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# Asymptotic Preserving scheme for a kinetic model describing incompressible fluids

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## Abstract

The kinetic theory of fluid turbulence modeling developed by Degond and Lemou in [6] is considered for further study, analysis and simulation. Starting with the Boltzmann like equation representation for turbulence modeling, a relaxation type collision term is introduced for isotropic turbulence. In order to describe some important turbulence phenomenology, the relaxation time incorporates a dependency on the turbulent microscopic energy and this makes difficult the construction of efficient numerical methods. To investigate this problem, we focus here on a multi-dimensional prototype model and first propose an appropriate change of frame that makes the numerical study simpler. Then, a numerical strategy to tackle the stiff relaxation source term is introduced in the spirit of *Asymptotic Preserving Schemes*. Numerical tests are performed in a one-dimensional framework on the basis of the developed strategy to confirm its efficiency.

## 1 Introduction

Modeling and simulation of turbulence in fluid flows is a challenging task. Though the research in this topic has been pursued for nearly a century, the ideal model is still elusive. We refer to the books of Pope [21] and Wilcox [23] for details of various turbulence models. The concepts from *Kinetic Theory of Gases* have been of significant use in some of the models for turbulence. Extending this approach further, Degond and Lemou [6] developed a model fully based on Boltzmann type equation and its expanded analogy for a description of turbulence. Chen *et al.* [3] also followed a similar approach in developing a different model. This work is a first step, following the approach of [6] for study, analysis and simulation of turbulence from kinetic theory.

In [6] it is proposed to describe a turbulent incompressible fluid flow through a probability distribution function  $f(t, x, v)$  of fluid elements (or structures) depending not only on time and on the position  $x$  of these elements but also on their velocity. In this description, the fluid elements are assumed to interact in order to bring the fluid to an isotropic distribution in velocity (isotropic turbulence) and this is described by a relaxation collision operator. On the other hand, the pressure in the fluid acts in order to maintain the probability nature of the distribution function in such a way that the incompressibility (divergence-free) condition is satisfied. In the relaxation term (collision kernel), the relaxation time can be tuned in a such way that the model

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incorporates some turbulence phenomenology and is able to describe two important regimes: the molecular viscosity regime and the turbulence (the so-called inertial range) regime which follows the well-known Kolmogorov law on the energy spectrum of the fluid structures. In particular, this relaxation time should depend on the turbulent velocity and, as we shall see, this makes more difficult the construction of efficient numerical schemes solving the corresponding kinetic model. Let us mention that this kind of relaxation operator has been studied in different frameworks such as rarefied gas dynamics in [2], [22], [19] or to describe wave-particle collision in plasma physics [8], [9], or for cometary flows modeling [10, 13].

We first give the basic kinetic equation introduced in [6]. The distribution function  $f(t, x, v)$ , with  $v \in \mathbb{R}^d$ ,  $x \in \Omega \subset \mathbb{R}^d$ ,  $d = 1, 2, 3$ ,  $t > 0$ , satisfies the following equation

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f - \nabla_x \mathcal{P} \cdot \nabla_v f = \frac{1}{\eta} Q_{u_f}(f) + \eta Q_{\text{diss}}(f), \quad (1)$$

where  $\eta > 0$  is a constant parameter. We shall now define the different terms in this equation. First the operator  $Q_{\text{diss}}$  is a dissipative operator which allows to take into account the dissipation of turbulence in a fluid, that is its tendency to become laminar. We shall omit it here in our preliminary numerical study and refer to [6] for a detailed description and different possible choices of such operators. Secondly, the operator  $Q_{u_f}$  is the relaxation collision operator that brings the distribution function to an isotropic state and is given by

$$Q_{u_f}(f)(v) = \frac{1}{\tau \left( \frac{|v - u_f|^2}{2} \right)} [\Pi f(v) - f(v)],$$

where  $u_f$  is defined from the following (nonlinear) equation

$$\int_{\mathbb{R}^d} \frac{(v - u_f)}{\tau \left( \frac{|v - u_f|^2}{2} \right)} f(v) dv = 0, \quad (2)$$

and the relaxation time  $\tau$  is a function of the microscopic energy  $\frac{|v - u_f|^2}{2}$  which will be made precise later on. The operator  $\Pi$  is the projection on to the space of isotropic functions around  $u_f$

$$\Pi f(v) = \frac{1}{|\mathbb{S}^{d-1}|} \int_{\omega \in \mathbb{S}^{d-1}} f(u_f + |v - u_f| \omega) d\omega,$$

where  $\mathbb{S}^{d-1}$  is the sphere in  $\mathbb{R}^d$  ( $|\mathbb{S}^{d-1}|$  denotes its measure). The distribution function  $f$  is constrained to satisfy

$$\int_{\mathbb{R}^d} f(v) dv = 1, \quad (3)$$

and this allows to see the pressure  $\mathcal{P}$  in (1) as a Lagrange multiplier of this constraint. Note that this kinetic model was also used in [6] to derive a  $k-\varepsilon$  type turbulence model or Modified  $k-\varepsilon$  turbulence model (M- $k-\varepsilon$ ).

Defining the mean velocity  $\bar{u} = \int_{\mathbb{R}^d} v f(v) dv$ , we remark that in general, we have  $u_f \neq \bar{u}$ . However when the relaxation time  $\tau$  is a (nonzero) constant function, then these two velocities coincide (see (2)). Now we emphasize that the collision operator  $Q_{u_f}(f)$  satisfies the important properties of conservation and leads to an entropy function (see [6] for details). In particular it preserves the mass, momentum and energy (see [6])

$$\int_{\mathbb{R}^d} \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} Q_{u_f}(f) dv = 0,$$

and it is worthwhile to see that the constraint (2) is necessary to ensure the momentum conservation property.

The goal of this work is to provide numerical simulations to efficiently solve the model (1)-(2)-(3) satisfied by  $f, u_f$  and  $\mathcal{P}$ . There are two main difficulties in constructing such efficient numerical schemes for this model. First, the determination of the velocity  $u_f$  from formula (2) requires the use of an iterative method like Newton- Raphson technique at each time step. This turns out to be numerically expensive and avoiding it would significantly accelerate the simulation, especially if one deals with multidimensional models. The second difficulty is with the stiffness of the collision operator. The usual way of solving stiffness problem is to use an implicit discretization for the collision operator, but this is not possible here since  $u_f$  should be determined before. We note that advancing the mean velocity  $\bar{u}$  is possible but advancing in time the velocity  $u_f$  is not possible since it is determined by (2). This means in particular that classical approaches based on Asymptotic Preserving schemes for a class of other stiff kinetic equations (see [15, 17, 11, 12, 16, 4, 1, 18, 14, 20]) do not work in this context.

The appellation *Asymptotic Preserving* has been introduced in [15] for numerical schemes that are stable with respect to a small parameter ( $\eta$  in this work) and degenerate into consistent numerical schemes for the limit model when  $\eta$  goes to zero. This class of numerical schemes is particularly well adapted to our framework since we want to solve (1)-(2)-(3) for arbitrary small values of  $\eta$ , which may lead to a severe constraint on the time step (it has to be of order  $\eta$  for stability) when a standard explicit numerical scheme is employed.

The above mentioned difficulties in designing an AP scheme are overcome in this framework with an efficient strategy as described in the following steps. First we perform a suitable change of frame to make the full implicitization of the collision term possible. Usually, this implicitation is at the heart of designing AP schemes. Unfortunately this is not sufficient to get such AP schemes in our context, since an additional stiff transport term arises after the change of frame (as we shall see in the next section) and therefore this requires a specific treatment. In fact, this new stiff term originates with the constraint (2) in the new frame. The numerical scheme we propose is then based on two main steps. Firstly, we focus on the stiffness of the collision operator and make it fully implicit by following for example the strategies in [17, 12]. As mentioned above, the obtained scheme still contains a stiff transport term. We then propose a suitable way to implicitize this new stiff term, by making a part of it implicit. We emphasize that the whole obtained implicit scheme is computationally explicit, which means that no additional computational step is needed to solve the implicit schemes compared to the resolution of an explicit one. In particular, this results in a first order numerical scheme which enjoys the Asymptotic Preserving property, and has the same computational cost as a standard explicit solver (it does not require any linear system to invert).

The rest of the paper is organized as follows. In section 2, we give a reformulation of the continuous kinetic model under study through a suitable change of frame. The asymptotic limit  $\eta \rightarrow 0$  is also studied. In section 3, a suitable time discretization is performed on the new formulation, and the spatial discretization is discussed. Finally, in section 6, some numerical results are presented in a one-dimensional setting to illustrate the behavior of the new scheme in various configurations.

## 2 A reformulation of the continuous kinetic model

In this section, the model is reformulated by using a suitable change of frame. In particular, the constraints (3) and (2) are reformulated in a more tractable way in view of the construction of efficient numerical schemes. The asymptotic limit  $\eta \rightarrow 0$  is then studied for this new formulation.

## 2.1 A change of frame

As underlined in the introduction above, the collision time  $\tau$  allows to incorporate some important turbulence phenomenology (viscosity regime and turbulence regime following the Kolmogorov law). For instance it is shown that in the molecular viscosity regime (small turbulent energies  $\xi$ , with  $\xi = |v - u_f|^2/2$ ), the relaxation time  $\tau$  must satisfy  $\tau(\xi) \sim C/\xi$  whereas in the turbulent regime (large energies  $\xi$ ),  $\tau$  should satisfy  $\tau(\xi) \sim C/\sqrt{\xi}$ . Then a general power-law is considered for  $\tau$  in [6],  $\tau(\xi) = \xi^\alpha$  with  $\xi = |v - u_f|^2/2$ ,  $-\frac{1}{2} \geq \alpha \geq -1$ , and extended  $k-\varepsilon$  models are derived from the corresponding kinetic equations. We will not detail this study here and refer to [6] for a thorough description of these aspects.

Here we will then consider  $\tau$  as a general function of the particle kinetic energy  $\xi = |v - u_f|^2/2$ , *i.e.*,  $\tau = \tau\left(\frac{|v - u_f|^2}{2}\right)$ . One of the main difficulties at the numerical level is to determine  $u_f$  which is the solution of the following nonlinear equation

$$\int_{\mathbb{R}^d} \frac{v - u_f}{\tau\left(\frac{|v - u_f|^2}{2}\right)} f dv = 0.$$

In the following we will use the notation  $u := u_f$ . Hence, for a known function  $f$ , the computation of  $u$  needs typically a Newton type algorithm, which can be very expensive. Since we are working on a model where most of the parameters depend on  $(v - u)$ , it makes sense to rewrite the equations after using the transformation  $v \rightarrow (v - u) = v'$  (see [7]). The greatest advantage of using this change of frame is that both  $\tau$  and the projector  $\Pi$  can now be considered independent of time and space variables. In particular, the projector in the new frame will commute with the transport operator and the time derivative, a fact of great importance in the construction of efficient numerical schemes. Using this transformation, the constraints (3) and (2) can be rewritten as:

$$\int_{\mathbb{R}^d} F dv = 1, \tag{4}$$

$$\int_{\mathbb{R}^d} \frac{v}{\tau\left(\frac{|v|^2}{2}\right)} F dv = 0, \tag{5}$$

where  $f(t, x, v) = F(t, x, v - u)$ . Moreover, since  $\int_{\mathbb{R}^d} v f dv = \bar{u}$ , we have

$$\int_{\mathbb{R}^d} v F dv = \bar{u} - u. \tag{6}$$

Let us now write the equation satisfied by  $F$ . The derivatives of  $F$  in terms of  $f$  can be written as

$$\begin{aligned} \partial_t f &= \partial_t F - \partial_t u \cdot \partial_v F, \\ \partial_x f &= \partial_x F - (\partial_x u) \partial_v F, \\ \partial_v f &= \partial_v F, \end{aligned}$$

with  $(\partial_x u)_{i,j} = \partial_{x_i} u_j$ . Using the above relations we can rewrite (1) as follows

$$\partial_t F + (v + u) \cdot \partial_x F - (E + (\partial_x u)^T v) \cdot \partial_v F = \frac{1}{\eta} \frac{1}{\tau\left(\frac{|v|^2}{2}\right)} (\Pi_0 F - F), \tag{7}$$

where  $E = \partial_t u + \partial_x \mathcal{P} + (\partial_x u)^T u$  and

$$\Pi_0 F = \frac{1}{|\mathbb{S}^{d-1}|} \int_{\mathbb{S}^{d-1}} F(|v|w) dw.$$

Hence, the unknowns  $(F, u, \mathcal{P}, E)$  satisfy the model given by the equations

$$\begin{cases} \partial_t F + (v + u) \cdot \partial_x F - (E + (\partial_x u)^T v) \cdot \partial_v F = \frac{1}{\eta} \frac{1}{\tau \left( \frac{|v|^2}{2} \right)} (\Pi_0 F - F), \\ \partial_t u + (u \cdot \partial_x) u = E - \partial_x \mathcal{P}, \\ \int_{\mathbb{R}^d} F dv = 1, \\ \int_{\mathbb{R}^d} \frac{v}{\tau} F dv = 0. \end{cases} \quad (8)$$

This model is equivalent to the original model (1)-(3)-(2) satisfied by  $(f, u, \mathcal{P})$ .

In the following, we rewrite the constraints  $\int_{\mathbb{R}^d} F dv = 1$  and  $\int_{\mathbb{R}^d} \frac{v}{\tau} F dv = 0$  in a more tractable way to construct numerical schemes.

Let us first consider the constraint  $\int_{\mathbb{R}^d} F dv = 1$ . Integrating the first equation of (8) with respect to  $v$  leads to

$$\partial_x \cdot \left( \int v F dv + u \right) = 0, \quad (9)$$

since  $-\int [(\partial_x u)^T v] \cdot \partial_v F dv = \partial_x \cdot u$ . Then, taking the divergence of the first moment of the first equation of (8) and summing the obtained equation to the divergence of the second equation of (8) leads to (using (9))

$$\partial_x \cdot \int v \otimes (v + u) \partial_x F dv - \partial_x \cdot \int v (E + (\partial_x u)^T v) \cdot \partial_v F dv + \partial_x \cdot [(u \cdot \partial_x) u] = \partial_x \cdot E - \Delta \mathcal{P}. \quad (10)$$

The left hand side gives  $\partial_x^2 : [\int (v + u) \otimes (v + u) F dv] + \partial_x \cdot E - \partial_x \cdot [(\partial_x \cdot u) u]$  which finally gives an equation for the pressure  $\mathcal{P}$

$$-\Delta \mathcal{P} = \partial_x^2 : \left[ \int (v + u) \otimes (v + u) F dv \right] - \partial_x \cdot [(\partial_x \cdot u) u], \quad (11)$$

where “:” denotes the contracted product of two tensors. We refer the reader to appendix 7.1 for the details of the computations.

Let us now consider the constraint  $\int_{\mathbb{R}^d} \frac{v}{\tau} F dv = 0$ . Multiplying the first equation of (8) by  $v/\tau$  and integrating with respect to  $v$  gives (using the constraint  $\int_{\mathbb{R}^d} \frac{v}{\tau} F dv = 0$ )

$$\int \frac{v}{\tau} (v + u) \cdot \partial_x F dv - \int \frac{v}{\tau} E \cdot \partial_v F dv - \int \frac{v}{\tau} [(\partial_x u)^T v] \cdot \partial_v F dv = -\frac{1}{\eta} \int \frac{v}{\tau^2} F dv.$$

This provides an explicit expression of  $E$

$$\left( \int \frac{v}{\tau} \otimes \partial_v F dv \right) E = \frac{1}{\eta} \int \frac{v}{\tau^2} F dv + \partial_x \cdot \int \frac{1}{\tau} v \otimes (v + u) F dv - \int \frac{v}{\tau} [(\partial_x u)^T v] \cdot \partial_v F dv. \quad (12)$$

Let us remark that the  $d \times d$  matrix  $\left( \int \frac{v}{\tau} \otimes \partial_v F dv \right)$  is invertible (see Appendix 7.2).

Finally, the system to solve is the following, satisfied by  $(F, u)$

$$\begin{cases} \partial_t F + (v + u) \cdot \partial_x F - (E + (\partial_x u)^T v) \cdot \partial_v F = \frac{1}{\eta} \frac{1}{\tau \left( \frac{|v|^2}{2} \right)} (\Pi_0 F - F), \\ \partial_t u + (u \cdot \partial_x) u = E - \partial_x \mathcal{P}, \\ -\Delta \mathcal{P} = \partial_x^2 : \left[ \int (v + u) \otimes (v + u) F dv \right] + \partial_x \cdot [(\partial_x \cdot u) u], \\ \left( \int \frac{v}{\tau} \otimes \partial_v F dv \right) E = \frac{1}{\eta} \int \frac{v}{\tau^2} F dv + \partial_x \cdot \int \frac{1}{\tau} v \otimes v F dv - \int \frac{v}{\tau} [(\partial_x u)^T v] \cdot \partial_v F dv. \end{cases} \quad (13)$$

**Remark 1.** The case  $\tau$  constant (equal to 1) leads to specific expression of  $E$  since the constraint (5) becomes  $\int_{\mathbb{R}} v F dv = 0$  and as a consequence the stiff term in the last equation of (13) vanishes. Then, in this configuration, the equation for  $E$  reduces to

$$\left( \int v \otimes \partial_v F dv \right) E = \partial_x \cdot \int v \otimes v F dv - \int v [(\partial_x u)^T v] \cdot \partial_v F dv,$$

and the equation for  $F$  becomes

$$\partial_t F + (v + u) \cdot \partial_x F - (E + (\partial_x u)^T v) \cdot \partial_v F = \frac{1}{\eta} (\Pi_0 F - F).$$

This equation has the same structure as the Vlasov-Poisson-BGK model in the hydrodynamic regime. Applying  $\Pi_0$  to this equation enables to derive a non stiff macro equation. At the numerical level, IMEX methods proposed in [1, 5, 11, 12, 14, 20] can be applied in a straightforward way.

**Remark 2.** The equation for  $F$  in (13) can be formulated in a conservative form

$$\partial_t F + v \cdot \partial_x F - E \cdot \partial_v F - (\partial_x u)^T : \partial_v (v F) + \partial_x \cdot (u F) = \frac{1}{\eta} \frac{1}{\tau \left( \frac{|v|^2}{2} \right)} (\Pi_0 F - F),$$

from which the conservation of mass  $\frac{d}{dt} \iint f(t, x, v) dx dv = 0$  is deduced easily.

## 2.2 Asymptotic models from the reformulation

The aim of this section is to derive the asymptotic model, *i.e.*, the model obtained formally from (1) when  $\eta$  goes to zero, or equivalently, from (13). It is of importance to understand the asymptotic limit at the continuous level when one wants to design Asymptotic Preserving schemes. To alleviate the notation, in the following we use brackets for the velocity integrals  $\langle \cdot \rangle := \int_{\mathbb{R}^d} \cdot dv$ .

First of all, we rewrite (13) using the following notation

$$AF = (v + u) \cdot \partial_x F - (\partial_x u)^T v \cdot \partial_v F, \quad (14)$$

so that the first equation of (13) reads

$$\partial_t F + AF - E \cdot \partial_v F = \frac{1}{\eta \tau} (\Pi_0 F - F).$$

where  $E$  is given by

$$\begin{aligned} E &= \frac{1}{\eta} \left( \int \frac{v}{\tau} \otimes \partial_v F dv \right)^{-1} \int \frac{v}{\tau^2} F dv \\ &\quad + \left( \int \frac{v}{\tau} \otimes \partial_v F dv \right)^{-1} \left[ \partial_x \cdot \int \frac{1}{\tau} v \otimes v F dv - \int \frac{v}{\tau} [(\partial_x u)^T v] \cdot \partial_v F dv \right]. \end{aligned}$$

This term makes an additional stiff term appear so that the operator

$$\mathcal{L}F := \mathcal{M} \cdot \partial_v F + \frac{1}{\tau} (\Pi_0 F - F)$$

with  $\mathcal{M} = \left( \int \frac{v}{\tau} \otimes \partial_v F dv \right)^{-1} \left( \int \frac{v}{\tau^2} F dv \right)$  needs to be studied. This is the purpose of the following proposition.

**Proposition 2.1.** *The null space of  $\mathcal{L}$  is the same as the null space of  $Q_0$ , with  $Q_0(F) = \frac{1}{\tau} (\Pi_0 F - F)$ , i.e., the space of functions  $\phi(|v|)$ , where the function  $\phi$  is arbitrary.*

*Proof.* Considering the relation  $\langle \mathcal{L}(F)F \rangle := \int_{\mathbb{R}^d} \mathcal{L}(F)F dv = 0$  gives

$$\left\langle \frac{1}{\tau} (\Pi_0 F - F)F \right\rangle = -\langle \mathcal{M} \partial_v F F \rangle, \quad (15)$$

with  $\mathcal{M} = \left( \int \frac{v}{\tau} \otimes \partial_v F dv \right)^{-1} \left( \int \frac{v}{\tau^2} F dv \right)$ . We now use the fact that  $\langle \frac{1}{\tau} (\Pi_0 F - F) \Pi_0 F \rangle = 0$ . Indeed, since  $\Pi_0$  is self-adjoint in  $L^2$  and commute with isotropic function like  $1/\tau$ , we have,

$$\left\langle \frac{1}{\tau} (\Pi_0 F - F) \Pi_0 F \right\rangle = \langle \Pi_0 \left( \frac{1}{\tau} (\Pi_0 F - F) \right) F \rangle = \left\langle \frac{1}{\tau} \Pi_0 (\Pi_0 F - F) F \right\rangle = 0.$$

Then, (15) becomes

$$\left\langle \frac{1}{\tau} (\Pi_0 F - F) (I - \Pi_0) F \right\rangle = -\langle \mathcal{M} \cdot \partial_v F F \rangle = 0,$$

so that

$$\left\| \frac{1}{\sqrt{\tau}} (\Pi_0 F - F) \right\|_{L^2}^2 = 0.$$

The function  $\tau$  being positive, we conclude that if  $\langle \mathcal{L}(F)F \rangle = 0$ , then  $F = \Pi_0 F$ , which means that  $F$  is isotropic, i.e.,  $F$  depends on  $|v|$ . □

We can now derive formally the asymptotic model from (13) satisfied by  $(G_0, \bar{u})$  which are the limits when  $\eta$  goes to zero of  $(F, u)$ . This is presented in the following proposition.

**Proposition 2.2.** *The asymptotic model obtained from (13) in the limit  $\eta$  goes to zero, satisfied by the limit  $(G_0, \bar{u})$  of  $(F, u)$ , writes*

$$\begin{cases} \partial_t G_0 + \bar{u} \cdot \partial_x G_0 = 0, \\ \partial_t \bar{u} + (\bar{u} \cdot \partial_x) \bar{u} = E_0 - \partial_x \mathcal{P}_0, \\ -\Delta \mathcal{P}_0 = \frac{1}{d} \Delta \int_{\mathbb{R}^d} |v|^2 G_0 dv + \partial_x^2 : (\bar{u} \otimes \bar{u}), \\ E_0 = -\frac{1}{d} \partial_x \int_{\mathbb{R}^d} |v|^2 G_0 dv. \end{cases} \quad (16)$$



*Proof.* We first derive the asymptotic model from the first equation of (13). To do that, we consider the Chapman-Enskog expansion of  $F$ :  $F = G_0 + \eta G_1 + \mathcal{O}(\eta^2)$ , where  $G_0 = \Pi_0 F$ . Inserting this expansion in the first equation of (13) leads to

$$\partial_t G_0 + AG_0 - E \cdot \partial_v G_0 = -\frac{1}{\tau} G_1 + \mathcal{O}(\eta). \quad (17)$$

Applying  $\Pi_0$  to this latter equation leads to the asymptotic equation for  $G_0$

$$\partial_t G_0 + \Pi_0 A G_0 - \Pi_0(E \cdot \partial_v G_0) = 0. \quad (18)$$

The term  $\Pi_0(E \cdot \partial_v G_0)$  vanishes whereas the term  $\Pi_0 A G_0$  becomes

$$\begin{aligned} \Pi_0 A G_0 &= \Pi_0 [(v + u) \cdot \partial_x G_0 - (\partial_x u)^T v \cdot \partial_v G_0] \\ &= u \cdot \partial_x G_0 - \Pi_0((\partial_x u)^T v \cdot \partial_v G_0). \end{aligned}$$

Rewriting the isotropic function  $G_0(v) := g(\xi)$  with  $\xi = |v|^2/2$ , we have  $\partial_v G_0 = v \partial_\xi g$  so that the last term becomes

$$\Pi_0((\partial_x u)^T v \cdot \partial_v G_0) = (\partial_x u)^T : \Pi_0(v \otimes v) \partial_\xi g.$$

Let us compute  $\Pi_0(v \otimes v)$

$$\begin{aligned} \Pi_0(v_i v_j) &= \frac{1}{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} |v|^2 \omega_i \omega_j d\omega \\ &= \frac{1}{\mathbb{S}^{d-1}} |v|^2 \delta_{i,j} \int_{\mathbb{S}^{d-1}} \omega_i^2 d\omega \\ &= \frac{1}{d \mathbb{S}^{d-1}} |v|^2 \delta_{i,j} \int_{\mathbb{S}^{d-1}} \sum_{i=1}^d \omega_i^2 d\omega \\ &= \frac{1}{d \mathbb{S}^{d-1}} |v|^2 \delta_{i,j} \int_{\mathbb{S}^{d-1}} d\omega \\ &= \frac{1}{d \mathbb{S}^{d-1}} |v|^2 \delta_{i,j} \mathbb{S}^{d-1} \\ &= \frac{1}{d} |v|^2 \delta_{i,j}. \end{aligned}$$

We then conclude  $(\partial_x u)^T \Pi_0(v \cdot \partial_v G_0) = \frac{1}{d} |v|^2 \partial_x \cdot u$ . When  $\eta$  goes to zero, we have  $u$  converging towards  $\bar{u}$  since  $\int v F dv = \bar{u} - u$  with  $F$  which tends towards  $G_0$  and  $\int v G_0 dv = 0$ ; hence, the incompressibility condition  $\partial_x \cdot \bar{u} = 0$  makes this term null when  $\eta$  goes to zero. The asymptotic model of the first equation of (13) then writes

$$\partial_t G_0 + \bar{u} \cdot \partial_x G_0 = 0,$$

which is the first equation of the asymptotic model (16).

To get the limit of the second equation of (13), we need to derive the limit of  $E$  and  $\partial_x \mathcal{P}$  when  $\eta$  goes to zero. The so-obtained model will be satisfied by the limit  $\bar{u}$  of  $u$ .

Let us first focus on the derivation of the limit of  $E$ . To do that, we apply  $(I - \Pi_0)$  to (17) and get (neglecting  $\mathcal{O}(\eta)$  terms)

$$(I - \Pi_0)AG_0 - E \cdot \partial_v G_0 = -\frac{1}{\tau}G_1.$$

We multiply by  $v$  and integrate with respect to  $v$  to get

$$\langle v(I - \Pi_0)AG_0 \rangle + E = 0.$$

Indeed, we used  $\langle (v/\tau)G_1 \rangle = 0$  since  $0 = \langle (v/\tau)F \rangle = \langle (v/\tau)G_0 \rangle + \eta \langle (v/\tau)G_1 \rangle$  and  $\langle (v/\tau)G_0 \rangle = 0$  (since  $G_0$  is even), so that we deduce  $\langle (v/\tau)G_1 \rangle = 0$ . The expression of the limit  $E_0$  of  $E$  when  $\eta$  goes to zero then becomes

$$\begin{aligned} E_0 &= -\langle v(I - \Pi_0)AG_0 \rangle \\ &= -\langle vAG_0 \rangle \\ &= -\langle v[(v+u) \cdot \partial_x G_0 - (\partial_x u)^T v \cdot \partial_v G_0] \rangle \\ &= -\langle v(v+u) \cdot \partial_x G_0 \rangle - \langle v(\partial_x u)^T v \cdot \partial_v G_0 \rangle \\ &= -\partial_x \cdot \langle v \otimes vG_0 \rangle \\ &= -\frac{1}{d}\partial_x \langle |v|^2 G_0 \rangle \end{aligned} \tag{19}$$

We then focus on the limit equation satisfied by the pressure  $\mathcal{P}_0$ , limit of  $\mathcal{P}$  when  $\eta$  goes to zero. Considering the third equation of (13) when  $F$  is replaced by its limit  $G_0$  and  $u$  by its limit  $\bar{u}$ , we have

$$\begin{aligned} -\Delta \mathcal{P}_0 &= \partial_x^2 : \langle (v + \bar{u}) \otimes (v + \bar{u})G_0 \rangle + \partial_x \cdot [(\partial_x \cdot \bar{u})\bar{u}] \\ &= \partial_x^2 : \langle (v + \bar{u}) \otimes (v + \bar{u})G_0 \rangle \\ &= \partial_x^2 : \langle v \otimes vG_0 \rangle + \partial_x^2 : (\bar{u} \otimes \bar{u}) \\ &= \frac{1}{d}\Delta \langle |v|^2 G_0 \rangle + \partial_x^2 : (\bar{u} \otimes \bar{u}). \end{aligned} \tag{20}$$

The asymptotic of the second equation of (13) satisfied by  $\bar{u}$  which is the limit of  $u$  when  $\eta$  goes to zero, finally writes

$$\partial_t \bar{u} + (\bar{u} \cdot \partial_x) \bar{u} = E_0 - \partial_x \mathcal{P}_0,$$

where  $E_0$  and  $\mathcal{P}_0$  satisfy (19) and (20). This corresponds to the second equation of (16), and with (19) and (20) correspond to the third and fourth equation of (16). This concludes the proof.  $\square$

### 3 Numerical schemes

In this section, we propose some numerical schemes for the kinetic models described previously. We divide our approach into the cases  $\tau$  constant and  $\tau$  non constant, the latter case being more complicated and needing different techniques. Our main goal is to provide semi-discretized (in time) numerical schemes which enjoy the Asymptotic Preserving (AP) property, that is: *(i)* they are stable with respect to the parameter  $\eta$ , *(ii)* they degenerate when  $\eta$  goes to zero towards numerical schemes which are consistent with the asymptotic model (16). Such schemes have been initiated in [15] and have become, during the last decade, very popular to approximate stiff kinetic equations.

One important problem here to derive an AP scheme is to make the relaxation operator implicit since the determination of  $u$  requires the knowledge of  $f$ . This involves a nonlinear system to solve which would be computationally very costly and difficult to handle. As presented in the previous section, one issue is to consider the change of variable  $v' \rightarrow v - u$  and the equation on  $F$  (7). However, an additional stiff term arises after this change of frame and the derivation of a numerical scheme where only the relaxation term  $\frac{1}{\tau}(\Pi_0 F - F)$  is considered implicit does not provide the AP property of this scheme, as we shall see. We will also look at an exponential integrator following [17] but in this case also, this is not sufficient to get the AP property. Finally, we propose a new strategy which is a modification of the latter approaches, that ensures the numerical scheme to degenerate as  $\eta$  goes to zero towards a consistent numerical scheme of the asymptotic model (16). We emphasize that this numerical scheme has the same computational cost as a standard explicit solver.

In the following, we present two strategies from the literature (see [11, 12, 18, 20]) and explain why they cannot be applied in our framework. We then propose our new scheme in two different versions. As we deal with a semi-discretization in time in this section, we introduce a time discretization  $t^n = n\Delta t, n \in \mathbb{N}, \Delta t > 0$  and we denote by  $F^n$  an approximation of  $F$  at time  $t^n$ .

Let us recall (13)

$$\begin{cases} \partial_t F + AF - E \cdot \partial_v F = \frac{1}{\eta} \frac{1}{\tau \left( \frac{|v|^2}{2} \right)} (\Pi_0 F - F), \\ \partial_t u + (u \cdot \partial_x) u = E - \partial_x \mathcal{P}, \\ -\Delta \mathcal{P} = \partial_x^2 : \langle (v + u) \otimes (v + u) F \rangle + \partial_x \cdot [(\partial_x \cdot u) u], \\ E = \frac{1}{\eta} \mathcal{A}(F)^{-1} \langle \frac{v}{\tau^2} F \rangle + \mathcal{A}(F)^{-1} \left[ \partial_x \cdot \langle \frac{v}{\tau} \otimes (v + u) F \rangle - \langle \frac{v}{\tau} [(\partial_x u)^T v] \cdot \partial_v F \rangle \right], \end{cases} \quad (21)$$

where  $A$  is given by (14) and  $\mathcal{A}(F)$  is the  $d \times d$  matrix defined by  $\mathcal{A}(F) = (\langle \frac{v}{\tau} \otimes \partial_v F \rangle)$ .

We introduce the following notations for the time discretization of the term  $E$

$$E^{n,n} = \frac{1}{\eta} \mathcal{A}(F^n)^{-1} \langle \frac{v}{\tau^2} F^n \rangle + E_1^n, \quad (22)$$

and

$$E^{n,n+1} = \frac{1}{\eta} \mathcal{A}(F^n)^{-1} \langle \frac{v}{\tau^2} F^{n+1} \rangle + E_1^n, \quad (23)$$

where  $E_1^n$  is given by

$$E_1^n = \mathcal{A}(F^n)^{-1} \left[ \partial_x \cdot \langle \frac{v}{\tau} \otimes (v + u^n) F^n \rangle - \langle \frac{v}{\tau} [(\partial_x u^n)^T v] \cdot \partial_v F^n \rangle \right]. \quad (24)$$

#### State of the art.

In this part, we look at two standard strategies to derive an AP numerical scheme for (21)

$$\partial_t F + AF - E \cdot \partial_v F = \frac{1}{\tau \eta} (\Pi_0 F - F), \quad (25)$$

where  $A$  is given by (14).

As we shall see, the presence of the stiff term in the left hand side is the major obstacle in a direct application of standard strategies.

The first idea would be to consider implicitizing the relaxation term

$$F^{n+1} = F^n - \Delta t A F^n + \Delta t E^{n,n} \cdot \partial_v F^n + \frac{\Delta t}{\tau \eta} (\Pi_0 F^{n+1} - F^{n+1}).$$

To get an expression of  $\Pi_0 F^{n+1}$ , we apply  $\Pi_0$  to this last equation

$$\Pi_0 F^{n+1} = \Pi_0 F^n - \Delta t \Pi_0 A F^n + \Delta t E^{n,n} \cdot \Pi_0 (\partial_v F^n).$$

This *macro* equation still involves a stiff term in the term  $E^{n,n}$  given by (22) which requires the time step  $\Delta t$  to be lower than  $\eta$ . This is due to the fact that the other stiff term has not been considered implicit.

A second idea to design an AP scheme is to use an exponential integrator ([17, 12]). From (25), one has

$$\partial_t (e^{t/(\eta\tau)} F) = -e^{t/(\eta\tau)} A F + \frac{e^{t/(\eta\tau)}}{\eta} E \cdot \partial_v F^n + \frac{e^{t/(\eta\tau)}}{\tau \eta} \Pi_0 F.$$

Integrating between  $t^n$  and  $t^{n+1}$  leads to

$$\begin{aligned} F^{n+1} &= e^{-\Delta t/(\eta\tau)} F^n - \int_{t^n}^{t^{n+1}} e^{(t-t^{n+1})/(\eta\tau)} A F dt + \int_{t^n}^{t^{n+1}} e^{(t-t^{n+1})/(\eta\tau)} E \cdot \partial_v F dt \\ &\quad + \int_{t^n}^{t^{n+1}} \frac{e^{(t-t^{n+1})/(\eta\tau)}}{\tau \eta} \Pi_0 F dt. \end{aligned}$$

The key point of this strategy is the approximation of the integrals (see [17]). For the two first integrals, we choose the left rectangle quadrature whereas the last integral is computed exactly after considering  $\Pi_0 F$  implicit. This leads to

$$\begin{aligned} F^{n+1} &= e^{-\Delta t/(\eta\tau)} F^n - \Delta t e^{-\Delta t/(\eta\tau)} A F^n \\ &\quad + \Delta t e^{-\Delta t/(\eta\tau)} E^{n,n} \cdot \partial_v F^n + (1 - e^{-\Delta t/(\eta\tau)}) \Pi_0 F^{n+1}. \end{aligned}$$

We can remark that when  $\eta$  goes to zero,  $F^{n+1}$  goes to  $\Pi_0 F^{n+1}$ , which was not the case in the previous approach. One needs now to compute  $\Pi_0 F^{n+1}$ . To do that, we apply  $\Pi_0$  to this last equation to get (recalling  $\Pi_0$  commutes with isotropic functions)

$$\begin{aligned} \Pi_0 F^{n+1} &= e^{-\Delta t/(\eta\tau)} \Pi_0 F^n - \Delta t e^{-\Delta t/(\eta\tau)} \Pi_0 (A F^n) \\ &\quad + \Delta t e^{-\Delta t/(\eta\tau)} E^{n,n} \cdot \Pi_0 (\partial_v F^n) + (1 - e^{-\Delta t/(\eta\tau)}) \Pi_0 F^{n+1}. \end{aligned}$$

from which we get  $\Pi_0 F^{n+1}$

$$\Pi_0 F^{n+1} = \Pi_0 F^n - \Delta t \Pi_0 (A F^n) + \Delta t E^{n,n} \cdot \Pi_0 (\partial_v F^n), \quad (26)$$

where  $E^{n,n}$  is given by (22). Here again, the *macro* equation (26) still contains a stiff term which cannot be considered implicit without requiring the inversion of a nonlocal operator. Let us remark that other choices of time integrals approximation do not lead to a semi-implicit Asymptotic Preserving scheme as well.

A slightly different strategy is then required since standard ones do not work. In the following, two schemes are proposed which enjoy the AP property. We point out that even if these new schemes involve an additional implicit term, namely  $E^{n,n+1}$  instead of  $E^{n,n}$ , they have the same computational cost as an explicit numerical scheme.

**New strategy: 1 (EXPO).**

In the sequel, we present a modification of the previous exponential scheme. The numerical scheme for  $F$  writes

$$F^{n+1} = e^{-\Delta t/(\eta\tau)} F^n - \Delta t e^{-\Delta t/(\eta\tau)} A F^n + \Delta t e^{-\Delta t/(\eta\tau)} E^{n,n+1} \cdot \partial_v F^n + (1 - e^{-\Delta t/(\eta\tau)}) \Pi_0 F^{n+1}. \quad (27)$$

The main problem now is to determine  $\langle (v/\tau^2) F^{n+1} \rangle$  in  $E^{n,n+1}$  given by (23) without requiring the inversion of a linear system. Integrating this latter equation with respect to  $v$ , after multiplying by  $(v/\tau^2)$ , leads to

$$\langle (v/\tau^2) F^{n+1} \rangle = \langle (v/\tau^2) e^{-\Delta t/(\eta\tau)} (F^n - \Delta t e^{-\Delta t/(\eta\tau)} A F^n) \rangle + \Delta t \langle (v/\tau^2) e^{-\Delta t/(\eta\tau)} E^{n,n+1} \cdot \partial_v F^n \rangle, \quad (28)$$

since  $\langle (v/\tau^2) (1 - e^{-\Delta t/(\eta\tau)}) \Pi_0 F^{n+1} \rangle = 0$ . Let us focus on the last term to deduce  $\langle (v/\tau^2) F^{n+1} \rangle$  in an explicit way

$$\begin{aligned} \langle (v/\tau^2) e^{-\Delta t/(\eta\tau)} E^{n,n+1} \cdot \partial_v F^n \rangle &= \frac{1}{\eta} \langle (v/\tau^2) e^{-\Delta t/(\eta\tau)} \mathcal{A}(F^n)^{-1} \langle \frac{v}{\tau^2} F^{n+1} \rangle \cdot \partial_v F^n \rangle + \langle (v/\tau^2) e^{-\Delta t/(\eta\tau)} E_1^n \cdot \partial_v F^n \rangle \\ &= \frac{1}{\eta} \langle \frac{e^{-\Delta t/(\eta\tau)}}{\tau^2} (v \otimes \partial_v F^n) \mathcal{A}(F^n)^{-1} \rangle \langle \frac{v}{\tau^2} F^{n+1} \rangle + \langle (v/\tau^2) e^{-\Delta t/(\eta\tau)} E_1^n \cdot \partial_v F^n \rangle, \end{aligned} \quad (29)$$

where  $E_1^n$  is given by (24). We can inject this term in (28) to get an expression of  $\langle (v/\tau^2) F^{n+1} \rangle$

$$\begin{aligned} \langle (v/\tau^2) F^{n+1} \rangle &= \left( I - \frac{\Delta t}{\eta} \langle \frac{e^{-\Delta t/(\eta\tau)}}{\tau^2} v \otimes \partial_v F^n \rangle \mathcal{A}(F^n)^{-1} \right)^{-1} \\ &\quad \times \left( \langle (v/\tau^2) e^{-\Delta t/(\eta\tau)} (F^n - \Delta t A F^n + \Delta t E_1^n \cdot \partial_v F^n) \rangle \right). \\ &= \eta \left( \eta I - \Delta t \langle \frac{e^{-\Delta t/(\eta\tau)}}{\tau^2} v \otimes \partial_v F^n \rangle \mathcal{A}(F^n)^{-1} \right)^{-1} \\ &\quad \times \left( \langle (v/\tau^2) e^{-\Delta t/(\eta\tau)} (F^n - \Delta t A F^n + \Delta t E_1^n \cdot \partial_v F^n) \rangle \right). \end{aligned} \quad (30)$$

Then, we have determined in an explicit way  $\langle (v/\tau^2) F^{n+1} \rangle$  and we can remark that it is of order  $\eta$ . Then, the term  $E^{n,n+1}$  can be computed with an explicit complexity thanks to

$$E^{n,n+1} = \frac{1}{\eta} \mathcal{A}(F^n)^{-1} \langle \frac{v}{\tau^2} F^{n+1} \rangle + E_1^n, \quad (31)$$

where  $E_1^n$  is given by (24).

The last thing to do is to express  $\Pi_0 F^{n+1}$ . To do that, we apply  $\Pi_0$  to (27) (recalling that  $\Pi_0$  commutes with isotropic functions)

$$\begin{aligned} \Pi_0 F^{n+1} &= e^{-\Delta t/(\eta\tau)} \Pi_0 F^n - \Delta t e^{-\Delta t/(\eta\tau)} \Pi_0 (A F^n) \\ &\quad + \Delta t e^{-\Delta t/(\eta\tau)} E^{n,n+1} \cdot \Pi_0 (\partial_v F^n) + (1 - e^{-\Delta t/(\eta\tau)}) \Pi_0 F^{n+1}, \end{aligned}$$

from which we can compute  $\Pi_0 F^{n+1}$

$$\Pi_0 F^{n+1} = \Pi_0 F^n - \Delta t \Pi_0 (A F^n) + \Delta t E^{n,n+1} \cdot \Pi_0 (\partial_v F^n). \quad (32)$$

This *macro* equation is the same as (26) except that the term  $\langle (v/\tau^2)F^{n+1} \rangle$  in  $E^{n,n+1}$  is now implicit. Thanks to (30), this term is of order  $\eta$  making the last term of (32) non-stiff.

To advance the equation for  $u$ , the stiff term contained in  $E$  also has to be considered implicit. This can be done easily since  $\langle v/\tau^2 F^{n+1} \rangle$  has been computed previously and  $E^{n,n+1}$  is known. Hence, the numerical scheme for the equation on  $u$  writes

$$u^{n+1} = u^n - \Delta t(u^n \cdot \partial_x)u^n + \Delta t E^{n,n+1} - \partial_x \mathcal{P}^n, \quad (33)$$

where  $E^{n,n+1}$  is computed from (23) and  $\mathcal{P}^n$  is computed from

$$-\Delta \mathcal{P}^n = \partial_x^2 : \left[ \int (v + u^n) \otimes (v + u^n) F^n dv \right] + \partial_x \cdot [(\partial_x \cdot u^n)u^n]. \quad (34)$$

From  $(F^n, u^n)$ , the algorithm to compute  $(F^{n+1}, u^{n+1})$  can be summarized as follows

- compute  $\mathcal{P}^n$  with (34),
- compute  $\langle (v/\tau^2)F^{n+1} \rangle$  with (30),
- compute  $E^{n,n+1}$  with (31),
- compute  $\Pi_0 F^{n+1}$  with (32),
- compute  $F^{n+1}$  with (27),
- compute  $u^{n+1}$  with (33).

Let us emphasize that the computational complexity of this algorithm is the same as the complexity of an explicit algorithm. The second step (computation of  $\langle (v/\tau^2)F^{n+1} \rangle$  with (30)) only requires the inversion of a  $d \times d$  matrix. The main properties of the so-obtained scheme (32)-(30)-(27)-(33)-(34)-(31) for (7) are listed in the following proposition.

**Proposition 3.1.** *The numerical scheme (32)-(30)-(33)-(27)-(34)-(31)*

- *is consistent and stable for (7);*
- *degenerates as  $\eta$  goes to zero towards a numerical scheme which is consistent with (16).*

*Proof.* We observe that formally, (27) degenerates when  $\eta$  goes to zero to  $F^{n+1} = \Pi_0 F^{n+1}$ .

Thanks to (30), the term  $E$  is of order one since the stiffness has been stabilized by considering  $E^{n,n+1}$  for its approximation. Hence, applying  $\Pi_0$  to (27) and considering the limit  $\eta$  goes to zero, the term  $E^{n,n+1} \cdot \Pi_0(\partial_v F^n)$  goes to zero since  $F^n$  converges towards  $\Pi_0 F^n$  and  $\Pi_0(\partial_v \Pi_0 F^n) = 0$ . Finally (32) degenerates as  $\eta$  goes to zero to

$$\Pi_0 F^{n+1} = \Pi_0 F^n - \Delta t \Pi_0(A \Pi_0 F^n),$$

with  $A F^n$  given by (14). The same computations as in the continuous case leads to

$$\Pi_0 F^{n+1} = \Pi_0 F^n - \Delta t u^n \cdot \partial_x \Pi_0 F^n,$$

which is an explicit discretization of the asymptotic model for  $G_0 := \Pi_0 F$  (first equation of (16)).

Applying now  $(I - \Pi_0)$  to (27), and since  $F^{n+1} = \Pi_0 F^{n+1} + \mathcal{O}(\eta)$ , one obtains

$$0 = -\Delta t e^{-\Delta t/(\eta\tau)}(I - \Pi_0)A(\Pi_0 F^n) + \Delta t e^{-\Delta t/(\eta\tau)}E^{n,n+1} \cdot (I - \Pi_0)\partial_v(\Pi_0 F^n).$$

Multiplying by  $ve^{-\Delta t/(\eta\tau)}/\Delta t$  and integrating with respect to  $v$  (following the computations of the continuous case) leads to

$$0 = -\langle v(I - \Pi_0)A(\Pi_0 F^n) \rangle + \langle vE^{n,n+1} \cdot \partial_v(\Pi_0 F^n) \rangle,$$

or, after an integration by parts (using  $\langle \Pi_0 F^n \rangle = 1$ )

$$E^{n,n+1} = -\langle v(I - \Pi_0)A(\Pi_0 F^n) \rangle = -\frac{1}{d}\partial_x \langle |v|^2 \Pi_0 F^n \rangle,$$

which is the same expression as the limit equation satisfied by  $E$  (last equation of (16)). The equation for the pressure  $\mathcal{P}$  does not present any difficulty since (34) degenerates into ( $u^n$  converges towards  $\bar{u}^n$  as  $\eta$  goes to zero)

$$-\Delta \mathcal{P}^n = \partial_x^2 : \left[ \int (v + \bar{u}^n) \otimes (v + \bar{u}^n) \Pi_0 F^n dv \right] + \partial_x \cdot [(\partial_x \cdot \bar{u}^n) \bar{u}^n].$$

Finally, the equation (33) for  $u$  becomes, with the previous notations

$$u^{n+1} = u^n - \Delta t(u^n \cdot \partial_x)u^n + \Delta tE^{n,n+1} - \Delta t\partial_x \mathcal{P}^n,$$

which degenerates when  $\eta$  goes to zero into

$$\bar{u}^{n+1} = \bar{u}^n - \Delta t(\bar{u}^n \cdot \partial_x)\bar{u}^n - \frac{\Delta t}{d}\partial_x \langle |v|^2 \Pi_0 F^n \rangle - \Delta t\partial_x \mathcal{P}^n,$$

which is an explicit time discretization of the asymptotic equation on  $\bar{u}$  (second equation of (16)). Finally, we check that this numerical scheme enjoys the Asymptotic Preserving property.  $\square$

### New strategy: 2 (DIMP).

Here, a slight modification of the previous scheme is presented. Instead of using exponential integrators, we use here a simple implicit scheme, including the same treatment as before of the stiff transport term. The numerical scheme for  $F$  writes

$$F^{n+1} = F^n - \Delta t A F^n + \Delta t E^{n,n+1} \cdot \partial_v F^n + \frac{\Delta t}{\eta\tau} (\Pi_0 F^{n+1} - F^{n+1}).$$

where  $A$  is given by (14) and  $E^{n,n+1}$  is given by (23), so that, with  $\delta = 1/(1 + \Delta t/(\tau\eta))$ , one obtains

$$F^{n+1} = \delta \left[ F^n - \Delta t A F^n + \Delta t E^{n,n+1} \cdot \partial_v F^n + \frac{\Delta t}{\eta\tau} \Pi_0 F^{n+1} \right]. \quad (35)$$

As previously, one needs to determine  $\langle (v/\tau^2) F^{n+1} \rangle$ ; this is done as in the previous case by integrating this latter equation against  $(v/\tau^2)$

$$\begin{aligned} \langle (v/\tau^2) F^{n+1} \rangle &= \left( I - \frac{\Delta t}{\eta} \left\langle \frac{\delta}{\tau^2} v \otimes \partial_v F^n \right\rangle \mathcal{A}(F^n)^{-1} \right)^{-1} \\ &\quad \times \left( \langle (\delta v/\tau^2) (F^n - \Delta t A F^n + \Delta t E_1^n \cdot \partial_v F^n) \rangle \right). \end{aligned} \quad (36)$$

Then, the term  $E^{n,n+1}$  can be computed thanks to

$$E^{n,n+1} = \frac{1}{\eta} \mathcal{A}(F^n)^{-1} \left\langle \frac{v}{\tau^2} F^{n+1} \right\rangle + E_1^n, \quad (37)$$

where  $E_1^n$  is given by (24).

The last thing to do is to express  $\Pi_0 F^{n+1}$ . To that, we apply  $\Pi_0$  to (35) and get an expression of  $\Pi_0 F^{n+1}$

$$\Pi_0 F^{n+1} = \Pi_0 F^n - \Delta t \Pi_0 (A F^n) + \Delta t E^{n,n+1} \cdot \Pi_0 \partial_v F^n. \quad (38)$$

The rest of the numerical scheme (computation of  $u^{n+1}$ ,  $E_1^n$  and  $\mathcal{P}^n$ ) is the same as in the previous version. Hence, from  $(F^n, u^n)$ , the algorithm to compute  $(F^{n+1}, u^{n+1})$  can be written as follows

- compute  $\mathcal{P}^n$  with (34),
- compute  $\langle (v/\tau^2) F^{n+1} \rangle$  with (36),
- compute  $E^{n,n+1}$  with (37),
- compute  $\Pi_0 F^{n+1}$  with (38),
- compute  $F^{n+1}$  with (35),
- compute  $u^{n+1}$  with (33).

As a conclusion, we derive the Asymptotic Preserving scheme (38)-(36)-(33)-(35)-(34)-(37) for (7), the properties of which are summarized in the following proposition.

**Proposition 3.2.** *The numerical scheme (38)-(36)-(33)-(35)-(34)-(37)*

- *is consistent and stable for (7);*
- *degenerates as  $\eta$  goes to zero towards a numerical scheme which is consistent with (16).*

*Proof.* The proof is similar to the previous one and is left to the reader.  $\square$

## 4 Spatial discretization

In this section, we focus on the one-dimensional case and we briefly detail the phase space discretization used to approximate the differential operators in (27) or in (35). Let us write the "EXPO" numerical scheme in the one-dimensional configuration (the DIMP version is similar)

$$F^{n+1} = e^{-\Delta t/(\eta\tau)} F^n - \Delta t e^{-\Delta t/(\eta\tau)} A F^n + \Delta t e^{-\Delta t/(\eta\tau)} E^{n,n+1} \partial_v F^n + (1 - e^{-\Delta t/(\eta\tau)}) \Pi_0 F^{n+1}, \quad (39)$$

where  $A F^n$  is given by

$$A F^n = (v + u^n) \partial_x F^n - \partial_x u^n v \partial_v F^n,$$

and  $E^{n,n+1}$  is given by

$$E^{n,n+1} = \frac{1}{\eta} \frac{\langle (v/\tau^2) F^{n+1} \rangle}{\langle (v/\tau) \partial_v F^n \rangle} + E_1^n,$$

with  $E_1^n$  given by

$$E_1^n = \frac{1}{\langle (v/\tau) \partial_v F^n \rangle} [\partial_x \langle (v^2/\tau) F^n \rangle - \partial_x u^n \langle (v^2/\tau) \partial_v F^n \rangle]. \quad (40)$$

If moreover periodic boundary conditions are used, one can show that  $\bar{u}$  is a constant. Indeed, considering the first moment of (1) enables to write an equation for  $\bar{u}$

$$\partial_t \bar{u} + \partial_x \langle v^2 f \rangle + \partial_x \mathcal{P},$$



and using the change of frame, we get

$$\partial_t \bar{u} + \partial_x \langle |v + u|^2 F \rangle + \partial_x \mathcal{P}. \quad (41)$$

The incompressibility condition  $\partial_x \bar{u} = 0$  leads to the equation satisfied by the pressure  $\mathcal{P}$  (applying  $\partial_x$  to (41))

$$-\partial_x^2 \mathcal{P} = \partial_x^2 \langle |v + u|^2 F \rangle.$$

Periodic boundary conditions enable us to write  $-\partial_x \mathcal{P} = \partial_x \langle |v + u|^2 F \rangle$  and we deduce from (41)  $\partial_t \bar{u} = 0$ . In addition to the incompressibility condition  $\partial_x \bar{u} = 0$ , we conclude that  $\bar{u}$  is constant. As a consequence, since  $\langle vF \rangle = \bar{u} - u$ ,  $u$  is then determined directly from this latter relation and the equation (33) is not needed in this simplified one-dimensional framework.

In the rest of this section, we describe the phase space discretization. We introduce a phase space uniform grid of size  $\Delta x > 0$  in the  $x$ -direction and  $\Delta v > 0$  in the  $v$ -direction:  $x_i = x_{\min} + i\Delta x$ ,  $i = 0, \dots, N_x$  and  $v_j = v_{\min} + j\Delta v$ ,  $j = 0, \dots, N_v$ , with  $N_x, N_v \in \mathbb{N}$  and  $x_{\min}, v_{\min} \in \mathbb{R}$ . In the sequel, we denote by  $F_{i,j}^n$  an approximation of  $F(t^n, x_i, v_j)$ .

We consider upwind (forward/backward difference depending on wave-speed) schemes. Typically, to approximate  $AF = (v + u)\partial_x F - (\partial_x u)v\partial_v F$  (see (14)) and  $E_1^n$  given by (40), we use the following approximations

$$\begin{aligned} (AF^n)_{i,j} \approx & (v_j + u_i)_+ (D_-^x F^n)_{i,j} + (v_j + u_i)_- (D_+^x F^n)_{i,j} \\ & + (-v_j (D^x u)_i)_+ (D_-^v F^n)_{i,j} + (-v_j (D^x u)_i)_- (D_+^v F^n)_{i,j}, \end{aligned}$$

and

$$(E_1^n)_{i,j} = \left( \frac{(D^x \langle (v^2/\tau) F^n \rangle)_{i,j} - (D^x u_f^n)_{i,j} \langle (v^2/\tau) (D^v F^n)_{i,j} \rangle}{\langle (v/\tau) (D^v F^n)_{i,j} \rangle} \right),$$

with the standard notation  $a_{\pm} = \frac{a \pm |a|}{2}$ , where  $D_{\pm}^{x,v}$  denotes a first order one-sided difference operator (for example  $(D_+^x F^n)_{i,j} = \frac{F_{i+1,j}^n - F_{i,j}^n}{\Delta x}$ ),  $D^x$  denotes a centered finite difference operator  $(D^x F^n)_{i,j} = \frac{F_{i+1,j}^n - F_{i-1,j}^n}{2\Delta x}$ . All the integrals with respect to  $v$  are approximated by a rectangle quadrature.

More sophisticated spatial discretizations can be used, such as WENO schemes to achieve high orders, but our goal here is to validate the whole strategy.

An important point is to ensure the conservations of  $\int_{\mathbb{R}} F dv$  and  $\int_{\mathbb{R}} (v/\tau) F dv$  at the discrete level. The previous spatial discretizations do not ensure these conservations. To overcome this problem, the scheme should be extended using some correction so that the above conservations are ensured irrespective of the discretization used. Let us mention that the above conservations can be ensured without an *a posteriori* correction by computing the discrete version of  $E$  and  $\mathcal{P}$  by following the same strategy as in the continuous level. However, this is more restrictive and we choose the first approach which needs an *a posteriori* correction which we detail below.

To do that, we propose to *a posteriori* modify the obtained  $F_{i,j}^{n+1}$  so that conservations are ensured. Say we use discretization  $\mathcal{B}$  to the scheme to get  $\tilde{F}^{n+1} = \mathcal{B}F^n$  following the time and spatial discretizations described above. In general, even if  $F^n$  satisfies the conservations

$\sum_j F_{i,j}^n \Delta v = 1$  and  $\sum_j (v_j/\tau_j) F_{i,j}^n \Delta v = 0$ , the obtained  $\tilde{F}^{n+1}$  does not preserve the properties which the corresponding equation in continuous level has. Then, we introduce an orthogonal projector  $\mathbf{P}$  onto  $\text{span}\{(1, (v/\tau)) F^n\}$  in  $L^2(dv/F^n)$ , such that

$$\mathbf{P}(F^{n+1} - F^n) = 0,$$

which means

$$\int_{\mathbb{R}} (F^{n+1} - F^n) \begin{pmatrix} 1 \\ v/\tau \end{pmatrix} dv = 0.$$

We hence modify  $\tilde{F}^{n+1}$  as follows

$$F^{n+1} = \tilde{F}^{n+1} + \mathbf{P}(F^n - \tilde{F}^{n+1}).$$

An example of such a projector can be given as

$$\mathbf{P}(g) = \langle g \rangle F^n + \frac{\langle \frac{v}{\tau} g \rangle \frac{v}{\tau} F^n}{\langle \frac{v^2}{\tau^2} F^n \rangle},$$

from which we easily deduce  $\langle \mathbf{P}(g) \rangle = \langle g \rangle$  and  $\langle \frac{v}{\tau} \mathbf{P}(g) \rangle = 0$ . At the discrete level, the integrals are replaced by the sums on the index  $j$ .

Introducing the defined projector  $\mathbf{P}$  along with the scheme gives the benefit of using any higher order upwind discretization for space and velocity in the scheme. This way of correction does not depend on the scheme used, *i.e.*, the scheme may be implicit or explicit.

## 5 Numerical results

This section is devoted to the validation of the numerical scheme presented above. Different schemes will be compared here in the one-dimensional configuration ( $d = 1$ ). First, the scheme (32)-(30)-(27) will be called "EXPO" and compared with explicit schemes such as Runge-Kutta schemes for (7). Let us remark that the "DIMP" scheme gives results which are very similar to those obtained by "EXPO" so that only results obtained by EXPO will be shown here. For these latter schemes, the time step  $\Delta t$  will be chosen to satisfy  $\Delta t \leq \eta$  which is not the case for "EXPO" scheme.

The numerical tests will be of increasing difficulty: firstly, the homogeneous case with constant and non constant  $\tau$  will be tested and lastly, the non homogeneous case with a constant and non constant  $\tau$  will be also validated.

### 5.1 Homogeneous, constant $\tau$ case

To illustrate the effect of the collision operator, we first consider the simple case ( $\tau = 1$ )

$$\partial_t F = \frac{1}{\eta} (\Pi_0 F - F), v \in \mathbb{R}. \quad (42)$$

The following initial condition is chosen  $F(t = 0, v) = F_0(v)$  such that  $\int v F dv = 0$  and  $\int F dv = 1$

$$F_0(v) = (1 - \alpha) M_1(v) + \alpha M_2(v), \quad \text{with } M_i(v) = \frac{1}{\sqrt{2\pi}} \exp(-|v - u_i|^2/2),$$

where  $u_i, i = 1, 2$  satisfies  $(1-\alpha)u_1 + \alpha u_2 = 0$ . The choice  $u_1 = -4, u_2 = 1$  and  $\alpha = -u_1/(u_2 - u_1)$  is made in the following tests. The following numerical parameters are used:  $N_v = 256, \Delta t = 0.1, v_{\max} = 16$ .

We are interested in entropies  $\langle \Theta(F) \rangle$ , where  $\Theta$  is a convex function. Indeed, for any convex function of  $F$ ,  $\Theta(F)$  decreases with time in (42): multiplying (42) by  $\Theta'(F)$  and integrating with respect to  $v$  leads to

$$\partial_t \langle \Theta(F) \rangle = \frac{1}{\eta} \langle \Theta'(F) (\Pi_0(F) - F) \rangle = \frac{1}{\eta} \langle [\Theta'(F) - \Theta'(\Pi_0 F)] (\Pi_0(F) - F) \rangle \leq 0,$$

since  $\Theta'(\Pi_0 F)$  is isotropic, we have  $\langle \Theta'(\Pi_0 F) (\Pi_0(F) - F) \rangle = 0$ .

We are also interested in the time dependent quantity  $\mathcal{E}(t) = \|F(t) - \Pi_0 F(t)\|_{L^2}$  for which we know an analytical solution. Indeed, since  $\Pi_0 F$  does not depend on time (applying  $\Pi_0$  to (42) directly implies this), we get  $(F - \Pi_0 F)(t, v) = e^{-t/(\eta\tau)}(F - \Pi_0 F)$ , hence  $\mathcal{E}(t) = e^{-t/(\eta\tau)}\mathcal{E}(0)$ .

In Figure 1, the time evolution of  $\mathcal{E}$  is plotted for different values of  $\eta$  and different schemes in time (explicit RK schemes and EXPO which refers to the scheme (32)-(30)-(27)). The curves "REF" corresponds to the analytical solution. We observe, in this simplified configuration, that the "EXPO" scheme captures the solution exactly whereas Runge-Kutta schemes produce an error of  $\Delta t^p$  where  $p$  is the order of the Runge-Kutta scheme used.

Figure 2 shows the time evolution of  $\mathcal{E}$  for different values of  $\eta$ . Here also, the "EXPO" scheme captures the solution exactly, the time step being fixed as 0.1.

Figure 3 displays the time evolution of  $\langle \Theta(F) \rangle$  for two choices of entropies  $\Theta(F) = F \log F$  and  $\Theta(F) = F^4$ , obtained using RK3 and EXPO schemes,  $\eta = 1$  and  $\eta = 0.1$ . We observe that these two quantities decrease with time as expected, the rate being stronger as  $\eta$  is smaller. Here again, the results obtained with RK3 and EXPO are very close.

In Figure 4, we plot the velocity dependence of the distribution function for different times, for  $\eta = 1$ . We observe the relaxation the function towards a symmetric state (which corresponds to isotropic function in 1D), in  $v$ .

## 5.2 Homogeneous, non constant $\tau$ case

We now consider the homogeneous case with a non constant  $\tau$ . Starting from (1) satisfied by  $f$

$$\partial_t f = \frac{1}{\eta\tau \left( \frac{|v-u|^2}{2} \right)} (\Pi_u f - f),$$

with  $u = u(t)$  and together with the constraints  $\langle f \rangle = 1$  and  $\langle (v/\tau)(|v-u|^2/2)f \rangle = 0$ , the change of frame  $v \rightarrow (v-u)$  leads to

$$\partial_t F - \partial_t u \partial_v F = \frac{1}{\eta\tau} (\Pi_0 F - F).$$

The first constraint  $\langle F \rangle = 1$  is automatically preserved by this model. The second constraint  $\langle (v/\tau)F \rangle = 0$  should be reformulated. Taking  $\langle (v/\tau) \rangle$  moment leads to

$$-\partial_t u \langle (v/\tau) \partial_v F \rangle = \frac{1}{\eta} \langle (v/\tau^2) (\Pi_0 F - F) \rangle = -\frac{1}{\eta} \langle (v/\tau^2) F \rangle,$$

which enables to determine  $\partial_t u$

$$\partial_t u = \frac{1}{\eta} \frac{\langle (v/\tau^2) F \rangle}{\langle (v/\tau) \partial_v F \rangle}.$$

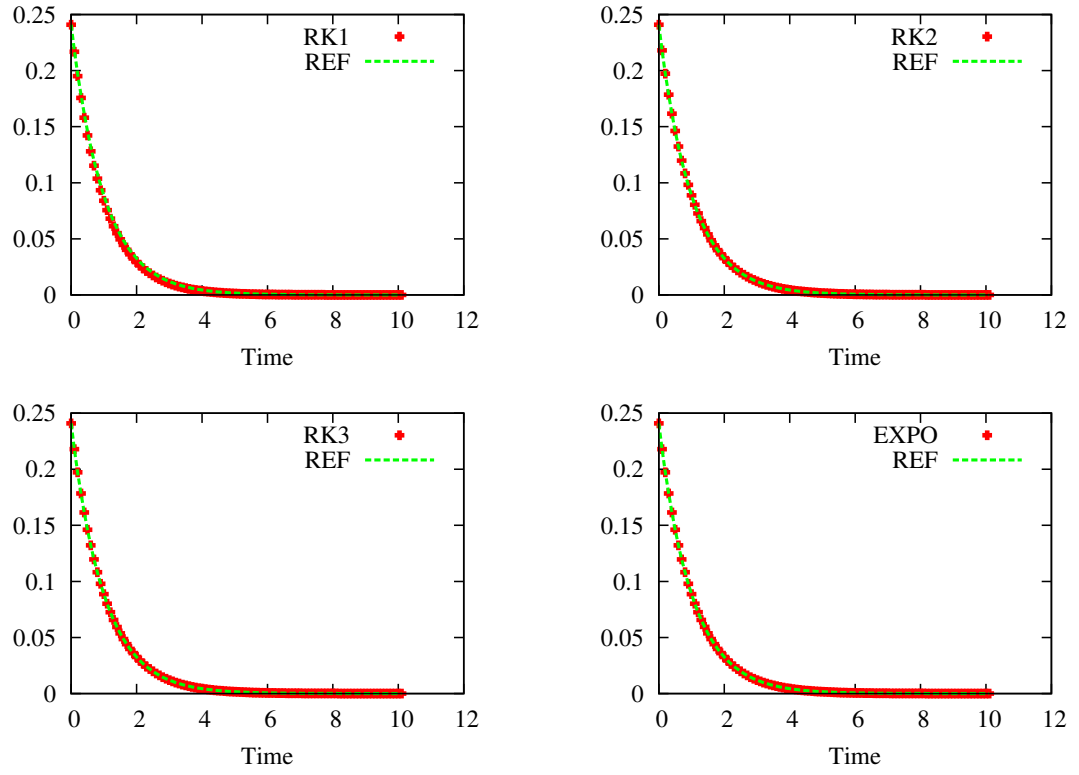


Figure 1:  $\tau = 1$  constant and homogeneous case: time evolution of  $\mathcal{E}(t) = \|(F - \Pi_0 F)(t)\|_{L^2}$ . Comparison of the different schemes for  $\eta = 1$ .

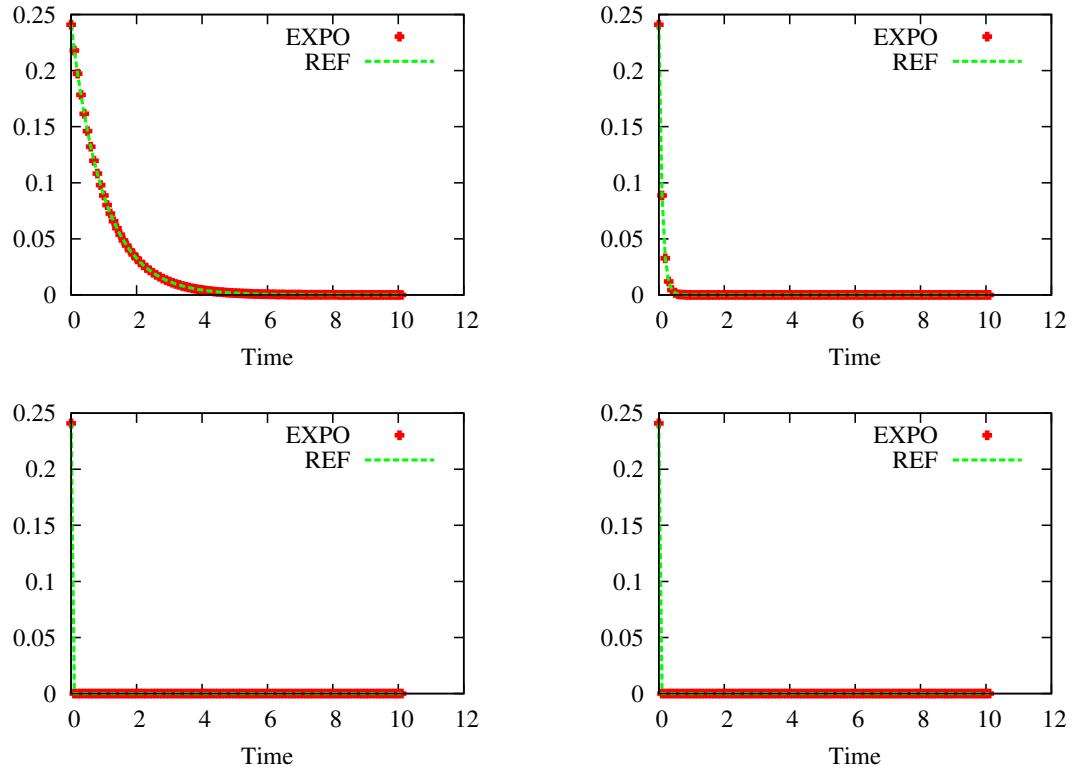


Figure 2:  $\tau = 1$  constant and homogeneous case: time evolution of  $\mathcal{E}(t) = \|(F - \Pi_0 F)(t)\|_{L^2}$ . EXPO scheme. From top-left to bottom-right  $\eta = 1, 10^{-1}, 10^{-2}, 10^{-4}$ .

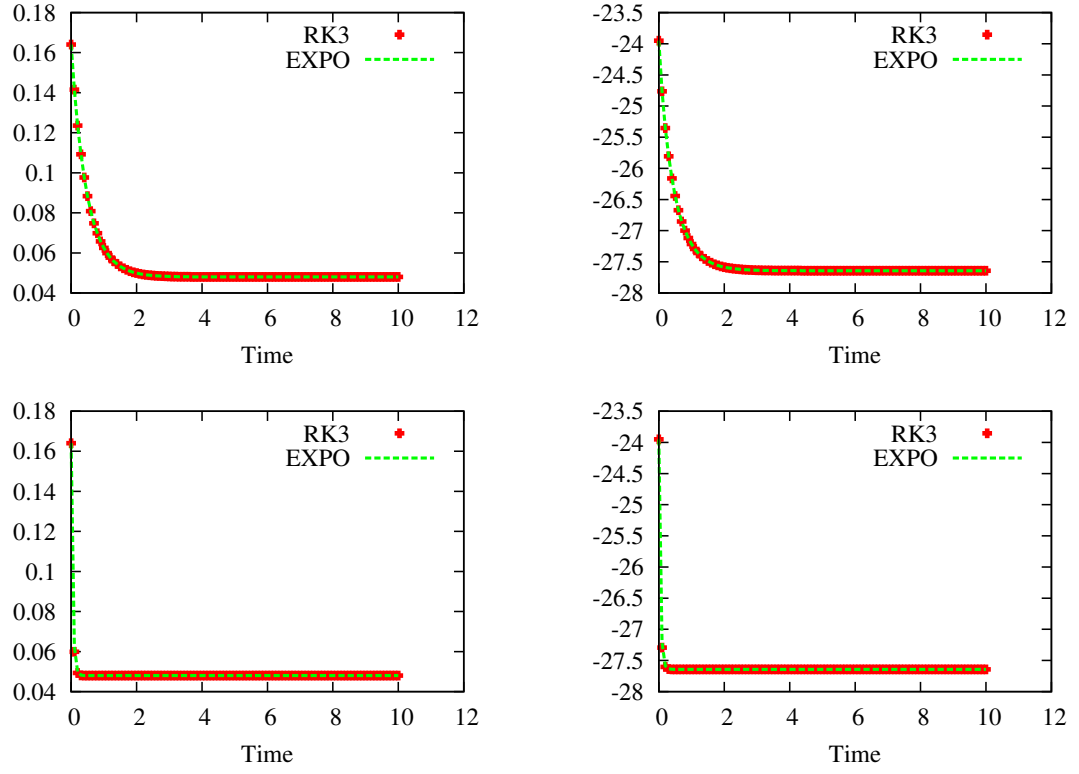


Figure 3:  $\tau = 1$  constant and homogeneous case: time evolution of  $\langle \Theta(F) \rangle$  for two choices of entropies  $\Theta(F) = F^4$  (left) and  $\Theta(F) = F \log(F)$  (right). RK3 and EXPO schemes for  $\eta = 1$  and  $\eta = 0.1$ .

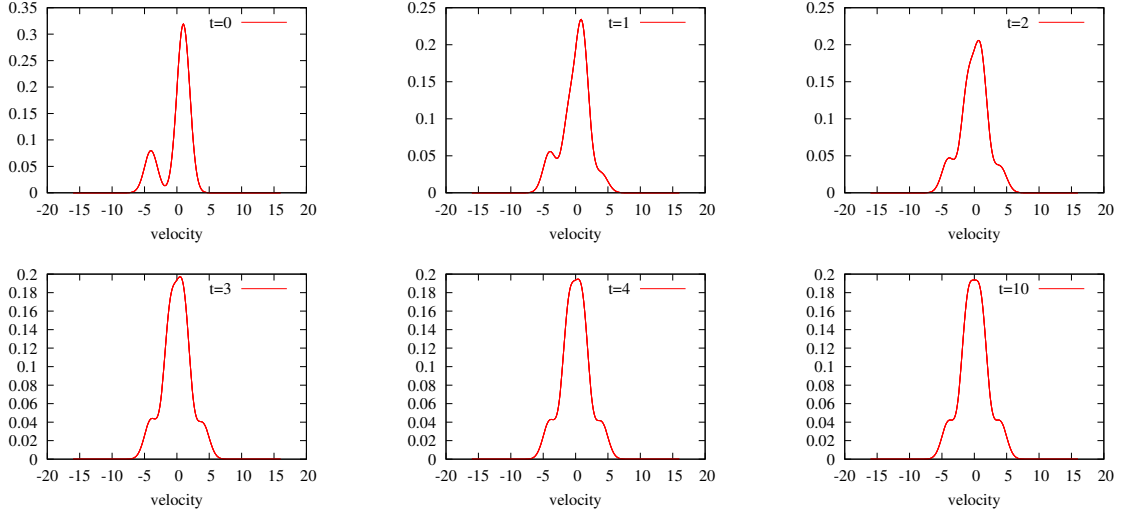


Figure 4:  $\tau = 1$  constant and homogeneous case: distribution function  $F(t, v)$  obtained by EXPO scheme for  $t = 0, 1, 2, 3, 4, 10$ .  $\eta = 1$ .

The model to solve in this context is then

$$\partial_t F - \frac{1}{\eta} \frac{\langle (v/\tau^2) F \rangle}{\langle (v/\tau) \partial_v F \rangle} \partial_v F = \frac{1}{\eta \tau} (\Pi_0 F - F), v \in \mathbb{R}, \quad (43)$$

originating from (1) in the homogeneous case. We detail the numerical scheme in this simplified configuration in Appendix 7.3.

The following initial condition is chosen  $F(t=0, v) = F_0(v)$  such that  $\int_{\mathbb{R}} (v/\tau) F dv = 0$  and  $\int_{\mathbb{R}} F dv = 1$

$$F_0(v) = (\alpha M_1(v) + (1 - \alpha) M_2(v))(1 + |v|),$$

with  $M_i(v) = \frac{1}{\sqrt{2\pi}} \exp(-|v - u_i|^2/2)$ ,  $u_i, i = 1, 2$  satisfies  $u_1 = 1$ ,  $u_2 = -2$  and  $\alpha = 7/9$  (this value of  $\alpha$  is chosen such that  $\int_{\mathbb{R}} (v/\tau) F_0(v) dv = 0$ ). The  $\tau$  function is  $\tau(\xi) = (1 + \sqrt{\xi})/\xi$  which is physically relevant.

The following numerical parameters are used:  $N_v = 256$ ,  $\Delta t = 0.1$ ,  $v_{\max} = 16$ .

In Figure 5, the time history of  $\mathcal{E}(t) = \|F(t) - \Pi_0 F(t)\|_{L^2}$  is plotted for different numerical schemes (explicit RK schemes and EXPO) for  $\eta = 1$ . The same conclusions as before follow since EXPO and RK's are nearly superimposed. Note that the rate of relaxation is quite different from the previous case. Figure 6 displays the distribution function  $F(t, v)$  for different times  $t = 0, 1, 2, 5, 100, 1000$  using EXPO scheme for  $\eta = 1$ . This illustrates the relaxation towards the isotropic state. The rate of isotropisation is not the same for all the velocities since we are dealing with a non constant relaxation time  $\tau$  (see also [19]). In Figure 7,  $\mathcal{E}$  is plotted for different values of  $\eta$  using the EXPO scheme and RK3. For this latter scheme, the time step is fixed as  $\Delta t = 0.01$  for stability, whereas  $\Delta t = 0.1$  for EXPO. We observe for  $\eta = 0.5$  that EXPO is very close to RK3, even if the time step is 10 times smaller for RK3. For  $\eta = 0.1$ , the two curves are not so close (Figure 8, right) and we also display the result obtained by EXPO with  $\Delta t = 0.01$ ; we then observe that the two curves are nearly the same, which confirms the convergence in time of EXPO. In Figure 8, results obtained with EXPO ( $\Delta t = 0.1$ ) are shown, where the value of  $\eta$

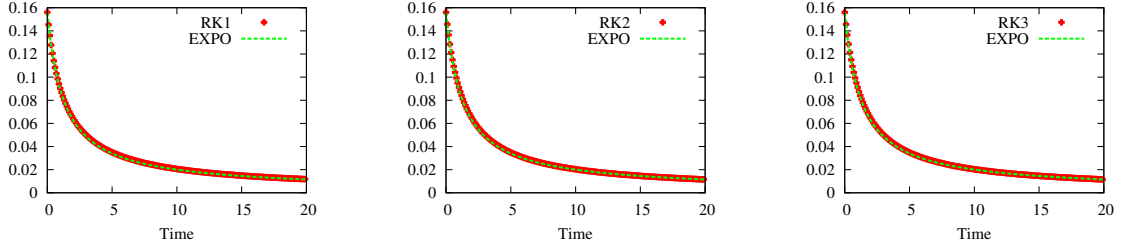


Figure 5:  $\tau$  non constant and homogeneous case: time evolution of  $\mathcal{E}$ . Comparison of the different schemes for  $\eta = 1$ .

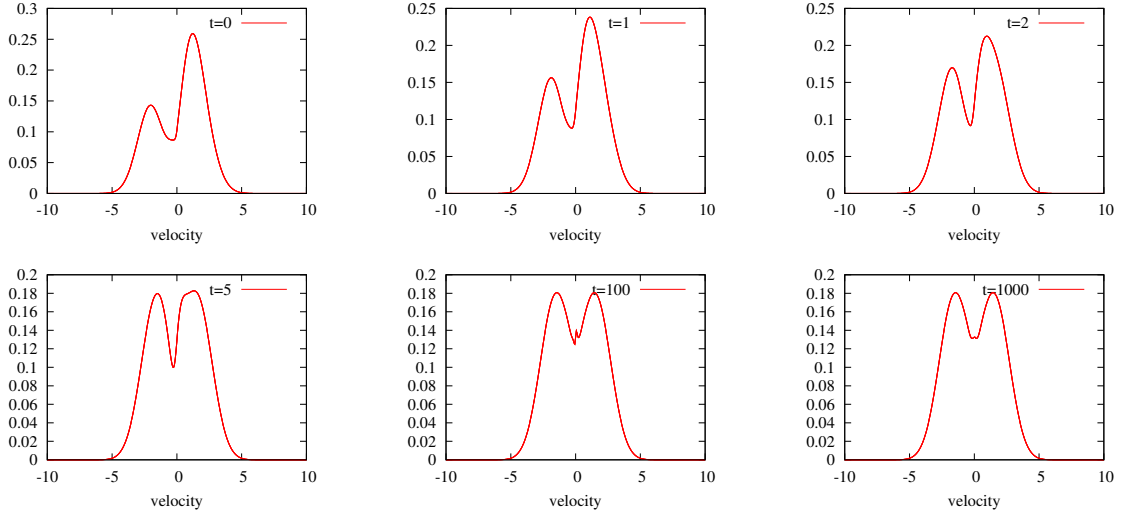


Figure 6:  $\tau$  non constant and homogeneous case: distribution function  $F(t, v)$  for  $t = 0, 1, 2, 5, 100, 1000$ .

is chosen equal to  $10^{-2}$  and  $10^{-4}$ ; we observe that at the first iteration, the solution is very close to its projection (since  $\mathcal{E}$  is nearly zero), even if the initial condition is not at equilibrium. This emphasizes the AP character of our numerical scheme.

### 5.3 Non homogeneous, constant $\tau$ case

In this subsection, we look at the non homogeneous case for constant  $\tau$ . We consider the following initial condition

$$F_0(x, v) = (1 - \alpha(x))M_1(x, v) + \alpha(x)M_2(x, v), x \in [0, 4\pi], v \in \mathbb{R},$$

with  $M_i(x, v) = \frac{1}{\sqrt{2\pi}} \exp(-|v - u_i(x)|^2/2)$ . We choose  $u_2(x) = 2, u_1(x) = \alpha(x)u_2(x)/(\alpha(x) - 1)$  and  $\alpha(x) = (1 + 0.3 \cos(x/2))/2$ .

The numerical parameters are the following:  $\Delta t = 0.2\Delta x/v_{\max}$ ,  $N_x = 64$ ,  $N_v = 256$  and  $v_{\max} = 16$ .



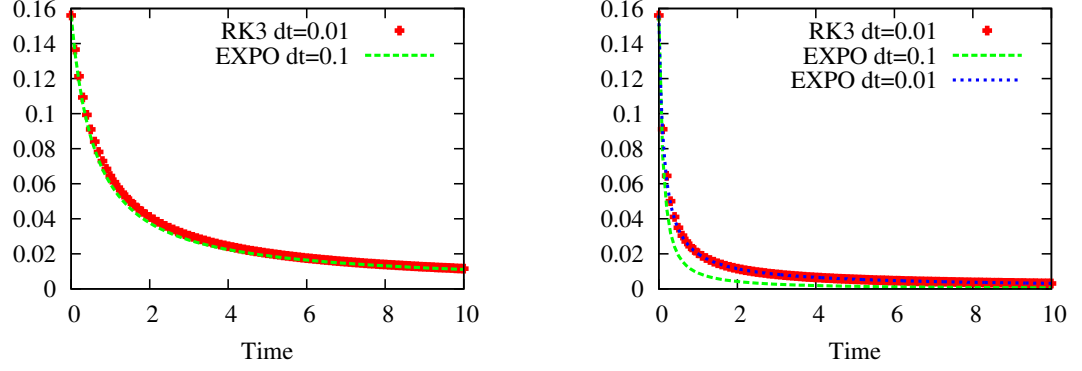


Figure 7:  $\tau$  non constant and homogeneous case: time evolution of  $\mathcal{E}(t) = \|(F - \Pi_0 F)(t)\|_{L^2}$ . RK3 (with  $\Delta t = 0.01$ ) and EXPO schemes for  $\eta = 0.5$  (left) and  $\eta = 0.1$  (right).

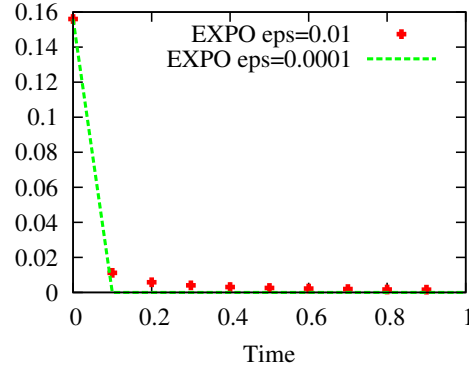


Figure 8:  $\tau$  non constant and homogeneous case: time evolution of  $\mathcal{E}$ . EXPO scheme for  $\eta = 10^{-2}$  and  $\eta = 10^{-4}$ .

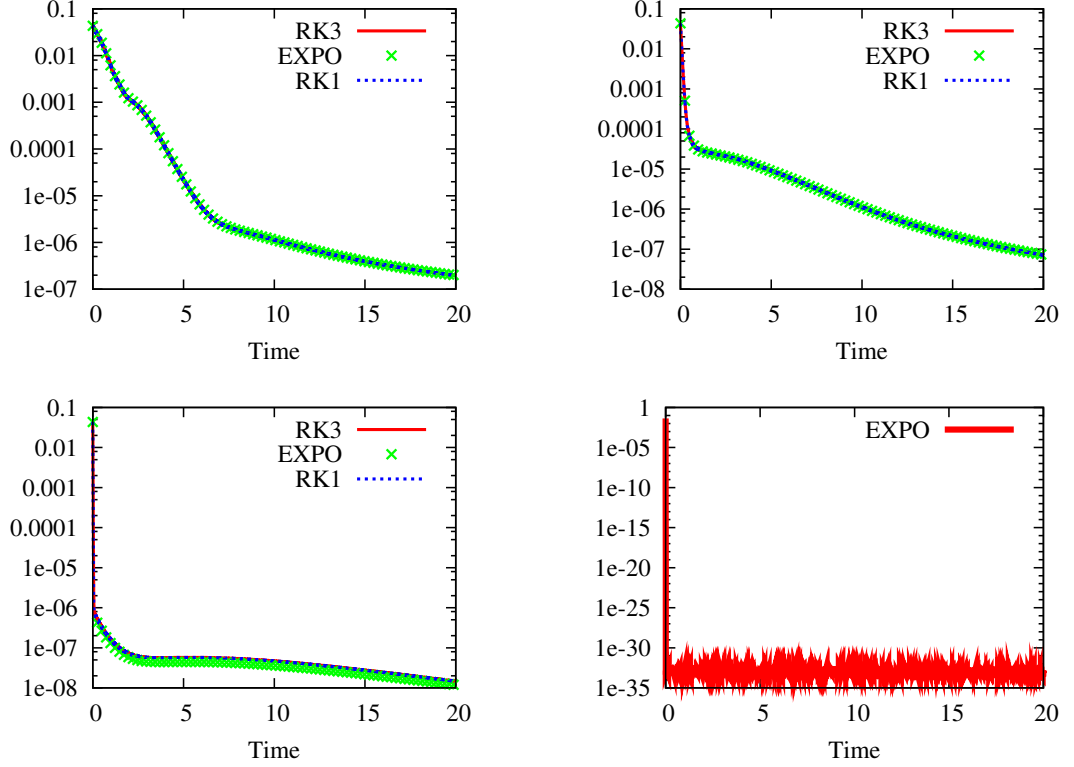


Figure 9:  $\tau$  constant and non homogeneous case: Time evolution of  $\mathcal{E}(t) = \|(F - \Pi_0 F)(t)\|_{L^2}$ . From top-left to bottom-right  $\eta = 1, 10^{-1}, 10^{-2}, 10^{-5}$ .

In Figure 9, the time history of  $\mathcal{E}(t) = \|(F - \Pi_0 F)(t)\|_{L^2}$  (the  $L^2$  norm is considered in  $x$  and  $v$ ) is shown, for different values of  $\eta$ . The new scheme "EXPO" is compared to RK3 and RK1. The new scheme turns out to be competitive in the kinetic regime since the results are very close to those obtained with RK3 or RK1. When  $\eta$  is too small ( $\eta = 10^{-5}$ ), the explicit scheme RK3 requires a very small time step so that only results obtained with EXPO scheme is shown. A very strong relaxation can be observed, confirming the fact that the solution is projected onto the set of isotropic functions at the first iteration.

#### 5.4 Non homogeneous, non constant $\tau$ case

In this subsection, we look at the non homogeneous case for a non constant  $\tau$ . We consider the following initial condition

$$F_0(x, v) = (\alpha(x)M_1(x, v) + (1 - \alpha(x))M_2(x, v))(1 + |v|), x \in [0, 4\pi], v \in \mathbb{R},$$

with  $M_i(x, v) = \frac{1}{\sqrt{2\pi}} \exp(-|v - u_i(x)|^2/2)$ . We choose  $u_2(x) = -2, u_1(x) = (1 + 0.3 \cos(x/2))/2$ , and  $\alpha(x) = (3u_2(x) + u_2(x)^3)/(3u_2(x) + u_2(x)^3 - 3u_1(x) - u_1(x)^3)$ .

The numerical parameters are the following:  $\Delta t = 0.2\Delta x/v_{\max}$ ,  $N_x = 512$ ,  $N_v = 512$  and  $v_{\max} = 16$ .

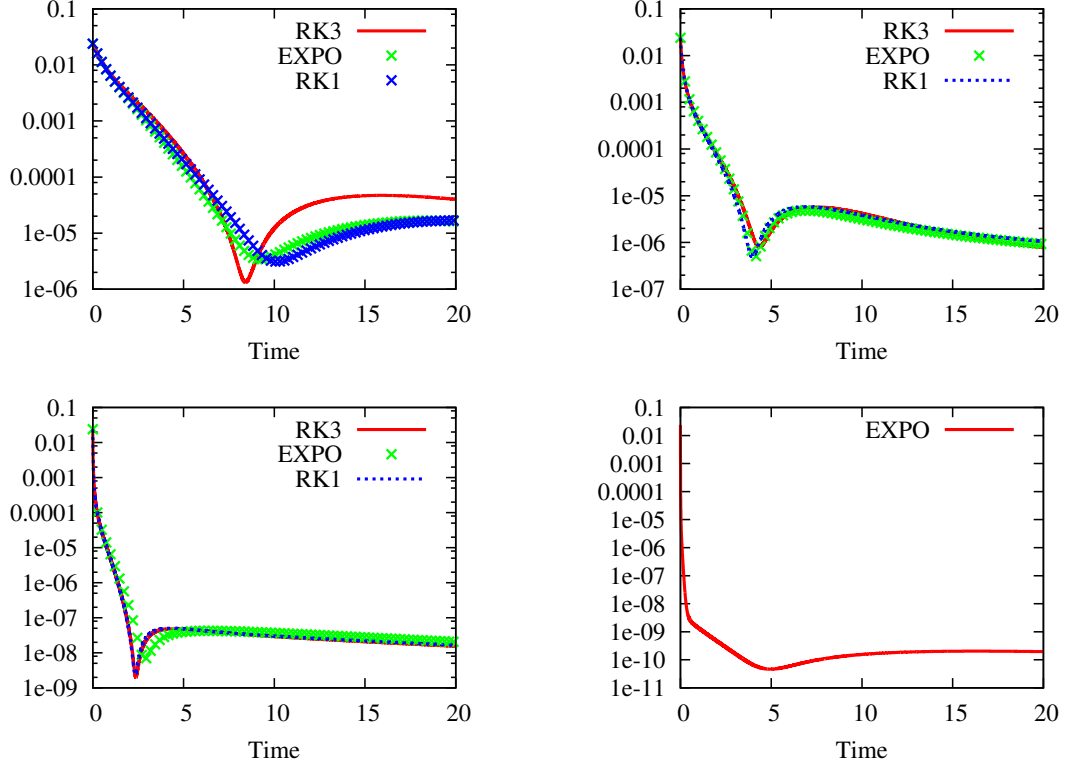


Figure 10:  $\tau$  non constant and non homogeneous case: time evolution of  $\mathcal{E}(t) = \|(F - \Pi_0 F)(t)\|_{L^2}$ . From top-left to bottom-right  $\eta = 1, 10^{-1}, 10^{-2}, 10^{-3}$ .

In Figure 10, the time history of  $\mathcal{E}(t) = \|(F - \Pi_0 F)(t)\|_{L^2}$  (the  $L^2$  norm is considered in  $x$  and  $v$ ), for different values of  $\eta$  and different numerical schemes (Runge-Kutta 1 and 3, EXPO) is shown. For  $\eta = 10^{-2}$ , comparison with explicit RK method needs  $\Delta t = 0.05 \Delta x / v_{\max}$  whereas the "EXPO" method still uses  $\Delta t = 0.2 \Delta x / v_{\max}$ . We observe that the three methods have a similar behavior which validates the EXPO method for  $\eta = 1, 10^{-1}, 10^{-2}$ . When  $\eta$  is smaller, explicit methods are too costly so that only results obtained by the EXPO method are presented. In particular, in Figure 11 when  $\eta = 10^{-4}$ , one can see that the initial condition is immediately projected onto isotropic functions.

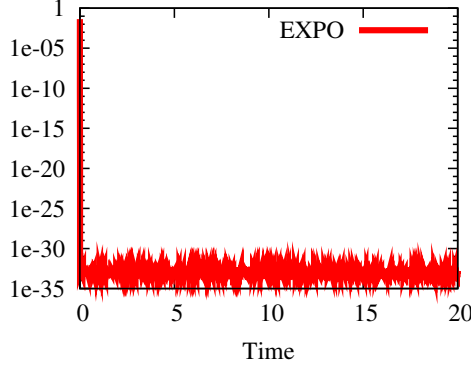


Figure 11:  $\tau$  non constant and non homogeneous case: time evolution of  $\mathcal{E}(t) = \|(F - \Pi_0 F)(t)\|_{L^2}$ .  $\eta = 10^{-4}$ .

## 6 Conclusion

In this work, a new scheme is proposed for the numerical simulation of a kinetic theory based model of turbulence. The derivation of this scheme needed two important tools: *(i)* the change of frame enables to avoid the delicate issue of the computation of  $u$ ; *(ii)* the improvement of standard AP schemes since the change of frame makes an additional stiff transport term appear. Even if this scheme is implicit to avoid the severe constraint coming from the stiff relaxation source term and the additional stiff transport term, its numerical cost is the same as an explicit scheme; moreover, it enjoys the Asymptotic Preserving property, *i.e.*, it degenerates when the stiff parameter goes to zero, to a numerical scheme consistent with the asymptotic model.

The current work presents first results obtained with a prototype one-dimensional model. More general configurations can be tackled in future, in order to study turbulence from kinetic theory in a deeper way. In addition, the spatial discretization can be improved to capture physical turbulence phenomena; as mentioned above, high order WENO schemes can be quite easily adapted to our time discretization strategy.

## 7 Appendix

### 7.1 Computation of pressure

The equation for the computation of pressure  $\mathcal{P}$  is presented here in detail. Starting from (10), we can obtain the following equality

$$\int v \otimes (v+u) \partial_x F dv - \int v (E + (\partial_x u)^T v) \cdot \partial_v F dv + [(u \cdot \partial_x) u] = \partial_x \cdot \left[ \int (v+u) \otimes (v+u) F dv \right] + E - [(\partial_x \cdot u) u]. \quad (44)$$

Using the relations

$$\begin{aligned} - \int v (E + (\partial_x u)^T v) \cdot \partial_v F dv &= E + (\partial_x u)^T \int v F dv + \partial_x \cdot u \int v F dv \\ \int v \otimes (v+u) \partial_x F dv &= \partial_x \cdot \int v \otimes (v+u) F dv - \partial_x \cdot u \int v F dv \end{aligned}$$

we can obtain

$$\begin{aligned}
& \int v \otimes (v + u) \partial_x F dv - \int v(E + (\partial_x u)^T v) \cdot \partial_v F dv + [(u \cdot \partial_x)u] = \\
& \partial_x \cdot \int v \otimes (v + u) F dv - \partial_x \cdot u \int v F dv + E + (\partial_x u)^T \int v F dv + \partial_x \cdot u \int v F dv + [(u \cdot \partial_x)u] = \\
& \partial_x \cdot \int (v + u) \otimes (v + u) F dv - \partial_x \cdot \int u \otimes (v + u) F dv + E + (\partial_x u)^T \int v F dv + [(u \cdot \partial_x)u] = \\
& \partial_x \cdot \int (v + u) \otimes (v + u) F dv - \partial_x \cdot (u \otimes u) - \partial_x \cdot \int u \otimes v F dv + E + (\partial_x u)^T \int v F dv + [(u \cdot \partial_x)u] = \\
& \partial_x \cdot \int (v + u) \otimes (v + u) F dv - \partial_x \cdot (u \otimes u) + E + [(u \cdot \partial_x)u] = \\
& \partial_x \cdot \int (v + u) \otimes (v + u) F dv - u(\partial_x \cdot u) + E.
\end{aligned}$$

Hence, from (10), we deduce by taking the divergence of (44) the equation for  $\mathcal{P}$

$$-\Delta \mathcal{P} = \partial_x^2 : \left[ \int (v + u) \otimes (v + u) F dv \right] - \partial_x \cdot [( \partial_x \cdot u) u].$$

## 7.2 Computation of the matrix involved in the term $E$

Let us consider the matrix  $\int \left( \frac{v}{\tau} \otimes \partial_v F \right) dv$

$$\begin{aligned}
\int \left( \frac{v}{\tau} \otimes \partial_v F \right) dv &= - \int \frac{1}{\tau} \left( I - v \otimes v \frac{\tau'}{\tau} \right) F dv \\
&= - \int \frac{F}{\tau} dv + \int \frac{v \otimes v}{\tau} \tau' F dv \\
&=: -S.
\end{aligned}$$

For an arbitrary vector  $\xi \in \mathbb{R}^d$ , let us consider the quantity  $(S\xi) \cdot \xi$

$$(S\xi) \cdot \xi = \int \frac{F}{\tau} dv |\xi|^2 - \int (v \cdot \xi)^2 \frac{\tau'}{\tau} F dv = \int \frac{F}{\tau} [|\xi|^2 - (v \cdot \xi)^2 \tau'] dv.$$

For a decreasing function  $\tau$  ( $\tau$  has a power law shape  $\tau(\xi) = \xi^\alpha$ ,  $-1 \leq \alpha \leq -1/2$ ), this quantity is positive, which ensures the invertibility of the matrix  $\int \left( \frac{v}{\tau} \otimes \partial_v F \right) dv$ .

## 7.3 Numerical scheme in the one-dimensional homogeneous case

Let us detail the steps of the algorithm in the simplified one-dimensional and homogeneous framework

$$\partial_t F - \frac{1}{\eta} \frac{\langle (v/\tau^2) F \rangle}{\langle (v/\tau) \partial_v F \rangle} \partial_v F = \frac{1}{\eta \tau} (\Pi_0 F - F), v \in \mathbb{R}.$$

First, we rewrite this equation

$$\partial_t (e^{t/(\eta\tau)} F) = \frac{e^{t/(\eta\tau)}}{\eta} \frac{\langle (v/\tau^2) F \rangle}{\langle (v/\tau) \partial_v F \rangle} \partial_v F + \frac{e^{t/(\eta\tau)}}{\eta \tau} \Pi_0 F,$$

so that the equation to compute  $F^{n+1}$  writes (following (27))

$$F^{n+1} = e^{-\Delta t/(\eta\tau)} F^n + \frac{\Delta t}{\eta} e^{-\Delta t/(\eta\tau)} \frac{\langle (v/\tau^2) F^{n+1} \rangle}{\langle (v/\tau) \partial_v F^n \rangle} \partial_v F^n + (1 - e^{-\Delta t/(\eta\tau)}) \Pi_0 F^{n+1}.$$

To compute  $\langle (v/\tau^2)F^{n+1} \rangle$ , we use the following relation (see (30)), obtained by multiplying the last equation by  $(v/\tau^2)$  and integrating with respect to  $v$

$$\begin{aligned}\langle (v/\tau^2)F^{n+1} \rangle &= \left[ 1 - \frac{\Delta t}{\eta} \frac{\langle (v/\tau^2)e^{-\Delta t/(\eta\tau)} \partial_v F^n \rangle}{\langle (v/\tau) \partial_v F^n \rangle} \right]^{-1} \langle (v/\tau^2)e^{-\Delta t/(\eta\tau)} F^n \rangle \\ &= \left[ \frac{\eta \langle (v/\tau) \partial_v F^n \rangle}{\eta \langle (v/\tau) \partial_v F^n \rangle - \Delta t \langle (v/\tau^2)e^{-\Delta t/(\eta\tau)} \partial_v F^n \rangle} \right] \langle (v/\tau^2)e^{-\Delta t/(\eta\tau)} F^n \rangle.\end{aligned}$$

The following equality enables us to compute  $\Pi_0 F^{n+1}$  (following (32))

$$\Pi_0 F^{n+1} = \Pi_0 F^n + \frac{\Delta t}{\eta} \frac{\langle (v/\tau^2)F^{n+1} \rangle}{\langle (v/\tau) \partial_v F^n \rangle} \Pi_0(\partial_v F^n).$$

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## References

- [1] Bennoune, M., Lemou, M. and Mieussens, L., *Uniformly stable numerical schemes for the Boltzmann equation preserving compressible Navier-Stokes asymptotics*, J. Comput. Phys., **227** (2008), pp. 3781-3803.
- [2] Bouchut, F. and Perthame, B., *The BGK model for small Prandtl number in the Navier-Stokes approximation*, J. Stat. Phys., **71** (1993), pp. 191-206.
- [3] Chen, H., Kandasamy, S., Orzag, S., Shock, R., Succi, S. and Yakhot, V., *Extended Boltzmann Kinetic Equation For Turbulent Flows*, J. Fluid Mech., **519** (2001), pp. 301-314.
- [4] Coron F. and Perthame, B., *Numerical passage from kinetic to fluid equations*, SIAM J. Numer. Anal., **28** (1991), pp. 26-42.
- [5] Crouseilles, N. and Lemou, M., *An asymptotic preserving scheme based on a micro-macro decomposition for collisional Vlasov equations: diffusion and high-field scaling limits*, Kinetic Related Models, **4** (2011), pp. 441-477.
- [6] Degond, P. and Lemou, M., *Turbulence Models for Incompressible Fluids Derived from Kinetic Theory*, J. Math. Fluid Mech., **4** (2002), pp. 257-284.
- [7] Degond, P. and Lemou, M., *On viscosity and thermal conduction of fluids with multivalued internal energy*, Eur. J. Mech. B-Fluids, **20** (2001), pp. 303-327.
- [8] Degond, P. and Peyrard, P. F., *Un modèle de collisions ondes-particules en physique des plasmas : application la dynamique des gaz*, C. R. Acad. Sci. Paris, Ser I, **323** (1996), pp. 209-214.
- [9] Degond, P., Lopez, J. L. and Peyrard, P. F., *On the macroscopic dynamics induced by a model wave-particle collision operator*, Continuum Mechanics and Thermodynamics, **10** (1998), pp. 153-178.

- [10] Degond, P., Lopez, J. L., Poupaud, F., Schmeiser, C., *Existence of solutions of a kinetic equation modeling cometary flows*, J. Stat. Phys., **96** (1999), pp. 361-376.
- [11] Dimarco, G. and Pareschi, L., *Asymptotic Preserving Implicit-Explicit Runge-Kutta Methods for Non-Linear Kinetic Equations*, SIAM Journal of Numerical Analysis, **51** (2013), pp. 1064-1087.
- [12] Dimarco, G. and Pareschi, L., *Exponential Runge-Kutta methods for stiff kinetic equations*. SIAM Journal of Numerical Analysis, **49** (2011), pp. 2057-2077.
- [13] Earl, J., Jokipii, J. R. and Morfill, G., *Cosmic ray viscosity*, Astrophysical Journal, **331** (1988).
- [14] Filbet F. and Jin, S., *A class of asymptotic-preserving schemes for kinetic equations and related problems with stiff sources*, J. of Comput. Phys., **229** (2010), pp. 7625-7648.
- [15] Jin, S., *Efficient asymptotic-preserving (AP) schemes for some multiscale kinetic equations*, SIAM Journal on Scientific Computing, **21** (1999), pp. 441-454.
- [16] Klar, A. and Schmeiser, C., *Numerical passage from radiative heat transfer to nonlinear diffusion models*, Math. Models Methods Appl. Sci., **11** (2001), pp. 749-767.
- [17] Lemou, M., *Relaxed micro-macro schemes for kinetic equations*, C.R. Acad. Sci. Paris, Ser. I, **348** (2010), pp. 455-460.
- [18] Lemou, M. and Mieussens, L., *A new asymptotic preserving scheme based on micro-macro formulation for linear kinetic equations in the diffusion limit*, SIAM J. Sci. Comput., **31** (2008), pp. 334-368.
- [19] Mieussens, L. and Struchtrup, H., *Numerical comparison of BGK-models with proper Prandtl number*, Phys. Fluids, **16** (2004), pp. 2797-2813.
- [20] Pieraccini, S. and Puppo, G., *Implicit-explicit schemes for BGK kinetic equations*, Journal of Scientific Computing, **32** (2007), pp. 1-28.
- [21] Pope, S. B., *Turbulent Flows*, Cambridge University Press, 2000.
- [22] Struchtrup, H., *The BGK-model with velocity-dependent collision time*, Cont. Mech. Thermodyn., **9** (1997), pp. 23-32.
- [23] Wilcox, D. C., *Turbulence Modeling for CFD*, D.C.W. Industries Inc., California, 1994.