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Diameter of Minimal Separators in Graphs*

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Abstract: We establish general relationships between the topological properties of graphs and their metric properties. For this purpose, we upper-bound the diameter of the *minimal separators* in any graph by a function of their sizes. More precisely, we prove that, in any graph G, the diameter of any minimal separator S in G is at most $\lfloor\frac{\ell(G)}{2}\rfloor \cdot (|S|-1)$ where $\ell(G)$ is the maximum length of an isometric cycle in G. We refine this bound in the case of graphs admitting a *distance preserving ordering* for which we prove that any minimal separator S has diameter at most $2(|S|-1)$. Our proofs are mainly based on the property that the minimal separators in a graph G are connected in some power of G.

Our result easily implies that the *treelength* $tl(G)$ of any graph G is at most $\lfloor\frac{\ell(G)}{2}\rfloor$ times its *treewidth* $tw(G)$. In addition, we prove that, for any graph G that excludes an *apex graph* H as a minor, $tw(G) \leq c_H \cdot tl(G)$ for some constant c_H only depending on H. We refine this constant when G has bounded genus. As a consequence, we obtain a very simple $O(\ell(G))$-approximation algorithm for computing the treewidth of n-node m-edge graphs that exclude an apex graph as a minor in $O(nm)$-time.

Key-words: Graph; Treewidth; Treelength; Hyperbolicity; Genus;

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Diamètre des séparateurs minimaux dans les graphes

Résumé : Nous établissons des relations générales entre les propriétés topologiques des graphes et leurs propriétés métriques. Dans ce but, nous bornons supérieurement le diamètre des *séparateurs minimaux* dans un graphe par une fonction de leur taille. Plus précisément, nous prouvons que, dans n'importe quel graphe G, le diamètre de tout séparateur minimal S dans G est au plus $\lfloor \frac{\ell(G)}{2} \rfloor \cdot (|S|-1)$ avec $\ell(G)$ la plus grande taille d'un cycle isométrique dans G. Nous améliorons cette borne dans le cas des graphes pour lesquels il existe un *ordre d'élimination isométrique*: nous prouvons que tout séparareur minimal S dans un tel graphe a pour diamètre au plus $2(|S|-1)$. Nos preuves sont principalement basées sur le fait que les séparateurs minimaux dans un graphe G sont connexes dans l'une de ses puissances.

Une conséquence facile de nos résultats est que pour tout graphe G, la *treelength* $tl(G)$ est au plus $\lfloor \frac{\ell(G)}{2} \rfloor$ fois sa *treewidth* $tw(G)$. En complément de cette relation, nous prouvons que, pour tout graphe G qui exclut un *apex graph* H comme mineur, $tw(G) \leq c_H \cdot tl(G)$ avec c_H une constante qui ne dépend que de H. Nous améliorons cette constante dans le cas où G est de genre borné. En conséquence de quoi, nous obtenons un algorithme très simple avec facteur d'approximation $O(\ell(G))$ pour calculer la treewidth des graphes qui excluent un apex graph comme mineur en temps $O(nm)$.

Mots-clés : Graphe; Treewidth; Treelength; Hyperbolicité; Genre;

1 Introduction

It turns out that for a vast range of graph problems, the borderline between tractable and intractable cases depends on the *tree-like properties* of the graphs. Classical parameters such as treewidth [RS86] aim to measure how close is the *structure* of a graph from the structure of a tree, and there are NP-hard problems that can be solved in polynomial-time on bounded-treewidth graphs. However in practice, many real-life graphs, such as the graphs of the Autonomous Systems of the Internet, have a large treewidth [dMSV11] *i.e.*, they are structurally far from a tree. This fact has motivated an alternative approach namely, studying the *metric* tree-likeness of a graph instead of the topological tree-likeness. A classical graph parameter that measures how close is a graph metric from a tree metric is the *hyperbolicity* [Gro87]. In this paper, we combine both approaches in order to establish new relationships between the topological properties of graphs and their metric properties.

A *tree-decomposition* of a graph [RS86] is a way to represent it by a family of subsets of its vertex-set organized in a tree-like manner and satisfying some connectivity properties. More formally, a tree decomposition of a graph $G = (V, E)$ is a pair $(T, \mathcal{X})$ where $\mathcal{X} = \{X_t \mid t \in V(T)\}$ is a family of subsets of V, called *bags*, and T is a tree, such that:

1. $\bigcup_{t \in V(T)} X_t = V$;
2. for any edge $uv \in E$, there is $t \in V(T)$ such that X_t contains both u and v;
3. for any vertex $v \in V$, the set $\{t \in V(T) \mid v \in X_t\}$ induces a subtree of T.

Classically, it is interesting to minimize the size of the bags. The *width* of $(T, \mathcal{X})$ equals to $\max_{t \in V(T)} |X_t| - 1$. The *treewidth* of G, denoted by $tw(G)$, is the minimum width over all tree-decompositions of G. Dourisboure and Gavoille introduced a metric aspect in tree-decompositions by studying the diameter of the bags [DG07]. The *diameter of a bag X* of a tree-decomposition is the maximum distance in G between any pair of vertices in X. The *length* of a tree-decomposition is the maximum diameter of its bags, and the *treelength* of G, denoted by $tl(G)$, is the minimum length over all tree-decomposition of G.

Tree-decompositions play an important role in the Graph Minor Theory [RS85]. They have mainly been studied for their algorithmic applications since they are the corner-stone of many dynamic programming algorithms for solving graph problems. For instance, many NP-hard problems can be solved in polynomial-time in the class of bounded-treewidth graphs (e.g., see [Bod88, Bod93]). In particular, problems that are expressible in Monadic Second Order Logic can be solved in linear time when the treewidth is bounded [Cou90]. Another framework where tree-decompositions play an important role is the *bi-dimensionality theory* that allowed the design of sub-exponential-time algorithms for many problems in the class of graphs excluding some fixed graph as a minor [DH08]. In all cases, computing tree-decompositions with small width is a prerequisite. Not surprisingly, computing the treewidth and optimal tree-decomposition is NP-hard in general graphs (more precisely, it is NP-hard in the class of bipartite and cobipartite graphs) [ACP87]. Fixed Parameter Tractable algorithms have been designed (e.g., [Bod96, BDD$^+$13]) but they are not efficient in practice. As far as we know, the best approximation algorithm has approximation ratio $O(\sqrt{\log tw(G)})$ for general graph G [FHL08]. There exist constant-ratio approximation algorithms in the case of bounded genus graphs [ST94, FHL08].

Computing the treelength of a graph is also NP-hard. More precisely, deciding whether a graph has treelength at most 2 is NP-hard [Lok10]. Moreover, it is hard to approximate within a factor smaller than 3/2 in weighted graphs [Lok10]. On the positive side, Dourisboure and Gavoille designed a 3-approximation algorithm for treelength, that performs in $O(nm)$-time in

n-node m-edge graphs [DG07]. Treelength has applications in the design of compact routing schemes [Dou05]. Moreover, graphs with bounded treelength also have bounded hyperbolicity[1] [CDE+12] and there are hard problems that can be solved efficiently on graphs with a bounded hyperbolicity. For instance, graphs with a bounded hyperbolicity admit a PTAS for the well-known Traveling Salesman problem [KL06]. There are also greedy routing schemes with a low additive stretch [BPK10], and approximation algorithms for several packing, covering, and augmentation problems [CE07] up to an additive constant that are devoted to these graphs.

The main goal of this work is to establish relationships between treewidth and treelength in order to take the algorithmic advantages from both sides. The treelength of a graph and its treewidth are uncomparable in general. Indeed, on the one hand for any cycle C_n with $n \geq 3$ vertices, $tw(C_n) = 2$ and $tl(C_n) = \lfloor \frac{n}{3} \rfloor$. This suggests that having a large treelength relies on large cycles. A subgraph H of a graph G is *isometric* if, for any two vertices of H, the distance between them is the same in H as in G. Note that the size of a longest isometric cycle in a graph can be computed in polynomial-time [Lok09]. It is known that the treelength of a graph G is bounded by the maximum length of a chordless cycle in G [DG07]. However, there are graphs such as grids with bounded-length isometric cycles and arbitrarily large treelength. We show that, in a graph G without long isometric cycles, $tl(G) = O(tw(G))$. In particular, this is the case in graphs with bounded hyperbolicity.

On the other hand, the complete graph with n vertices has treewidth $n-1$ and treelength 1. Another interesting example is the graph H obtained by adding a universal vertex to a square-grid with n^2 vertices, for which it holds $tw(H) = n+1$ and $tl(H) = 2$. Note that such graphs have a large genus. On the other hand, it is known that $tw(G) < 12 \cdot tl(G)$ for any planar graph G [Die09, DG09]. Therefore, it is natural to ask whether having a treewidth arbitrarily larger than the treelength requires a large genus. In this report, we prove that $tw(G) = O(tl(G))$ in bounded-genus graphs.

Altogether, our results allow us to design a very simple algorithm to compute new bounds for the treewidth in general graphs.

1.1 Our contributions

We introduce a very generic method to upper-bound the diameter of *minimal separators* in graphs. More precisely, we prove that minimal separators in a graph G induce connected subsets in some of its power G^j, where j only depends on the length of cycles in some arbitrary *cycle basis* of G. We deduce from our method that, for any graph G with longest isometric cycle of size $\ell(G)$ and for any minimal separator S in G, the diameter of S is upper bounded by $\lfloor \frac{\ell(G)}{2} \rfloor \cdot (|S|-1)$. Thus it easily follows that, for any graph G which is not a tree, $tl(G) \leq \lfloor \frac{\ell(G)}{2} \rfloor \cdot (tw(G)-1)^2$. Moreover, this implies that $tl(G) \leq (2\delta+1) \cdot (tw(G)-1)$ in any δ-hyperbolic graph.

We then refine our bound in several particular graph classes (the formal definition of these classes are postponed to the technical sections of the paper). For any graph G in the class of null-homotopic graphs (including the class of dismantable graphs), we prove that $tl(G) \leq tw(G)$. In the class of graphs G that admit a distance preserving ordering, we prove that $tl(G) < 2 \cdot tw(G)$. We emphasize that this latter class contains the cobipartite graphs for which computing the treewidth is NP-hard. Thus, combined with the 3-approximation for treelength [DG07], our

[1] Roughly, the hyperbolicity δ of a simple connected graph $G = (V, E)$, as defined by Gromov [Gro87], is the smallest value such that $\forall u, v, x, y \in V, d(u,v) + d(x,y) \leq \max\{d(u,x) + d(v,y), \mathrm{d}(u,y) + \mathrm{d}(v,x)\} + 2\delta$. This parameter can be computed in $O(n^{3.69})$-time [FIV12].

[2] Very recently and independently of this work, Diestel and Muller proved that $tl(G) \leq \ell(G)(tw(G)-1)$ [DM14]. Unlike our results which apply to any minimal separator in a graph, theirs rely on minimal separators in a specific tree-decomposition called an atomic tree-decomposition.

results provide a polynomial-time algorithm for computing a new non-trivial lower-bound for treewidth.

Finally, we consider lower-bounds for treelength. We prove that, for any graph excluding an apex graph H as a minor, there is a constant c_H such that $tw(G) \leq c_H \cdot tl(G)$. The constant c_H only depends on H. In the particular case of graphs with bounded genus g, we prove that $tw(G) = O(g^{3/2}) \cdot tl(G)$ where the "big O" notation hides a small constant. As a consequence, the 3-approximation for treelength [DG07] also allows to approximate the treewidth in the class of graphs that exclude an apex graph as a minor and in the class of bounded-genus graphs.

2 Preliminaries

All graphs considered in this paper are simple (*i.e.*, without loops or multiple edges), connected and finite. Given a graph $G = (V, E)$, the number $|V|$ of vertices will always be denoted by n and the number of edges $|E|$ by m.

Minimal separators A set $S \subseteq V$ is a *minimal separator* if there exist $a, b \in V \setminus S$ such that any path from a to b intersects S and, for any proper subset $S' \subset S$, there is a path from a to b which does not intersect S'. We name any such a set S an a-b minimal separator.
A connected component $C \subseteq V \setminus S$ of $G[V \setminus S]$ is *full* with respect to S if any node in S has a neighbour in C. Any a-b minimal separator has at least two full components: the one containing a and the one containing b. Conversely, any separator having at least two full components is a minimal separator.
A graph is said *well connected* if each of its minimal separators induces a connected subgraph [DVM86].

Cycle space The set $\mathcal{C}$ of Eulerian subgraphs of G is called the *cycle space* of G. It is well known that $\mathcal{C}$ forms a vector space over $\mathbb{F}_2$ where the addition of two subgraphs of $\mathcal{C}$ is their symmetric difference. That is, the addition $G_1 \oplus \cdots \oplus G_i$ of several subgraphs of G is equal to the subgraph consisting of the edges that appear an odd number of times in $G_1, \cdots, G_i$. A graph is said *null homotopic* if its cycle space admits a basis constituted of triangles.

Theorem 1 *[DVM86] Any connected null-homotopic graph is well-connected.*

In this paper, we will extend the class of null-homotopic graphs as follows.

Definition 1 *Let $l \geq 3$. We define $\mathcal{G}_l$ as the class of graphs whose cycle space is generated by all cycles of length at most l.*

Note that $\mathcal{G}_3$ is exactly the class of null-homotopic graphs. Moreover, the class $\mathcal{G}_l$ contains all graphs with no isometric cycle longer than l. Thus by varying the parameter l, classes $\mathcal{G}_l$ include all graphs and they form an inclusion wise increasing hierarchy.

Diameter and Graph powers For any $X \subseteq V$, let $diam_G(X)$ denote the maximum distance in G between any pair of vertices in X. Last, for any $j \geq 1$, the graph G^j is obtained from G by adding an edge between any two distinct nodes that are at distance at most j in G.

3 Diameter of Minimal Separators in Graphs

In this section, we prove that the diameter of any minimal separator S in a graph G is upper-bounded by an $O(\ell(G) \cdot |S|)$, where $\ell(G)$ is the length of a longest isometric cycle in G. We then strengthen our results in some particular graph classes that are defined by the existence of some *elimination ordering* of their vertices. Last, we introduce a new graph invariant $wc(G)$ which is the least integer j such that G^j is well-connected. We prove that the diameter of any minimal separator S in a graph G is upper-bounded by an $O(wc(G) \cdot |S|)$.

3.1 Case of general graphs

We start proving some properties of graphs in the class $\mathcal{G}_l$. This will lead us to the main result in this section (Theorem 2).

Lemma 1 *Let $l \geq 3$, the class $\mathcal{G}_l$ as defined in Definition 1 is stable under edge-contraction.*

Proof. Let $G \in \mathcal{G}_l$, let $e = \{u, v\} \in E(G)$ and let $G' = G/e$ be obtained from G by removing nodes u, v and adding a new vertex x_e such that $N_{G'}(x_e) = (N_G(u) \cup N_G(v)) \setminus \{u, v\}$. This defines a non-expansive mapping $\varphi_e : V(G) \to V(G')$ such that $\varphi_e(u) = \varphi_e(v) = x_e$ and $\varphi_e(y) = y$ when $y \notin e$. We extend the mapping to subgraphs as follows; given H a subgraph of G, the subgraph $\varphi_e(H)$ has as vertex-set $\varphi_e(V(H))$ and it has as edge set $\{\{\varphi_e(x), \varphi_e(y)\} \mid \{x, y\} \in E(H) \setminus e\}$. Observe that for any cycle C of G, $\varphi_e(C)$ is a cycle of G' unless it is a triangle containing e, in which case $\varphi_e(C)$ is an edge. Furthermore, we have that for any cycle C of G, the length of $\varphi_e(C)$ is at most the length of C. It can be checked that i addition, every cycle of G' is the image by φ_e of some cycle C of G.

Let C' be an induced cycle of G'. We want to prove that C' is a sum of cycles of length at most l in G'. Note that it will prove the lemma because induced cycles generate the whole cycle space. To prove it, let C be a cycle of G satisfying $\varphi_e(C) = C'$. By Definition 1, C is a sum of cycles of length at most l in G. So, let $\mathcal{C} = \{C_1, \ldots, C_k\}$ be a set of cycles of length at most l in G whose sum equals C. Let $\mathcal{C}' = \{\varphi_e(C_i) \mid C_i \in \mathcal{C}$, and $(e \notin E(C_i)$ or C_i is not a triangle $)\}$. Equivalently, $\mathcal{C}'$ is the set of all the non-edge images by φ_e of some cycle in $\mathcal{C}$. By the previous remark, $\mathcal{C}'$ is a set of cycles of length at most l in G'. We claim that $C' = \bigoplus_{C'_j \in \mathcal{C}'} C'_j$.

By contradiction, suppose that $C' \neq \bigoplus_{C'_j \in \mathcal{C}'} C'_j$, and let $H' = \bigoplus_{C'_j \in \mathcal{C}'} C'_j$. The graph H' is an Eulerian subgraph of G' by construction. Moreover, let $e' \in E(G')$ be an edge that is not incident to x_e. We claim that $e' \in E(H')$ if and only if $e' \in E(C')$. Indeed, the edge e' is also an edge of G that is not incident to vertices u, v. Therefore, it cannot be contained in any triangle $C_i \in \mathcal{C}$ containing e. This implies that such an edge e' is contained in as many cycles in $\mathcal{C}'$ as it is contained in $\mathcal{C}$. So, any edge not incident to x_e belongs to H if and only if it belongs to $E(C)$, hence to $E(C')$. A consequence for the subgraph H is that it contains (at least) all edges of C' not containing x_e as an endpoint. This implies that its edge-set is nonempty because C' is a cycle by the hypothesis. As a result, since there is no proper subgraph of a cycle which is Eulerian unless it is a singleton, and $H' \neq C'$ by the hypothesis, there must exist $e' \in E(H') \setminus E(C')$.

This edge e' must have at least one endpoint $u' \notin V(C')$ because C' is induced by the hypothesis. However, H' being Eulerian we must have that vertex u' has even degree in H', where the degree is defined as the number of edges incident to the node. So, any node $u' \in V(H') \setminus V(C')$ must be incident in H' to at least two distinct edges. None of these two edges is in $E(C')$ by the choice of node u', hence they are both incident to x_e. This implies $u' = x_e$. As a result, x_e is the only node in $V(H') \setminus V(C')$, and there are at least two distinct nodes in $V(C')$ that are adjacent to x_e in H'. But $x_e \notin V(C')$ implies that C' is a subgraph of H', hence H' is composed of C' plus additional edges incident to x_e. This is a contradiction because in such a case, there are vertices in $V(C')$ with odd degree in H'. ■

Definition 2 *Let $\mathcal{C}$ be a set of cycles in a graph G. The intersection graph of $\mathcal{C}$ is a graph with vertex-set $\mathcal{C}$ such that there is an edge between any two cycles sharing at least one edge in G.*

Lemma 2 *Let G be a graph, C be a cycle of G. For any set of cycles $\mathcal{C}$ that is inclusion wise minimal w.r.t. the property of generating C, the intersection graph of $\mathcal{C}$ is connected.*

Proof. By contradiction, let $\mathcal{C}_1 \subsetneq \mathcal{C}$ be a connected component of the intersection graph, and let H be the Eulerian subgraph of G generated by the component. By inclusion wise minimality of the set $\mathcal{C}$, we have that H is not trivial *i.e.*, it has to contain at least one edge. Since no proper non-trivial subgraph of a cycle is Eulerian, and $H \neq C$ by inclusion wise minimality of the set $\mathcal{C}$, then H must contain an edge $e \notin E(C)$. Moreover since we have that $\bigoplus_{C' \in \mathcal{C}} C' = C$ by the hypothesis, this implies the existence of some $C'_e \in \mathcal{C} \setminus \mathcal{C}_1$ satisfying $e \in E(C'_e)$, thus contradicting the fact that $\mathcal{C}_1$ is a connected component of the intersection graph of $\mathcal{C}$. ■

Lemma 3 *Let $l \geq 3$, let $G \in \mathcal{G}_l$ and let S be a minimal separator in G. Either S is a cut-vertex, or there are two distinct nodes $x, y \in S$ such that $\mathrm{d}_G(x,y) \leq \lfloor \frac{l}{2} \rfloor$.*

Proof. Suppose that S is not a cut-vertex. If the subgraph induced by S contains at least one edge $\{x,y\}$, then we are done as in such case $\mathrm{d}_G(x,y) = 1 \leq \lfloor \frac{l}{2} \rfloor$. So, we will assume S to be a stable. Let A, B be distinct full components of $G \setminus S$ and let $s, t \in S$ be distinct. By connectivity, there is a st-path whose all internal vertices are contained in A, and in the same way there is a st-path whose all internal vertices are contained in B. Let C be a cycle composed of two such paths. Because $G \in \mathcal{G}_l$, there is some set $\mathcal{C}$ of cycles of length at most l whose sum equals C. So, we can choose such a set $\mathcal{C}$ and we assume that it is inclusion wise minimal w.r.t. the property of generating C. By Lemma 2, this implies that the intersection graph of $\mathcal{C}$ is connected.

Then, we assign to each cycle in $\mathcal{C}$ the list of connected components of $G \setminus S$ that it intersects. Observe that no cycle can have an empty list because otherwise it would imply the existence of a cycle, hence of an edge, in the induced subgraph $G[S]$. Furthermore, we claim that there is at least one cycle whose list contains more than one connected component of $G \setminus S$. By contradiction, suppose that all cycles are assigned a unique connected component of $G \setminus S$. This defines a coloring of the intersection graph of $\mathcal{C}$. By construction the cycle C intersects the two distinct components A, B so, there exist two cycles in $\mathcal{C}$ coloured differently in the intersection graph. Thus by connectivity there are two adjacent cycles in this graph coloured differently. By Definition 2, both cycles must share at least one edge. But then it must be contained in S and no such an edge exists by the hypothesis. A contradiction.

Finally, let $C' \in \mathcal{C}$ intersect at least two distinct connected components of $G \setminus S$. Then there are $x, y \in S \cap V(C')$ and so, since the length of C' is at most l by construction we deduce that $\mathrm{d}_G(x,y) \leq \lfloor \frac{l}{2} \rfloor$. ■

Lemma 4 *Let $G = G_1 \cup G_2$ be such that $V(G_1 \cap G_2) = \{x,y\}$ and $E(G_1) \cap E(G_2) = \emptyset$. If $G_1, G_2 \in \mathcal{G}_l$ for some $l \geq 3$ and $\mathrm{d}_{G_1}(x,y) + \mathrm{d}_{G_2}(x,y) \leq l$, then $G \in \mathcal{G}_l$.*

Proof. Let C be a cycle in G. We will prove that it is a sum of cycles of length at most l in G. If it is a cycle in G_1 (resp. in G_2), then we are done as it is the sum of cycles of length at most l by Definition 1. Else, it must contain the pair x, y and it can be decomposed into: a xy-path in G_1, and a xy-path in G_2. Let C_l be obtained from the union of a xy-shortest-path in G_1 with a xy-shortest-path in G_2. Note that C_l has length $\mathrm{d}_{G_1}(x,y) + \mathrm{d}_{G_2}(x,y) \leq l$ by the hypothesis. Furthermore, $H = C \oplus C_l$ is an Eulerian subgraph of G. Let H_1, H_2 be the respective subgraphs of H that are induced by the edges in G_1, G_2 (possibly empty). Note that $E(H_1) \cap E(H_2) = \emptyset$ by construction. We claim that both graphs H_1, H_2 are Eulerian subgraphs. Indeed, on the one hand the subsets $V(H_1) \setminus \{x,y\}, V(H_2) \setminus \{x,y\}$ are disjoint and so, any node $\neq x,y$ in one of these graphs, say in H_1, has the same (even) degree in H_1 as in H. On the other hand, by construction each node amongst x,y is incident exactly to one edge in $E(C) \cap E(G_1)$ (resp. in $E(C) \cap E(G_2)$) and to one edge in $E(C_l) \cap E(G_1)$ (resp. in $E(C_l) \cap E(G_2)$). As a result, nodes x,y have degree either null or equal to 2 in H_1, and similarly they have degree either null or equal to 2 in H_2, which is even in both cases. Consequently, both H_1, H_2 are sums of cycles of length at most l by the hypothesis because they are respective Eulerian subgraphs of $G_1, G_2 \in \mathcal{G}_l$. Hence $H = H_1 \cup H_2$ is also a sum of cycles of length at most l in G. This concludes the proof because $C = H \oplus C_l$. ■

Theorem 2 *Let $l \geq 3$. For any graph $G \in \mathcal{G}_l$, every minimal separator in G induces a connected subgraph in the power $G^{\lfloor \frac{l}{2} \rfloor}$.*

Proof. By contradiction, let $G \in \mathcal{G}_l$, let S be a minimal separator in G falsifying the property. We first make adjacent every two vertices in S that are at distance at most $\lfloor \frac{l}{2} \rfloor$ in G. We claim that the resulting graph is still in $\mathcal{G}_l$. Indeed, let $x,y \in S$ non-adjacent and at distance at most $\lfloor \frac{l}{2} \rfloor$ in G, let $G_1 = G$ and let G_2 be the edge-graph with vertex-set x,y. Since we have that $G_1 \in \mathcal{G}_l$ by the hypothesis, that $G_2 \in \mathcal{G}_3 \subseteq \mathcal{G}_l$ and that $\mathrm{d}_{G_1}(x,y) + \mathrm{d}_{G_2}(x,y) \leq \lfloor \frac{l}{2} \rfloor + 1 \leq l$, then we deduce from Lemma 4 that $G_1 \cup G_2 \in \mathcal{G}_l$. The same argument can be applied iteratively because adding an edge in G cannot increase the distances between nodes in S. So, the claim is proved.

We then contract each connected component of the subgraph induced by S in a single node, thus contracting S to obtain a stable set S', and the resulting graph G' still belongs to $\mathcal{G}_l$ by Lemma 2. Furthermore, the stable set S' is a minimal separator in G' by construction. Since S falsifies the property of the theorem, we have that all nodes in S' are pairwise at distance at least $\lfloor \frac{l}{2} \rfloor + 1$, but then it contradicts Lemma 3. ■

To close this section, let us emphasize some straightforward consequences of Theorem 2.

Corollary 1 *Let $G \in \mathcal{G}_l, l \geq 3$, then any minimal separator S in G has diameter at most $\lfloor \frac{l}{2} \rfloor \cdot (|S| - 1)$.*

Corollary 2 *Let G be a graph that is not a tree, any minimal separator in G induces a connected subset in the power $G^{\lfloor \frac{\ell(G)}{2} \rfloor}$, where $\ell(G)$ denotes the length of a longest isometric cycle in G.*

Proof. It follows from Theorem 2 combined with the fact that isometric cycles generate the cycle space. ■

Corollary 3 *For any δ-hyperbolic graph G, any minimal separator in G induces a connected subset in the power $G^{2\delta+1}$.*

Proof. It follows from Corollary 2 combined with the fact that an isometric cycle in a δ-hyperbolic graph has length at most $4\delta + 3$. ■

3.2 Graphs with distance-preserving elimination ordering

In this section, we strengthen the results of previous section in the case of graphs admitting a distance-preserving elimination ordering. We say that G admits a *distance-preserving elimination ordering* if there exists a total order of V, denoted by $v_1, v_2, \ldots, v_n$, such that, for any $1 \leq i \leq n$, the graph $G_i = G \setminus \{v_1, \ldots, v_i\}$ is an isometric subgraph of G.
A graph is *dismantable* if, for any $1 \leq i < n$, there exists $j > i$, such that $N[u_i] \setminus \{u_1, \cdots, u_{i-1}\} \subseteq N[u_j]$. It is easy to check that, if a graph is dismantable, then it admits a distance-preserving elimination ordering. Furthermore, we have by a result from [BC08] that *every* graph is an isometric subgraph of some dismantable graph, hence there are dismantable graphs with arbitrarily long isometric cycles.
Another example of graphs that admit a distance-preserving elimination ordering is the class of cobipartite graphs. Note that this implies that computing the treewidth is NP-hard in the class of graphs that admits a distance-preserving elimination ordering.

We first show that, if G is a graph that admits a distance-preserving elimination ordering, then $G \in \mathcal{G}_4$. This is an improvement over Corollary 2 in this graph class.

Lemma 5 *A graph that admits a distance-preserving elimination ordering has its cycle space generated by all its triangles and quadrangles.*

Proof. It is enough to prove the result for induced cycles, because they generate the cycle space. Let $(u_1, u_2, \ldots, u_n)$ be a distance-preserving elimination ordering of G. By contradiction, amongst all induced cycles falsifying the property let C maximize the least index j such that $u_j \in C$. Note that C is a cycle of $G_{j-1} = G[\{v_j, \cdots, v_n\}]$ by the hypothesis. Moreover, all cycles contained into G_j are generated by triangles and quadrangles of G because of the maximality of index j. Let $x, y \in V(C)$ be the two neighbours of u_j in cycle C. By the hypothesis, x, y are not adjacent because C is induced. So, because $x, y, u_j \in G_{j-1}$ which has a distance-preserving elimination ordering, there is $u_i, i > j$ such that x, y are adjacent to u_i. Moreover, $u_i \notin C$ because otherwise C would be the quadrangle (u_j, x, u_i, y, u_j), thus contradicting the fact that it falsifies the property. As a result, $C = Q \oplus C'$, with Q the quadrangle (u_j, x, u_i, y, u_j) and C' is the cycle of G_j obtained from C by replacing the path x, u_j, y with x, u_i, y. Furthermore, cycle C' is a sum of induced cycles of G_j that are themselves a sum of triangles and quadrangles by maximality of j. Hence so is cycle C, which contradicts the fact that it falsifies the property. ■

Corollary 4 *Let G be a graph that admits a distance-preserving elimination ordering. Every minimal separator S in G induces a connected subgraph in the square graph G^2.*

Note that the result of Lemma 5 is the best possible that one can expect for this class of graphs, in the sense that they are graphs admitting a distance-preserving elimination ordering that are not well-connected, hence not in $\mathcal{G}_3$ either. This can be easily shown with a cycle with 4 vertices.

Corollary 5 *Let G be a graph that admits a distance-preserving elimination ordering. For any minimal separator S in G, $diam_G(S) \leq 2(|S| - 1)$.*

To conclude this section, our results can be strenghtened in the case of dismantable graphs. Indeed, it was already noticed in [DVM86] that dismantable graphs are null-homotopic, but the proof was left to the reader. We give it here for self-containment.

Lemma 6 *A dismantable graph is null-homotopic and so, well-connected.*

Proof. Let G be a dismantable graph. We prove that cycles of G are generated by its triangles, which proves that G is null-homotopic. The fact that G is well-connected follows from Theorem 1.

It is enough to prove that induced cycles are generated by triangles. Let $(u_1, u_2, \ldots, u_n)$ be a dismantling ordering of G. By contradiction, amongst all induced cycles falsifying the property, let C maximize the least index j such that $u_j \in C$. Let $x, y \in V(C)$ be the two neighbours of u_j in cycle C, and let u_i, with $i > j$, be a dominator of u_j in G_{j-1}. We have that $u_i \notin C$ because C is induced and it has length at least 4 by the hypothesis. As a result, $C = T_1 \oplus T_2 \oplus C'$, with T_1 the triangle induced by nodes u_i, x, u_j; with T_2 the triangle induced by nodes u_i, y, u_j; and with C' a cycle of G_j obtained from C by replacing the path x, u_j, y with x, u_i, y. Furthermore, cycle C' is a sum of induced cycles of G_j that are themselves a sum of triangles of G_j by maximality of j. Hence, so is cycle C, which contradicts the fact that it falsifies the property. ■

Note that Lemma 6 extends a previous result from [JKW03] where it was shown that bridged graphs (which are dismantable by [Che97]) are well-connected.

3.3 A new graph invariant

We conclude this section by introducing a new graph invariant $wc(G)$, that is defined as the least integer j such that G^j is well-connected. Interestingly, this invariant $wc(G)$ is strongly related to the least integer j such that any minimal separator in G induces a connected subset of G^j. We finally provide upper-bounds on $wc(G)$ for graphs in $\mathcal{G}_l$. This will show that our approach in this section extends the one from previous Section 3.1.

We will use the following result.

Lemma 7 (Theorem 1.3,[DVM86]) *A connected graph G is well-connected if and only if, for every pair of connected subgraphs G_1, G_2 whose union is G, the graph $G_1 \cap G_2$ is connected.*

Theorem 3 *Let G be connected, j be a positive integer such that G^j is well-connected. Every minimal separator in G induces a connected subgraph in the power G^{2j-1}.*

Proof. Let S be a minimal separator in G and let A, B be two distinct full components of $G \setminus S$. Let S_1, S_2 be an arbitrary bipartition of nodes in S. We claim that $\mathrm{d}_G(S_1, S_2) \leq 2j-1$. Note that it will prove that $G^{2j-1}[S]$ is connected because the bipartition is chosen arbitrarily.

For any $r \geq 0$ and any $X \subseteq V$, let $B_r(X)$ denote the set of vertices at distance at most r from a vertex in X. We say that two subgraphs *touch* each other if they intersect or if there exists an edge between one node of each subgraph. To prove the claim, let us define the following sets:

$$X_1 = \left[B_{\lfloor \frac{j}{2} \rfloor}(S_1) \cap A\right] \cup \left[B_{\lceil \frac{j}{2} \rceil - 1}(S_1) \cap (V(G) \setminus A)\right],$$

$$X_2 = \left[B_{\lfloor \frac{j}{2} \rfloor}(S_2) \cap (V(G) \setminus A)\right] \cup \left[B_{\lceil \frac{j}{2} \rceil - 1}(S_2) \cap A\right],$$

$$X = X_1 \cup X_2,$$

and we set G_1, G_2 the subgraphs of G^j that are respectively induced by the subsets $A \cup X$ and $V(G) \setminus (A \setminus X)$. Since we have that A is a full component of $G \setminus S$ by the hypothesis, it follows that $G[A \cup S]$ is connected and so, G_1 is connected in G^j because all nodes in $X \setminus S$ are adjacent to some node in S in G^j. Similarly, we have that B is a full component by the hypothesis, hence $G[B \cup S]$ is connected, that implies $G[V(G) \setminus A]$ is connected because all connected components of $G \setminus (A \cup B \cup S)$ touch the separator S in G. Therefore, G_2 is also connected in G^j.

There are two cases to be considered.

1. Suppose that subsets $A \setminus X$ and $V(G) \setminus (A \cup X)$ touch in the graph power G^j, and let $u \in A \setminus X, v \in V(G) \setminus (A \cup X)$ be adjacent in G^j. If there is a uv-shortest-path in G that intersects both S_1, S_2, then we obtain $\mathrm{d}_G(S_1,S_2) \leq \mathrm{d}_G(u,v) - 2 \leq j-2$, which proves the claim.

   Otherwise, we can assume w.l.o.g. that $\mathrm{d}_G(u,v) \geq \mathrm{d}_G(u,S_1) + \mathrm{d}_G(v,S_1) \geq \left(\lfloor \frac{j}{2} \rfloor + 1\right) + \left(\left(\lceil \frac{j}{2} \rceil - 1\right) + 1\right) \geq j+1$, thus contradicting the fact that nodes u, v are adjacent in G^j.

2. Let us assume for the remaining of the proof that $A \setminus X$ and $V(G) \setminus (A \cup X)$ do not touch in the graph power G^j. That is, X is a separator of G^j. We have: $G^j = G_1 \cup G_2$, G_1, G_2 are connected and G^j is well-connected by the hypothesis. Therefore, $X = G_1 \cap G_2$ induces a connected subgraph in G^j by Lemma 7. In such case, let $x_1 \in X_1, x_2 \in X_2$ be adjacent in G^j. Up to replacing one vertex amongst x_1, x_2 with some node in S onto a x_1x_2-shortest-path in G, we can assume w.l.o.g. $x_1, x_2 \in A \cup S$. Let $y_1 \in S_1, y_2 \in S_2$ be such that $\mathrm{d}_G(x_1,y_1) \leq \lfloor \frac{j}{2} \rfloor$ and $\mathrm{d}_G(x_2,y_2) \leq \lceil \frac{j}{2} \rceil - 1$. Then we have that $\mathrm{d}_G(S_1,S_2) \leq \mathrm{d}_G(y_1,y_2) \leq \mathrm{d}_G(y_1,x_1) + \mathrm{d}_G(x_1,x_2) + \mathrm{d}_G(x_2,y_2) \leq \lfloor \frac{j}{2} \rfloor + j + \lceil \frac{j}{2} \rceil - 1 \leq 2j-1$. This finally proves the claim.

■

Next, we bound the diameter of minimal separators using the fact that some power of a graph is well-connected.

Corollary 6 *Let G be connected, j be a positive integer such that G^j is well-connected. For any minimal separator S in G, $diam_G(S) \leq (2j-1)(|S|-1)$.*

Corollary 6 as it can be shown with a cycle C_6 of length 6. Indeed, on the one hand it can be checked that the square graph of a C_6 is null-homotopic (*e.g.*, see Theorem 5) and so, well-connected. On the other hand, all of the diametral pairs of the cycle are minimal separator of diameter $3 = 2*2-1$.

We now show that some sort of converse result also holds for Theorem 3. Namely, we prove that if any minimal separator in G induces a connected subset of G^j, then G^j is well-connected.

Theorem 4 *Let G be connected, let $j \geq 1$ be such that any minimal separator in G induces a connected subset of G^j. Then G^j is well-connected.*

Proof. Let S' be a minimal separator in G^j, let A', B' be two distinct full components of $G^j \setminus S'$. We want to prove that S' induces a connected subset of G^j. To prove it, let $S = N_G(B')$, let $A = A' \cup (S' \setminus S)$ and let $B_1, \ldots, B_p$ be the connected components of $G[B']$. Finally, for any $1 \leq i \leq p$ let $S_i = N_G(B_i) \subseteq S$.

We first claim that A induces a connected subset of the graph G. Indeed, by construction this subset contains all vertices at distance at most $j-1$ in G from A' *i.e.*, $S' \setminus N_G(B)$ and so, all vertices in A' are in the same connected component of $G[A]$ because A' induces a connected subset of G^j by the hypothesis. This shows that $G[A]$ is connected because all vertices in $S' \setminus N_G(B)$ are connected to A' by a path of length at most $j-1$ whose all vertices are contained into A.

Furthermore, the subset S being exactly the set of vertices at distance j in G from A', it follows that each node in S is adjacent to some node in A. Consequently, each subset S_i is a minimal separator in G, with A, B_i two distinct full components of $G \setminus S_i$. This implies that each subset S_i induces a connected subgraph of G^j by the hypothesis. We claim that it implies that S also induces a connected subgraph of G^j. By contradiction, assume the existence of a bipartition $\mathcal{S}^1, \mathcal{S}^2$ of S such that both subsets do not touch in G^j. Let $\mathcal{B}^i, i \in \{1,2\}$ be the subset of B' which is adjacent to $\mathcal{S}^i$ in G. We observe that $\mathcal{B}^1, \mathcal{B}^2$ is a bipartition of B' because each subset

S_i being connected in G^j, it is fully contained either in $\mathcal{S}^1$ or in $\mathcal{S}^2$. However, there is a pair $x \in \mathcal{B}^1, y \in \mathcal{B}^2$ that is at distance at most j in G because B' is connected in G^j by the hypothesis. Furthermore a xy-shortest-path in G must intersect $\mathcal{S}^1, \mathcal{S}^2$ because each connected component of B' in B is fully contained either in $\mathcal{B}^1$ or in $\mathcal{B}^2$. As a result, there is $x' \in \mathcal{S}^1, y' \in \mathcal{S}^2$ satisfying $\mathrm{d}_G(x', y') \leq j - 2$, a contradiction.

The proof that S' is connected in G^j finally follows from the fact that each node in $S' \setminus S$ is at distance at most $j-1$ in G from some node in S, hence adjacent to this node in G^j. ■

The combination of Theorem 4 with Theorem 2 gives us that for any graph $G \in \mathcal{G}_l$, we have that $G^{\lfloor \frac{l}{2} \rfloor}$ is well-connected. Hence, we have that $wc(G) \leq \lfloor \frac{l}{2} \rfloor$ for any graph $G \in \mathcal{G}_l$. We refine this bound as follows:

Theorem 5 *Let $G \in \mathcal{G}_l$. Then $G^{\lceil \frac{l}{3} \rceil}$ is null-homotopic and so, well-connected.*

Proof. We prove that cycles of $G^{\lceil \frac{l}{3} \rceil}$ are generated by its triangles, which proves that $G^{\lceil \frac{l}{3} \rceil}$ is null-homotopic. The fact that $G^{\lceil \frac{l}{3} \rceil}$ is well-connected follows from Theorem 1.

Let $j \geq 1$ and let $e = \{u, v\}$ be an edge of G^j. The *length* of e denotes the distance between u and v in G. Given a cycle C of $G^{\lceil \frac{l}{3} \rceil}$, its number of edges in $G^{\lceil \frac{l}{3} \rceil}$ is denoted by $l(C)$ and let $w(C) = \sum_{\{x,y\} \in E(C)} \mathrm{d}_G(x, y)$ be the sum of the lengths of all its edges. Note that $w(C) \geq l(C)$.

The proof is by induction on $(l(C), w(C))$ in lexicographic order. The base cases are when $l(C) = 3$, in which case the cycle C is a triangle. Thus from now on assume $l(C) > 3$, hence $w(C) > 3$. There are two cases.

- If there is a chord in C, then we can split it in two smaller cycles C_1, C_2 with $l(C_1), l(C_2) < l(C)$; by the induction hypothesis, both cycles are a sum of triangles and so is cycle C.

- Else, C is induced and we associate to every edge $e = \{x, y\} \in E(C)$ a xy-shortest-path P_e in G. Note that, any vertex z of P_e is adjacent to both x and y in $G^{\lceil \frac{l}{3} \rceil}$. Indeed, $dist_G(z, x), dist_G(z, y) \leq dist_G(x, y) \leq \lceil \frac{l}{3} \rceil$. Therefore, no node in C can be contained as an internal node of some P_e because otherwise there would be a chord in C.

  However, it may be that there are two edges $e \neq e'$ of C such that $P_e, P_{e'}$ have a common internal node, and we will first solve this subcase. Note that if $P_e, P_{e'}$ share a common internal node, then it implies that the sets $e = \{x, y\}, e' = \{x', y'\}$ touch in the graph power G^j, *i.e.*, we have w.l.o.g. either $x = x'$ or $\{x, x'\} \in E(C)$.

  - Case 1: $x = x'$. Let $z \in P_e \cap P_{e'}$ be a common internal node to both paths, let T_1, T_2 be the triangles in $G^{\lceil \frac{l}{3} \rceil}$ that are respectively induced by x, y, z and x, y', z. The cycle C' is obtained from C by replacing the path y, x, y' with the path y, z, y' that exists in $G^{\lceil \frac{l}{3} \rceil}$. In such case, we have that $l(C) = l(C')$ while $w(C') < w(C)$, hence we can apply the induction hypothesis to conclude that C' is a sum of triangles. Since we have that $C = T_1 \oplus T_2 \oplus C'$, then it implies that the same holds for cycle C.

  - Case 2: $\{x, x'\} \in E(C)$. Let $z \in P_e \cap P_{e'}$ be a common interval node to both paths. We consider the triangles T_1, T_2, T_3 in $G^{\lceil \frac{l}{3} \rceil}$ that are respectively induced by x, y, z and x', y', z and x, x', z. Let C' be the cycle obtained from C by replacing the path y, x, x', y' with the path y, z, y'. Since $l(C') = l(C) - 1$, we can apply the induction hypothesis to conclude that C' is a sum of triangles. Since we have that $C = T_1 \oplus T_2 \oplus T_3 \oplus C'$, then it implies that the same holds for cycle C.

Finally, assume for the remaining of the proof that all shortest-paths P_e in G have their internal nodes that are pairwise disjoint. For any edge $e = \{x, y\} \in E(C)$, write $P_e = (x = z_0, z_1, z_2, \ldots, z_{k-1}, y = z_k), k \leq \lceil \frac{l}{3} \rceil$. Since P_e is a shortest-path in G, then all sets $x, z_i, z_{i+1}, i < k$, induce a triangle T_i^e in $G^{\lceil \frac{l}{3} \rceil}$. So, we can define $C_e = \bigoplus_{1 \leq i \leq k-1} T_i^e$. Note that C_e is the cycle induced by P_e and the edge e. Let C' be obtained from C by replacing every edge $e \in E(C)$ with the path P_e. By construction, C' is a cycle in G and so, it is the sum of cycles of length at most l in G by the hypothesis. Furthermore, we claim that each cycle of length at most l in G is a sum of triangles in $G^{\lceil \frac{l}{3} \rceil}$. Indeed, the treelength of a cycle of length l is $\lceil \frac{l}{3} \rceil$ by [DG07], hence $G^{\lceil \frac{l}{3} \rceil}$ contains a chordal supergraph of any cycle of length at most l in G. As a result, we have that C' is a sum of triangles in $G^{\lceil \frac{l}{3} \rceil}$. The same holds for C as it equals $C' \oplus \left(\bigoplus_{e \in E(C)} C_e \right)$.

■

4 Relating treewidth with treelength

4.1 Preliminaries

In this section, we recall some useful definitions and known results that will be used in the sequel.

A graph is *chordal* if all its induced cycles have length at most 3.
Let $G = (V, E)$ be a graph. A *triangulation* of G is any chordal supergraph $H = (V, E \cup F)$ of G. A triangulation $H = (V, E \cup F)$ is *minimal* if, for any $f \in F$, $H' = (V, E \cup F \setminus \{f\})$ is not chordal.
A *tree-decomposition* of G consists of a pair $(T, \mathcal{X})$ where T is a tree and $\mathcal{X} = (X_t)_{t \in V(T)}$ is a family of subsets of V, called *bags*, indexed by the nodes of T and that satisfies the following three properties.

1. $\bigcup_{t \in V(T)} X_t = V$;
2. for any $\{u, v\} \in E$, there is $t \in V(T)$ with $u, v \in X_t$;
3. for any $u \in V$, the set of bags containing u induces a subtree of T.

The diameter of a bag X is the maximum distance in G between any two nodes of X. The *length* of $(T, \mathcal{X})$ equals the maximum diameter of its bags. The *treelength* of G, denoted by $tl(G)$, is the minimum length over all tree-decompositions of G. Equivalently, the treelength of G is the least integer j such that G^j contains a chordal supergraph of G.

The *width* of $(T, \mathcal{X})$ equals the maximum size of its bags minus one. The *treewidth* of G, denoted by $tw(G)$, is the minimum width over all tree-decompositions of G. Equivalently, the treewidth of G is the minimum over all minimal triangulations H of G of $\omega(H) - 1$, where $\omega(H)$ is the maximum size of a clique in H.

Let $G = (V, E)$ be a chordal graph. The clique-graph $\mathcal{C}(G)$ of G is the weighted graph $\mathcal{C}(G) = (V_C, E_C, w)$, where $w : E_c \to \mathbb{N}$, V_C is the set of maximal cliques of G, $\{C_1, C_2\} \in E_C$ if $C_1 \cap C_2 \neq \emptyset$ and $w(\{C_1, C_2\}) = |C_1 \cap C_2|$. A clique-tree of G is a tree $T_C = (V_C, F)$ such that for each vertex $x \in V$, the set of maximal cliques containing x induces a subtree of T_C. Note that (T_C, V_C) is a tree-decomposition of G.

Theorem 6 *[GHP95] Let $G = (V, E)$ be a chordal graph. Any maximum weighted spanning tree of $\mathcal{C}(G)$ is a clique-tree of G.*

Let S, T be two minimal separators in G. S *crosses* T if there are two components C, D of $G \setminus T$ that S intersects. If S does not cross T, then S is said *parallel* to T.

Theorem 7 *[PS97] H is a minimal triangulation of G if and only if H is obtained by completing all sets of a maximal set of pairwise parallel minimal separators in G.*

4.2 Upper-bounds for treelength

Using the results recalled in Section 4.1, we are now able to upper-bound the treelength of a graph by a linear function depending on the size of its minimal separators. We then show that the treelength of a graph is upper-bounded by a function that is linear in its treewidth.

Lemma 8 *Let G be a graph and $\mathcal{S}$ be a maximal set of pairwise parallel minimal separators in G. If there is a constant $c_{\mathcal{S}}$ such that $diam(S) \leq c_{\mathcal{S}}(|S|-1)$ for any $S \in \mathcal{S}$, then $tl(G) \leq \max\{1\} \cup \{c_{\mathcal{S}} \cdot (|S|-1) \mid S \in \mathcal{S}\}$.*

Proof. Let H be the supergraph of G obtained by completing all sets of $\mathcal{S}$. By Theorem 7, H is a minimal triangulation of G. By Theorem 6, H admits a clique-tree T_C. Moreover, T_C corresponds to a reduced tree-decomposition of G where each clique of H induces a bag. Let Ω be any maximal clique in H, i.e., Ω is any bag of the tree-decomposition T_C. Let $x, y \in \Omega$. By definition of H, either $\{x, y\} \in E(G)$ or there is a minimal separator $S \in \mathcal{S}$ that contains both x and y. In the latter case, $\mathrm{d}(x, y) \leq diam_G(S) \leq c_{\mathcal{S}}(|S|-1)$. ■

Theorem 8 *Let G be a graph with treewidth $tw(G)$. If every minimal separator in G induces a connected subgraph in its power G^j, then $tl(G) \leq \max\{1, j \cdot (tw(G)-1)\}$.*

Proof. Let H be a minimal triangulation of G with maximum clique-size $tw(G)+1$. By Theorem 7, there is a maximal set $\mathcal{S}$ of pairwise parallel minimal separators of G such that H results from the completion of all elements in $\mathcal{S}$. Note that any $S \in \mathcal{S}$ induces a clique-minimal separator in H and therefore S is strictly contained in a maximal clique in H. Hence, $\max_{S \in \mathcal{S}} |S| \leq tw(G)$. By Lemma 8, $tl(G) \leq \max\{1\} \cup \{c_{\mathcal{S}} \cdot (|S|-1) \mid S \in \mathcal{S}\} \leq \max\{1, j \cdot (tw(G)-1)\}$. ■

Corollary 7 *Let G be a connected graph which is not a tree, then $tl(G) \leq c \cdot (tw(G)-1)$, where*

- *$c = 2$ if G admits a distance-preserving elimination ordering;*
- *$c = 2j-1$, where j is the least integer such that G^j is well-connected;*
- *$c = \left\lfloor \frac{\ell(G)}{2} \right\rfloor$, where $\ell(G)$ denotes the length of a longest isometric cycle of G;*
- *$c = 2\delta + 1$ if G is δ-hyperbolic.*

Proof. First item follows from Corollary 4 combined with Theorem 8. Second item follows from Theorem 3 combined with Theorem 8. Third item follows from Corollary 2 combined with Theorem 8. Last item follows from Corollary 3 combined with Theorem 8. ■

Note that it is NP-hard to compute the treelength of a graph [Lok10], but there exist 3-approximation polynomial-time algorithms to compute it [DG07]. Moreover, the hyperbolicity of a graph can be computed in polynomial-time [Gro87, CCL12, FIV12]. Hence, the previous result gives a new way to compute lower-bounds for treewidth.

4.3 Lower-bound in case of bounded-genus graphs

In this section, we prove that the treewidth of a graph is upper-bounded by a function of its treelength and of its genus. Our result is mainly based on the result from [DHT06] stating that any graph with large treewidth and genus contains a large "grid-like" graph as a contraction. We use their terminology.

A *partially triangulated* $(r \times r)$*-grid* is any graph that contains an $(r \times r)$-grid as a subgraph and is a subgraph of some planar triangulation of the same $(r \times r)$-grid. A (r,k)*-gridoid* G is a partially triangulated $(r \times r)$-grid G' in which k extra edges have been added[3].

Theorem 9 *[DHT06] Let G be a graph with genus g and $tw(G) > 4k(g+1)$ with $k \geq 12g$, then G contains a $(k-12g, g)$-gridoid as a contraction.*

We prove that such a gridoid has large treelength and, since the treelength is contraction-closed, such a graph has large treelength too.

Lemma 9 *Let G be a partially triangulated $(r \times r)$-grid. Then, $tl(G) \geq r/\left(3\sqrt{2}\right) - 1$.*

Proof. The result holds if $r \leq 3$ because in such a case $tl(G) \geq 1 > r/\left(3\sqrt{2}\right) - 1$. Else, let G' be the $(r \times r)$-grid from which G is obtained by planar triangulation. Let V' be the set of vertices that are at distance at least $\left\lfloor \frac{r-1}{3} \right\rfloor$ from the external face of G'. The vertices of V' induce a partially triangulated $(r' \times r')$-grid F in G, $r = 2\left\lfloor \frac{r-1}{3} \right\rfloor + r'$, such that the external face has not been triangulated. Moreover, F is isometric in G. Hence, $tl(G) \geq tl(F)$. We show that $tl(F) \geq r/\left(3\sqrt{2}\right) - 1$.

Let S be a balanced separator in F. That is, any connected component of $F \setminus S$ has size at most ${r'}^2/2$. Let $D-1$ be the maximum distance in F between any two vertices of S. We claim that there exists an induced subgraph H of F that is a partially triangulated $(D \times D)$-grid and such that $S \subseteq V(H)$ and the maximum distance between two vertices of S is at least $D-1$. Indeed, let x be a top vertex of S in F, let y be a leftmost vertex of S in F, and let H be the subgrid of F with side D and top row containing x and leftmost column containing y. Clearly, $S \subseteq V(H)$. Furthermore for any $D < r$, $F \setminus H$ is connected and has ${r'}^2 - D^2$ nodes. Since ${r'}^2 - D^2 \leq {r'}^2/2$, we get that $D^2 \geq \lceil {r'}^2/2 \rceil$ and so, $D \geq r'/\sqrt{2} \geq r/\left(3\sqrt{2}\right)$.

It is well known that any tree-decomposition of a graph F has a bag which is a balanced separator of F. By previous paragraph, it has diameter at least $r/\left(3\sqrt{2}\right) - 1$ and therefore, $tl(G) \geq r/\left(3\sqrt{2}\right) - 1$. ■

Lemma 10 *Let G be a (r,k)-gridoid, then $tl(G) \geq r/\left(12\sqrt{2(2k+1)}\right) - 1 - 1/\left(6\sqrt{2}\right)$.*

Proof. The result holds if $r < 10\sqrt{2k+1}$ because in such case $tl(G) \geq 1 > r/\left(12\sqrt{2(2k+1)}\right) - 1 - 1/\left(6\sqrt{2}\right)$. Else, let S be the set of endpoints of at most k edges whose removal in G yields a partially r-triangulated grid. Note that $|S| \leq 2k$. Also, let G' be the $(r \times r)$-grid whose $G \setminus S$ is a partial triangulation. Let finally $4 \leq x \leq r$ be an integer. There are $(r-x+1)^2$ distinct $(x \times x)$-grids as subgraphs in G', that give us as many distinct partially x-triangulated grids as subgraphs in G. Furthermore, each node in S belongs to at most x^2 such subgraphs. Therefore assuming $(r-x+1)^2 - 2k \cdot x^2 \geq 1$, there is one of these partially x-triangulated grids, say H, that does not contain any node incident to one of the k extra edges.

[3] Note that the notion of (r,k)-gridoid is more general in [DHT06].

Consider the partially triangulated x'-grid R which is in the center of H, with $x = 2 \cdot \lfloor \frac{x-1}{3} \rfloor + x'$. That is, R is a subgraph of H and any node of R is at distance at least $\lfloor \frac{x-1}{3} \rfloor$ from a node of $G \setminus H$ (it is possible because H does not contain an extremity of an extra edge). Therefore, R is isometric in G and $tl(R) \leq tl(G)$. By Lemma 9, $tl(R) \geq x' / \left(3\sqrt{2}\right) - 1 \geq x / \left(6\sqrt{2}\right) - 1$.

It remains to maximize x satisfying the inequality $(r - x + 1)^2 - 2k \cdot x^2 \geq 1$ so that we maximize the above lower-bound for $tl(R)$. The polynomial $(r - X + 1)^2 - 2k \cdot X^2 - 1 = r^2 + X^2 + 1 - 2r \cdot X + 2r - 2X - 2k \cdot X^2 - 1 = -\left[(2k-1) \cdot X^2 + 2(r+1) \cdot X - r(r+2)\right]$ has for reduced discriminant $(r+1)^2 + r(r+2)(2k-1) = 2k \cdot r(r+2) + 1$, hence its roots are equal to $\left\{ -\frac{\sqrt{2k \cdot r(r+2)+1}+r+1}{2k-1}, \frac{\sqrt{2k \cdot r(r+2)+1}-r-1}{2k-1} \right\}$. Since this polynomial is nonnegative only between its roots, the value maximizing x is:

$$\left\lfloor \frac{\sqrt{2k \cdot r(r+2)+1} - r - 1}{2k-1} \right\rfloor \geq \frac{\sqrt{2k \cdot r(r+2)+1} - r - 1}{2k-1} - 1 + \frac{1}{2k-1}$$

$$= \frac{r(r+2)}{\sqrt{2k \cdot r(r+2)+1} + r + 1} - 1 + \frac{1}{2k-1} > \frac{r(r+2)}{2\sqrt{2k \cdot r(r+2)+1}} - 1$$

$$> \frac{1}{2}\sqrt{\frac{r(r+2)}{2k+1}} - 1 > \frac{r}{2\sqrt{2k+1}} - 1 \geq 4.$$

■

Theorem 10 *Let G be a graph with genus g and $tw(G) > 4k(g+1)$ with $k \geq 12g$. Then*

$$tl(G) = \Omega(tw(G)/g^{3/2}).$$

Proof. By Theorem 9, G contains a $(k - 12g, g)$-gridoid R as a contraction. By Lemma 10, $tl(R) \geq \frac{k-12g}{12\sqrt{2(2g+1)}} - 1 - \frac{1}{6\sqrt{2}}$. Thus, by setting $k = tw(G)/(4(g+1)) - 1$, we obtain that $tl(R) = \Omega(tw(G)/g^{3/2})$. The result then follows from the fact that treelength is contraction-closed. ■

An *apex graph* is a graph such that the removal of one vertex creates a planar graph. Similar techniques allow us to deal with graphs that exclude an apex graph as minor.

Let Γ_k be the graph obtained from a $(k \times k)$-grid by triangulating its internal faces such that all internal vertices become of degree 6, all non-corner external vertices are of degree 4, and then one corner of degree two is joined by edges with all vertices of the external face [FGT11].

Theorem 11 *[FGT11] For every apex graph H, there is a $c_H > 0$ such that every connected H-minor-free graph of treewidth at least $c_H \cdot k$ contains Γ_k as a contraction.*

Theorem 12 *Let H be any apex graph and G be a connected H-minor-free graph of treewidth at least $c_H \cdot k$, where c_H is the constant of Theorem 11. Then $tl(G) \geq tw(G)/(3c_H \cdot \sqrt{2}) - 1$.*

Proof. By Theorem 11, G contains Γ_k as a contraction. Moreover, Γ_k is a partially triangulated grid. The result follows from Lemma 9 and the fact that treelength is contraction-closed. ■

By combining Corollary 7, Theorems 10, 12 and the fact that the treelength can be 3-approximated in $O(n \cdot m)$-time [DG07], we get the following corollary.

Corollary 8 *There is an algorithm that, in any n-node m-edge graph, computes in $O(n \cdot m)$-time an integer t^* such that:*

- $t^* \leq 3 \cdot \left\lfloor \frac{\ell(G)}{2} \right\rfloor \cdot tw(G)$ *where $\ell(G)$ is the size of a longest isometric cycle in G;*
- $t^* = \Omega(tw(G)/g^{3/2})$ *where g is the genus of G;*
- $t^* \geq tw(G)/(3c_H \cdot \sqrt{2}) - 1$ *if G excludes some apex graph H as a minor.*

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