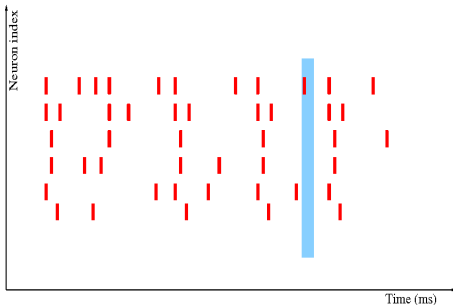


Probability of characteristic spatio-temporal patterns

The probability of a spike pattern
depends on the **network history** (transition probabilities).

Two "small" problems.

Handling temporality and memory.

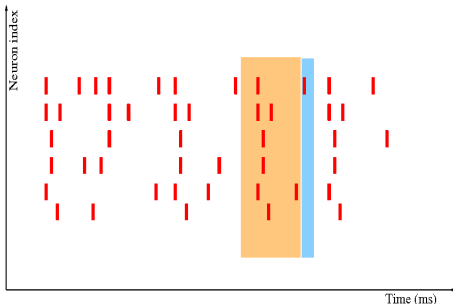


Probability of characteristic spatio-temporal patterns

The probability of a spike pattern
depends on the **network history** (transition probabilities).

Two "small" problems.

Handling temporality and memory.

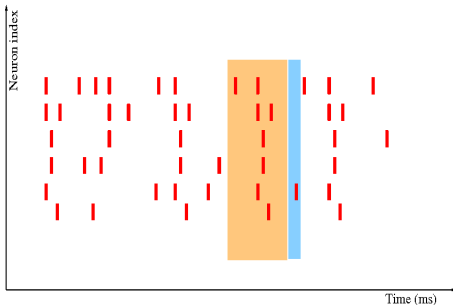


Probability of characteristic spatio-temporal patterns

The probability of a spike pattern
depends on the **network history** (transition probabilities).

Two "small" problems.

Handling temporality and memory.

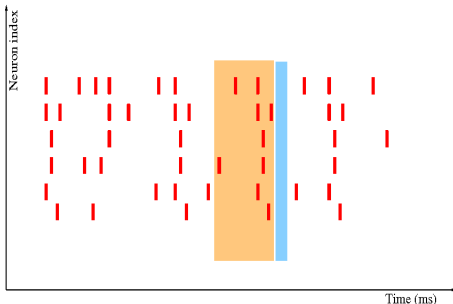


Probability of characteristic spatio-temporal patterns

The probability of a spike pattern
depends on the **network history** (transition probabilities).

Two "small" problems.

Handling temporality and memory.

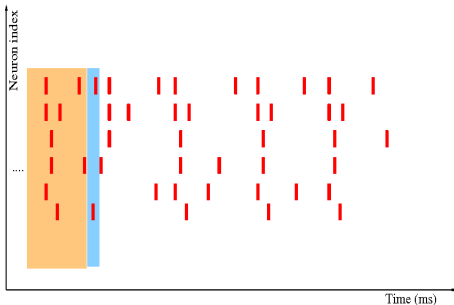


Probability of characteristic spatio-temporal patterns

The probability of a spike pattern
depends on the **network history** (transition probabilities).

Two "small" problems.

Handling temporality and memory.

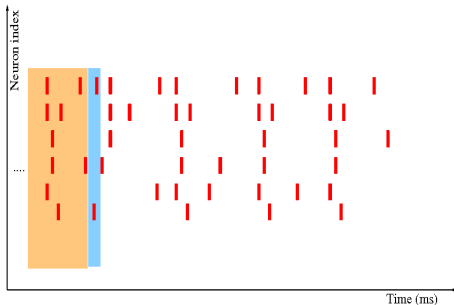


Probability of characteristic spatio-temporal patterns

The probability of a spike pattern
depends on the **network history** (transition probabilities).

Two "small" problems.

Handling temporality and memory.



Probability of characteristic spatio-temporal patterns

Given a set of hypotheses on transition probabilities there exists a mathematical framework to solve the problem.

Handling memory.

- Markov chains
- Variable length Markov chains
- Chains with complete connections

...

- **Gibbs distributions.**

- Probability distribution on (bi-infinite) rasters:
 $\mu[\omega_m^n], \forall m < n \in \mathbb{Z}$

- Probability distribution on (bi-infinite) rasters:
 $\mu[\omega_m^n], \forall m < n \in \mathbb{Z}$
- Conditional probabilities with memory depth D :
 $P_n[\omega(n) \mid \omega_{n-D}^{n-1}]$.

- Probability distribution on (bi-infinite) rasters:
 $\mu[\omega_m^n], \forall m < n \in \mathbb{Z}$
- Conditional probabilities with memory depth D :
 $P_n[\omega(n) \mid \omega_{n-D}^{n-1}]$.

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{array} \right)$$

- Probability distribution on (bi-infinite) rasters:
 $\mu[\omega_m^n], \forall m < n \in \mathbb{Z}$
- Conditional probabilities with memory depth D :
 $P_n[\omega(n) \mid \omega_{n-D}^{n-1}]$.
- Generating arbitrary depth D blocks probabilities:

- Probability distribution on (bi-infinite) rasters:

$$\mu [\omega_m^n], \forall m < n \in \mathbb{Z}$$

- Conditional probabilities with memory depth D :

$$P_n [\omega(n) \mid \omega_{n-D}^{n-1}].$$

- Generating arbitrary depth D blocks probabilities:

$$\mu [\omega_m^{m+D}] =$$

- Probability distribution on (bi-infinite) rasters:
 $\mu [\omega_m^n], \forall m < n \in \mathbb{Z}$
- Conditional probabilities with memory depth D :
 $P_n [\omega(n) \mid \omega_{n-D}^{n-1}]$.

- Generating arbitrary depth D blocks probabilities:

$$\mu [\omega_m^{m+D}] = P_{m+D} [\omega(m+D) \mid \omega_m^{m+D-1}] \mu [\omega_m^{m+D-1}]$$

- Probability distribution on (bi-infinite) rasters:
 $\mu [\omega_m^n], \forall m < n \in \mathbb{Z}$
- Conditional probabilities with memory depth D :
 $P_n [\omega(n) \mid \omega_{n-D}^{n-1}]$.

- Generating arbitrary depth D blocks probabilities:

$$\mu [\omega_m^{m+D}] = P_{m+D} [\omega(m+D) \mid \omega_m^{m+D-1}] \mu [\omega_m^{m+D-1}]$$

$$\mu [\omega_m^n] = \prod_{l=m+D}^n P_l [\omega(l) \mid \omega_{l-D}^{l-1}] \mu [\omega_m^{m+D-1}],$$
$$\forall m < n \in \mathbb{Z}$$

Chapman-Kolmogorov relation

$$\mu[\omega_m^n] = \prod_{l=m+D}^n P_l \left[\omega(l) \mid \omega_{l-D}^{l-1} \right] \mu \left[\omega_m^{m+D-1} \right], \quad \forall m < n \in \mathbb{Z}$$

$$\mu[\omega_m^n] = \prod_{l=m+D}^n P_l \left[\omega(l) \mid \omega_{l-D}^{l-1} \right] \mu \left[\omega_m^{m+D-1} \right], \quad \forall m < n \in \mathbb{Z}$$

$$\phi_l \left(\omega_{l-D}^l \right) = \log P_l \left[\omega(l) \mid \omega_{l-D}^{l-1} \right]$$

$$\mu[\omega_m^n] = \prod_{l=m+D}^n P_l \left[\omega(l) \mid \omega_{l-D}^{l-1} \right] \mu \left[\omega_m^{m+D-1} \right], \quad \forall m < n \in \mathbb{Z}$$

$$\phi_l \left(\omega_{l-D}^l \right) = \log P_l \left[\omega(l) \mid \omega_{l-D}^{l-1} \right]$$

$$\mu[\omega_m^n] = \exp \sum_{l=m+D}^n \phi_l \left(\omega_{l-D}^l \right) \mu \left[\omega_m^{m+D-1} \right]$$

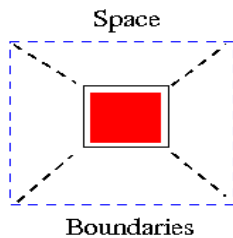
$$\mu[\omega_m^n] = \prod_{l=m+D}^n P_l \left[\omega(l) \mid \omega_{l-D}^{l-1} \right] \mu \left[\omega_m^{m+D-1} \right], \quad \forall m < n \in \mathbb{Z}$$

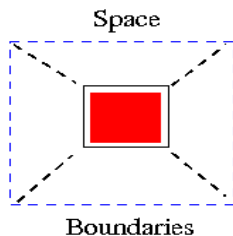
$$\phi_l \left(\omega_{l-D}^l \right) = \log P_l \left[\omega(l) \mid \omega_{l-D}^{l-1} \right]$$

$$\mu[\omega_m^n] = \exp \sum_{l=m+D}^n \phi_l \left(\omega_{l-D}^l \right) \mu \left[\omega_m^{m+D-1} \right]$$

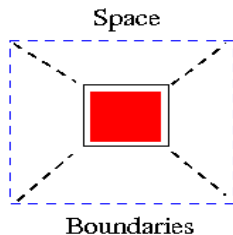
$$\mu \left[\omega_m^n \mid \omega_m^{m+D-1} \right] = \exp \sum_{l=m+D}^n \phi_l \left(\omega_{l-D}^l \right)$$

Gibbs distribution





$$\forall \Lambda \subset \mathbb{Z}^d, \mu(\{S\} \mid \partial\Lambda) = \frac{1}{Z_{\Lambda, \partial\Lambda}} e^{-\beta H_{\Lambda, \partial\Lambda}(\{S\})}$$



$$\forall \Lambda \subset \mathbb{Z}^d, \mu(\{S\} \mid \partial\Lambda) = \frac{1}{Z_{\Lambda, \partial\Lambda}} e^{-\beta H_{\Lambda, \partial\Lambda}(\{S\})}$$

$$f(\beta) = -\frac{1}{\beta} \lim_{\Lambda \uparrow \infty} \frac{1}{|\Lambda|} \log Z_{\Lambda, \partial\Lambda}$$

(*free energy density*)

Gibbs distribution





$$\forall m, n, \quad \mu \left[\omega_m^n \mid \omega_m^{m+D-1} \right] = \exp \sum_{l=m+D}^n \phi_l \left(\omega_{l-D}^l \right)$$

(*normalized potential*)

Gibbs distribution



$$\forall m < n, \quad A < \frac{\mu[\omega_m^n]}{\exp \sum_{l=m+D}^n H(\omega_{l-D}^l) \exp -(n-m)P(H)} < B$$

(non normalized potential)

- $P(H)$ is called "**topological pressure**" and is formally equivalent to free energy density.
- *Does not require time-translation invariance (stationarity).*
- In the stationary case (+ assumptions) a **Gibbs state** is also an **equilibrium state**.

$$\sup_{\nu \in M_{inv}} h(\nu) + \nu(H) = h(\mu) + \mu(H) = P(H)$$

.

This formalism allows to handle the spatio-temporal case

$$\begin{aligned} H(\omega_0^D) &= \sum_{i=1}^N h_i \omega_i(0) + \sum_{i,j=1}^N J_{ij}^{(0)} \omega_i(0) \omega_j(0) \\ &\quad + \sum_{i,j=1}^N J_{ij}^{(1)} \omega_i(0) \omega_j(1) \\ &\quad + \sum_{i,j,k=1}^N J_{ijk}^{(2)} \omega_i(0) \omega_j(1) \omega_k(2) + \dots \end{aligned}$$

even numerically.

J.C. Vasquez, A. Palacios, O. Marre, M.J. Berry II, B. Cessac, J. Physiol. Paris, , Vol 106, Issues 3–4, (2012).

H. Nasser, O. Marre, and B. Cessac, J. Stat. Mech. (2013) P03006.

H. Nasser, B. Cessac, Entropy (2014), 16(4), 2244-2277.

This formalism allows to handle the spatio-temporal case

$$\begin{aligned} H(\omega_0^D) &= \sum_{i=1}^N h_i \omega_i(0) + \sum_{i,j=1}^N J_{ij}^{(0)} \omega_i(0) \omega_j(0) \\ &+ \sum_{i,j=1}^N J_{ij}^{(1)} \omega_i(0) \omega_j(1) \\ &+ \sum_{i,j,k=1}^N J_{ijk}^{(2)} \omega_i(0) \omega_j(1) \omega_k(2) + \text{?????} \end{aligned}$$

even numerically.

J.C. Vasquez, A. Palacios, O. Marre, M.J. Berry II, B. Cessac, J. Physiol. Paris, , Vol 106, Issues 3–4, (2012).

H. Nasser, O. Marre, and B. Cessac, J. Stat. Mech. (2013) P03006.

H. Nasser, B. Cessac, Entropy (2014), 16(4), 2244-2277.

Two small problems.

Exponential number of possible terms.

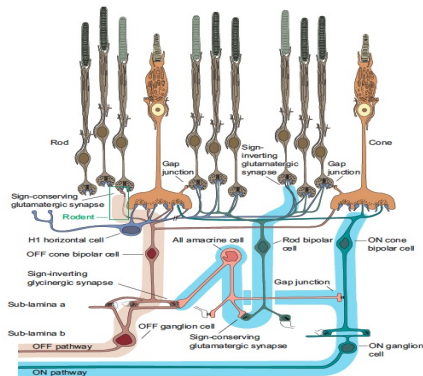
Two small problems.

Exponential number of possible terms.

Contrarily to what happens usually in physics, we do not know
what should be
the **right potential**.

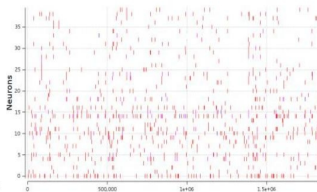
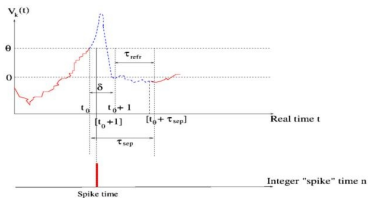
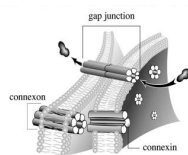
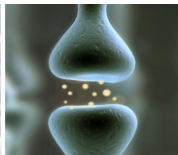
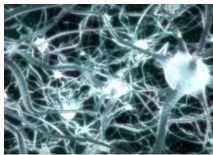
Can we have a *reasonable idea* of what could be the spike statistics by studying a *neural network model* ?

An Integrate and Fire neural network model with chemical and electric synapses



An Integrate and Fire neural network model with chemical and electric synapses

R.Cofré, B. Cessac: "Dynamics and spike trains statistics in conductance-based Integrate-and-Fire neural networks with chemical and electric synapses", *Chaos, Solitons and Fractals*, 2013.



An Integrate and Fire neural network model with chemical and electric synapses

Sub-threshold dynamics:

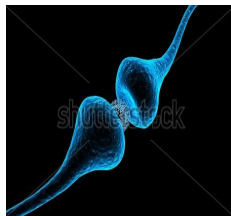
$$C_k \frac{dV_k}{dt} = -g_{L,k}(V_k - E_L)$$

An Integrate and Fire neural network model with chemical and electric synapses

Sub-threshold dynamics:

$$C_k \frac{dV_k}{dt} = -g_{L,k}(V_k - E_L)$$

$$- \sum_j g_{kj}(t, \omega)(V_k - E_j)$$

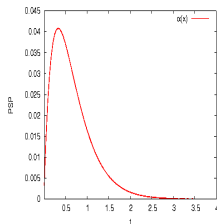


An Integrate and Fire neural network model with chemical and electric synapses

Sub-threshold dynamics:

$$C_k \frac{dV_k}{dt} = -g_{L,k}(V_k - E_L)$$

$$- \sum_j g_{kj}(t, \omega)(V_k - E_j)$$

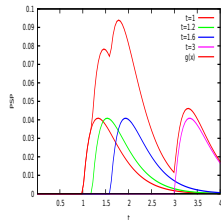


An Integrate and Fire neural network model with chemical and electric synapses

Sub-threshold dynamics:

$$C_k \frac{dV_k}{dt} = -g_{L,k}(V_k - E_L)$$

$$- \sum_j g_{kj}(t, \omega)(V_k - E_j)$$

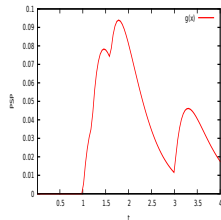


An Integrate and Fire neural network model with chemical and electric synapses

Sub-threshold dynamics:

$$C_k \frac{dV_k}{dt} = -g_{L,k}(V_k - E_L)$$

$$- \sum_j g_{kj}(t, \omega)(V_k - E_j)$$



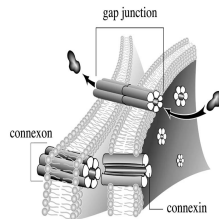
An Integrate and Fire neural network model with chemical and electric synapses

Sub-threshold dynamics:

$$C_k \frac{dV_k}{dt} = -g_{L,k}(V_k - E_L)$$

$$- \sum_j g_{kj}(t, \omega)(V_k - E_j)$$

$$- \sum_j \bar{g}_{kj}(V_k - V_j)$$



An Integrate and Fire neural network model with chemical and electric synapses

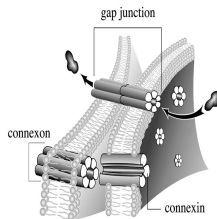
Sub-threshold dynamics:

$$C_k \frac{dV_k}{dt} = -g_{L,k}(V_k - E_L)$$

$$- \sum_j g_{kj}(t, \omega)(V_k - E_j)$$

$$- \sum_j \bar{g}_{kj}(V_k - V_j)$$

$$+ i_k^{(ext)}(t) + \sigma_B \xi_k(t)$$



$$C \frac{dV}{dt} + [G(t, \omega) - \bar{G}] V = I(t, \omega),$$

$$G_{kl}(t, \omega) = \left[g_{L,k} + \sum_{j=1}^N g_{kj}(t, \omega) \right] \delta_{kl} \stackrel{\text{def}}{=} g_k(t, \omega) \delta_{kl}.$$

$$I(t, \omega) = I^{(cs)}(t, \omega) + I^{(\text{ext})}(t) + I^{(B)}(t)$$

$$I_k^{(cs)}(t, \omega) = \sum_j W_{kj} \alpha_{kj}(t, \omega), \quad W_{kj} \stackrel{\text{def}}{=} G_{kj} E_j.$$

$$\begin{cases} dV &= (\Phi(t, \omega)V + f(t, \omega))dt + \frac{\sigma_B}{c} \mathcal{I}_N dW(t), \\ V(t_0) &= v, \end{cases}$$

$$\Phi(t, \omega) = C^{-1} (\bar{G} - G(t, \omega))$$

$$f(t, \omega) = C^{-1} I^{(cs)}(t, \omega) + C^{-1} I^{(ext)}(t)$$

Homogeneous Cauchy problem

$$\begin{cases} \frac{dV(t,\omega)}{dt} = \Phi(t,\omega)V(t,\omega), \\ V(t_0) = v, \end{cases}$$

Homogeneous Cauchy problem

$$\begin{cases} \frac{dV(t,\omega)}{dt} = \Phi(t,\omega)V(t,\omega), \\ V(t_0) = v, \end{cases}$$

Theorem

$\Phi(t, \omega)$ square matrix with bounded elements.

$$M_0(t_0, t, \omega) = \mathcal{I}_N$$

$$M_k(t_0, t, \omega) = \mathcal{I}_N + \int_{t_0}^t \Phi(s, \omega) M_{k-1}(s, t) ds, \quad t \leq t_1,$$

converges uniformly in $[t_0, t_1]$.

Brockett, R. W., "Finite Dimensional Linear Systems", John Wiley and Sons, 1970.

Homogeneous Cauchy problem

$$\begin{cases} \frac{dV(t,\omega)}{dt} = \Phi(t,\omega)V(t,\omega), \\ V(t_0) = v, \end{cases}$$

Theorem

$\Phi(t,\omega)$ square matrix with bounded elements.

$$M_0(t_0, t, \omega) = \mathcal{I}_N$$

$$M_k(t_0, t, \omega) = \mathcal{I}_N + \int_{t_0}^t \Phi(s, \omega) M_{k-1}(s, t) ds, \quad t \leq t_1,$$

converges uniformly in $[t_0, t_1]$.

Brockett, R. W., "Finite Dimensional Linear Systems", John Wiley and Sons, 1970.

Flow

$$\Gamma(t_0, t, \omega) \stackrel{\text{def}}{=} \lim_{k \rightarrow \infty} M_k(t_0, t, \omega)$$

Homogeneous Cauchy problem

If $\Phi(t, \omega)$ and $\Phi(s, \omega)$ commute

$$\Gamma(t_0, t, \omega) = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\int_{t_0}^t \Phi(s, \omega) ds \right)^k = e^{\int_{t_0}^t \Phi(s, \omega) ds}$$

Homogeneous Cauchy problem

If $\Phi(t, \omega)$ and $\Phi(s, \omega)$ commute

$$\Gamma(t_0, t, \omega) = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\int_{t_0}^t \Phi(s, \omega) ds \right)^k = e^{\int_{t_0}^t \Phi(s, \omega) ds}$$

This holds only in two cases :

- $\bar{G} = 0$;

$$\Gamma(t_0, t, \omega) = e^{-\frac{1}{c} \int_{t_0}^t G(s, \omega) ds}$$

B. Cessac, J. Math. Neuroscience, 2011.

Homogeneous Cauchy problem

If $\Phi(t, \omega)$ and $\Phi(s, \omega)$ commute

$$\Gamma(t_0, t, \omega) = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\int_{t_0}^t \Phi(s, \omega) ds \right)^k = e^{\int_{t_0}^t \Phi(s, \omega) ds}$$

This holds only in two cases :

- $\bar{G} = 0$;

$$\Gamma(t_0, t, \omega) = e^{-\frac{1}{c} \int_{t_0}^t G(s, \omega) ds}$$

B. Cessac, J. Math. Neuroscience, 2011.

- $G(t, \omega) = \kappa(t, \omega) \mathcal{I}_N$

Homogeneous Cauchy problem

In general:

$$\Gamma(t_0, t, \omega) = \mathcal{I}_N + \sum_{n=1}^{+\infty} \sum_{\substack{X_1 = (B, A(s_1, \omega)) \\ X_2 = (B, A(s_2, \omega)) \\ \vdots \\ X_n = (B, A(s_n, \omega))}} \int_{t_0}^t \cdots \int_{t_0}^{s_{n-1}} \prod_{k=1}^n X_k d_{s_1} \cdots d_{s_n}.$$

$$B = C^{-1}\bar{G}; \quad A(t, \omega) = -C^{-1}G(t, \omega)$$

Definition: An **exponentially bounded flow** is a two parameter (t_0, t) family $\{\Gamma(t_0, t, \omega)\}_{t \leq t_0}$ of flows such that, $\forall \omega \in \Omega$:

- 1 $\Gamma(t_0, t_0, \omega) = \mathcal{I}_N$ and $\Gamma(t_0, t, \omega)\Gamma(t, s, \omega) = \Gamma(t_0, s, \omega)$ whenever $t_0 \leq t \leq s$;
- 2 For each $v \in \mathbb{R}^N$ and $\omega \in \Omega$, $(t_0, t) \rightarrow \Gamma(t_0, t, \omega)v$ is continuous for $t_0 \leq t$;
- 3 There is $M > 0$ and $m > 0$ such that :

$$\|\Gamma(s, t, \omega)\| \leq Me^{-m(t-s)}, s \leq t. \quad (1)$$

Proposition

Let σ_1 be the largest eigenvalue of \bar{G} . If:

$$\sigma_1 < g_L,$$

then the flow Γ in our model has the exponentially bounded flow property.

Proposition

Let σ_1 be the largest eigenvalue of \bar{G} . If:

$$\sigma_1 < g_L,$$

then the flow Γ in our model has the exponentially bounded flow property.

Remark The typical electrical conductance values are of order 1 nano-Siemens, while the leak conductance of retinal ganglion cells is of order 50 micro-Siemens. *Therefore, this condition is compatible with the biophysical values of conductances in the retina.*

Theorem

If $\Gamma(t_0, t, \omega)$ is an exponentially bounded flow, there is a unique strong solution for $t \geq t_0$ given by:

$$V(t_0, t, \omega) = \Gamma(t_0, t, \omega)v + \int_{t_0}^t \Gamma(s, t, \omega)f(s, \omega)ds + \frac{\sigma_B}{c} \int_{t_0}^t \Gamma(s, t, \omega)dW(s)$$

R. Wooster, "Evolution systems of measures for non-autonomous stochastic differential equations with Levy noise", Communications on Stochastic Analysis, vol 5, 353-370, 2011

Membrane potential decomposition

$$V(t, \omega) = V^{(d)}(t, \omega) + V^{(noise)}(t, \omega),$$

Membrane potential decomposition

$$V(t, \omega) = V^{(d)}(t, \omega) + V^{(noise)}(t, \omega),$$

$$V^{(d)}(t, \omega) = V^{(cs)}(t, \omega) + V^{(ext)}(t, \omega),$$

Membrane potential decomposition

$$V(t, \omega) = V^{(d)}(t, \omega) + V^{(noise)}(t, \omega),$$

$$V^{(d)}(t, \omega) = V^{(cs)}(t, \omega) + V^{(ext)}(t, \omega),$$

$$V^{(cs)}(t, \omega) = \frac{1}{c} \int_{-\infty}^t \Gamma(s, t, \omega) \chi(s, \omega) I^{(cs)}(s, \omega) ds,$$

Membrane potential decomposition

$$V(t, \omega) = V^{(d)}(t, \omega) + V^{(noise)}(t, \omega),$$

$$V^{(d)}(t, \omega) = V^{(cs)}(t, \omega) + V^{(ext)}(t, \omega),$$

$$V^{(cs)}(t, \omega) = \frac{1}{c} \int_{-\infty}^t \Gamma(s, t, \omega) \chi(s, \omega) I^{(cs)}(s, \omega) ds,$$

$$V^{(ext)}(t, \omega) = \frac{1}{c} \int_{-\infty}^t \Gamma(s, t, \omega) \chi(s, \omega) I^{(ext)}(s, \omega) ds,$$

Membrane potential decomposition

$$V(t, \omega) = V^{(d)}(t, \omega) + V^{(noise)}(t, \omega),$$

$$V^{(d)}(t, \omega) = V^{(cs)}(t, \omega) + V^{(ext)}(t, \omega),$$

$$V^{(cs)}(t, \omega) = \frac{1}{c} \int_{-\infty}^t \Gamma(s, t, \omega) \chi(s, \omega) I^{(cs)}(s, \omega) ds,$$

$$V^{(ext)}(t, \omega) = \frac{1}{c} \int_{-\infty}^t \Gamma(s, t, \omega) \chi(s, \omega) I^{(ext)}(s, \omega) ds,$$

$$V^{(noise)}(t, \omega) = \frac{\sigma_B}{c} \int_{\tau_k(t, \omega)}^t \Gamma(s, t, \omega) dW(s).$$

Transition probabilities

Pb: to determine $P [\omega(n) \mid \omega_{-\infty}^{n-1}]$

Transition probabilities

Pb: to determine $P [\omega(n) \mid \omega_{-\infty}^{n-1}]$

Fix ω , n and $t < n$. Set:

$$\hat{\theta}_k(t, \omega) = \theta - V_k^{(d)}(t, \omega), \quad (1)$$

Pb: to determine $P [\omega(n) \mid \omega_{-\infty}^{n-1}]$

Fix ω , n and $t < n$. Set:

$$\hat{\theta}_k(t, \omega) = \theta - V_k^{(d)}(t, \omega), \quad (1)$$

Neuron k emits a spike at integer time n ($\omega_k(n) = 1$) if:

$$\exists t \in [n - 1, n], V_k^{(noise)}(t, \omega) = \hat{\theta}_k(t, \omega).$$

Pb: to determine $P [\omega(n) \mid \omega_{-\infty}^{n-1}]$

Fix ω , n and $t < n$. Set:

$$\hat{\theta}_k(t, \omega) = \theta - V_k^{(d)}(t, \omega), \quad (1)$$

Neuron k emits a spike at integer time n ($\omega_k(n) = 1$) if:

$$\exists t \in [n - 1, n], V_k^{(\text{noise})}(t, \omega) = \hat{\theta}_k(t, \omega).$$

"First passage" problem, in N dimension, with a time dependent boundary $\hat{\theta}_k(t, \omega)$. (general form unknown).

Conditional probability

Without electric synapses the probability of $\omega(n)$ conditionally to $\omega_{-\infty}^{n-1}$ can be approximated by:

$$P[\omega(n) | \omega_{-\infty}^{n-1}] = \prod_{k=1}^N P[\omega_k(n) | \omega_{-\infty}^{n-1}],$$

with $P[\omega_k(n) | \omega_{-\infty}^{n-1}] =$

$$\omega_k(n) \pi(X_k(n-1, \omega)) + (1 - \omega_k(n)) (1 - \pi(X_k(n-1, \omega))),$$

where

$$X_k(n-1, \omega) = \frac{\theta - V_k^{(det)}(n-1, \omega)}{\sigma_k(n-1, \omega)},$$

and

$$\pi(x) = \frac{1}{\sqrt{2\pi}} \int_x^{+\infty} e^{-\frac{u^2}{2}} du.$$

- $\phi(\omega) = \log P [\omega(n) \mid \omega_{-\infty}^{n-1}]$ defines a (infinite range) normalized potential defining a *unique Gibbs distribution*.

Conditional probability

- $\phi(\omega) = \log P [\omega(n) \mid \omega_{-\infty}^{n-1}]$ defines a (infinite range) normalized potential defining a *unique Gibbs distribution*.
- It depends explicitly on *networks parameters and external stimulus*.

Conditional probability

- $\phi(\omega) = \log P [\omega(n) \mid \omega_{-\infty}^{n-1}]$ defines a (infinite range) normalized potential defining a *unique Gibbs distribution*.
- It depends explicitly on *networks parameters and external stimulus*.
- Its definition holds for a time-dependent stimulus (*non stationary*).

Conditional probability

- $\phi(\omega) = \log P [\omega(n) | \omega_{-\infty}^{n-1}]$ defines a (infinite range) normalized potential defining a *unique Gibbs distribution*.
- It depends explicitly on *networks parameters and external stimulus*.
- Its definition holds for a time-dependent stimulus (*non stationary*).
- It is similar to the so-called *Generalized Linear Model* used for retina analysis, although with a more complex structure.

Conditional probability

- $\phi(\omega) = \log P [\omega(n) \mid \omega_{-\infty}^{n-1}]$ defines a (infinite range) normalized potential defining a *unique Gibbs distribution*.
- It depends explicitly on *networks parameters and external stimulus*.
- Its definition holds for a time-dependent stimulus (*non stationary*).
- It is similar to the so-called *Generalized Linear Model* used for retina analysis, although with a more complex structure.
- The general form (with electric synapses) is *yet unknown*.

Back to our second "small" problem

Back to our second "small" problem

Is there a **Maximum Entropy potential** corresponding to ϕ (in the stationary case) ?

Back to our second "small" problem

One can make a Taylor expansion of $\phi(\omega)$.

Back to our second "small" problem

Using $\omega_i(n)^k = \omega_i(n)$, $k \geq 1$ one ends up with a potential of the form:

$$\phi(\omega) = \sum_{i=1}^N h_i \omega_i(0) + \sum_{i,j=1}^N J_{ij}^{(0)} \omega_i(0) \omega_j(0) + \dots$$

Back to our second "small" problem

The expansion is *infinite* although one can approximate the *infinite range potential* ϕ by a finite range approximation (*finite memory*), giving rise to a finite expansion.

Back to our second "small" problem

The coefficients of the expansion are *non linear functions* of the network parameters and stimulus.

They are therefore somewhat *redundant*.

Back to our second "small" problem

Rodrigo Cofré, Bruno Cessac, "Exact computation of the maximum-entropy potential of spiking neural-network models", *Phys. Rev. E* 89, 052117.

Given a set of *stationary* transition probabilities $P[\omega(D) | \omega_0^{D-1}] > 0$ there is a unique (up to a constant) *Maximum Entropy* potential, written as a linear combination of spike interactions terms with a minimal number of terms (*normal form*). This potential can be explicitly (and algorithmically) computed.

Hints: Using variable change one can eliminate terms in the potential ("normal" form).

The construction is based on equivalence between Gibbs potentials (*cohomology*) and *periodic orbits expansion*.

Back to our second "small" problem

However, there is still a number of terms **growing exponentially with the number of neurons and the memory depth.**

These terms are **generically** non zero.

Back to the retina

- Neuromimetic models have typically $O(N^2)$ parameters where N is the number of neurons.

Back to the retina

- Neuromimetic models have typically $O(N^2)$ parameters where N is the number of neurons.
- The equivalent MaxEnt potential has *generically* a number of parameters growing exponentially with N , non linear and redundant functions of the network parameters (synaptic weights, stimulus).

- Neuromimetic models have typically $O(N^2)$ parameters where N is the number of neurons.
- The equivalent MaxEnt potential has *generically* a number of parameters growing exponentially with N , non linear and redundant functions of the network parameters (synaptic weights, stimulus).

⇒

- Intractable determination of parameters;
- Stimulus dependent parameters;
- Overfitting.

BUT

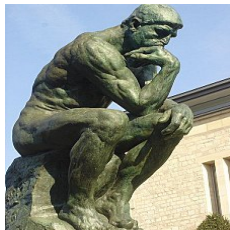
- Neuromimetic models have typically $O(N^2)$ parameters where N is the number of neurons.
- The equivalent MaxEnt potential has *generically* a number of parameters growing exponentially with N , non linear and redundant functions of the network parameters (synaptic weights, stimulus).

⇒

- Intractable determination of parameters;
- Stimulus dependent parameters;
- Overfitting.

BUT *Real neural networks are not generic*

MaxEnt approach might be useful if there is some hidden law of nature/ symmetry which cancels most terms in the expansion.



Acknowledgment

Neuromathcomp team

- Rodrigo Cofré (pHd, September 2014)
- Dora Karvouniari (M2)
- Pierre Kornprobst (CR1 INRIA)
- Sélim Kraria (IR)
- Gaia Lombardi (M2 → Paris)
- Hassan Nasser (pHd → Startup)
- Daniela Pamplona (PostDoc)
- Geoffrey Portelli (Post Doc)
- Vivien Robinet (Post Doc → MCF Kourou)
- Horacio Rostro (pHd → Docent Mexico)
- Wahiba Taouali (Post Doc → Post Doc INT Marseille)
- Juan-Carlos Vasquez (pHd → Post Doc Bogota)

Princeton University

- Michael J. Berry II

ANR KEOPS

- Maria-José Escobar (CN Valparaiso)
- Adrian Palacios (CN Valparaiso)
- Cesar Ravelo (CN Valparaiso)
- Thierry Viéville (INRIA Mnemosyne)

Revisión FP7 project

- Luca Bernondini (IIT Genova)
- Matthias Hennig (Edinburgh)
- Alessandro Maccionne (IIT Genova)
- Evelyne Sernagor (Newcastle)

Institut de la Vision

- Olivier Marre
- Serge Picaud

Can we hear the shape of a Maximum entropy potential

Two distinct potentials $\mathcal{H}^{(1)}, \mathcal{H}^{(2)}$ of range $R = D + 1$ correspond to the same Gibbs distribution (are “equivalent”), if and only if there exists a range D function f such that (Chazottes-Keller (2009)):

$$\mathcal{H}^{(2)}\left(\omega_0^D\right) = \mathcal{H}^{(1)}\left(\omega_0^D\right) - f\left(\omega_0^{D-1}\right) + f\left(\omega_1^D\right) + \Delta, \quad (2)$$

where $\Delta = \mathcal{P}(H^{(2)}) - \mathcal{P}(H^{(1)})$.

Can we hear the shape of a Maximum entropy potential

Summing over periodic orbits we get rid of the function f

$$\sum_{n=1}^R \phi(\omega \sigma^n l_1) = \sum_{n=1}^R \mathcal{H}^*(\omega \sigma^n l_1) - R\mathcal{P}(\mathcal{H}^*), \quad (3)$$

We eliminate equivalent constraints.

Can we hear the shape of a Maximum entropy potential

Conclusion

Given a set of transition probabilities $P \left[\omega(D) \mid \omega_0^{D-1} \right] > 0$ there is a unique, up to a constant, MaxEnt potential, written as a linear combination of constraints (average of spike events) with a minimal number of terms. This potential can be explicitly (and algorithmically) computed.