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# Self-optimizing Strategies for Dynamic Vertical Sectorization in LTE Networks

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**Abstract**—Vertical Sectorization (VS) consists in creating vertically separated sectors in the original cell using an Active Antenna Systems (AAS) supporting two distinct beams with different downtilts. The total transmit power is split between the two sectors, while the frequency bandwidth can be reused by each sector, creating additional interference between the two sectors. For low traffic demand, VS may lead to performance degradation, while for high traffic demand in both sectors, VS is likely to bring about important capacity gains. Hence intelligent activation policy of VS is needed to fully benefit from this feature. In this paper, we propose an approach taking advantage of the more focused downtilted beam. A dynamic alpha fair bandwidth sharing is proposed for low and medium load. It is autonomously replaced by full bandwidth reuse for high load scenarios using a threshold-based controller. A flow-level dynamic simulator is used to numerically validate the proposed mechanisms.

**Keywords**—Vertical Sectorization, Active Antenna Systems, AAS, Self-Organizing Networks, SON, Interference Coordination

## I. INTRODUCTION

We consider a cell of a Long Term Evolution (LTE) network as depicted in Figure 1. A vertically sectorized cell is split into an inner sector the antenna of which is vertically downtilted by  $\theta_i^\circ$  and an outer sector with vertical tilt  $\theta_o^\circ$ , and  $\theta_i^\circ > \theta_o^\circ$ . The tilts are generally performed electrically, allowing to set them dynamically as we activate/deactivate VS. In order to preserve the energy consumption, the inner and outer cells share equally the total transmit power available for the sector [1],[2].

Previous work on the VS feature include its performance evaluation with static configurations for different scenarios [1]–[4]. Optimization algorithms for the antenna tilts used in the VS feature are proposed in [5] and [6]. An optimal activation of the VS feature according to the traffic distribution in the cell is given in [1].

The performance gains expected from VS are mainly brought by the downtilted inner antenna which gives the inner users a more focused coverage, but also by reusing the whole bandwidth twice in the same cell. The gains brought by the total bandwidth reuse come at the price of splitting the power between inner and outer cells. Also, since the two beams use the same bandwidth, the inner and outer cells interfere with each other. As a consequence, a Signal to Interference plus Noise Ratio (SINR) degradation is observed when activating

VS. Because of this SINR loss, activating VS can degrade performance when there is not enough traffic in the inner cell.

Indeed, the bandwidth reuse is only useful if there are enough users in both inner and outer cells to take advantage of it. So a controller must be designed to dynamically activate VS according to the traffic distribution in the cell. In [1], a threshold-based SON controller for the activation of VS according to an estimation of inner/outer cell loads is analytically derived and calibrated with realistic simulations. Their proposed controller is in the form of a decision boundary in the (inner, outer) load plane, which delimits the region where activating VS increases the Mean User Throughput (MUT) from the region where it actually degrades it. Such a controller is useful in order to avoid MUT degradation in low traffic demand scenarios. However, the more focused inner cell coverage is still beneficial for the inner users even in low traffic scenarios.

Instead of switching off the VS feature at low inner load, we propose in this paper to apply a bandwidth sharing between inner and outer cells. The inner and outer cells will share the total available bandwidth and the total transmit power with the same proportions so that the transmit power per Hertz remains the same as in the case where VS is OFF. The sharing proportions are optimized according to an alpha-fair utility of user throughputs. We also provide a simple load threshold-based Self-Organizing Network (SON) controller to automatically switch between full reuse and bandwidth sharing implementations of the VS feature.

The contributions in this paper can be summarized as follows:

- An interference-free implementation of VS which provides better performance than the traditional full reuse VS implementation in low to medium load scenarios.
- Alpha-fair bandwidth sharing self-optimizing algorithms for this implementation of VS based on convex optimization with closed-form expressions for  $\alpha \in \{0, 1\}$ . For other values of  $\alpha$ , Stochastic Approximation (SA) algorithms are given in order to ensure convergence despite the fluctuations in LTE wireless environment.
- A threshold-based SON controller which autonomously switches between the bandwidth sharing and the full reuse implementations of the VS feature according to the traffic demand in the cell.

The remainder of the paper is organized as follows. Section II presents the VS feature system model. Section III

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describes the bandwidth sharing implementation and the self-optimization mechanisms that achieve its alpha-fair optimality. Section IV introduces the SON controller for dynamic switching between bandwidth sharing and full reuse. Section V presents some numerical results validating the proposed approach using a flow level event-based simulator. Section VI concludes the paper and the proofs for the theoretical results are given in the Appendices.

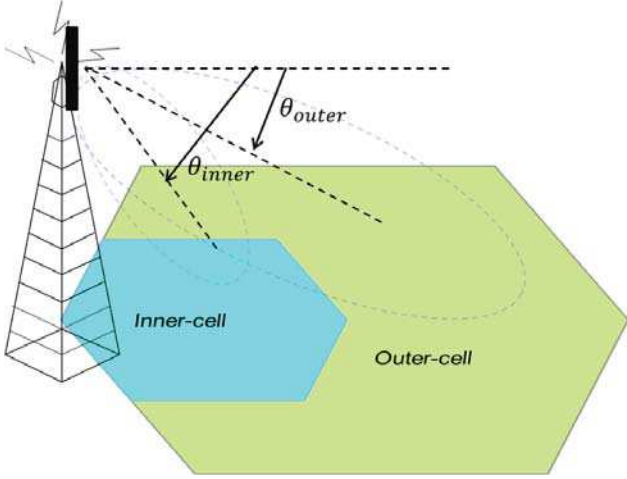


Fig. 1. Illustration of vertical sectorization

## II. SYSTEM MODEL

We consider the downlink of a LTE network serving elastic traffic. Users arrive in the network to download a file and leave the network as soon as their download is complete. The data rate of a user  $u$  is approached with a modified Shannon capacity formula [7]

$$R_u = \eta W \min(4.4, 0.6 \log_2(1 + S_u)) \quad (1)$$

where  $W$  is the total bandwidth available to user  $u$ ,  $\eta$  is the proportion of time that user  $u$  is scheduled, and  $S_u$  is the SINR of user  $u$ .  $\eta$  depends on the number of users that are served simultaneously with user  $u$ . The level of the SINR depends among others on whether VS is activated and which implementation of the feature is adopted.

By denoting  $P^s$  the transmit power of Base Station (BS)  $s$ ,  $h_u^s$  the signal attenuation from BS  $s$  to user  $u$  and  $N_0$  the thermal noise, the SINR of a user when the VS is not implemented can be written as

$$S_u = \frac{P^s h_u^s}{N_0 + \sum_{c \neq s} P^c h_u^c} \quad (2)$$

where  $s = \operatorname{argmax}_c P^c h_u^c$  is the best serving cell for user  $u$ . The sum over  $c \neq s$  represents the interfering signals from the neighboring BSs.

When VS is implemented, the classical way [1] is to fully reuse the bandwidth, so the inner and outer cells interfere with

each other. The SINR of a user  $u$  served by the inner cell is then

$$S_u = \frac{P^i h_u^i}{N_0 + P^o h_u^o + \sum_{c \neq s} P^c h_u^c} \quad (3)$$

where  $P^i, P^o$  are respectively the transmit powers of inner and outer cells, and  $h_u^i, h_u^o$  are the pathlosses from respectively the inner and outer cell antennas to user  $u$ . It is noted that a similar expression can be given for a user served by the outer cell. Since the bandwidth is fully reused, the total transmit power has to be shared between inner and outer cells, e.g. using equal split  $P^i = P^o = P^s/2$ .

Since the power for the inner and outer cells is reduced compared to the case where VS is not implemented, a SINR degradation may be observed due to reduced useful signal and increased interference for all the users. A better power split of the transmit power between the inner and outer cells is challenging because of the complex dependencies between the transmit powers and the average data rates of the users.

It has been shown in [1] that the full bandwidth reuse implementation of VS only improves performance over no VS when there is enough traffic in the inner cell area. So the authors in [1] proposed a SON controller which governs the activation of the VS feature only when needed. However, even when the traffic demand in the inner region is low, the users that are served in the inner region can still benefit from the stronger signal transmitted by the downtilted inner antenna.

The VS feature implementation considered here is to share the total available bandwidth between inner and outer cells (no reuse). In this case, not only the inner and outer cells do not interfere with each other, but also the transmit power per Hertz remains unchanged. Indeed, if we denote by  $\delta \in [0, 1]$  the fraction of bandwidth allocated to the inner cell, the same fraction of the transmit power is also allocated to the inner cell, so that  $P^i = \delta P^s$  and  $P^i/W_\delta = (\delta P^s)/W_\delta = P^s/W$  where  $W_\delta = \delta W$  is the bandwidth allocated to the inner cell.

The SINR of a user  $u$  served by the inner cell in the case of bandwidth sharing can be written as

$$S_u = \frac{P^s h_u^i}{N_0 + \sum_{c \neq s} P^c h_u^c} \quad (4)$$

It is noted that this SINR expression is given for the whole bandwidth  $W$  but it is also equivalent to the SINR per Hertz (both the numerator and the denominator divided by  $W$ ).

Equation (4) shows how the SINR is improved over the full bandwidth reuse case (3); the interference is reduced because the outer cell does not interfere any more and the useful signal is increased because the power per bandwidth is not halved. This SINR improvement comes at the loss of bandwidth which in turn impacts the data rate of user  $u$  as follows

$$R_u = \eta \delta W \min(4.4, 0.6 \log_2(1 + S_u)) \quad (5)$$

The choice of  $\delta$  drives the performance of this approach, so we formulate and solve the optimization problem for  $\delta$  in the next section. In the following we denote by  $\bar{R}_u$  the data rate of user  $u$  when his serving cell is allocated the whole bandwidth so that his actual data rate is  $R_u = \delta \bar{R}_u$ . If the

user  $u$  was served by the outer cell, his data rate would be  $R_u = (1 - \delta)\bar{R}_u$ .

### III. ALPHA-FAIR BANDWIDTH SHARING

In the bandwidth sharing implementation of VS, the sharing proportions have to match the actual proportions of traffic served by the inner and outer cells. The class of alpha-fair utilities of user throughputs [8],[9] offer a wide range of criteria for choosing these sharing proportions.

In the following, the bandwidth split factor  $\delta$  is dynamically optimized for any new user configuration, i.e. at every event (arrival or departure). The optimization problem is thus described for one instance of user configuration where the number and positions of users are fixed.

Let us denote by  $\mathcal{U}_i$  and  $\mathcal{U}_o$  the sets of users in inner and outer cells respectively. The alpha-fair utility of users' throughputs is given by [9]

$$U_\alpha(\delta) = \begin{cases} \sum_{u \in \mathcal{U}_i} \log(\delta \bar{R}_u) + \sum_{u \in \mathcal{U}_o} \log((1 - \delta)\bar{R}_u) & \alpha = 1 \\ \sum_{u \in \mathcal{U}_i} \frac{(\delta \bar{R}_u)^{1-\alpha}}{1-\alpha} + \sum_{u \in \mathcal{U}_o} \frac{((1-\delta)\bar{R}_u)^{1-\alpha}}{1-\alpha} & \alpha \neq 1 \end{cases} \quad (6)$$

When  $\alpha = 0$ , this utility reduces to the sum of users throughputs. This choice of  $\alpha = 0$  is not interesting as no fairness is enforced among the users. The case  $\alpha = 1$  corresponds to the well-known proportional fair utility. It can also be shown using queuing theory that the case  $\alpha = 2$  corresponds to the sum of the file transfer times in the network.

It can be shown that the utility functions in (6) are concave in  $\delta$  (See Appendix A). By using the Karush-Kuhn-Tucker (KKT) conditions on optimality for convex functions, simple gradient descent algorithms [10] can be used to find the optimal  $\delta$  efficiently.

For  $\alpha = 0$  which is unfair, the solution is rather simple and no gradient descent is needed. The utility function is linear in  $\delta$  so its maximum is attained at one of the ends  $\delta = 0$  or  $\delta = 1$ , so a simple evaluation of the utility at those ends gives the maximum.

For  $\alpha = 1$  (Proportional Fair), a closed form expression of the optimal  $\delta$  is also given below. Indeed, the KKT conditions are equivalent to finding  $\delta$  satisfying the following equation

$$\frac{\partial U_\alpha(\delta)}{\partial \delta} = \frac{N_i}{\delta} - \frac{N_o}{1 - \delta} = 0 \quad (7)$$

where  $N_i = |\mathcal{U}_i|$  and  $N_o = |\mathcal{U}_o|$  are the number of users in the inner and outer cells respectively. The solution to (7) can be easily derived and reads

$$\delta = \frac{N_i}{N_i + N_o} \quad (8)$$

when there is at least one user in the cell ( $N_i + N_o > 0$ ). This simple solution does not depend on the particular channel quality of the users present in the cell rendering it particularly effective.

For  $\alpha \notin \{0, 1\}$ , the gradient descent algorithm is needed. However, in a real network the values of  $\bar{R}_u$  which are used by the algorithms are generally fluctuating because of the random

nature of wireless channels. Since a good estimate (e.g. average over a long time interval) of these values cannot be waited before the optimization is performed, a SA algorithm can be applied in which a new estimate of  $\bar{R}_u$  is used at every step until convergence of the algorithm. The SA algorithm is of the form

$$\delta[k + 1] = \delta[k] + \epsilon \frac{\partial \hat{U}_\alpha(\delta[k])}{\partial \delta} \quad (9)$$

where  $k$  is the step index,  $\epsilon$  a small step size and  $\hat{U}_\alpha$  the estimate available at step  $k$ . If  $\epsilon$  is sufficiently small and the consecutive estimates of the gradient form a Martingale Difference sequence, this algorithm converges to a neighborhood of the optimal  $\delta$  (see [11] for more details on convergence proofs of SA).

The proportional fair utility ( $\alpha = 1$ ) brings both fairness and simple implementation. Indeed,  $\delta$  can be updated in one step at every event using (8). Because of these advantages, the Proportional Fair (PF) utility is adopted in the remainder of this paper, and in particular for the numerical results.

The bandwidth sharing approach allows to significantly improve the SINR but at the price of reduced bandwidth reuse. It is thus expected to perform better than the full reuse case only when the traffic demand is low and where more bandwidth is not needed. With regard to this observation, a switching mechanism is also needed to enable the full bandwidth reuse when the traffic demand gets higher.

### IV. THRESHOLD-BASED SON CONTROLLER

The bandwidth sharing implementation of the VS feature indirectly allows to balance the loads between inner and outer cells by applying fairness among all users in the sector. So this solution is naturally robust to load disparity in the cell. The full bandwidth reuse implementation on the other hand is only efficient when the traffic demand in both inner and outer cells is high enough.

With these observations, we propose a simple SON controller based on a load threshold ( $\rho_{th}$ ) to automatically switch between the two implementations of the VS feature according to the traffic demand in the cell. The load is defined as the average over time of the proportion of transmission resources (Physical Resource Blocks (PRBs) in LTE) used.

By denoting  $\rho_i$  and  $\rho_o$  the inner and outer cells loads respectively, we have for the bandwidth sharing implementation

$$\rho_{i/o}^{\text{sharing}} = \frac{\text{number of PRBs used by inner/outer}}{\text{Total number of PRBs allocated to inner/outer}}, \quad (10)$$

and for the full reuse implementation we have

$$\rho_{i/o}^{\text{reuse one}} = \frac{\text{number of PRBs used by inner/outer}}{\text{Total number of PRBs}}, \quad (11)$$

since each sector can use the whole bandwidth. If elastic traffic is considered and all the resources available are used whenever there is a user to serve, the loads correspond to the proportion of time there are users in the cell.

The SON controller for automatic selection of the implementation of the VS feature among bandwidth sharing and full reuse is presented in Algorithm 1.

**Algorithm 1** SON controller algorithm*Initialization:*Activate VS feature with **bandwidth sharing***loop:***for**  $k \in \mathbb{N}, k > 0$  **do**Estimate the inner and outer loads during time interval  $k$  using (10) or (11)**if** VS uses **bandwidth sharing** and  $\max(\rho_i^{\text{no reuse}}, \rho_o^{\text{no reuse}}) \geq \rho_{th}$  **then**Activate VS feature with **reuse one****if** VS uses **reuse one** and  $\max(\rho_i^{\text{reuse one}}, \rho_o^{\text{reuse one}}) < \rho_{th}$  **then**Activate VS feature with **bandwidth sharing**

The maximum between the inner and outer loads is considered in the algorithm in order to avoid attaining congestion in either cell before switching to full bandwidth reuse. The values for the thresholds can be learned from experience. A simple learning algorithm would be to increase (resp. decrease) the threshold if switching to full reuse (resp. bandwidth sharing) results in a performance loss.

## V. NUMERICAL RESULTS

We consider a trisector LTE network surrounded by six interfering macro sites as shown in Figure 2. The performance evaluation concerns only the central sectors (colored blue in Figure 2). In the case where the VS feature is implemented, it is deployed only on those 3 central sectors.

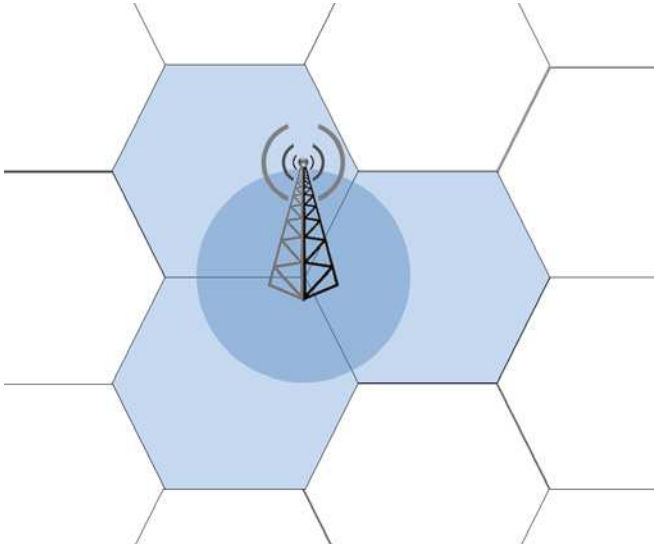


Fig. 2. Simulation scenario network layout

We also consider elastic traffic in which users arrive in the network according to a Poisson process. The arrival rate is denoted  $\lambda$  so that the inter-arrival times are exponentially distributed with mean  $1/\lambda$ . Each user downloads a file of exponentially distributed size and leaves the network as soon as the download is complete. Simulation parameters are resumed in Table I.

TABLE I. NETWORK AND TRAFFIC CHARACTERISTICS

Network parameters	
Number of interfering macros	$6 \times 3$ sectors
Macro Cell layout	hexagonal trisector
Intersite distance	500 m
Bandwidth	10MHz
Inner Cell Antenna Tilt	$18^\circ$
Outer Cell Antenna Tilt	$12^\circ$
Antenna Tilt Type	Electrical only
Channel characteristics	
Thermal noise	-174 dBm/Hz
Macro Path loss (d in km)	$128 + 36.8 \log_{10}(d)$ dB
Traffic characteristics	
Traffic spatial distribution	uniform
Service type	FTP
Average file size	6 Mbits

We evaluate user performance (MUT and Cell-Edge Throughput (CET)) as well as network performance (maximum cell loads) with varying traffic demand for the four following cases:

- **Baseline:** The VS feature is not implemented, the total transmit power for each sector is 46dBm. This case is colored black in the Figures.
- **VS reuse one:** The VS feature is implemented with full bandwidth reuse, the total transmit power is split equally among inner and outer sectors (43dBm each), and each cell uses the whole bandwidth. This case is colored red in the Figures.
- **VS bandwidth sharing:** The VS feature is also implemented but this time with no reuse, the bandwidth is shared between inner and outer cells according to Equation (8) which optimizes the proportional fair utility of users throughputs. This case is colored blue in the Figures.
- **SON controller:** The VS feature is implemented following Algorithm 1 which switches automatically between the two implementations according to the traffic demand. The value for  $\rho_{th}$  is set to 70% after observing the results from a first run of the three previous cases. This case is colored magenta in the Figures.

The results are presented for a global user arrival rate in the 3 sectors varying from 1 user/s to 10 users/s. Figure 3 shows the MUT, Figure 4 - the CET and Figure 5 - the maximum

load observed in all the sectors (3 sectors for the baseline, 6 sectors when the VS feature is enabled).

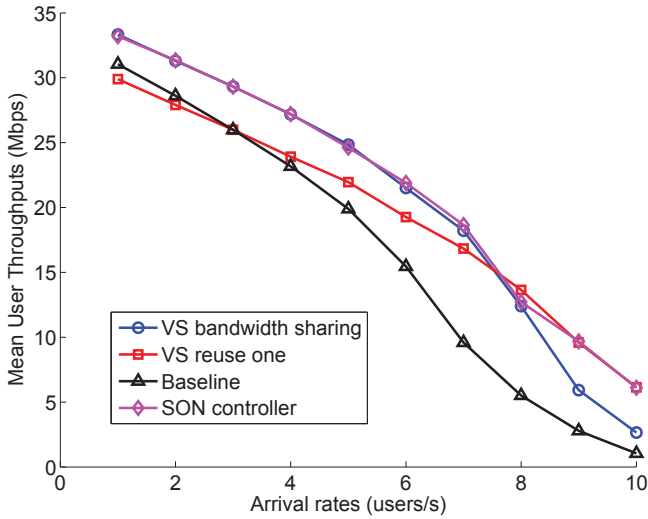


Fig. 3. Mean user throughputs for increasing arrival rates

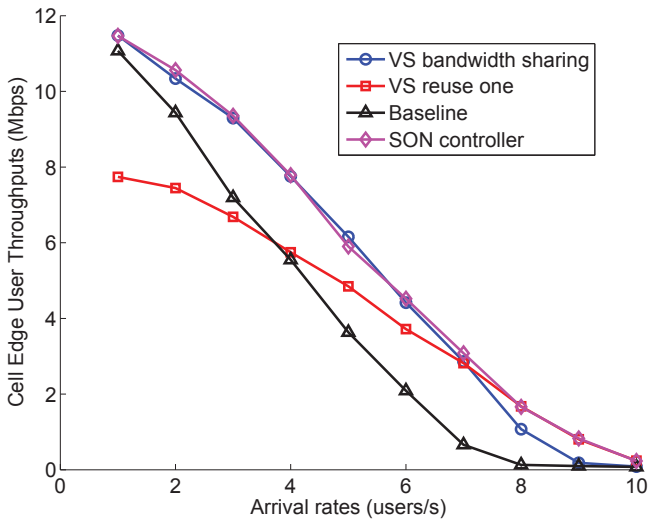


Fig. 4. Cell edge user throughputs for increasing arrival rates

The figures readily show that **VS reuse one** improves performance (MUT and CET) over Baseline only when the traffic demand is high enough (here  $\lambda \geq 3$ ). **VS bandwidth sharing** on the other hand takes advantage of the higher inner cell signal strength to improve performance (MUT and CET) over the Baseline at all loads. **VS bandwidth sharing** also provides better MUT and CET than **VS reuse one** for arrival rates less than 7.5 users/s (low to medium load scenarios).

The stability region of a sector is defined as the maximum traffic demand that can be handled by that sector with a load strictly less than one. If the traffic demand is inside the stability region, the mean number of users simultaneously present in the

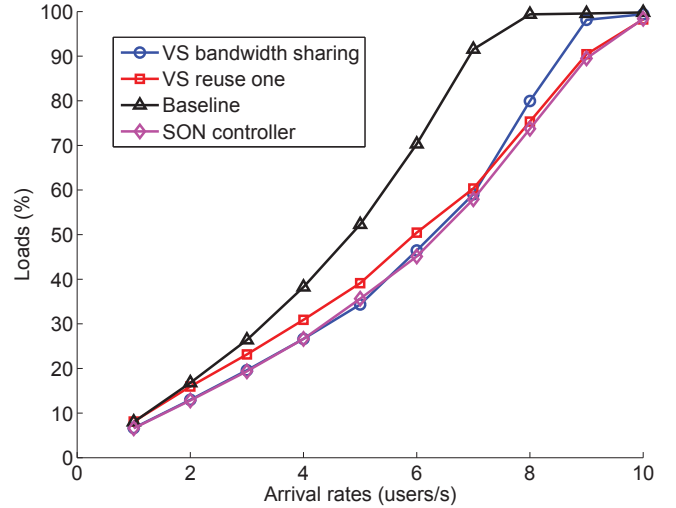


Fig. 5. Maximum loads for increasing arrival rates

cell remains bounded. As shown in Figure 5, **VS bandwidth sharing** improves the stability of the system over the Baseline. However at high load, full bandwidth reuse is needed. Thus **VS reuse one** provides a larger stability region than **VS bandwidth sharing**.

As shown by all the performance results (MUT, CET and stability region), the SON controller provides the best of both worlds: the interference gain of **VS bandwidth sharing** in the medium to low load traffic demand scenarios, and the increased bandwidth of **VS reuse one** for high traffic demands.

These numerical results show the improved performance of the bandwidth sharing implementation of VS feature, and how the SON controller improves performance by switching from VS with bandwidth sharing to VS with reuse one at high loads.

## VI. CONCLUSION

The paper has shown that a bandwidth sharing implementation of the VS feature can bring improved performance for medium to low traffic demands provided that the sharing proportions are optimized. Alpha-fair self-optimizing algorithms have been proposed for the bandwidth sharing proportions between the inner and outer cells. A threshold-based SON controller has also been proposed to automatically enable the full bandwidth reuse for high traffic demands. Flow-level simulations show the benefit of the proposed approaches. An improvement of 10% for the MUT and 50% for the CET has been observed for VS with bandwidth sharing compared to VS with reuse one at low load. The SON controller offers the best performance for all load scenarios.

APPENDIX A  
PROOF OF CONCAVITY OF  $U_\alpha(\delta)$

Since the concavity is preserved under linear transformation and non-negative sums [10], it suffices to prove that

$$f_\alpha(x) = \begin{cases} \log x & \alpha = 1 \\ \frac{x^{1-\alpha}}{1-\alpha} & \alpha \neq 1 \end{cases} \quad (12)$$

is concave. This function is twice differentiable in  $x > 0$  and its second derivative with regard to  $x$  is

$$\frac{\partial^2 f_\alpha(x)}{\partial x^2} = \begin{cases} \frac{-1}{x^2} & \alpha = 1 \\ -\alpha x^{-\alpha-1} & \alpha \neq 1 \end{cases} \quad (13)$$

So  $\frac{\partial^2 f_\alpha(x)}{\partial x^2} \neq 0$  for all  $\alpha \geq 0$ . As a consequence,  $f_\alpha(x)$  is concave by the second order conditions on convexity.

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