Kinetic quasi-velocities in unilaterally constrained Lagrangian mechanics with impacts and friction

Bernard Brogliato

Kinetic quasi-velocities in unilaterally constrained Lagrangian mechanics with impacts and friction

Bernard Brogliato

Abstract Quasi-velocities computed with the kinetic metric of a Lagrangian system are introduced, and the quasi-Lagrange equations are derived with and without friction. This is shown to be very well suited to systems subject to unilateral constraints (hence varying topology) and impacts. Energetical consistency of a generalized kinematic impact law is carefully studied, both in the frictionless and the frictional cases. Some results concerning the existence and uniqueness of solutions to the so-called contact linear complementarity problem, when friction is present, are provided.

Keywords Bilateral holonomic constraints · Unilateral constraints · Complementarity conditions · Coulomb friction · Tangential restitution · Painlevé paradox · Quasi-velocities · Quasi-Lagrange dynamics · Kinematic impact law · Moreau's impact law · Kinetic angles · Kinetic metric

1 Introduction

Finite-dimensional Lagrangian mechanical systems subject to bilateral, or unilateral constraints with or without Coulomb's friction, have received a considerable attention. It is known that the choice of the generalized coordinates may be a crucial step for either feedback control design, or numerical simulation. Many different ways of transforming the Lagrange dynamics into more suitable "canonical" forms have been proposed. Among these some are based on quantities known as *quasi-velocities* (or non-holonomic velocities, or generalized velocities, or generalized speeds, or pseudo-velocities, or kinematic characteristics) [32, 38, 46, 47, 49, 59, 61, 71]. Others use generalized coordinate transformations [64]. The coordinate partitioning methods have almost always been introduced for systems with a set of bilateral (holonomic or non-holonomic) constraints. The case of unilateral constraints

B. Brogliato (✉)
Bipop team-project, ZIRST Montbonnot, INRIA, 655 avenue de l'Europe, 38334 Saint-Ismier cedex, France
e-mail: bernard.brogliato@inria.fr

and impacts deserves, as is shown in this paper, a specific analysis because of the varying set of active constraints.

The issues related to impacts in multibody systems constitute an important field of research because of the many applications and challenges; see [48, 72] and [16, Chaps. 4, 5, 6] for reviews of single and multiple collisions modeling approaches. Among these, so-called multiple impacts are particularly tough to model. Frémond [28] briefly analyzed the rocking block and pointed out the necessity of "distance effects" in multiple impacts, dealing later on with chains of balls [22, 24]. The kinetic quasi-velocity change of variables described in this paper was introduced in [14], inspired by ideas in [41, 42, 50]. A generalized impact kinematic impact law based on this particular state space transformation was also pointed out in [14, Eq. (6.86)]; see also [16, Eq. (6.96)]. Glocker [35] later interpreted Frémond's idea as a restitution matrix, recovering (as far as chains of balls are concerned) the normal part of the generalized law presented in the sequel of this paper: this restitution matrix should not in general be diagonal, but should contain off-diagonal entries representing the "distance effects". More recently, it has been shown that even more may sometimes be needed [19], because the normal part of the restitution law may not be sufficient to describe some motions like perfect rocking of a planar block. Recent results on multiple impacts with or without friction have also been obtained in [11, 55, 65, 66, 75, 80–82, 87].

In this paper, it is shown that the kinetic quasi-velocity change of variables is very well suited for unilaterally constrained Lagrangian systems, and allows one to analyze these systems as a generalized multi-constrained particle with which two Riemannian metrics are associated: one in a normal space and one in a tangent space. In the frictionless case, the analogy with a particle is clear. In the case with friction, the analogy is less obvious because of normal/tangential coupling terms. It is nevertheless still possible to derive a general criterion for energy consistency at impacts, involving "small couplings" and "small friction". This criterion (see Theorem 2 below) is not just an existence result but is a constructive result since all the bounds can be computed for a given system.

The paper is organized as follows: Sect. 2 presents the quasi-Lagrangian dynamics in the frictionless (Sect. 2.1) and frictional (Sect. 2.2) cases, and ends with several examples in Sect. 2.3. Section 3 is dedicated to the analysis of kinematic impact laws, in the frictionless (Sect. 3.1) and frictional (Sect. 3.2) cases. Issues of energetical consistency, uniqueness of coefficients of restitution, link with Moreau's impact law, tangential restitution and Coulomb's friction at impacts are analyzed. In passing, we provide a correct version of a result stated in [19], where some crucial assumptions are missing. Also a general and constructive criterion for the energetical consistency when normal restitution and Coulomb's friction are used, is proposed. In Sect. 4, the contact LCP well-posedness is studied. Conclusions end the paper in Sect. 5, and some useful mathematical results are recalled in Appendix A.

Notation and definitions [7, 25, 51, 78] $w \in \mathbb{R}^n_+$ if all components of w are non-negative. I generically denotes the identity matrix of appropriate dimension. A real matrix $A \in \mathbb{R}^{n\times n}$ is positive definite if $x^T Ax > 0$ for all $x \neq 0$, it is non-negative if all its entries are non-negative, its Moore–Penrose pseudo-inverse is denoted as $A^\dagger$, it is a P-matrix if all its principal minors are positive, it is a Z-matrix if its off-diagonal entries are all non-positive, it is a K-matrix if it is a Z-matrix and also a P-matrix. It is a co-positive matrix if $x^T Ax \geqslant 0$ for all $x \in \mathbb{R}^n_+$, and a strictly co-positive matrix if $x^T Ax > 0$ for all $x \in \mathbb{R}^n_+$, $x \neq 0$. Let $A \in \mathbb{R}^{n\times m}$ be a real matrix, then its Euclidean norm is $\|A\|_2 = (\sum_{i=1}^n \sum_{j=1}^m A_{ij}^2)^{\frac{1}{2}}$. Let $A \in \mathbb{R}^{n\times n}$ be a real matrix, then $Q_A \stackrel{\Delta}{=} \{z \in \mathbb{R}^n | 0 \leqslant z \perp Az \geqslant 0\}$. Its dual is $Q_A^* = \{z \in \mathbb{R}^n | z^T x \geqslant 0 \text{ for all } x \in Q_A\}$. Eigenvalues are denoted as $\lambda_i(A)$. The smallest eigenvalue of A

is denoted $\lambda_{\min}(A)$, the largest one as $\lambda_{\max}(A)$, whereas $\sigma_{\max}(A)$ is its largest singular value ($\sigma_i^2(A) = \lambda_i(AA^T) = \lambda_i(A^T A)$). The multi-valued sign function $\mathbb{R} \to [-1, 1]$ is defined as $\mathrm{sgn}(x) = 1$ if $x > 0$, $\mathrm{sgn}(x) = -1$ if $x < 0$, $\mathrm{sgn}(0) = [-1, 1]$. Let $\mathbb{R}^{n\times n} \ni M = M^T > 0$, then $\|x\|_M \triangleq \sqrt{x^T Mx}$, and $\langle x, y\rangle_M = x^T My$, for all $x, y \in \mathbb{R}^n$. $[a_{ij}]$ denotes the matrix with entries a_{ij}. $\mathcal{R}(A) = \{z|\exists y$ such that $z = Ay\}$ is the range of the matrix A. For a given convex, non- empty set $K \subseteq \mathbb{R}^n$, and a matrix $M = M^T > 0$, its normal cone at x in the metric defined by M is denoted $N_K^M(x) = \{w \in \mathbb{R}^n | \langle w, z - x\rangle_M \leqslant 0$ for all $z \in K\}$. If the metric is the Euclidean one (i.e. $M = I$), then we denote the normal cone as $N_K(\cdot)$, and $N_K^M(\cdot) = M^{-1} N_K(\cdot)$. The indicator function of a set K is defined as $\psi_K(x) = 0$ if $x \in K$, $\psi_K(x) = +\infty$ if $x \notin K$. Let $f(\cdot)$ be a convex, proper, lower semi-continuous function. Then $\partial f(\cdot)$ is its sub-differential of convex analysis. In particular, for a convex non-empty K, one has $\partial\psi_K(\cdot) = N_K(\cdot)$.

2 The kinetic quasi-velocities

Let us consider a Lagrangian system with n degrees of freedom, configuration space $\mathcal{C}$, n-dimensional generalized coordinate vector q with independent coordinates q_i, $i \in \{1, n\}$, inertia matrix $M(q) = M^T(q) > 0$, and potential energy $U(q)$. The tangent space to $\mathcal{C}$ at q is denoted as $T_q\mathcal{C} \ni \dot{q}$, its dual space is denoted as $T_q^*\mathcal{C} \ni F$, where F are the forces that work on $\dot{q}\,dt$. In this paper, two kinds of time-independent constraints are considered: holonomic bilateral constraints, and unilateral constraints. The first take the form $f(q) = 0$, $f : \mathcal{C} \mapsto \mathbb{R}^m$, while the second take the form $h(q) \geqslant 0$, $h : \mathcal{C} \mapsto \mathbb{R}^p$. It is understood that the coordinates q_i are independent when the system is unconstrained, i.e., the system may be treated as a generalized particle, evolving freely on the configuration manifold $\mathcal{C}$. Obviously, the addition of bilateral holonomic constraints may add some dependency between some coordinates. Generally speaking, *quasi-velocities* v are defined as

$$v = A(q)\dot{q} \tag{1}$$

where v has dimension n, and $A(q)$ is invertible, but not necessarily integrable. In other words, there does not necessarily exist any *quasi-position* $\bar{q} = g(q)$ such that $\frac{d\bar{q}}{dt} = \frac{\partial g}{\partial q}(q)\dot{q}$, so that $A(q)$ is not the Jacobian of any mapping $g(q)$. It is clear that v may correspond to some non-holonomic constraints, hence the name non-holonomic velocities that is sometimes given to quasi-velocities. A very generic form of the unconstrained dynamics follows [9, 10]:

$$\begin{aligned} \dot{q} &= A^{-1}(q)v \\ \bar{M}(q)\dot{v} &= H(q, v, t) \end{aligned} \tag{2}$$

where $H(q, v, t)$ gathers inertial forces (centrifugal, Coriolis), forces that derive from the potential energy (gravity, elasticity), and external forces (control inputs, disturbances). Notice that the mass matrix $\bar{M}(q)$ in (2) is not necessarily equal to $M(q)$. This is the case for instance if (2) represents the Newton–Euler dynamics of an unconstrained 3-dimensional rigid body with rotation parameterization using Euler angles. In other cases, $\bar{M}(q)$ may be the identity, showing in passing that in general the dynamics in (2) is not in the Lagrange formalism. Quasi-velocities may also be defined as the projection of the instantaneous angular and gravity center velocities on the axes of some suitable frame [47]. Usually quasi-velocities are applied to decouple and simplify, in a way, the dynamics, with diagonalized Lagrangian dynamics and simplified kinetic energy form [21, 38]. This is also the case, as shown below, of the quasi-velocity that is analyzed in this paper. Clearly, one does not have in general $v \in T_{\bar{q}}\mathcal{C}$ for some generalized coordinate $\bar{q}$, except if $A(q)$ is a Jacobian in which case $v = \dot{\bar{q}}$.

The basic idea for defining the kinetic quasi-velocities as introduced in [14, 16] for the case $m = 0$ and $p = 1$, is first to embed the configuration manifold $\mathcal{C}$ with the kinetic metric that is the metric defined with the inertia matrix $M(q)$: given two vectors x and y in $\mathbb{R}^n$, their inner product is $\langle x, y\rangle_q = x^T M(q)y$. Thus equipped, the configuration space is a Riemannian manifold. This idea is quite natural since the kinetic energy is, together with $\mathcal{C}$ and $U(q)$, a fundamental ingredient of the Lagrange dynamics under study [56], so that $M(q)$ appears as *the* metric of the system. It has been used in several analysis [9, 10, 12, 21, 27, 53, 57, 65]. This is not to be confused with so-called *inertial quasi-velocities* defined from some factorization of $M(q) = m(q)m^T(q)$ and $A(q) = m^T(q)$ in (1), or variants of this [38, 44].

The kinetic quasi-velocities are designed as follows. To simplify the notation and without loss of generality, we assume that the m bilateral constraints are denoted as $f_i(q) \triangleq h_i(q) = 0$ for $i \in \{1, \dots, m\}$, and the p unilateral constraints are $h_i(q) \geqslant 0$ for $i \in \{m + 1, \dots, m + p\}$, with $m + p \leqslant n$. We also assume that all the constraints $h_i(q)$ are functionally independent at any $q \in \mathcal{C}$, that is the $(m + p) \times n$ gradient matrix $\nabla h(q)$ has full column rank $m + p$. This in particular precludes that the gradients vanish in the domain of interest on $\mathcal{C}$. The $m + p$ normal unitary vectors to the co-dimension 1 constraints manifolds $\Sigma_i = \{q \in \mathcal{C} | h_i(q) = 0\}$ equipped with the kinetic metric are defined as

$$\mathbf{n}_{q,i} = \frac{M^{-1}(q)\nabla h_i(q)}{\sqrt{\nabla h_i^T(q)M^{-1}(q)\nabla h_i(q)}} \tag{3}$$

Clearly, the normal vectors $\mathbf{n}_{q,i}$ are independent. If $m + p < n$ we have to complete the set $(\mathbf{n}_{q,m+1}, \dots, \mathbf{n}_{q,m+p})$ by $n - m - p$ mutually independent vectors $\mathbf{t}_{q,i}$ in order to make a basis. The $\mathbf{t}_{q,i}$ vectors are chosen such that $\langle \mathbf{t}_{q,i}, \mathbf{n}_{q,j}\rangle_M = \mathbf{t}_{q,i}^T M(q)\mathbf{n}_{q,j} = 0$ for all $i \in \{1, \dots, n - m - p\}$, $j \in \{1, \dots, m + p\}$. We notice that $\mathbf{t}_{q,i}^T M(q)\mathbf{n}_{q,j} = \frac{\mathbf{t}_{q,i}^T \nabla h_i(q)}{\sqrt{\nabla h_i^T(q)M^{-1}(q)\nabla h_i(q)}}$ so that the vectors $\mathbf{t}_{q,i}^T$ are orthogonal to the kinetic gradients $\mathbf{n}_{q,j}$ in the kinetic metric, and orthogonal to the Euclidean gradients $\nabla h_i(q)$ in the Euclidean metric. One may choose unitary vectors $\mathbf{t}_{q,i}$, i.e. $\mathbf{t}_{q,i}^T M(q)\mathbf{t}_{q,i} = 1$. Therefore, the vectors $\mathbf{t}_{q,i}$ span $T_q\mathcal{C}$ whereas the vectors $\mathbf{n}_{q,i}$ span the normal cone $N_\Phi(q)$ to the admissible domain Φ of $\mathcal{C}$. This admissible domain for q is defined as follows: $\Phi \triangleq \Phi_b \times \Phi_u$ with $\Phi_b = \{q \in \mathcal{C} | h_i(q) = 0,\ i \in \{1, \dots, m\}\}$ and $\Phi_u = \{q \in \mathcal{C} | h_i(q) \geqslant 0,\ i \in \{m + 1, \dots, m + p\}\}$. Thus, Φ_b is the bilateral holonomic constraints manifold with co-dimension m, $\Phi_b = \bigcap_{i=1}^m \Sigma_i$, whereas Φ_u is the admissible domain defined by the unilateral constraints, $\Phi_u = \bigcap_{i=m+1}^{m+p} \Phi_{u,i}$, with $\Phi_{u,i} = \{q \in \mathcal{C} | h_i(q) \geqslant 0,\ i \in \{m + 1, \dots, m + p\}\}$. For obvious reasons, we assume that Φ_u contains a ball of radius > 0. One has $N_\Phi(q) = N_{\Phi_b}(q) \times N_{\Phi_u}(q)$, where $N_{\Phi_b}(q) = \{w \in \mathbb{R}^n | w = \sum_{i=1}^m \alpha_i \mathbf{n}_{q,i},\ \alpha_i \in \mathbb{R}\}$. It is in fact possible to use more geometrical arguments to characterize the bilateral constraints following, e.g. the developments for passive decomposition in [53, Eq. (10) (11)], see also [57] and [10]. Concerning the geometrical interpretation in case of unilateral constraints, see [35].

Let us now define the set of *active* unilateral constraints as $\mathcal{I}_a(q) = \{i \in \{m + 1, \dots, m + p\} | h_i(q) = 0\}$. Then the normal cone to Φ_u at q in the kinetic metric is given by:[1]

[1]This is the set denoted $\mathcal{T}_C^\perp$ in [35, Eq. (5.11)].

$$N^M_{\Phi_u}(q) = \left\{ w \in \mathbb{R}^n | \langle w, q' - q \rangle_M \leqslant 0 \text{ for all } q' \in \Phi_u \right\}$$
$$= \left\{ w \in \mathbb{R}^n | w = \sum_{i \in \mathcal{I}_a(q)} -\alpha_i \mathbf{n}_{q,i}, \ \alpha_i \geqslant 0 \right\} \tag{4}$$

If $\mathcal{I}_a(q) = \emptyset$ (the system evolves strictly inside Φ_u), then following [69] we adopt the convention that $N^M_{\Phi_u}(q) = \{0\}$. The minus sign pre-multiplying α_i comes from the fact that one considers the outwards normal cone, which is generated by the vectors $-\mathbf{n}_{q,i}$. In (4), the dual space $T^*_q\mathcal{C}$ is identified with $\mathbb{R}^n$. The expression in (4) holds when Φ_u is for instance a prox-regular set (including convex sets), and constraints are independent at all $q \in \Phi$; see Theorem 6.14 in [79]. This in turn is closely related to the so-called Clarke or tangential regularity of Φ_u, which implies the absence of re-intrant corners [79, §6.B]. One classically defines the polar cone to $N^M_{\Phi_u}(q)$, which is the tangent cone defined as [69]:

$$T_{\Phi_u}(q) = \left\{ w \in \mathbb{R}^n | \langle w, \mathbf{n}_{q,i} \rangle_M = \frac{1}{\|h_i(q)\|_{M^{-1}}} w^T \nabla h_i(q) \geqslant 0, \text{for all } i \in \mathcal{I}_a(q) \right\} \tag{5}$$

where this time $T_q\mathcal{C}$ is identified with $\mathbb{R}^n$, and $\|\nabla h_i(q)\|_{M^{-1}} = \sqrt{\nabla h_i^T(q) M^{-1}(q) \nabla h_i(q)}$. Thus, $N^M_{\Phi_u}(q) \subset T^*_q\mathcal{C}$ while $T_{\Phi_u}(q) \subset T_q\mathcal{C}$. Actually $T_{\Phi_u}(q)$ is generated by p vectors $\mathbf{t}_{q,i}$ such that $\langle \mathbf{t}_{q,i}, \mathbf{n}_{q,i} \rangle_q = 0$. The normal and tangent cones replace, for unilaterally constrained systems, the classical constraint subspace and its orthogonal subspace. They are both convex polyhedral cones [69].

Let us define the $n \times n$ matrix $\Xi(q) = \begin{pmatrix} \mathbf{n}_q^T \\ \mathbf{t}_q^T \end{pmatrix}$, where $\mathbf{n}_q = (\mathbf{n}_{q,m+1}, \ldots, \mathbf{n}_{q,m+p})$ and $\mathbf{t}_q = (\mathbf{t}_{q,1}, \ldots, \mathbf{t}_{q,m})$. The kinetic quasi-velocities are defined as

$$v \triangleq \begin{pmatrix} \dot{q}_{\text{norm}} \\ \dot{q}_{\text{tan}} \end{pmatrix} = \Xi(q) M(q) \dot{q} \tag{6}$$

where the notation norm and tan come from the fact that v in (6) is the Euclidean projection of the generalized momentum $p = M(q)\dot{q}$ on the basis $\mathbf{n}_q$ and $\mathbf{t}_q$ (equivalently the projection of $\dot{q}$ on $\mathbf{n}_q$ and $\mathbf{t}_q$ in the kinetic metric). One could therefore call the kinetic quasi-velocities, the *mass-projected momentum*. From (6), $\dot{q}_{\text{norm}} = \mathbf{n}_q^T M(q)\dot{q}$ has dimension $m + p$ and $\dot{q}_{\text{tan}} = \mathbf{t}_q^T M(q)\dot{q}$ has dimension $n - m - p$. Notice that the $(m+p) \times n$ matrix $\mathbf{n}_q^T M(q)$ has rows $\frac{\nabla h_i^T(q)}{\|\nabla h_i(q)\|_{M^{-1}}}$. Thus, $\dot{q}_{\text{norm},i} = \frac{\nabla h_i^T(q)\dot{q}}{\|\nabla h_i(q)\|_{M^{-1}}}$, and $\mathbf{n}_q = M^{-1}(q) \nabla h(q) \,\text{diag}(\frac{1}{\|\nabla h_i(q)\|_{M^{-1}}})$.

Remark 1 The use of the kinetic metric for the study of multiple impacts was perhaps first advocated in [41]. It is also implicitly present in Moreau's works [68, 69] where the tangent and normal cones are defined in a generic way, independently of the metric; see also [35, Sect. 4] for a detailed analysis. It is also used in mathematical proofs for convergence of numerical schemes [27]. It is clear that as far as one analyses the system at a fixed q (like for impacts), then $M(q)$ is constant and the metric is Euclidean.

2.1 Frictionless Lagrangian systems

The frictionless Lagrange dynamics is given by

$$M(q)\ddot{q} + F(q, \dot{q}, t) = \nabla h(q)\lambda_n = \sum_{i=1}^{m} \nabla h_i(q)\lambda_{n,i} + \sum_{i=m+1}^{m+p} \nabla h_i(q)\lambda_{n,i} \tag{7}$$

where other ingredients like complementarity conditions and impact law are disregarded for the moment, and we recall that $h_i(q) = 0$ for all $i \in \{1, \dots, m\}$. The $m + p$ vector λ_n groups Lagrange multipliers associated with the constraints. Let us denote $F_u(q) = \sum_{i=m+1}^{m+p} \nabla h_i(q)\lambda_{n,i}$ the generalized contact force associated with the unilateral constraints. Notice that for a given q one has $F_u(q) \in -M(q)N^M_{\Phi_u}(q) = N_{\Phi_u}(q)$ due to the non-negativity of the multipliers, in particular $F_u(q) = 0$ if q is in the interior of Φ_u. Let us now perform the kinetic quasi-velocity transformation of the Lagrange dynamics (7). First notice that

$$\begin{pmatrix} \ddot{q}_{\text{norm}} \\ \ddot{q}_{\tan} \end{pmatrix} = \Xi(q)M(q)\ddot{q} + \frac{d}{dt}\big(\Xi(q)M(q)\big)\dot{q} \tag{8}$$

Pre-multiplying both sides of (7) by $\Xi(q)$ one obtains

$$\begin{pmatrix} \ddot{q}_{\text{norm}} \\ \ddot{q}_{\tan} \end{pmatrix} + \Xi(q)F(q,\dot{q},t) - \frac{d}{dt}\big(\Xi(q)M(q)\big)\dot{q} = \begin{pmatrix} \mathbf{n}_q^T \nabla h(q)\lambda_n \\ \mathbf{t}_q^T \nabla h(q)\lambda_n \end{pmatrix} \tag{9}$$

Let us define $\bar{\lambda}_n$ such that $\bar{\lambda}_{n,i} \triangleq \|\nabla h_i(q)\|_{M^{-1}}\lambda_{n,i}$, i.e. $\bar{\lambda}_n = \text{diag}(\|\nabla h_i(q)\|_{M^{-1}})\lambda_n$.[2] From the definition of $\mathbf{t}_{q,i}$, it follows that $\mathbf{t}_q^T \nabla h(q)\lambda_n = 0$, therefore, (9) becomes:

$$\begin{aligned} &\ddot{q}_{\text{norm}} + F_{\text{norm}}(q, \dot{q}_{\text{norm}}, \dot{q}_{\tan}, t) = \mathbf{n}_q^T M(q)\mathbf{n}_q \bar{\lambda}_n \\ &\ddot{q}_{\tan} + F_{\tan}(q, \dot{q}_{\text{norm}}, \dot{q}_{\tan}, t) = 0 \end{aligned} \tag{10}$$

with obvious definitions for $F_{\text{norm}}(q, \dot{q}_{\text{norm}}, \dot{q}_{\tan}, t)$ and $F_{\tan}(q, \dot{q}_{\text{norm}}, \dot{q}_{\tan}, t)$. The terms indexed by tan are not affected by the contact force and may be thought of as some kind of tangential dynamics. We may choose to call the first line of (10) the *quasi-normal dynamics* and the second line the *quasi-tangential dynamics*. The dynamics in (6) (10) is consequently a particular case of (2). It is clear that (10) usually is not a Lagrange dynamics since $\bar{M}$ is constant (the identity) whereas non-linear inertial forces do not vanish (such dynamics are sometimes called Lagrange's equations in quasi-velocities, or Boltzmann–Hamel equations [10], and they may be written in a Lagrangian-like form [26, 30]). The so-called Delassus' matrix defined when $m = 0$ (only unilateral constraints) is equal to $\nabla h^T(q)M^{-1}(q)\nabla h(q)$ [1]. The matrix $\mathbf{n}_q^T M(q)\mathbf{n}_q$ may be seen as a normalized Delassus' matrix,[3] whose diagonal entries are equal to 1. It is positive definite if and only if $\mathbf{n}_q$ has full rank $m + p$. Notice that we can split $\dot{q}_{\text{norm}}$ as $\dot{q}_{\text{norm}} = \begin{pmatrix} \dot{q}^b_{\text{norm}} \\ \dot{q}^u_{\text{norm}} \end{pmatrix}$ with $\dot{q}^b_{\text{norm}} \in \mathbb{R}^m$ corresponds to bilateral constraints, and $\dot{q}^u_{\text{norm}} \in \mathbb{R}^p$ corresponds to unilateral constraints. Similarly, one has

$$\mathbf{n}_q^T M(q)\mathbf{n}_q = \begin{pmatrix} \mathbf{n}_q^{b,T} M(q)\mathbf{n}_q^b & \mathbf{n}_q^{b,T} M(q)\mathbf{n}_q^u \\ \mathbf{n}_q^{u,T} M(q)\mathbf{n}_q^b & \mathbf{n}_q^{u,T} M(q)\mathbf{n}_q^u \end{pmatrix} \tag{11}$$

Thus, the first line in (10) can be rewritten as

$$\begin{aligned} &\ddot{q}^b_{\text{norm}} + F^b_{\text{norm}}(q, \dot{q}_{\text{norm}}, \dot{q}_{\tan}, t) = \mathbf{n}_q^{b,T} M(q)\mathbf{n}_q^b \bar{\lambda}_n^b + \mathbf{n}_q^{b,T} M(q)\mathbf{n}_q^u \bar{\lambda}_n^u \\ &\ddot{q}^u_{\text{norm}} + F^u_{\text{norm}}(q, \dot{q}_{\text{norm}}, \dot{q}_{\tan}, t) = \mathbf{n}_q^{u,T} M(q)\mathbf{n}_q^b \bar{\lambda}_n^b + \mathbf{n}_q^{u,T} M(q)\mathbf{n}_q^u \bar{\lambda}_n^u \end{aligned} \tag{12}$$

[2] Notice that the assumption that the constraints are functionally independent, guarantees that the norms $\|\nabla h_i(q)\|_{M^{-1}}$ never vanish, so $\text{diag}(\|\nabla h_i(q)\|_{M^{-1}})$ is positive definite.

[3] The Delassus' operator is sometimes called the *fundamental matrix* [12].

where $\bar{\lambda}_n^b$ collects the first m multipliers $\bar{\lambda}_{n,i}$ and $\bar{\lambda}_n^u$ collects the last p multipliers $\bar{\lambda}_{n,i}$. Since $\dot{q}_{\text{norm}}^b = 0$ at all times because the system evolves on the co-dimension m manifold $\{q \in \mathcal{C} | h_i(q) = 0, \nabla h_i^T(q)\dot{q} = 0, i \in \{1, \ldots, m\}\}$, the first equation in (12) is equal to $F_{\text{norm}}^b(q, \dot{q}_{\text{norm}}^u, \dot{q}_{\text{tan}}, t) = \mathbf{n}_q^{b,T} M(q)\mathbf{n}_q^b \bar{\lambda}_n^b + \mathbf{n}_q^{b,T} M(q)\mathbf{n}_q^u \bar{\lambda}_n^u$. If the $m \times m$ matrix $\mathbf{n}_q^{b,T} M(q)\mathbf{n}_q^b$ is invertible one may obtain λ_n^b from this equation and insert it into the second equation in (12) to obtain a dynamics that no longer depends on $\bar{\lambda}_n^b$. This modifies the unilateral part of the dynamics (and in particular one obtains a new Delassus' matrix in (17)); see [20] for detailed analysis of the couplings between unilateral and bilateral constraints.

2.1.1 *The kinetic energy*

The kinetic energy of our system is $T(q, \dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q}$. Clearly $\dot{q}_{\text{norm}}^b$ does not play any role in the kinetic energy, being zero. We will see later that the same applies to $\dot{q}_{\text{tan}}$ when one considers the kinetic energy variation at an impact. Let us assume that $\Xi(q)$ has full rank n. One has

$$\begin{aligned} \Xi(q)M(q)\Xi^T(q) &= \begin{pmatrix} \mathbf{n}_q^T \\ \mathbf{t}_q^T \end{pmatrix} M(q)(\mathbf{n}_q \mathbf{t}_q) \\ &= \begin{pmatrix} \mathbf{n}_q^T M(q)\mathbf{n}_q & 0 \\ 0 & \mathbf{t}_q^T M(q)\mathbf{t}_q \end{pmatrix} \end{aligned} \tag{13}$$

from which one deduces the inverse matrix:

$$\Xi^{-T}(q)M^{-1}(q)\Xi^{-1}(q) = \begin{pmatrix} (\mathbf{n}_q^T M(q)\mathbf{n}_q)^{-1} & 0 \\ 0 & (\mathbf{t}_q^T M(q)\mathbf{t}_q)^{-1} \end{pmatrix} \tag{14}$$

which holds provided the normalized Delassus' matrix has full rank n. Now we have:

$$\begin{aligned} T(q, \dot{q}) &= \frac{1}{2}\dot{q}^T M(q)\dot{q} \\ &= \frac{1}{2}\dot{q}^T M(q)\Xi^T(q)\Xi^{-T}(q)M^{-1}(q)\Xi^{-1}(q)\Xi(q)M(q)\dot{q} \\ &= \frac{1}{2}v^T \begin{pmatrix} (\mathbf{n}_q^T M(q)\mathbf{n}_q)^{-1} & 0 \\ 0 & (\mathbf{t}_q^T M(q)\mathbf{t}_q)^{-1} \end{pmatrix} v \\ &= \frac{1}{2}\dot{q}_{\text{norm}}^T \left(\mathbf{n}_q^T M(q)\mathbf{n}_q\right)^{-1} \dot{q}_{\text{norm}} + \frac{1}{2}\dot{q}_{\text{tan}}^T \left(\mathbf{t}_q^T M(q)\mathbf{t}_q\right)^{-1} \dot{q}_{\text{tan}} = T(q, v) \end{aligned} \tag{15}$$

Now one may use (11) and the Schur complement [18, Sect. A.5] to deduce:

$$T(q, \dot{q}) = \frac{1}{2}\dot{q}_{\text{norm}}^{u,T} G^{-1}(q)\dot{q}_{\text{norm}}^u + \frac{1}{2}\dot{q}_{\text{tan}}^T \left(\mathbf{t}_q^T M(q)\mathbf{t}_q\right)^{-1} \dot{q}_{\text{tan}} \tag{16}$$

with

$$G(q) = \mathbf{n}_q^{u,T} M(q)\mathbf{n}_q^u - \mathbf{n}_q^{u,T} M(q)\mathbf{n}_q^b \left(\mathbf{n}_q^{b,T} M(q)\mathbf{n}_q^b\right)^{-1} \mathbf{n}_q^{b,T} M(q)\mathbf{n}_q^u \tag{17}$$

If $m = 0$ (no bilateral constraints) and $p = 1$, then $\dot{q}_{\text{norm}}^u = \dot{q}_{\text{norm}}$ and one recovers the result in [16, Eq. (6.11)] that $T(q, \dot{q}) = \frac{1}{2}\dot{q}_{\text{norm}}^2 + \frac{1}{2}\dot{q}_{\text{tan}}^T(\mathbf{t}_q^T M(q)\mathbf{t}_q)^{-1}\dot{q}_{\text{tan}}$.

It is noteworthy that the basis $(\mathbf{n}_q, \mathbf{t}_q)$ is not ortho-normal, because the vectors $\mathbf{n}_{q,i}$, $i \in \{1, \ldots, m+p\}$, and $\mathbf{t}_{q,i}$, $i \in \{1, \ldots, n-m-p\}$ are not necessarily orthogonal to one another (except if the constraints are orthogonal). Thus, despite the quasi-mass matrix $\bar{M}(q)$ in (10) is the identity, the kinetic energy in (16) does not have the simple form $2T(q, v) = v^T v$ as is for instance the case in [10, Eq. (20)].

Remark 2 Let $h^u(q) = (h_{m+1}(q), \ldots, h_{m+p}(q))^T$. The multiplier vector λ_n^u satisfies the complementarity conditions $\lambda_n^u \geqslant 0$, $h^u(q) \geqslant 0$, $(\lambda_n^u)^T h^u(q) = 0$, written compactly as $0 \leqslant \lambda_n^u \perp h^u(q) \geqslant 0$. If $\dot{q}_{\text{norm}}$ is integrable, i.e. there exists q_{norm} such that $\frac{d}{dt} q_{\text{norm}} = \dot{q}_{\text{norm}}$, then the complementarity conditions can be written as $0 \leqslant q_{\text{norm}} \perp \lambda_n^u \geqslant 0$. It is, however, always true that on a contact time interval where $h_i(q) = 0$ persistently, then $\dot{q}_{\text{norm,i}}(t^+) \geqslant 0$ and $\ddot{q}_{\text{norm},i}(t^+) \geqslant 0$, where we implicitly suppose that velocities and accelerations are right-continuous functions of time. This will allow us to construct the so-called *contact linear complementarity problem* (contact LCP) to calculate λ_n^u.

2.1.2 The normal contact forces power

Let us denote $F_c(q) = \nabla h(q)\lambda_n$. Let us investigate now the power performed by the generalized contact force: $\mathcal{P}_n = F_c^T(q)\dot{q}$ where $\dot{q}$ is assumed to be compatible with the bilateral and the unilateral constraints (i.e. we consider virtual velocities $\dot{q}$ such that the virtual displacement $\delta q = \dot{q}dt$ is compatible with the constraints and such that the virtual work is $\mathcal{W}_n = \mathcal{P}_n \, dt$). Then from the above developments we obtain:

$$\begin{aligned} \mathcal{P}_n &= F_c^T(q)\dot{q} = \lambda_n^T \nabla h^T(q)\dot{q} = \lambda_n^T \operatorname{diag}\left(\left\|\nabla h_i(q)\right\|_{M^{-1}}\right)\mathbf{n}_q^T M(q)\dot{q} \\ &= \bar{\lambda}_n^T \mathbf{n}_q^T M(q)\dot{q} = \bar{\lambda}_n^T \dot{q}_{\text{norm}} = \bar{\lambda}_n^{u,T} \dot{q}_{\text{norm}}^u \end{aligned} \tag{18}$$

where we used that $\dot{q}_{\text{norm}}^b = 0$ always. Now one has $0 \leqslant \lambda_n^u \perp h^u(q) \geqslant 0$, therefore, if the system lies in the interior of the admissible domain Φ_u one has $\lambda_n^u = 0$ and $\mathcal{P}_n = 0$. If the system evolves smoothly on a part of the boundary $\text{bd}(\Phi_u)$ that is finitely represented by the active constraints indexed in $\mathcal{I}_a$, one has $0 \leqslant \dot{q}_{\text{norm},i} \perp \lambda_{n,i}^u \geqslant 0$ for all $i \in \mathcal{I}_a$. Consequently, in this case also $\mathcal{P}_n = 0$. Since the constraints are all perfect, the power developed by the contact forces outside possible impacts is always zero, as expected. The interest of (18) is to highlight the fact that the "forces" that perform work on the quasi-velocities $\dot{q}_{\text{norm}}^u$ are the multipliers $\bar{\lambda}_n^u$. This may be useful when dealing with impacts where the work done by the normal force during the collision has to be calculated (compare (18) and (47)).

Let us denote $F_{\text{norm}}^c(q) \stackrel{\Delta}{=} \mathbf{n}_q^T M(q)\mathbf{n}_q\bar{\lambda}_n$ and $D_{\text{n}}(q) = (\mathbf{n}_q^T M(q)\mathbf{n}_q)^{-1}$. Then from (18), one gets

$$\mathcal{P}_n = \bar{\lambda}_n^T \dot{q}_{\text{norm}} = \left\langle F_{\text{norm}}^c(q), \dot{q}_{\text{norm}} \right\rangle_{D_{\text{n}}} \tag{19}$$

Let us also denote $D_{\text{t}}(q) = (\mathbf{t}_q^T M(q)\mathbf{t}_q)^{-1}$.[4] As a result, one finds that the frictionless Lagrangian system (7) with a set of holonomic bilateral and unilateral constraints is equivalently represented as a generalized particle with dynamics

$$\begin{aligned} &\ddot{q}_{\text{norm}} + F_{\text{norm}}(q, \dot{q}_{\text{norm}}, \dot{q}_{\text{tan}}, t) = F_{\text{norm}}^c(q) \\ &\ddot{q}_{\text{tan}} + F_{\text{tan}}(q, \dot{q}_{\text{norm}}, \dot{q}_{\text{tan}}, t) = 0 \end{aligned} \tag{20}$$

[4]Clearly, if the vectors $\mathbf{t}_{q,i}$ are chosen mutually orthogonal then $D_{\text{t}}(q) = I$.

and the kinetic metric $D(q) = \mathrm{diag}(D_{\mathrm{n}}(q), D_{\mathrm{t}}(q))$ (see (15)), while $\dot{q}_{\mathrm{norm}}\,dt$ performs work on $F^c_{\mathrm{norm}}(q)$ in the metric of $D_{\mathrm{n}}(q)$.

2.1.3 The kinetic angles

The kinetic angles play a crucial role in systems with multiple unilateral constraints [1, 6, 31, 74]. The kinetic angle between two active constraints i and j at q is given by

$$\theta_{ij}(q) = \pi - \arccos \frac{\nabla f_i(q)^T M^{-1}(q) \nabla f_j(q)}{\sqrt{\nabla f_i(q)^T M^{-1}(q) \nabla f_i(q)}\sqrt{\nabla f_j(q)^T M^{-1}(q) \nabla f_j(q)}} \tag{21}$$

Kinetic angles are quantities that reflect the couplings between the inertial properties and the geometrical properties of the system with unilateral constraints. It readily follows that

$$\mathbf{n}_q^T M(q) \mathbf{n}_q = \left[\cos(\pi - \theta_{ij})\right] = -\left[\cos(\theta_{ij})\right] \tag{22}$$

In particular, the diagonal entries are $-\cos(\theta_{ii}) = 1$. One sees from (10) (11) that the normal part of the dynamics is strongly influenced by the kinetic angles matrix. This is known to play a crucial role in existence and uniqueness of solutions [6], continuous dependence on initial data [1, 31, 74], couplings between unilateral and bilateral constraints [20], and dynamics at impacts [19].

2.2 Systems with Coulomb's friction

To simplify the analysis, from now on we assume that $m = 0$, that is there are no bilateral constraints, and all the unilateral constraints are rough with Coulomb's frictional effects. The Lagrange dynamics with tangential effects at the contacts have the generic form:

$$M(q)\ddot{q} + F(q, \dot{q}, t) = \nabla h^u(q) \lambda_n^u + H_{\mathrm{T}}(q) \lambda_{\mathrm{t}} \tag{23}$$

where $\nabla h^u(q)\lambda_n^u = \sum_{i=1}^p \nabla h_i(q) \lambda_{n,i}$, λ_{t} collects the p multipliers that represent tangential contact forces at the contact points, and $H_{\mathrm{T}}(q)$ is an $n \times p$ transformation matrix from the local frames at the contact points to the configuration space [1, Chap. 3] [34, 55, 77]. Premultiplying both sides of (23) by $\Xi(q)$, one obtains:

$$\begin{aligned} \ddot{q}_{\mathrm{norm}} - \frac{d}{dt}\left(\mathbf{n}_q^T M(q)\right)\dot{q} + \mathbf{n}_q^T F(q, \dot{q}, t) &= \mathbf{n}_q^T M(q) \mathbf{n}_q \bar{\lambda}_n^u + \mathbf{n}_q^T H_{\mathrm{T}}(q) \lambda_{\mathrm{t}} \\ \ddot{q}_{\mathrm{tan}} - \frac{d}{dt}\left(\mathbf{t}_q^T M(q)\right)\dot{q} + \mathbf{t}_q^T F(q, \dot{q}, t) &= \mathbf{t}_q^T H_{\mathrm{T}}(q) \lambda_{\mathrm{t}} \end{aligned} \tag{24}$$

It is remarkable in (24) that there is no reason in general that $\mathbf{n}_q^T H_{\mathrm{T}}(q) = 0$, i.e. $\mathbf{n}_q$ is not in general an annihilator of $H_{\mathrm{T}}(q)$. This means that the quasi-tangential dynamics may influence the quasi-normal dynamics, but the reverse never holds since by construction of the basis $(\mathbf{n}_q, \mathbf{t}_q)$ one has $\mathbf{t}_q^T \nabla h(q) = 0$. We will see later that this explains some of the difficulties encountered when facing Painlevé paradoxes.

2.2.1 The contact LCP

Let us construct the contact complementarity problem for (24). This is done using the complementarity conditions: $0 \leqslant \lambda_n^u \perp \ddot{q}_{\mathrm{norm}} \geqslant 0$ on phases of persistent contact with $h^u(q) = 0$

and $\dot{q}_{\mathrm{norm}} = 0$:

$$0 \leqslant \bar{\lambda}_n^u \perp \mathbf{n}_q^T M(q)\mathbf{n}_q \bar{\lambda}_n^u + \mathbf{n}_q^T H_{\mathrm{T}}(q)\lambda_{\mathrm{t}} + \frac{d}{dt}\left(\mathbf{n}_q^T M(q)\right)\dot{q} - \mathbf{n}_q^T F(q,\dot{q},t) \geqslant 0 \tag{25}$$

Inserting Coulomb's model $\lambda_{\mathrm{t},i} \in -\mu_i \lambda_{n,i} \operatorname{sgn}(v_{\mathrm{t},i})$, where $v_{\mathrm{t},i} = H_{\mathrm{T},i}^T(q)\dot{q}$ is the relative tangential velocity at the contact point i, one obtains a complementarity problem with unknown $\bar{\lambda}_n$. Here, $H_{\mathrm{T},i}$ is the ith column of H_{T} and sgn is the multi-valued sign function. To obtain (25), we also used the fact that since λ_n^u and $\bar{\lambda}_n^u$ are related with a diagonal positive definite matrix, stating the orthogonality and the non-negativity conditions with one or the other vector is equivalent. We may rewrite the contact LCP with Coulomb's friction as

$$\begin{aligned}0 \leqslant \bar{\lambda}_n^u \perp &\left(\mathbf{n}_q^T M(q)\mathbf{n}_q - \mathbf{n}_q^T H_{\mathrm{T}}(q) \operatorname{diag}\left(\frac{\mu_i \xi_i}{\|\nabla h_i(q)\|_{M^{-1}}}\right)\right)\bar{\lambda}_n^u + \frac{d}{dt}\left(\mathbf{n}_q^T M(q)\right)\dot{q} \\ &- \mathbf{n}_q^T F(q,\dot{q},t) \geqslant 0\end{aligned} \tag{26}$$

with $\xi_i \in \operatorname{sgn}(v_{\mathrm{t},i})$. The existence and uniqueness of solutions to this problem depends on the matrix $D(q,\mu,v_{\mathrm{t}}) \triangleq (\mathbf{n}_q^T M(q)\mathbf{n}_q - \mathbf{n}_q^T H_{\mathrm{T}}(q) \operatorname{diag}(\frac{\mu_i \xi_i}{\|\nabla h_i(q)\|_{M^{-1}}}))$ being or not a P-matrix. In the following, we shall denote $b(q,\dot{q},t) \triangleq \frac{d}{dt}(\mathbf{n}_q^T M(q))\dot{q} - \mathbf{n}_q^T F(q,\dot{q},t)$.

Remark 3 One has $\dot{q}_{\tan} = \mathbf{t}_q^T M(q)\dot{q}$ and $v_{\mathrm{t}} = H_{\mathrm{T}}^T(q)\dot{q}$. As alluded to above, $\mathbf{t}_q$ spans $T_q\mathcal{C}$ so that $\dot{q}_{\tan}$ is a projection of the generalized momentum on $T_q\mathcal{C}$. The vector v_{t} collects the tangential velocities in local Euclidean frames at the contact points. There is in general no reason that $\dot{q}_{\tan}$ and v_{t} have the same physical units. There are even examples where only normal local velocities are defined, while the quasi-tangent dynamics exists (see the aligned chain of balls example below).

2.2.2 *The tangential contact forces power*

Let us compute the virtual power developed by the tangential forces. Let us denote $F^{\mathrm{t}}(q) = \Xi(q)H_{\mathrm{T}}(q)\lambda_{\mathrm{t}} = \begin{pmatrix} F^{\mathrm{t}}_{\mathrm{norm}}(q) \\ F^{\mathrm{t}}_{\tan}(q)\end{pmatrix}$. Then:

$$\begin{aligned}\mathcal{P}_{\mathrm{t}} &= \dot{q}^T H_{\mathrm{T}}(q)\lambda_{\mathrm{t}} = v^T \Xi^{-T}(q)M^{-1}(q)\Xi^{-1}(q)\Xi(q)H_{\mathrm{T}}(q)\lambda_{\mathrm{t}} \\ &= \left\langle v, F^{\mathrm{t}}(q)\right\rangle_D = \left\langle \dot{q}_{\mathrm{norm}}, F^{\mathrm{t}}_{\mathrm{norm}}(q)\right\rangle_{D_n} + \left\langle \dot{q}_{\tan}, F^{\mathrm{t}}_{\tan}(q)\right\rangle_{D_t}\end{aligned} \tag{27}$$

Thus, the total virtual power of the contact forces of the dynamics in (24) is equal to

$$\mathcal{P} = \left\langle F^c_{\mathrm{norm}}(q), \dot{q}_{\mathrm{norm}}\right\rangle_{D_{\mathrm{n}}} + \left\langle \dot{q}_{\mathrm{norm}}, F^{\mathrm{t}}_{\mathrm{norm}}(q)\right\rangle_{D_{\mathrm{n}}} + \left\langle \dot{q}_{\tan}, F^{\mathrm{t}}_{\tan}(q)\right\rangle_{D_{\mathrm{t}}} \tag{28}$$

The positive definite matrices $D_{\mathrm{n}}(q)$ and $D_{\mathrm{t}}(q)$ define natural metrics for the system analyzed in kinetic quasi-velocities. The coupling between normal and tangential directions appears in the second term in (28). There is no orthogonality of the quasi-generalized contact forces $\begin{pmatrix} F^{\mathrm{t}}_{\mathrm{norm}}(q) \\ F^{\mathrm{t}}_{\tan}(q)\end{pmatrix}$ and $\begin{pmatrix} F^{c}_{\mathrm{norm}}(q) \\ 0\end{pmatrix}$ in the inner product defined by the metric $D(q)$. This is in contrast with what happens at the local kinematics level [1] at the p contact points. Let us denote the ortho-normal local frame at contact point i as $(n_i, t_{i,1}, t_{i,2})$, with $n_i \in \mathbb{R}^3$, $t_{i,j} \in \mathbb{R}^3$. One has $\langle n_i, t_{i,j}\rangle = 0$ in the Euclidean metric. Each contact force can be denoted as $F_i^c = F_{i,n}^c + F_{i,t}^c$ with $F_{i,n}^c = f_{i,n} n_i$ and $F_{i,t}^c = f_{i,t,1} t_{i,1} + f_{i,t,2} t_{i,2}$. The p Coulomb's cones are denoted as $\mathbf{C}_i$, with $F_i^c \in \mathbf{C}_i$. Let $U_i \in \mathbb{R}^3$ be the local velocity, decomposed naturally as $U_i = U_{i,n} + U_{i,t} = u_{i,n} n_i + u_{i,t,1} t_{i,1} + u_{i,t,2} t_{i,2}$. We may thus define virtual local velocities that are compatible with the constraints, and the virtual power at contact i is given by $\mathcal{P}_i = \langle U_i, F_i^c \rangle = \langle U_{i,n}, F_{i,n}^c \rangle + \langle U_{i,t}, F_{i,t}^c \rangle$, while

$$\mathcal{P} = \sum_{i=1}^{p} \mathcal{P}_{i,n} + \mathcal{P}_{i,t} = \mathcal{P}_n + \mathcal{P}_\mathrm{t} \tag{29}$$

Thus, in the local kinematics there is a decoupling between tangential and normal virtual powers, which does not transport very well into generalized frameworks, because of the term $\mathbf{n}_q^T H_\mathrm{T}(q)$ in (24). Notice that if $u_{i,n} = \nabla h_i^T(q)\dot{q}$, then the multipliers λ_n in (7) satisfy $\lambda_{n,i} = f_{i,n}$, and thus $\mathcal{P}_n$ in (18) and $\mathcal{P}_n$ in (29) are the same.

2.3 Examples

In this section, several simple systems are presented and recast into the kinetic quasi-velocity framework.

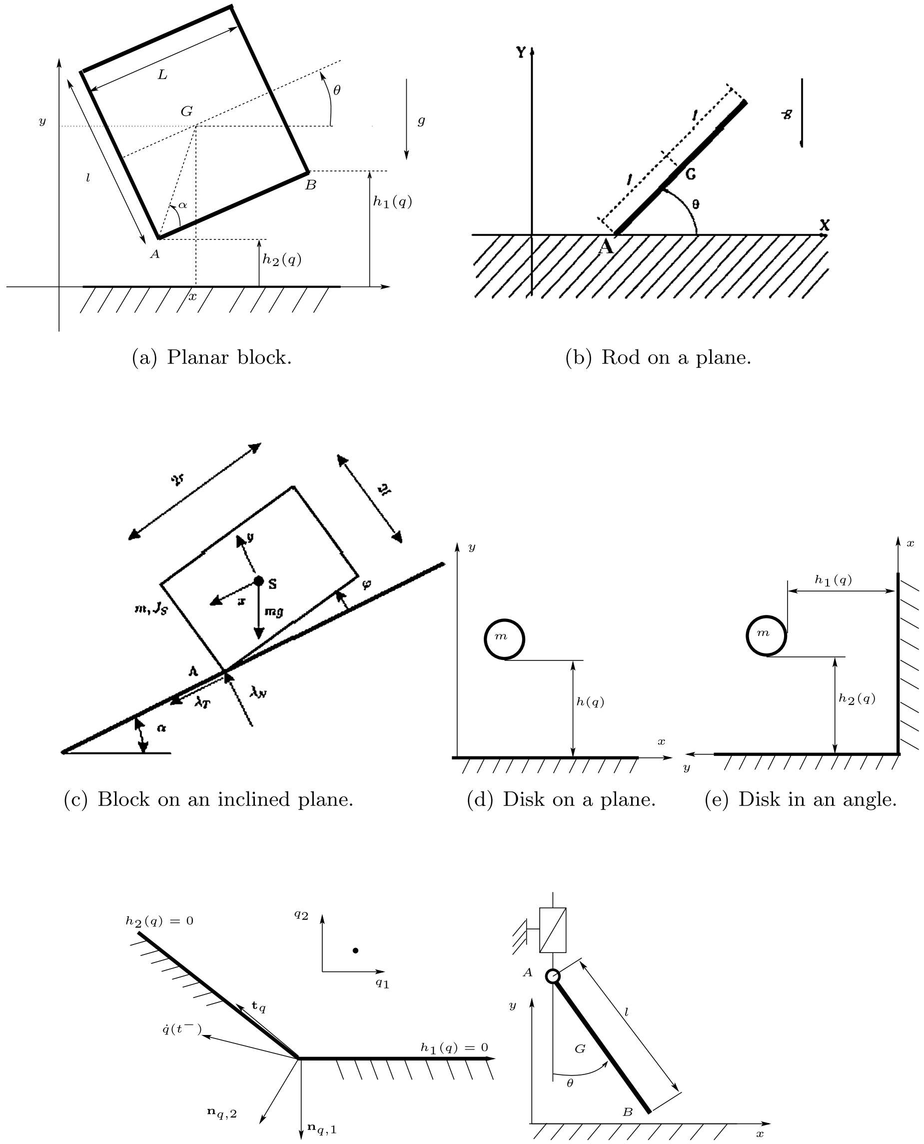
2.3.1 Chains of balls

Let us consider a chain of N aligned balls with masses m_i, radii R and positions q_i. There are $N-1$ unilateral constraints $h_i(q) = q_{i+1} - q_i - 2R \geqslant 0$. Thus $\nabla h_i(q) = (0 \ldots 0\ -1\ 1\ 0 \ldots 0)^T$, where the -1 is at the ith place and the 1 is at the $(i+1)$ place, $\sqrt{\nabla h_i^T M^{-1} \nabla h_i} = \sqrt{\frac{m_i + m_{i+1}}{m_i m_{i+1}}}$, $\mathbf{n}_{q,i} = \sqrt{\frac{m_i m_{i+1}}{m_i + m_{i+1}}}(0 \ldots 0\ \frac{-1}{m_i}\ \frac{1}{m_{i+1}}\ 0 \ldots 0)^T$, $\mathbf{t}_q = \frac{1}{\sqrt{\sum_{i=1}^N m_i^2}} \times (m_1\ m_2 \ldots m_N)^T$. One finds $\dot{q}_{\mathrm{norm},i} = \sqrt{\frac{m_i m_{i+1}}{m_i + m_{i+1}}}(\dot{q}_{i+1} - \dot{q}_i)$, and $\dot{q}_\mathrm{tan} = \frac{1}{\sqrt{\sum_{i=1}^N m_i^2}} \sum_{i=1}^N m_i \dot{q}_i$. The normalized Delassus matrix has entries $(i, i+1)$ and $(i+1, i)$ given by $\mathbf{n}_{q,i}^T M \mathbf{n}_{q,i+1} = \frac{-1}{m_i} \sqrt{\frac{m_i m_{i+1}^2 m_{i+2}}{(m_i + m_{i+1})(m_{i+1} + m_{i+2})}}$, with diagonal entries equal to 1, and other entries zero.

2.3.2 Rocking block

Let us consider the planar system in Fig. 1(a). The block is a three degrees of freedom planar homogeneous solid, with generalized coordinates $q = (x\ y\ \theta)^T$, where x and y and the horizontal and vertical positions of the center of gravity G, θ is the angular position. When tangential effects act at the contact points, the dynamics is [19]:

$$\begin{aligned}
m\ddot{x}(t) &= \lambda_{\mathrm{t},1}(t) + \lambda_{\mathrm{t},2}(t) \\
m\ddot{y}(t) &= \lambda_{n,1}(t) + \lambda_{n,2}(t) - mg \\
I_G\ddot{\theta}(t) &= \lambda_{n,1}(t)\left(\frac{l}{2}\sin\big(\theta(t)\big) + \frac{L}{2}\cos\big(\theta(t)\big)\right) + \lambda_{n,2}(t)\left(\frac{l}{2}\sin\big(\theta(t)\big) - \frac{L}{2}\cos\big(\theta(t)\big)\right) \\
&\quad + \left(\frac{l}{2}\cos(\theta) - \frac{L}{2}\sin(\theta)\right)\lambda_{\mathrm{t},1} + \left(\frac{l}{2}\cos(\theta) + \frac{L}{2}\sin(\theta)\right)\lambda_{\mathrm{t},2}
\end{aligned} \tag{30}$$



(a) Planar block. (b) Rod on a plane.

(c) Block on an inclined plane. (d) Disk on a plane. (e) Disk in an angle.

(f) Particle in an angle. (g) Constrained pendulum.

Fig. 1 Various systems

There are two unilateral constraints:

$$
\begin{aligned}
h_1(q) &= y - \frac{l}{2}\cos(\theta) + \frac{L}{2}\sin(\theta) \geqslant 0 \\
h_2(q) &= y - \frac{l}{2}\cos(\theta) - \frac{L}{2}\sin(\theta) \geqslant 0
\end{aligned}
\tag{31}
$$

One computes

$$\dot{q}_{\text{norm},1} = \frac{\dot{y} + (\frac{l}{2}\sin(\theta) + \frac{L}{2}\cos(\theta))\dot{\theta}}{\sqrt{\frac{1}{m} + \frac{1}{4I_G}(l\sin(\theta) + L\cos(\theta))^2}}$$
$$\dot{q}_{\text{norm},2} = \frac{\dot{y} + (\frac{l}{2}\sin(\theta) - \frac{L}{2}\cos(\theta))\dot{\theta}}{\sqrt{\frac{1}{m} + \frac{1}{4I_G}(l\sin(\theta) + L\cos(\theta))^2}}, \qquad \dot{q}_{\tan} = \sqrt{m}\dot{x} \tag{32}$$

The calculation of the dynamics in (24) is lengthy, however, we are not interested in the explicit calculation of the non-linear terms but rather of the right-hand side. To simplify the presentation, we adopt the notation $c\theta$ for $\cos(\theta)$ and $s\theta$ for $\sin(\theta)$, and $f(\theta) = \frac{1}{m} + \frac{1}{4I_G}(l\sin(\theta) + L\cos(\theta))^2$. We obtain:

$$\begin{aligned}
&\ddot{q}_{\text{norm},1} + F_{\text{norm},1} \\
&\quad = \frac{1}{\sqrt{f(\theta)}}\left(1 + \left(\frac{l}{2}s\theta + \frac{L}{2}c\theta\right)^2 \quad 1 + \frac{l^2}{4}s^2\theta - \frac{L^2}{4}c^2\theta\right)\lambda_n \\
&\quad\quad + \frac{1}{\sqrt{f(\theta)}}\left(\frac{1}{4}\left(l^2 - L^2\right)c\theta s\theta + \frac{lL}{4}\left(c^2\theta - s^2\theta\right) \quad \frac{1}{4}(l^2 - L^2)c\theta s\theta + \frac{lL}{4}\right)\lambda_{\text{t}} \\
&\ddot{q}_{\text{norm},2} + F_{\text{norm},2} \\
&\quad = \frac{1}{\sqrt{f(\theta)}}\left(1 + \frac{l^2}{4}s^2\theta - \frac{L^2}{4}c^2\theta \quad 1 + \left(\frac{l}{2}s\theta - \frac{L}{2}c\theta\right)^2\right)\lambda_n \\
&\quad\quad + \frac{1}{\sqrt{f(\theta)}}\left(\frac{1}{4}\left(l^2 + L^2\right)s\theta c\theta - \frac{lL}{4} \quad \frac{1}{4}\left(l^2 - L^2\right)c\theta s\theta + \frac{lL}{4}\left(s^2\theta - c^2\theta\right)\right)\lambda_{\text{t}} \\
&\ddot{q}_{\tan} = \sqrt{m}(\lambda_{\text{t},1} + \lambda_{\text{t},2})
\end{aligned} \tag{33}$$

The various terms appearing in the right-hand side of (24) can be easily identified from (33).

2.3.3 Rod on a plane

We now deal with the system depicted in Fig. 1(b). It has three degrees of freedom and $q = (x\ y\ \theta)^T$, with x and$y$ the coordinate of the gravity center. Its dynamics is given by:

$$\begin{aligned}
m\ddot{x} &= \lambda_{\text{t}} \\
m\ddot{y} &= -mg + \lambda_n \\
I\ddot{\theta} &= -l\cos(\theta)\lambda_n + l\sin(\theta)\lambda_{\text{t}}
\end{aligned} \tag{34}$$

The unilateral constraint is the signed distance between the tip A and the x-axis: $h(q) = y - l\sin(\theta) \geqslant 0$. One obtains:

$$\begin{aligned}
&\dot{q}_{\text{norm}} = \frac{m}{1 + 3\cos^2(\theta)}\left(\dot{y} - l\cos(\theta)\dot{\theta}\right), \qquad \dot{q}_{\tan,1} = \sqrt{m}\dot{x}, \\
&\dot{q}_{\tan,2} = \frac{1}{\sqrt{I + ml^2\cos^2(\theta)}}\left(l\cos(\theta)\dot{y} + \dot{\theta}\right)
\end{aligned} \tag{35}$$

Also, $H_{\text{T}} = (1\ 0\ l\sin(\theta))^T$.

2.3.4 Planar box over an inclined plane

Let us consider the system depicted in Fig. 1(c). One has $q = (x\ y\ \varphi)^T$, $h(q) = y - l\cos(\varphi) - r\sin(\varphi) \geqslant 0$, so that $\nabla h(q) = (0\ 1\ \ -l\sin(\varphi) - r\cos(\varphi))^T$. The mass matrix is $M = \mathrm{diag}(m, m, J)$ with $J = k^2 m$, where k is the radius of gyration of the block. One has

$$\mathbf{n}_q = \frac{1}{\sqrt{\frac{1}{m} + \frac{(l\sin(\varphi)+r\cos[\varphi))^2}{J}}}\left(0\ \frac{1}{m}\ \frac{-l\sin(\varphi) - r\cos(\varphi)}{J}\right)^T$$

$\mathbf{t}_{q,1} = \frac{1}{\sqrt{m}}(1\ \ 0\ \ 0)^T$, $\mathbf{t}_{q,2} = \frac{1}{\sqrt{J+m(l\sin(\varphi)+r\cos(\varphi))^2}}(0\ \ l\sin(\varphi) + r\cos(\varphi)\ \ 1)^T$, $\dot{q}_{\mathrm{norm}} = \frac{\dot{y}-l\sin(\varphi)-r\cos(\varphi)}{\sqrt{\frac{1}{m}+\frac{(l\sin(\varphi)+r\cos[\varphi))^2}{J}}}$, $\dot{q}_{\tan,1} = \sqrt{m}\dot{x}$, $\dot{q}_{\tan,2} = \frac{1}{\sqrt{J+m(l\sin(\varphi)+r\cos(\varphi))^2}}(m(l\sin(\varphi) + r\cos(\varphi))\dot{y} + \dot{\varphi})$. Also, $H_{\mathrm{T}}(q) = (x\ 0\ \ -l)^T$.

2.3.5 Disk over a plane

Let us consider a planar system made of a disk constrained as in Fig. 1(d). The disk has mass m, moment of inertia I, radius $R > 0$, and $q = (x\ y\ \theta)^T$. The unilateral constraint is $h(q) = y - R \geqslant 0$. One has $\mathbf{n}_q = \frac{1}{\sqrt{m}}(0\ 1\ 0)^T$, $\mathbf{t}_{q,1} = \frac{1}{\sqrt{m}}(1\ 0\ 0)^T$, $\mathbf{t}_{q,2} = \frac{1}{\sqrt{I}}(0\ 0\ 1)^T$, and $H_{\mathrm{T}} = \begin{pmatrix} 0 \\ 1 \\ R \end{pmatrix}$. Thus, $\dot{q}_{\mathrm{norm}} = \sqrt{m}\dot{y}$, $\dot{q}_{\tan,1} = \sqrt{m}\dot{x}$, $\dot{q}_{\tan,2} = \sqrt{I}\dot{\theta}$. The dynamics is given by

$$\begin{aligned} \ddot{q}_{\mathrm{norm}} &= \frac{1}{\sqrt{m}}\lambda_n \\ \ddot{q}_{\tan,1} &= \frac{1}{\sqrt{m}}\lambda_{\mathrm{t}} \\ \ddot{q}_{\tan,2} &= \frac{R}{\sqrt{I}}\lambda_{\mathrm{t}} \end{aligned} \tag{36}$$

The tangential relative velocity at the contact point is $v_{\mathrm{t}} = \dot{x} + R\dot{\theta} = \frac{1}{\sqrt{m}}\dot{q}_{\tan,1} + \frac{R}{\sqrt{I}}\dot{q}_{\tan,2}$.

2.3.6 Disk in an angle

Let us consider the system depicted in Fig. 1(e). The disk has mass m, moment of inertia I, radius $R > 0$, and $q = (x\ y\ \theta)^T$. The two unilateral constraints are $h_1(q) = y - R \geqslant 0$ and $h_2(q) = x - R \geqslant 0$. One has $\mathbf{n}_{q,1}^T = (0\ \frac{1}{\sqrt{m}}\ 0)$, $\mathbf{n}_{q,2}^T = (\frac{1}{\sqrt{m}}\ 0\ 0)$, and $\mathbf{t}_q^T = (0\ 0\ \frac{1}{\sqrt{I}})$, and $H_{\mathrm{T}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ R & R \end{pmatrix}$. Thus, $\dot{q}_{\mathrm{norm},1} = \sqrt{m}\dot{y}$, $\dot{q}_{\mathrm{norm},2} = \sqrt{m}\dot{x}$, $\dot{q}_{\tan} = \sqrt{I}\dot{\theta}$. The dynamics (24) is

$$\begin{aligned} \ddot{q}_{\mathrm{norm},1} &= \frac{1}{\sqrt{m}}(\lambda_{n,1} + \lambda_{\mathrm{t},2}) \\ \ddot{q}_{\mathrm{norm},2} &= \frac{1}{\sqrt{m}}(\lambda_{n,2} + \lambda_{\mathrm{t},1}) \\ \ddot{q}_{\tan} &= \frac{1}{\sqrt{I}}R(\lambda_{\mathrm{t},1} + \lambda_{\mathrm{t},2}) \end{aligned} \tag{37}$$

2.3.7 Constrained pendulum

Consider now the system in Fig. 1(g). There is one bilateral constraint represented by a prismatic joint (the horizontal position x_A of point A) and one unilateral constraint (the vertical position of the tip B), thus $n=3$, $m=p=1$. Let x and y be the coordinates of the gravity center, supposed to be at the middle of the bar, with mass m. One has $q=(x\ y\ \theta)^T$, $x_A = x - \frac{l}{2}\sin(\theta)$, $y_B = y - \frac{l}{2}\cos(\theta)$. The mass matrix is $M = \mathrm{diag}(m, m, \frac{ml^2}{12})$, $h_1(q) = x - \frac{l}{2}\sin(\theta)$, $h_2(q) = y - \frac{l}{2}\cos(\theta)$. After few calculations, one finds:

$$
\begin{aligned}
&\mathbf{n}_q^b = \frac{1}{\sqrt{1+3\cos^2(\theta)}}\begin{pmatrix} \frac{1}{\sqrt{m}} \\ 0 \\ \frac{-6\cos(\theta)}{\sqrt{m}l} \end{pmatrix}, \qquad \mathbf{n}_q^u = \frac{1}{\sqrt{1+3\sin^2(\theta)}}\begin{pmatrix} 0 \\ \frac{1}{\sqrt{m}} \\ \frac{6\sin(\theta)}{\sqrt{m}l} \end{pmatrix} \\
&\mathbf{t}_q = \frac{1}{\sqrt{2m}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \qquad \mathbf{n}_q^T M \mathbf{n}_q = \begin{pmatrix} 1 & \frac{-3\sin(\theta)\cos(\theta)}{\sqrt{1+3\cos^2(\theta)}\sqrt{1+3\sin^2(\theta)}} \\ \frac{-3\sin(\theta)\cos(\theta)}{\sqrt{1+3\cos^2(\theta)}\sqrt{1+3\sin^2(\theta)}} & 1 \end{pmatrix} \\
&\dot{q}^b_{\mathrm{norm}} = \frac{\sqrt{m}}{\sqrt{1+3\cos^2(\theta)}}\left(\dot{x} - \frac{l\cos(\theta)}{2}\dot{\theta}\right), \qquad \dot{q}^u_{\mathrm{norm}} = \frac{\sqrt{m}}{\sqrt{1+3\sin^2(\theta)}}\left(\dot{y} + \frac{l\sin(\theta)}{2}\dot{\theta}\right) \\
&\dot{q}_{\tan} = \sqrt{\frac{m}{2}}(\dot{x} - \dot{y})
\end{aligned}
\tag{38}
$$

The Delassus' matrix shows couplings between the unilateral and the bilateral constraints, which vanish when the bar is either horizontal ($\theta = \frac{\pi}{2}$), or vertical ($\theta = 0$). Friction may act at both the bilateral and the unilateral contacts, in the tangential directions $v_{\mathrm{t},2} = \dot{x}_B = \dot{x} - \frac{l}{2}\sin(\theta)\dot{\theta}$ and $v_{\mathrm{t},1} = \dot{y}_A = \dot{y} + \frac{l}{2}\cos(\theta)\dot{\theta}$. Thus,

$$
H_{\mathrm{T}}(q) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ \frac{l}{2}\cos(\theta) & -\frac{l}{2}\sin(\theta) \end{pmatrix}
$$

It is easy to see that the term $\mathbf{n}_q^T H_{\mathrm{T}}(q)$ is non-zero, showing that quasi-tangential forces act on quasi-normal dynamics.

2.4 Discussion and comparison with other transformations

Most of the works reviewed below deal with the bilateral (holonomic or non-holonomic) constraints only. This makes a big difference since the normal cone in (4), which accounts for unilateral constraints, and hence topology changes, does not appear. The constraint manifold is therefore always constant of co-dimension m.

Comparing (16) with the kinetic energy expressions when inertial quasi-velocities analyzed in [3, 38, 44, 45] are used, shows that they are not equivalent to the kinetic quasi-velocities because they all yield $T(q,\dot{q}) = \frac{1}{2}v^T v$. It is known also that Kane's generalized speeds are only functions of q, $\dot{q}$ and time (like projections of instantaneous angular velocities on some frame axes), whereas the inertial and kinetic quasi-velocities also depend on the dynamical parameters of the system. It is also noteworthy that the kinetic quasi-velocities are fundamentally generalized quantities since they are defined from any set of generalized coordinates and velocities. Due to this it is not possible in general to assign physical units to the kinetic quasi-velocities, which have to be understood from a geometrical point of view.

Various methods to achieve *coordinate partitioning*, which roughly speaking allows one to split the dynamics into a part that depends on the contact force (a "normal" part) and one part that does not (a "tangential" part), have been proposed. The transformation proposed in [64] decouples the system with holonomic bilateral constraints. This is a quite convenient framework for control purposes [67]. However, it is a generalized coordinate transformation (hence it preserves the Lagrange structure of the dynamics), usually defined only locally on $\mathcal{C}$, and whose velocity splitting is different from the one in (6): indeed the "tangential" velocity may jump at impacts because of couplings with the "normal" dynamics [15, Eq. (63)], which is not the case for the quantity $\dot{q}_{\text{tan}}$.

There also exists many techniques using direct projections of the Lagrange dynamics (7) on $T_q\mathcal{C}$ based on annihilation matrices, or null space matrices [2, 4, 8, 10, 59, 86]. It consists of finding a matrix $P(q)$ (the annihilator) such that $P(q)\nabla h(q) = 0$ for all $q \in \mathcal{C}$ in the bilateral manifold $\{q | h_i(q) = 0, i \in \{1, \ldots, m\}\}$. $P(q)$ is constructed from a basis of $T_q\mathcal{C}$ [8], just as $\mathbf{t}_q$ is. Other methods [2, 4] use the projector $I - (\frac{\partial h(q)}{\partial q})^\dagger \frac{\partial h(q)}{\partial q}$ to obtain dynamics independent of the contact force. This may be seen as a particular case of more general projectors as defined in [12, Eq. (31) (32)], who proposes to use projections in the kinetic metric for the analysis of systems with bilateral constraints. It can be verified that the constrained velocity u_β in [12, Eq. (27)] (called $H_c\dot{q}$ in [65, 66]) is equal to $\mathbf{n}_q \operatorname{diag}(\|\nabla h_i(q)\|_{M^{-1}})(\nabla h^T(q)M^{-1}(q)\nabla h(q))^{-1} \operatorname{diag}(\|\nabla h_i(q)\|_{M^{-1}})\dot{q}_{\text{norm}}$. Hence, it coincides with our kinetic quasi-velocity $\dot{q}_{\text{norm}}$ if there is a single constraint (i.e. $u_\beta = \dot{q}_{\text{norm}}\mathbf{n}_q$) or if the constraints are mutually orthogonal in the kinetic metric so that the matrix $(\nabla h^T(q)M^{-1}(q)\nabla h(q))^{-1}$ is diagonal. Similar arguments and projectors are used in [13, 65, 66]; see also [58]. Actually these papers consider orthogonal (in the kinetic metric) projections onto the co-dimension m constrained manifold, whereas we rather consider projections on each vector of the basis $(\mathbf{n}_q, \mathbf{t}_q)$. In [65, Sect. 5], the same projection of the generalized momentum as in (6) is used (without normalization of $\mathbf{n}_{q,i}$). It is pointed out the advantage of such a projection over the orthogonal projection used in [65, Sects. 2, 3] or [12] for impact modeling. As clarified here, the key is the existence of the normal cone in (4) for unilaterally constrained Lagrangian systems.

3 Application to kinematic impact law design and analysis

Let us assume for simplicity that there are no bilateral constraints (i.e. $m = 0$). Thus, $G(q) = D_{\text{n}}^{-1}(q)$ and in the sequel we shall use both notations equally. It is nevertheless possible to consider $m > 0$, using the matrix $G(q)$ in (17) and assuming that it is positive definite. We also assume that $\dot{q}_{\text{norm}}$ is constructed from the active constraints at the impact time t, i.e. with the constraints whose index belongs to $\mathcal{I}_a(q(t))$. We denote $p' = \operatorname{card}(\mathcal{I}_a(q(t)))$.[5] From classical arguments in frictionless non-smooth mechanics, the impact dynamics at an instant t such that there is at least one $i \in \{1, \ldots, p'\}$ such that $\dot{q}_{\text{norm},i}(t^-) < 0$ and $h_i(q(t)) = 0$[6] is given by (using (10)):

$$\begin{aligned} \dot{q}_{\text{norm}}(t^+) - \dot{q}_{\text{norm}}(t^-) &= \mathbf{n}_q^T M(q)\mathbf{n}_q \bar{p}_n(t) \\ \dot{q}_{\text{tan}}(t^+) - \dot{q}_{\text{tan}}(t^-) &= 0, \end{aligned} \tag{39}$$

[5] For simplicity of notation the dependence of p' on q is not recalled.

[6] This is equivalently stated as $\dot{q}(t^-) \in -T_{\Phi_u}(q(t))$ [69].

where $\bar{p}_{n,i}(t) = \|\nabla h_i(q)\|_{M^{-1}} p_{n,i}(t)$, i.e. $\bar{p}_n(t) = \mathrm{diag}(\|\nabla h_i(q)\|_{M^{-1}}) p_n(t)$, and $p_{n,i}(t)$ is the impulse of the contact force multiplier $\lambda_{n,i}$ at the impact instant t. More rigorously $\lambda_{n,i}$ is a measure at t and $p_{n,i}(t)$ is its density with respect to the Dirac measure at the atom t. The role played by the projection of the generalized momentum on the basis $\mathbf{t}_q$ clearly appears in (39): the quasi-velocities $\dot{q}_{\tan}$ are conserved at the impacts when friction is absent (the constraints are said *perfect*). It is important to notice that despite there may be $\dot{q}_{\mathrm{norm},i}(t^-) = 0$ for some $i \in \mathcal{I}_a(q(t))$, all the terms $\dot{q}_{\mathrm{norm},i}$, $i \in \{1, \ldots, m\}$ may undergo a jump because of the inertial couplings between the constraints, as reflected by the normalized Delassus' matrix $\mathbf{n}_q^T M(q) \mathbf{n}_q$, which is not diagonal in general. This is for instance the case for chains of balls, or the planar rocking block (see [19] for detailed analysis of these two cases).

It readily follows from the impact dynamics in (39) and (16) that the kinetic energy loss $T_L(t) \triangleq T(q, \dot{q}(t^+)) - T(q, \dot{q}(t^-))$ at a time t of impact is given by:

$$T_L(t) = \frac{1}{2}\dot{q}_{\mathrm{norm}}^{u,T}(t^+) G^{-1}(q) \dot{q}_{\mathrm{norm}}^{u}(t^+) - \frac{1}{2}\dot{q}_{\mathrm{norm}}^{u,T}(t^-) G^{-1}(q) \dot{q}_{\mathrm{norm}}^{u}(t^-) \tag{40}$$

where q denotes $q(t)$, and due to the absence of bilateral constraints $G(q) = \mathbf{n}_q^{u,T} M(q) \mathbf{n}_q^u$. From now on, we will drop the upper-script u since there are no bilateral constraints. The framework in (39) is suitable to formulate a kinematic impact law as [14, 16, 19]:

$$v(t^+) = \begin{pmatrix} \dot{q}_{\mathrm{norm}}(t^+) \\ \dot{q}_{\tan}(t^+) \end{pmatrix} = -\mathcal{E} \begin{pmatrix} \dot{q}_{\mathrm{norm}}(t^-) \\ \dot{q}_{\tan}(t^-) \end{pmatrix} \tag{41}$$

where $\mathcal{E}$ is a generalized $n \times n$ restitution matrix. Its entries will be named the coefficients of restitution. Let us decompose it as

$$\mathcal{E} = \begin{pmatrix} \mathcal{E}_{\mathrm{nn}} & \mathcal{E}_{\mathrm{nt}} \\ \mathcal{E}_{\mathrm{tn}} & \mathcal{E}_{\mathrm{tt}} \end{pmatrix} \tag{42}$$

with obvious dimensions of the four submatrices: $\mathcal{E}_{\mathrm{nn}} \in \mathbb{R}^{p' \times p'}$, $\mathcal{E}_{\mathrm{tt}} \in \mathbb{R}^{(n-p') \times (n-p')}$. In the frictionless case, one has $\dot{q}_{\tan}(t^+) = \dot{q}_{\tan}(t^-)$ for any pre-impact velocity $\dot{q}_{\mathrm{norm}}(t^-)$, so necessarily $\mathcal{E}_{\mathrm{tn}} = 0$ and $\mathcal{E}_{\mathrm{tt}} = -I$. The restitution law in (41) is very general as the next result shows.

Proposition 1 *Suppose that at least one component of $\dot{q}_{\mathrm{norm}}(t^-)$ or of $\dot{q}_{\tan}(t^-)$ is non-zero. Then given any post-impact kinetic quasi-velocity, there exists a value of $\mathcal{E}$ such that* (41) *is satisfied. If at least one component of $\dot{q}_{\mathrm{norm}}(t^-)$ is negative, then there exists a value of $\mathcal{E}_{\mathrm{nn}}$ such that $\dot{q}_{\mathrm{norm}}(t^+) = -\mathcal{E}_{\mathrm{nn}} \dot{q}_{\mathrm{norm}}(t^-)$.*

Proof Without loss of generality suppose that $v_1(t^-) \neq 0$ while $v_i(t^-) = 0$ for all $i \geqslant 2$. Then it suffices to choose $\epsilon_{i1} = -\frac{v_i(t^+)}{v_1(t^-)}$. □

Remark 4 The fact that the quantity $\dot{q}_{\tan}$ is continuous at impacts (implying a constraint on the restitution matrix in (41) (42), like $\mathcal{E}_{\mathrm{tn}} = 0$ and $\mathcal{E}_{\mathrm{tt}} = -I$) was first noticed, in the case when a single constraint is given by $q_1 \geqslant 0$, in [5], and in a more general setting in [83].

Remark 5 Usually the unilateral constraints are constructed from the gap functions at the contact points, yielding the so-called local kinematics [1, Sect. 3.3]. In such a case, the quantities $\nabla h_i^T(q)\dot{q}$ are the gap functions time-derivatives. Hence, setting $\mathcal{E}_{\mathrm{nn}} = \mathrm{diag}(e_{\mathrm{n},i})$ boils down to applying a Newton's restitution law at each impact point.

In the following, we examine the relation with Moreau's impact law, the dissipativity conditions, and the issue related to the uniqueness of the restitution coefficients.

3.1.1 Moreau's impact law

This impact law has been proposed in [69], and is used in granular matter simulation [70]. It has been deeply analyzed in [35]. This law can be written as [62]:

$$\dot{q}\left(t^{+}\right)=\dot{q}\left(t^{-}\right)-(1+e)\,\mathrm{proj}\left[N_{\Phi_u}^{M}(q);\dot{q}\left(t^{-}\right)\right] \tag{43}$$

where the projection is understood here as the Euclidean orthogonal projection in view of the definition of the normal cone in (4). The coefficient e is a global restitution coefficient. Let us write (43) in terms of the vectors $\mathbf{n}_{q,i}$. For this, let us use Moreau's lemma of the two cones in the kinetic metric. Any vector $z \in \mathbb{R}^n$ can be decomposed uniquely as $z = x + y$ with $x = \mathrm{proj}[N_{\Phi_u}^{M}(q);\dot{q}(t^-)] \in N_{\Phi_u}^{M}(q)$ and $y = \mathrm{proj}[T_{\Phi_u}(q);\dot{q}(t^-)] \in T_{\Phi_u}(q)$, with $x^T M(q) y = 0$ [40, Theorem 3.2.5]. The velocity $\dot{q}(t^-)$ may be decomposed this way with $y = \frac{\dot{q}(t^+)+e\dot{q}(t^-)}{1+e}$ and $x = \frac{\dot{q}(t^-)-\dot{q}(t^+)}{1+e}$, so that $\dot{q}(t^+)$ satisfies (43) [62]. Let us define the index set $\mathcal{I}_a'(q) \subseteq \mathcal{I}_a(q)$ such that $\mathrm{proj}[N_{\Phi_u}^{M}(q);\dot{q}(t^-)] = \sum_{i\in\mathcal{I}_a'(q)} \alpha_i \mathbf{n}_{q,i}$, $\alpha_i \geqslant 0$. By the two cones lemma $\mathcal{I}_a'(q)$ is uniquely defined. Using the second line of (5) and Moreau's viability lemma [16, Proposition 5.1], which states that $\dot{q}(t^-) \in -T_{\Phi_u}(q)$, one deduces that the index set $\mathcal{I}_a'(q)$ can also be computed from the signs of the projections of $\dot{q}(t^-)$ on the vectors $\mathbf{n}_{q,i}$. For instance, let $e = 0$. If $\dot{q}(t^-) \in N_{\Phi_u}^{M}(q)$ then $\mathcal{I}_a'(q) = \mathcal{I}_a(q)$. If $\dot{q}(t^-) \notin N_{\Phi_u}^{M}(q)$, then $\mathcal{I}_a'(q) \subset \mathcal{I}_a(q)$. Next, we define $\dot{q}_{\mathrm{norm}}' = [\dot{q}_{\mathrm{norm},i}]^T$ for $i \in \mathcal{I}_a'(q)$. The new vector $\dot{q}_{\mathrm{tan}}'$ is defined accordingly. Moreau's law in (43) is equivalently written as

$$\begin{aligned} &\dot{q}_{\mathrm{norm}}'\left(t^{+}\right)=-\mathcal{E}_{\mathrm{nn}}\dot{q}_{\mathrm{norm}}'\left(t^{-}\right), \quad \mathcal{E}_{\mathrm{nn}}=\mathrm{diag}(e)\\ &\dot{q}_{\mathrm{tan}}'\left(t^{+}\right)=\dot{q}_{\mathrm{tan}}'\left(t^{-}\right) \end{aligned} \tag{44}$$

and we recover that Moreau's law, under the constraint $\dot{q}(t^-) \in -T_{\Phi_u}(q)$ and equal restitution coefficients, is equivalent to Newton's law applied at each contact [35]. Moreau's law is always kinetically and kinematically consistent, and energetically consistent whenever $e \in [0, 1]$ [35, §5]. It is clear that the impact law in (41) significantly enlarges the scope of Moreau's law which spans only a half-line of the admissible post-impact velocities subspace. Chains of aligned balls are analyzed in [19] (the 4-ball chain is treated but the analysis easily generalized to n-ball chains). In such systems, one always has $\dot{q}_{\mathrm{tan}}$ that is continuous at the impact because of linear momentum conservation. Depending on the contact properties (stiffness ratio, elasticity coefficient—linear or non-linear elasticity), the outcome may be quite different from what is predicted by Moreau's law. Extensions of Moreau's law are analyzed in [35, 55]. In the case of no tangential effects, the law in [55] allows for different coefficients at each contact, i.e. $\mathcal{E}_{nn} = \mathrm{diag}(e_i)$ in (44) (see Eq. (5.103) in [55]); see also [41, Eq. (1.3)] for the same idea. The above proof that (44) holds, and is equivalent to Proposition 5.6 in [35]. Filling-in the restitution matrix in (41) also generalizes Frémond's idea of introducing "distance effects" in multibody systems [24, 28], which led Glocker to introduce what he named Frémond matrices. It is, however, noteworthy that the restitution matrix defined in [35, Eq. (5.36)] corresponds to the normal part $\mathcal{E}_{\mathrm{nn}}$ only (except for the normalization terms, see (46)), which is not sufficient to model the rocking block for which a quasi-tangential restitution $\mathcal{E}_{\mathrm{tt}}$ is necessary, as shown in [19] (the rocking block is in fact the example treated in [28], and the necessity to introduce a tangential coefficient was also pointed out in [70]). In case of a single contact, (43) is equivalently rewritten as $\dot{q}(t^+) = \dot{q}(t^-) - (1+e)\dot{q}^T(t^-)M(q)\mathbf{n}_q\mathbf{n}_q$, that is

$$\dot{q}(t^+) = \left(I - (1+e)\frac{M^{-1}(q)\nabla h(q)\nabla h^T(q)}{\nabla h^T(q)M^{-1}(q)\nabla h(q)}\right)\dot{q}(t^-) \tag{45}$$

In a more general setting, since the normal cone is closed convex, the projection in (43) belongs to the face of $N^M_{\Phi_u}(q)$ exposed by $\dot{q}(t^-) - \text{proj}[N^M_{\Phi_u}(q); \dot{q}(t^-)]$ [40, p. 50]. The expression in (45) is used in [81, 82] with $e = 1$, who study a kind of binary collisions operator associating each constraint with such a linear mapping between post and pre-impact velocities.

Example 1 Let us consider a chain of three aligned balls with masses m_1, m_2, and m_3, radius $R > 0$, coordinates q_1, q_2, and q_3. The initial velocities are $\dot{q}_1(t^-) = 1$ m/s, $\dot{q}_2(t^-) = \dot{q}_3(t^-) = 0$ m/s. The two unilateral constraints are $h_1(q) = q_2 - q_1 + 2R \geqslant 0$, $h_2(q) = q_3 - q_2 + 2R \geqslant 0$. Initially, the three balls are in contact so that $\mathcal{I}_a(q(t)) = \{1, 2\}$. One calculates that $\dot{q}_{\text{norm},1} = \frac{1}{\sqrt{\frac{1}{m_1}+\frac{1}{m_2}}}(\dot{q}_2 - \dot{q}_1)$, $\dot{q}_{\text{norm},2} = \frac{1}{\sqrt{\frac{1}{m_2}+\frac{1}{m_3}}}(\dot{q}_3 - \dot{q}_2)$. The quasi-tangent vector can be chosen as $\mathbf{t}_q^T = \sqrt{\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3}}(1\ 1\ 1)$, so that $\dot{q}_{\tan} = \sqrt{\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3}}(m_1\dot{q}_1 + m_2\dot{q}_2 + m_3\dot{q}_3)$. One has also $-\mathbf{n}_{q,1}^T\dot{q}(t^-) = \sqrt{\frac{1}{m_1} + \frac{1}{m_2}}(\frac{1}{m_1}) > 0$ and $-\mathbf{n}_{q,2}^T\dot{q}(t^-) = 0$, consequently $\mathcal{I}'_a = \mathcal{I}_a = \{1, 2\}$. Thus, applying (44) yields $\dot{q}_2(t^+) - \dot{q}_1(t^+) = -e(\dot{q}_2(t^-) - \dot{q}_1(t^-))$, $\dot{q}_3(t^+) - \dot{q}_2(t^+) = -e(\dot{q}_3(t^-) - \dot{q}_2(t^-))$, and $m_1\dot{q}_1(t^+) + m_2\dot{q}_2(t^+) + m_3\dot{q}_3(t^+) = m_1\dot{q}_1(t^-) + m_2\dot{q}_2(t^-) + m_3\dot{q}_3(t^-)$, the linear momentum conservation. Let $e = 1$ and $m_1 = m_2 = m_3$, then $\dot{q}_1(t^+) = -\frac{1}{3}$ m/s, $\dot{q}_2(t^+) = \dot{q}_3(t^+) = \frac{2}{3}$ m/s. One checks that energy is conserved. This solution is the one given by Moreau's law; see [16, Example 5.7]. Similarly, $e = 0$ yields $\dot{q}_1(t^+) = \dot{q}_2(t^+) = \dot{q}_3(t^+) = \frac{1}{3}$ m/s, and one checks again that this is Moreau's law solution [16, Example 5.7]. Moreau's law is unable to make balls 2 and 3 separate, as it spans only a small portion of the post-impact velocities subspace [35, 72]. From Proposition 1, we infer that given the above pre-impact velocity one can find $\mathcal{E}_{\text{nn}}$ such that any post-impact $\dot{q}_{\text{norm}}(t^+)$ can be reached, by adding if necessary off-diagonal coefficients of restitution. This is closely linked with "distance effects", which are due to waves traveling inside the chains during the impact, and responsible for the dispersion of energy.

Example 2 Let us now consider a particle in the plane with $m = 1$, with two unilateral constraints as depicted in Fig. 1(f). At the impact time which occurs at the corner one has $\mathcal{I}_a(q) = \{1, 2\}$. Let us consider the pre-impact velocity $\dot{q}(t^-) = (-1\ \frac{1}{2})^T$. One has $-\mathbf{n}_{q,2}^T\dot{q}(t^-) > 0$ and $-\mathbf{n}_{q,1}^T\dot{q}(t^-) < 0$. Therefore $\mathcal{I}'_a(q) = \{2\}$. We can simply define $\mathbf{t}_q$ as a vector orthogonal to $\mathbf{n}_{q,2}$ as shown on the figure. Then Moreau's law states $\dot{q}_{\text{norm},2}(t^+) = -e\dot{q}_{\text{norm},2}(t^-) = e$ while $\dot{q}_{\tan}(t^+) = \dot{q}_{\tan}(t^-)$. If $e = 0$ the particle slides on the boundary defined by $h_2(q) = 0$.

Finally, let us notice that the restitution matrix introduced in [35, Eq. (5.36)] is designed from the un-normalized normal vectors $\nabla h_i(q)$. Let us denote its entries ϵ_{ij}, while the entries of $\mathcal{E}_{\text{nn}}$ are $e_{\text{n},ij}$. Then $\epsilon_{ii} = e_{\text{n},ii}$ (diagonal elements are the same), but

$$\epsilon_{ij} = \frac{\|\nabla h_i(q)\|_{M^{-1}}}{\|\nabla h_j(q)\|_{M^{-1}}} e_{\text{n},ij} \tag{46}$$

This can be shown from the equalities $\dot{q}_{\text{norm},i}(t^+) = -\sum_{j=1}^{p} e_{\text{n},ij}\dot{q}_{\text{norm},i}(t^-)$, and $\nabla h_i^T(q)\dot{q}(t^+) = -\sum_{j=1}^{p} \epsilon_{ij}\nabla h_i^T(q)\dot{q}(t^-)$.

3.1.2 Energy loss at impacts

There are various equivalent ways to express the quantity $T_L(t)$, using (39) and (40):

$$T_L(t) = \frac{1}{2}\left(\dot{q}_{\text{norm}}(t^+) + \dot{q}_{\text{norm}}(t^-)\right)^T \bar{p}_n \tag{47}$$

which we may define as being the Thomson and Tait formula [16], or

$$T_L(t) = \frac{1}{2}\dot{q}_{\text{norm}}^T(t^-)(\mathcal{E}_{\text{nn}} - I)^T G^{-1}(q)(\mathcal{E}_{\text{nn}} + I)\dot{q}_{\text{norm}}(t^-) \tag{48}$$

or, using the symmetry of $G(q)$ [19]:[7]

$$T_L(t) = \frac{1}{2}\dot{q}_{\text{norm}}^T(t^-)\left(\mathcal{E}_{\text{nn}}^T G^{-1}(q)\mathcal{E}_{\text{nn}} - G^{-1}(q)\right)\dot{q}_{\text{norm}}(t^-) \tag{49}$$

or, following [36] and with $\xi = \dot{q}_{\text{norm}}(t^+) + \mathcal{E}_{\text{nn}}\dot{q}_{\text{norm}}(t^-)$:

$$T_L(t) = \frac{1}{2}\bar{p}_n^T\left(2\xi - (I - \mathcal{E}_{\text{nn}})G(q)\bar{p}_n\right) \tag{50}$$

A first general result may be stated.

Proposition 2 *Suppose that* $-(\mathcal{E}_{\text{nn}} - I)^T G^{-1}(q)(\mathcal{E}_{\text{nn}} + I)$ *or* $-(\mathcal{E}_{\text{nn}}^T G^{-1}(q)\mathcal{E}_{\text{nn}} - G^{-1}(q))$ *are co-positive matrices. Then* $T_L(t) \leqslant 0$*. If they are strictly co-positive, then* $T_L(t) < 0$ *for any non-zero pre-impact velocity.*

Proof Due to the impact conditions, one has $\dot{q}_{\text{norm}}(t^-) \leqslant 0$, in other words the p dimensional vector $-\dot{q}_{\text{norm}}(t^-)$ belongs to $\mathbb{R}_+^p$. From the definition of co-positivity, the results follow. □

Notice that in view of the definition of co-positivity, the conditions of the proposition are necessary and sufficient. The major issue with Proposition 2 is that it is rather difficult in general to characterize the co-positivity of a matrix [39]. Moreover, the co-positivity conditions imply some dependence of $\mathcal{E}_{\text{nn}}$ on $G(q)$, i.e. the restitution coefficients are not *a priori* independent on the system's configuration. Independence holds only if the constraints are orthogonal one to each other, in which case $G(q) = I$. Explicit criteria exist for symmetric matrices and $p \leqslant 4$, or when the matrices have special structures.

Proposition 3 *Suppose that all the unilateral constraints are orthogonal to one another, in the kinetic metric, and that* $\mathcal{E}_{\text{nn}} = \text{diag}(e)$*. Then* $e^2 = \frac{T(t^+)}{T(t^-)}$.

Proof Under the stated conditions, one has $G(q) = I$, and using (49) the result follows. □

This result is obvious in the case $p = 1$ (a single constraint) using Eq. (6.11) in [16]; see [66] for some experimental validation. Let us now state various characterizations of the kinetic energy loss.

[7] $x^T\mathcal{E}_{\text{nn}}^T G^{-1}(q)x = x^T(\mathcal{E}_{\text{nn}}^T G^{-1}(q))^T x = x^T G^{-1}(q)\mathcal{E}_{\text{nn}}x$ for any vector $x \in \mathbb{R}^p$.

Proposition 4 *Suppose that* $\mathcal{E}_{\text{nn}}^T G^{-1}(q)\mathcal{E}_{\text{nn}} = \epsilon A(q)$ *for some matrix* $A(q) = A^T(q)$ *and real* ϵ. *Then there exists* ϵ^* *such that* $\mathcal{E}_{\text{nn}}^T G^{-1}(q)\mathcal{E}_{\text{nn}} - G^{-1}(q) < 0$ *for all* $\epsilon \in [0, \epsilon^*)$.

Proof The identity matrix perturbed by a small enough symmetric matrix is still positive definite [51, Exercise 8, p. 218]. One has for any $x \in \mathbb{R}^p$: $x^T(-\mathcal{E}_{\text{nn}}^T G^{-1}(q)\mathcal{E}_{\text{nn}} + G^{-1}(q))x = z^T(I + \epsilon G^{-\frac{1}{2}}(q)A(q)G^{-\frac{1}{2}}(q))z$ with $z = G^{\frac{1}{2}}(q)x$, where $G^{\frac{1}{2}}(q)$ is the symmetric square root of $G(q)$. Thus, the result follows. □

Proposition 5 *Suppose that* $\mathcal{E}_{\text{nn}} = G(q)\mathcal{E}_{\text{nn}}^T G^{-1}(q)$ *and* $G(q)$ *is positive definite. Then a necessary and sufficient condition for* $T_L(t) \leqslant 0$ *for any vector* $\dot{q}_{\text{norm}}(t^-)$ *is that* $|\lambda_{\max}(\mathcal{E}_n)| \leqslant 1$.

Proof From (49), we have $T_L(t) \leqslant 0$ for any $\dot{q}_{\text{norm}}(t^-)$ if and only if $\mathcal{E}_{\text{nn}}^T G^{-1}(q)\mathcal{E}_{\text{nn}} \leqslant G^{-1}(q)$. Let $G^{\frac{1}{2}}(q)$ be the symmetric positive definite square root of $G(q)$. This inequality is equivalent to $G^{\frac{1}{2}}(q)\mathcal{E}_{\text{nn}}^T G^{-1}(q)\mathcal{E}_{\text{nn}} G^{\frac{1}{2}}(q) \leqslant I$, using Proposition 8.1.2 (xi) and (xiii) in [7]. Let us denote $B(q) = G^{\frac{1}{2}}(q)\mathcal{E}_{\text{nn}}^T G^{-\frac{1}{2}}(q)$. By the assumption of the proposition, we have $G^{-\frac{1}{2}}(q)\mathcal{E}_{\text{nn}} G^{\frac{1}{2}}(q) = G^{\frac{1}{2}}(q)\mathcal{E}_{\text{nn}}^T G^{-\frac{1}{2}}(q)$ so $B(q) = B^T(q)$, and since $B^T(q) = G^{-\frac{1}{2}}(q)\mathcal{E}_{\text{nn}} G^{\frac{1}{2}}(q)$ we obtain $B^2(q) \leqslant I$. Using [7, Lemma 8.4.1], it follows that equivalently $\lambda_{\max}(B^2(q)) \leqslant 1$, because $B^2(q) = B(q)B^T(q)$ is positive semi definite and symmetric. Now we have that $B^2(q) = G^{\frac{1}{2}}(q)(\mathcal{E}_{\text{nn}}^T)^2 G^{-\frac{1}{2}}(q)$, and since it is a symmetric matrix one obtains $B^2(q) = G^{-\frac{1}{2}}(q)\mathcal{E}_{\text{nn}}^2 G^{\frac{1}{2}}(q)$. Therefore, $B^2(q)$ and $\mathcal{E}_{\text{nn}}^2$ are similar matrices so they have the same eigenvalues [51, Proposition 1, p. 152]. Therefore, $\lambda_{\max}(\mathcal{E}_{\text{nn}}^2(q)) \leqslant 1$. Since the eigenvalues of $\mathcal{E}_{\text{nn}}^2$ are the squares of those of $\mathcal{E}_{\text{nn}}$, the result follows. □

Proposition 6 *Let* $G(q) > 0$. *Then* $T_L(t) \leqslant 0$ *if* $\sigma_{\max}(\mathcal{E}_{\text{nn}}) \leqslant \frac{1}{\sqrt{\lambda_{\max}(G(q))\lambda_{\max}(G^{-1}(q))}}$, *which implies that* $\sigma_{\max}(\mathcal{E}_{\text{nn}}) \leqslant 1$.

Proof The proof begins similarly to the proof of Proposition 5, and we obtain that $T_L(t) \leqslant 0 \Leftrightarrow B(q)B^T(q) \leqslant I$ with $B(q) = G^{\frac{1}{2}}(q)\mathcal{E}_{\text{nn}}^T G^{-\frac{1}{2}}(q)$. By [7, Lemma 8.4.1], one has equivalently $\lambda_{\max}(B(q)B^T(q)) = \sigma_{\max}^2(B(q)) \leqslant 1$. From [7, Corollary 9.6.5], one has $\sigma_{\max}(B(q)) \leqslant \sigma_{\max}(G^{\frac{1}{2}}(q))\sigma_{\max}(G^{-\frac{1}{2}}(q))\sigma_{\max}(\mathcal{E}_{\text{nn}})$. Therefore, $\sigma_{\max}(G^{\frac{1}{2}}(q))\sigma_{\max}(G^{-\frac{1}{2}}(q)) \times \sigma_{\max}(\mathcal{E}_{\text{nn}}) \leqslant 1$ implies that $\sigma_{\max}^2(B(q)) \leqslant 1$. From the symmetry and positive definiteness of $G(q)$ and of its square root, one has $\sigma_{\max}(G^{\frac{1}{2}}(q)) = \sqrt{\lambda_{\max}(G(q))}$, so the result follows. For the last statement notice that $I = G^{-\frac{1}{2}}(q)G^{\frac{1}{2}}(q)$, so again from [7, Corollary 9.6.5] $1 \leqslant \sigma_{\max}(G^{\frac{1}{2}}(q))\sigma_{\max}(G^{-\frac{1}{2}}(q)) = \sqrt{\lambda_{\max}(G(q))\lambda_{\max}(G^{-1}(q))}$. □

If $\mathcal{E}_{\text{nn}}$ is symmetric, one has $\sigma_i(\mathcal{E}_{\text{nn}}) = |\lambda_i(\mathcal{E}_{\text{nn}})|$, $1 \leqslant i \leqslant p'$. Thus, $\sigma_{\max}(\mathcal{E}_{\text{nn}}) = \text{sprad}(\mathcal{E}_{\text{nn}})$, where sprad is the spectral radius, i.e. the largest $|\lambda_i(\mathcal{E}_{\text{nn}})|$, $1 \leqslant i \leqslant p'$. One can therefore replace $\sigma_{\max}(\mathcal{E}_{\text{nn}})$ by $\text{sprad}(\mathcal{E}_{\text{nn}})$ in Proposition 6. If $\mathcal{E}_{\text{nn}}$ has only non-negative eigenvalues, then $\text{sprad}(\mathcal{E}_{\text{nn}}) = \lambda_{\max}(\mathcal{E}_{\text{nn}})$. The condition imposed in Proposition 5 holds if for instance $\mathcal{E}_{\text{nn}} = \text{diag}(e)$. In fact, $\mathcal{E}_{\text{nn}} = G(q)\mathcal{E}_{\text{nn}}^T G^{-1}(q)$ is equivalent to $G^{-1}(q)\mathcal{E}_{\text{nn}} = \mathcal{E}_{\text{nn}}^T G^{-1}(q)$, which allows us to rewrite (49) as

$$T_L(t) = \frac{1}{2}\dot{q}_{\text{norm}}^T(t^-)\big[\big(\mathcal{E}_{\text{nn}}^T\mathcal{E}_{\text{nn}} - I\big)G^{-1}(q)\big]\dot{q}_{\text{norm}}(t^-) \tag{51}$$

This shows clearly that the kinetic angles play a crucial role in the multiple impact energetical behavior. The expression in (51) is very similar to the classical energy loss expression for two frictionless particles that undergo a single impact (see, for instance, [16, Eq. (4.44)]. We may therefore see (39) and (51) as the true generalization of the simple frictionless shock between two particles, to multi-constraint frictionless Lagrange systems considered as a point moving on $\mathcal{C}$. The kinetic angles matrix $G(q)$ is a measure of the distortion due not only to the inertia matrix $M(q)$ but also to the unilateral constraints. Another characterization of energy loss is as follows, which starts from (49).

Proposition 7 *Assume that $G(q)$ is positive definite. If $\|\mathcal{E}_{\mathrm{nn}}\|_2 < \frac{1}{\sqrt{\|G^{-1}(q)\|_2\|G(q)\|_2}}$, then the matrix $G^{-1}(q) - \mathcal{E}_{\mathrm{nn}}^T G^{-1}(q)\mathcal{E}_{\mathrm{nn}}$ is positive definite, and $T_L(t) \leqslant 0$ in (49).*

Proof Using [23, Theorem 2.11] (see Theorem 3 in Appendix A) with $A \triangleq G^{-1}(q) - \mathcal{E}_{\mathrm{nn}}^T G^{-1}(q)\mathcal{E}_{\mathrm{nn}}$ and $M \triangleq G^{-1}(q)$, one obtains that $M - A = \mathcal{E}_{\mathrm{nn}}^T G^{-1}(q)\mathcal{E}_{\mathrm{nn}}$, from which it follows that provided $\|G(q)\|_2\|\mathcal{E}_{\mathrm{nn}}^T G^{-1}(q)\mathcal{E}_{\mathrm{nn}}\|_2 < 1$ then A is positive definite. From [7, Proposition 9.3.5] one has $\|\mathcal{E}_{\mathrm{nn}}^T G^{-1}(q)\mathcal{E}_{\mathrm{nn}}\|_2 \leqslant \|\mathcal{E}_{\mathrm{nn}}\|_2^2\|G^{-1}(q)\|_2$. So the condition $\|G(q)\|_2\|\mathcal{E}_{\mathrm{nn}}\|_2^2\|G^{-1}(q)\|_2 < 1$ guarantees that A is positive definite. □

Obviously, in Proposition 4, the value ϵ^* may be linked with the entries of $\mathcal{E}_{\mathrm{nn}}$. Roughly, it says that if the restitution coefficients are small enough the energy loss is guaranteed. When all constraints are orthogonal to one another ($G(q) = I$) and choosing $\mathcal{E} = \mathrm{diag}(e_{\mathrm{n},i})$, then one finds from Proposition 7 that $\sum_{i=1}^p e_{\mathrm{n},i}^2 \leqslant p'$. If all coefficients are equal to one another, then it follows that $|e_{\mathrm{n}}| \leqslant 1$. The interest of Propositions 6 and 7 is that they clearly link the restitution matrix to the kinetic angles, i.e. to the inertial/geometrical couplings that exist in the system, with computable upper-bounds. Propositions 5 and 6 state in a correct way Proposition 1 in [19], which wrongly asserts that $|\lambda_{\max}(\mathcal{E}_{\mathrm{nn}})| \leqslant 1$ is a sufficient condition for $T_L(t) \leqslant 0$ under symmetry of $\mathcal{E}_{\mathrm{nn}}$. The energy consistency of an extended frictionless Moreau's law with $\mathcal{E}_{\mathrm{nn}} = \mathrm{diag}(e_i)$ is analyzed in [55, §7.1] [36], starting from the Thomson and Tait formula (47), or from (48), or from (50). Actually, one may use Propositions 7.1 and 7.2 in [55] to analyze (51). The condition of Proposition 5 is quite close to the commuting conditions of [55, p. 159]. The conditions of Proposition 7 may be conservative as shown in Remark 6. Finally, let us remind that in the case Poisson coefficients are used (kinetic impact law), one obtains similar expressions for the loss of kinetic energy (see Eq. (43) in [37]). The quadratic forms in (48)–(50) therefore possess a general interest for both kinematic and kinetic impact laws.

Remark 6 Let us compare the criteria obtained in Propositions 2 and 7, when $p = 2$. We denote

$$G(q) = \begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix}$$

hence

$$G^{-1}(q) = \frac{1}{1-a^2}\begin{pmatrix} 1 & -a \\ -a & 1 \end{pmatrix}$$

and $|a| < 1$. Proposition 7 yields $\|\mathcal{E}_{\mathrm{nn}}\|_2 < \frac{1-a^2}{2+a^2} \leqslant \frac{1}{2}$. The co-positivity conditions of Proposition 2 for 2×2 matrices and diagonal $\mathcal{E}_{\mathrm{nn}}$ yield[8] $|e_{n,1}| \leqslant 1$, $|e_{n,2}| \leqslant 1$, and

[8]Necessary and sufficient conditions for co-positivity of $A = A^T$ are $a_{11} \geqslant 0$, $a_{22} \geqslant 0$, $a_{12} + \sqrt{a_{11}a_{22}} \geqslant 0$ [39].

$-ae_{n,1}e_{n,2}+a+\sqrt{(e_{n,1}^2-1)(e_{n,2}^2-1)}\geqslant 0$, which is equivalent to $(1-a^2)e_{n,1}^2e_{n,2}^2-e_{n,1}^2-e_{n,2}^2+2a^2e_{n,1}e_{n,2}+1-a^2\geqslant 0$. Suppose that $a=0$. Clearly, the second criterion holds for any $|e_{n,1}|\leqslant 1$, $|e_{n,2}|\leqslant 1$ and is less conservative than the first one which implies that $e_{n,1}^2+e_{n,2}^2\leqslant \frac{1}{4}$, which in turn implies that $e_{n,1}\leqslant \frac{1}{2}$ and $e_{n,2}\leqslant \frac{1}{2}$. Proposition 7, however, proposes a criterion that is easy to check, whereas as alluded to above, co-positivity is in general hard to check.

3.1.3 Uniqueness of the coefficients or restitution

As shown in [19, Sect. 7] on a 4-ball chain, a major discrepancy between single and multiple impacts is that for the latter the kinematic, kinetic, and energetic consistency constraints do not yield a unique set of coefficients of restitution for given pre-impact velocity.

Proposition 8 *Let a frictionless impact occur at t. It is necessary and sufficient that*:

- (i) $\mathcal{E}_{\rm nn}$ *is non-negative* (*kinematic consistency*),
- (ii) $G^{-1}(q)(I+\mathcal{E}_{\rm nn})$ *is non-negative* (*kinetic consistency*),
- (iii) $G^{-1}(q)-\mathcal{E}_{\rm nn}^TG^{-1}(q)\mathcal{E}_{\rm nn}$ *is co-positive* (*energetic consistency*),

for $\mathcal{E}_{\rm nn}$ to be an admissible restitution matrix for any pre-impact velocity $\dot{q}_{\rm norm}(t^-)$.

Proof (i) assures that $\dot{q}_{\rm norm}(t^+)\geqslant 0$ for any $\dot{q}_{\rm norm}(t^-)\leqslant 0$ (kinematic consistency), (ii) guarantees that $\bar{p}_n\geqslant 0$ (kinetic consistency), and (iii) is the energetical constraint. □

Let us note that similar conditions have been stated in [76, Sect. 3.3] and [19]. Clearly, if $G^{-1}(q)$ is itself non-negative, then (i) satisfied implies that (ii) is satisfied (kinematic consistency implies kinetic consistency). This is the case for a chain of three aligned balls [72].

Corollary 1 *Suppose that the system is conservative. Then it is necessary and sufficient that*:

- (i) $\mathcal{E}_{\rm nn}$ *is non-negative* (*kinematic consistency*),
- (ii) $\mathcal{E}_{\rm nn}^TG^{-1}(q)\mathcal{E}_{\rm nn}(I+\mathcal{E}_{\rm nn})$ *is non-negative* (*kinetic consistency*),
- (iii) $G^{-1}(q)=\mathcal{E}_{\rm nn}^TG^{-1}(q)\mathcal{E}_{\rm nn}$ (*energetic consistency*),

for $\mathcal{E}_{\rm nn}$ to be an admissible restitution matrix for any pre-impact velocity $\dot{q}_{\rm norm}(t^-)$. Then the eigenvalues of $\mathcal{E}_{\rm nn}$ have modulus equal to one.

Proof The proof of (i) and (ii) is a direct consequence of Proposition 8. (iii) Let $B(q)=G^{\frac{1}{2}}(q)\mathcal{E}_{\rm nn}G^{-\frac{1}{2}}(q)$. One has $B^T(q)B(q)=I$. Being a unitary (orthogonal) matrix $B(q)$ has its eigenvalues on the unit circle [7, Proposition 5.5.25]. Since $B(q)$ and $\mathcal{E}_{\rm nn}$ are similar, they have the same eigenvalues [51, Proposition 1]. □

If $\mathcal{E}_{\rm nn}$ has real eigenvalues, then they are equal to one under Corollary 1 conditions. The conditions (i) (ii) (iii) of Corollary 1 make a non-linear equation for $\mathcal{E}_{\rm nn}$. There are p'^2 unknowns (the restitution coefficients), with $\frac{p'(p'+1)}{2}$ equalities from (iii) and $2p'^2$ inequalities from (i) and (ii). Since $p'^2-\frac{p'(p'+1)}{2}>0$ for all $p'\geqslant 2$, there are more unknowns than equalities: the problem of calculating the coefficients of restitution (the entries of $\mathcal{E}_{\rm nn}$) from Corollary 1 seems to be under-determined, consequently the uniqueness of the coefficients may not hold. Suppose that (i) is satisfied. From (22), if all the kinetic angles satisfy $\theta_{ij} \geqslant \frac{\pi}{2}$, then (ii) is satisfied. When $p' = 1$ (single impact) then (i) $e_n \geqslant 0$, (ii) $e_n^2(1 + e_n) \geqslant 0$, (iii) $e_n^2 = 1$, so that $e_n = 1$ is the only solution (the same holds if the constraints are orthogonal so that $G(q) = I$, which implies that $e_{n,i} = 1$ for all $\in \{1, \ldots, p'\}$).

Proposition 8 and Corollary 1 are stated for all admissible pre-impact velocities. However, there are typical problems (chains of balls, rocking block) for which some components $\dot{q}_{\text{norm},i}(t^-) = 0$. As alluded to in [19], this may explicitly yield non-uniqueness of the coefficients of restitution. To illustrate this point, let us consider chains of balls (Sect. 2.3.1) with $N = 3$, $\dot{q}_1(t^-) = 1$ m/s, $\dot{q}_2(t^-) = \dot{q}_3(t^-) = 0$ m/s, and equal masses m. One has:

$$\begin{aligned}
&\dot{q}_{\text{norm},1} = \sqrt{\frac{m}{2}}(\dot{q}_2 - \dot{q}_1), \qquad \dot{q}_{\text{norm},2} = \sqrt{\frac{m}{2}}(\dot{q}_3 - \dot{q}_2), \qquad \dot{q}_{\tan} = \frac{1}{\sqrt{3}}(\dot{q}_1 + \dot{q}_2 + \dot{q}_3) \\
&G(q) = \begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix}, \qquad \bar{\lambda}_n = \frac{4}{3}\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}\begin{pmatrix} e_{n,1} + 1 \\ e_{n,21} \end{pmatrix}
\end{aligned} \tag{52}$$

It is noteworthy that $\dot{q}_{\text{norm}}(t^+) = \binom{e_{n,1}}{e_{n,21}}$, i.e. $e_{n,2}$ and $e_{n,12}$ do not play any role in this collision. The kinematic and kinetic constraints are satisfied for any non-negative coefficients. The conservation of kinetic energy yields that the two coefficients lie on the ellipse:

$$e_{n,1}^2 + e_{n,21}^2 - e_{n,1}e_{n,21} = 1 \tag{53}$$

The non-negativity implies that the portion of this ellipse in $\mathbb{R}_+^2$ contains all (an infinity) the admissible restitution law parameters. It is therefore a peculiarity of multiple impacts that the kinematic, kinetic, and energetic consistencies do not allow one to always uniquely determine the restitution coefficients. This, in passing, raises the following question: What is the mechanical meaning of such coefficients? *Such uniqueness issues stem from the fact that multiple impacts intrinsically are a multiscale phenomenon, where the elasticity properties at each contact strongly influence the impact outcome [14, 16, 42, 72].* For a fixed energetical behavior (e.g. conservative systems) varying the stiffnesses ratios and the elasticity coefficient (linear, Hertz elasticity, or else) may induce drastic changes of the post-impact velocity [72, Chaps. 5, 6]. Kinematic impact laws do not convey enough information on the contact process to *a priori* detect, which one of the outcomes is the right one. Some parameter-fitting process seems unavoidable.

This analysis motivates the next definition.

Definition 1 One says that the system has the U-property if the set of conditions (i) (ii) (iii) in Corollary 1 yields a unique restitution matrix $\mathcal{E}_{\text{nn}}$ (equivalently a unique set of restitution coefficients).

The orthogonality conditions $\mathbf{n}_{q,i}^T M(q)\mathbf{n}_{q,j} = 0$ for all $i \neq j$ (equivalently $G(q) = I$) guarantee the U-property. It would be quite interesting to link the U-property with discontinuity of the trajectories with respect to initial conditions and the study of limits of compliant models as the stiffnesses diverge to infinity. In other words, are the following equivalences true?

$$\begin{aligned}
\text{U-property} \quad &\Leftrightarrow \quad \text{continuity w.r.t. initial data} \\
&\Leftrightarrow \quad \text{uniqueness of limit for infinite stiffnesses}
\end{aligned}$$

Elements of answers may exist in the literature, using the results about a particle hitting a corner [42, 50, 72, 84], where explicit formulas giving the particle evolution when it collides the angle are provided. This analysis, however, quickly become quite cumbersome and seem to be limited to simple dynamics with $p = 2$. A general characterization of continuity w.r.t. initial data is made in [74]. Continuity holds when the kinetic angles between the p constraints are all $\leqslant \frac{\pi}{2}$ and the coefficients of restitution with each constraint are zero, or if the constraints are all pairwise orthogonal (in the kinetic metric). In such cases, the first equivalence holds. Indeed by continuation from the neighborhood of the singularities of the boundary of the admissible domain $\Phi = \{q \in \mathcal{C} | h(q) \geqslant 0\}$, one may define a unique post-impact velocity for any pre-impact data. Therefore, $\mathcal{E}_{\mathrm{nn}}$ is unique.

One may think that (10) straightforwardly shows that a frictionless Lagrangian system is equivalent, *via* the kinetic quasi-velocities transformation, to a particle. Even if we disregard the non-linear terms in the left-hand side in (10), the above shows that inertial couplings between the constraints render the impact problem quite different from the single collision case.

3.2 Systems with tangential effects

Some systems like chains of balls possess an intrinsic "invariance at impacts" of the quasi-tangential velocity $\dot{q}_{\tan}$ (in this case, the total linear momentum of the chain). However, in general some tangential effects exist at the contact points, in the locally tangent planes spanned by $(t_{i,1}, t_{i,2})$ (see Sect. 2.2.2 for the notations).

3.2.1 Tangential restitution

The relation between local tangential coefficients of restitution and Coulomb's friction at the impulse level is discussed in [36, Sect. 4.5]. It is pointed out that despite some few experimental cases show the necessity of introducing tangential restitution coefficients, in general tangential effects are primarily due to Coulomb's friction. The relation between local tangential and quasi-tangential coefficients of restitution is established for the planar rocking block when there is perfect sticking at the two impact points in [19, Sect. 6]. In fact, sticking imposes some constraints on the tangential velocities, and the tangential coefficients (local and quasi) impose also some constraints on tangential velocities. The mixture of all those constraints create the relations between the various coefficients.

The starting point is to consider the local kinematics at the contact points [1, Sect. 3.3], which dictates the form of $H_{\mathrm{T}}(q)$ in (23) and (24). When the local frames consist of a single normal vector, like in aligned chains of balls, then $H_{\mathrm{T}}(q) = 0$ and there are no tangential effects. Otherwise, $H_{\mathrm{T}}(q) \neq 0$. It is noteworthy that since $\dot{q}_{\tan}$ is an $n - p'$ vector, and since p_{t} is a p' vector (if all the contact points are rough, otherwise it has dimension less than p'), in general the tangent quasi-velocity and the local-kinematics tangent velocity do not have a clear, one-to-one, correspondence.

A first condition is the consistency of the impact law (41) (42) and of the shock dynamics

$$\begin{aligned} \dot{q}_{\mathrm{norm}}(t^+) - \dot{q}_{\mathrm{norm}}(t^-) &= \mathbf{n}_q^T M(q) \mathbf{n}_q \bar{p}_n(t) + \mathbf{n}_q^T H_{\mathrm{T}}(q) p_{\mathrm{t}} \\ \dot{q}_{\tan}(t^+) - \dot{q}_{\tan}(t^-) &= \mathbf{t}_q^T H_{\mathrm{T}}(q) p_{\mathrm{t}} \end{aligned} \tag{54}$$

Inserting (41) into (54), one obtains

$$-\underbrace{\begin{pmatrix} I+\mathcal{E}_{\text{nn}} & \mathcal{E}_{\text{nt}} \\ \mathcal{E}_{\text{tn}} & I+\mathcal{E}_{\text{tt}} \end{pmatrix}}_{=\bar{\mathcal{E}}} \begin{pmatrix} \dot{q}_{\text{norm}}(t^-) \\ \dot{q}_{\tan}(t^-) \end{pmatrix} = \underbrace{\begin{pmatrix} G(q) & \mathbf{n}_q^T H_{\text{T}}(q) \\ 0 & \mathbf{t}_q^T H_{\text{T}}(q) \end{pmatrix}}_{\stackrel{\Delta}{=}\bar{G}(q)} \begin{pmatrix} \bar{p}_n \\ p_{\text{t}} \end{pmatrix} \tag{55}$$

In the frictionless case, $\bar{p}_n$ can always be calculated (independently of its sign) as $\bar{p}_n = -G^{-1}(q)(I+\mathcal{E}_{\text{nn}})\dot{q}_{\text{norm}}(t^-)$, for any value of $\dot{q}_{\text{norm}}(t^-)$, because $G(q)$ has full rank. We recall that $\bar{\mathcal{E}}$ is $n \times n$, while $\bar{G}(q)$ is $n \times 2p'$ since we suppose that friction acts at all the contacts. The equation in (55) has a solution for the impulse if and only if:

$$\bar{\mathcal{E}}\dot{q}_{\text{norm}}\big(t^-\big) \in \mathcal{R}\big(\bar{G}(q)\big) \tag{56}$$

A sufficient condition is that $\mathcal{R}(\bar{\mathcal{E}}) \subseteq \mathcal{R}(\bar{G}(q))$. Taking into account that the dimensions of p_{t} and of $\dot{q}_{\tan}$ usually are different, there is no systematic way to translate this range condition into conditions on $\mathcal{E}$ and the impact geometry in $\bar{G}(q)$. Case by case analysis seem unavoidable.

Implicitly defined restitution operator Inspired by Moreau [62, 69] and others later [28, 29, 36, 55], we may impose $\bar{p}_n \in -N_{V_n(q)}(\Gamma_n(\dot{q}_{\text{norm}}(t^+) + \Lambda_n\dot{q}_{\text{norm}}(t^-)))$ and $p_{\text{t}} \in -N_{V_{\text{t}(q)}}(\Gamma_{\text{t}}(\dot{q}_{\tan}(t^+) + \Lambda_{\text{t}}\dot{q}_{\tan}(t^-)))$, for some matrices Λ_n, Λ_{t}, Γ_n, Γ_{t}, and convex sets $V_n(q)$ and $V_{\text{t}}(q)$. Inserting this into (54), one obtains the generalized equation:

$$v\big(t^+\big) - v\big(t^-\big) \in -\bar{G}(q) \begin{pmatrix} N_{V_n(q)}(\Gamma_n(\dot{q}_{\text{norm}}(t^+) + \Lambda_n\dot{q}_{\text{norm}}(t^-))) \\ N_{V_{\text{t}(q)}}(\Gamma_{\text{t}}(\dot{q}_{\tan}(t^+) + \Lambda_{\text{t}}\dot{q}_{\tan}(t^-))) \end{pmatrix} \tag{57}$$

Defining the convex set $W(q) \stackrel{\Delta}{=} V_n(q) \times V_{\text{t}}(q)$ and $\Lambda = \text{diag}(\Lambda_n, \Lambda_{\text{t}})$, $\Gamma = \text{diag}(\Gamma_n, \Gamma_{\text{t}})$, we get

$$v\big(t^+\big) - v\big(t^-\big) \in -\bar{G}(q)\, N_{W(q)}\big(\Gamma\big(v\big(t^+\big) + \Lambda v\big(t^-\big)\big)\big) \tag{58}$$

The existence and uniqueness of a solution $v(t^+)$ to the generalized equation in (58) depend on the matrices $\bar{G}(q)$, Γ, Λ, and on the convex sets $V_n(q)$ and $V_{\text{t}}(q)$. Suppose for instance that there exists a symmetric positive definite matrix P such that $P\bar{G}(q) = \Gamma^T$, and let us denote R its symmetric square root: $R^2 = P$.[9] Then using the chain rule from convex analysis [78], one obtains that (58) is equivalent to

$$z - Rv\big(t^-\big) \in -N_{\bar{W}(q)}\big(z + R\Lambda v\big(t^-\big)\big) \tag{59}$$

where $z = Rv(t^+)$ is the unknown,[10] and $\bar{W}(q) = \{x \mid \bar{G}^T Rx \in W(q)\}$ is a convex set (which we assume to be non-empty, which is equivalent to $\mathcal{R}(\bar{G}(q)) \cap W(q) \neq \emptyset$). Then by classical arguments from convex analysis, one infers that the unique solution of the generalized equation (58) is given by:

---

[9]Recall that q is a constant during the shock, so that all the matrices that depend on q only are also constant. Thus, P and H may depend on q as well.

[10]The idea of doing such a change of coordinates was first introduced in [17] in the context of maximal monotone differential inclusions.

$$
\begin{aligned}
v(t^+) &= -\Lambda v(t^-) + R^{-1}\operatorname{proj}\left[\bar{W}(q); R(\Lambda+I)v(t^-)\right] \\
&= -\Lambda v(t^-) + R^{-1}z^* \\
\text{with } z^* &= \operatorname{argmin}_{z\in\bar{W}(q)} \frac{1}{2}\left(z - R(\Lambda+I)v(t^-)\right)^T\left(z - R(\Lambda+I)v(t^-)\right)
\end{aligned} \tag{60}
$$

where proj denotes the orthogonal projection in the Euclidean metric. The proof for (59) and (60) is given in Appendix D. Since Γ is free to the designer, fulfilling the above conditions is always possible. However it is another, much more difficult matter to design an impact law which possesses a mechanical meaning. Depending on $\bar{W}(q)$, Γ, P and $v(t^-)$, (60) implicitly defines a restitution operator of the same type as (43).

3.2.2 *Energetic consistency (restitution matrix)*

The minimum requirement one may impose on an impact law like the one in (41) (42) is that it satisfies the three fundamental consistencies: kinematic, kinetic, and energetic. The energetic consistency may be checked using (15) and (54). Indeed one has three basic ingredients:

$$
T(q,v) = \frac{1}{2}v^T D(q)v, \qquad v(t^+) = -\mathcal{E}v(t^-), \qquad v(t^+) - v(t^-) = \bar{G}(q)\bar{p} \tag{61}
$$

with $\bar{p} = \binom{\bar{p}_n}{p_t}$ and $D(q) = \operatorname{diag}(D_n(q), D_t(q))$. The main discrepancy between the frictionless case and the case with tangential effects, is that $\bar{G}(q)$ may not be positive definite, even if the constraints are independent. Actually, $G(q)$ is always at least positive semi definite, but $\bar{G}(q)$ may be indefinite, or negative definite. Similarly to the frictionless case in (47)–(50), several expressions of the kinetic energy loss may be written:

$$
T_L(t) = \frac{1}{2}\bar{p}^T\bar{G}^T(q)(I-\mathcal{E})v(t^-) \tag{62}
$$

or

$$
T_L(t) = \frac{1}{2}v^T(t^-)(I+\mathcal{E})^T D(q)(\mathcal{E}-I)v(t^-) \tag{63}
$$

or

$$
T_L(t) = \frac{1}{2}v^T(t^-)\left(\mathcal{E}^T D(q)\mathcal{E} - D(q)\right)v(t^-) \tag{64}
$$

Deriving a similar proposition as Proposition 4, it follows that small enough restitution coefficients guarantee energetic consistency. The extension of Proposition 7 is as follows.

Proposition 9 *Assume that $D(q)$ is positive definite. If $\|\mathcal{E}\|_2 < \frac{1}{\sqrt{\|D^{-1}(q)\|_2\|D(q)\|_2}}$, then the matrix $D(q) - \mathcal{E}^T D(q)\mathcal{E}$ is positive definite, and $T_L(t) \leqslant 0$ in* (64).

One can use the block-diagonal form of $D(q)$ to refine the result since one has $\|D(q)\|_2 = \sqrt{\|D_n(q)\|_2^2 + \|D_t(q)\|_2^2}$ and $\|D^{-1}(q)\|_2 = \sqrt{\|D_n^{-1}(q)\|_2^2 + \|D_t^{-1}(q)\|_2^2}$. Modeling tangential effects with constant tangential restitution coefficients is in most of the cases a very crude approximation of the real mechanical phenomenon.

3.2.3 Energetic consistency (implicit restitution operator)

Starting from $T_L(t) = \frac{1}{2}(v(t^+) + v(t^-))^T D(q)(v(t^+) - v(t^-))$, a natural way to deal with its dissipativity is to force the operator defined in (57) to be maximal monotone. Using (59), one obtains

$$T_L(t) \in -\frac{1}{2}\big(z(t^+) + z(t^-)\big)^T R^{-1} D(q) R\, N_{\bar{W}(q)}\big(z(t^+) + R\Lambda R^{-1} z(t^-)\big) \tag{65}$$

At this stage , we are left with the analysis of the monotonicity of $y \mapsto R^{-1}D(q)R\, N_{\bar{W}(q)}(y + (R\Lambda R^{-1} - I)z(t^-))$. Notice that another path that consists of using (60) can be followed. Let $\Lambda = \Lambda^T$. If one assumes that $0 \in \bar{W}(q)$, then the projection is contractive because $\bar{W}(q)$ is convex, so that $\mathrm{proj}[\bar{W}(q);\, R(\Lambda + I)v(t^-)] \leqslant \|R(\Lambda + I)v(t^-)\|$, and one obtains:

$$\begin{aligned} T_L(t) = {} & \frac{1}{2} v^T\big(t^-\big)\big[\Lambda D \Lambda - D\big] v\big(t^-\big) \\ & + \frac{1}{2}\,\mathrm{proj}^T\big[\bar{W}(q);\, R(\Lambda + I)v\big(t^-\big)\big] R^{-1} D R^{-1}\,\mathrm{proj}\big[\bar{W}(q);\, R(\Lambda + I)v\big(t^-\big)\big] \\ & - v^T\big(t^-\big)\Lambda D R^{-1}\,\mathrm{proj}\big[\bar{W}(q);\, R(\Lambda + I)v\big(t^-\big)\big] \\ \leqslant {} & -\frac{1}{2}\lambda_{\min}(D - \Lambda D \Lambda)\big\|v\big(t^-\big)\big\|^2 \\ & + \frac{1}{2}\big(\lambda_{\max}\big(R^{-1} D R^{-1}\big)\big\|R(\Lambda + I)\big\|_2 + 2\big\|R^{-1} D \Lambda\big\|_2\big)\big\|R(\Lambda + I)\big\|_2\big)\big\|v\big(t^-\big)\big\|^2 \end{aligned} \tag{66}$$

It appears that Λ plays the role of a restitution matrix that should be small enough. We do not investigate further implicitly defined kinematic restitution operators, because this has been thoroughly analyzed by Moreau, Frémond, and others, while such type of formulations is not the main topic of this paper.

Remark 7 (Necessity of $\mathcal{E}_{\mathrm{tt}} \neq -I$) The rocking block of Sect. 2.3.2 has been analyzed in [19] with the above generalized kinematic law. It has been shown that Coulomb friction is unable to model some rocking motions, which necessitate the introduction of a quasi-tangential restitution coefficient that makes $\dot{q}_{\tan}$ jump. The obtained impact law is in turn equivalent to the angular velocity restitution that is widely used in earthquake engineering.

3.2.4 Coulomb's friction at contact points

A condition for the tangent quasi-velocities $\dot{q}_{\tan,i}$ to be able to correctly represent the local tangential effects is that their number, which is always $n - p'$, is larger or equal to the number of independent local tangent velocities, at impact times. To illustrate this, we may consider the disk in an angle and the rocking block examples in Sect. 2.3. For the disk system, $n = 3$, $p' = 2$, there are two tangential velocities $v_{\mathrm{t},1}$ and $v_{\mathrm{t},2}$, which may be zero (stick) or non-zero (slip) independently. For the block system, $n = 3$, $p' = 2$, one always has $v_{\mathrm{t},1} = v_{\mathrm{t},2}$ when the two corners are in contact. One expects that restitution applied to $\dot{q}_{\tan}$ will be a crude approximation of Coulomb's friction for the disk, whereas it may be a better model for the block. Let us consider the disk example of Sect. 2.3.5 with $n = 3$, $p = 1$. As shown by several authors a correct model for the tangential velocity restitution is to apply $v_{\mathrm{t}}(t^+) = -e_{\mathrm{t}} v_{\mathrm{t}}(t^-)$, where the tangential restitution coefficient is related to Coulomb's friction in a certain manner (see [16, pp. 133–134] for a short review and references). Let us choose $\mathcal{E}_{\mathrm{tt}} = \mathrm{diag}(e_{\mathrm{t}})$ and $\mathcal{E} = \mathrm{diag}(\mathcal{E}_{\mathrm{nn}}, \mathcal{E}_{\mathrm{tt}})$. Then the mechanical meaning of the restitution law (41) is clear. Notice from (36) that one has $\mathbf{n}_q^T H_{\mathrm{T}} = 0$, which means that the normal and the tangential quasi-dynamics are decoupled. This is not the case in neither (33) nor (37).

We may assume in a rather classical way that at each contact point one has

$$p_{\mathrm{t},i} \in -\mu_i p_{n,i} \ \mathrm{sgn}\big(v_{\mathrm{t},i}\big(t^+\big) + e_{\mathrm{t},i} v_{\mathrm{t},i}\big(t^-\big)\big) \tag{67}$$

for some coefficient $e_{\mathrm{t},i}$, which is a way to write the Coulomb's model, at the impulse level. Adopting the normal restitution rule $\dot{q}_{\mathrm{norm}}(t^+) = -\mathcal{E}_{\mathrm{nn}}\dot{q}_{\mathrm{norm}}(t^-)$, we obtain the following inclusions:

$$\begin{aligned} &-(I+\mathcal{E}_{\mathrm{nn}})\dot{q}_{\mathrm{norm}}\big(t^-\big) \in G(q)\bar{p}_n - \mathbf{n}_q^T H_{\mathrm{T}}(q)[\bar{\mu}][\bar{p}_n]\,\mathrm{Sgn}\big(v_{\mathrm{t}}\big(t^+\big) + \mathcal{E}_{\mathrm{tt}} v_{\mathrm{t}}\big(t^-\big)\big) \\ &\dot{q}_{\tan}\big(t^+\big) - \dot{q}_{\tan}\big(t^-\big) \in -\mathbf{t}_q^T H_{\mathrm{T}}(q)[\bar{\mu}][\bar{p}_n]\,\mathrm{Sgn}\big(v_{\mathrm{t}}\big(t^+\big) + \mathcal{E}_{\mathrm{tt}} v_{\mathrm{t}}\big(t^-\big)\big) \end{aligned} \tag{68}$$

with:

$$[\bar{\mu}] = \mathrm{diag}\left(\frac{\mu_i}{\|\nabla h_i(q)\|_{M^{-1}}}\right) \in \mathbb{R}^{p'\times p'}, \qquad [\bar{p}_n] = \mathrm{diag}(\bar{p}_{n,i}), \qquad \mathcal{E}_{\mathrm{tt}} = \mathrm{diag}(e_{\mathrm{t},i}),$$

$$v_{\mathrm{t}} = H_{\mathrm{T}}^T(q)\varXi^T(q)v \quad (v \text{ is in } (6)),$$

$$\mathrm{Sgn}\big(v_{\mathrm{t}}\big(t^+\big) + \mathcal{E}_{\mathrm{tt}} v_{\mathrm{t}}\big(t^-\big)\big) = \big(\mathrm{sgn}\big(v_{\mathrm{t},1}\big(t^+\big) + e_{\mathrm{t},1} v_{\mathrm{t},1}\big(t^-\big)\big), \ldots, \mathrm{sgn}\big(v_{\mathrm{t},p}\big(t^+\big) + e_{\mathrm{t},p} v_{\mathrm{t},p}\big(t^-\big)\big)\big)^T$$

The unknowns of the generalized equation (68) are the p' impulses $\bar{p}_{n,i}$, and the $n-p'$ quasi-velocities $\dot{q}_{\tan}(t^+)$. The generalized equation in (68) is quite complex, in part because it is non-associated (i.e. the multi-valued part is not the sub-differential of a convex non-smooth potential function). It may be analyzed as follows. Suppose that all components $v_{\mathrm{t},i}(t^+) + e_{\mathrm{t},i} v_{\mathrm{t},i}(t^-) > 0$, i.e. the contact points are all in the same sliding status. In order to verify if such a mode is the mode of the system, one constructs the following system:

$$\begin{aligned} &\text{(a)} \quad -(I+\mathcal{E}_{\mathrm{nn}})\dot{q}_{\mathrm{norm}}\big(t^-\big) = \big(G(q) - \mathbf{n}_q^T H_{\mathrm{T}}(q)[\bar{\mu}]\big)\bar{p}_n \\ &\text{(b)} \quad \dot{q}_{\tan}\big(t^+\big) - \dot{q}_{\tan}\big(t^-\big) = -\mathbf{t}_q^T H_{\mathrm{T}}(q)[\bar{\mu}]\bar{p}_n \\ &\text{(c)} \quad H_{\mathrm{T}}^T(q)M^{-1}(q)\varXi^{-1}(q)\begin{pmatrix} -\mathcal{E}_{\mathrm{nn}}\dot{q}_{\mathrm{norm}}(t^-) \\ \dot{q}_{\tan}(t^+) \end{pmatrix} \\ &\qquad + \mathcal{E}_{\mathrm{tt}} H_{\mathrm{T}}^T(q)M^{-1}(q)\varXi^{-1}(q)\begin{pmatrix} \dot{q}_{\mathrm{norm}}(t^-) \\ \dot{q}_{\tan}(t^-) \end{pmatrix} > 0 \\ &\text{(d)} \quad \bar{p}_n \geqslant 0. \end{aligned} \tag{69}$$

If the system in (69) has a unique solution for $\bar{p}_n$ and $\dot{q}_{\tan}(t^+)$, then the impact is in this "all-sliding" mode. Suppose that one has $\bar{p}_n = -(G(q) - \mathbf{n}_q^T H_{\mathrm{T}}(q)[\bar{\mu}])^{-1}(I+\mathcal{E}_{\mathrm{nn}})\dot{q}_{\mathrm{norm}}(t^-) \geqslant 0$. If this is the case, then (69) boils down to finding $\dot{q}_{\tan}(t^+)$ such that:

$$\begin{aligned} &\dot{q}_{\tan}\big(t^+\big) = \dot{q}_{\tan}\big(t^-\big) + \mathbf{t}_q^T H_{\mathrm{T}}(q)[\bar{\mu}]\big(G(q) - \mathbf{n}_q^T H_{\mathrm{T}}(q)[\bar{\mu}]\big)^{-1}(I+\mathcal{E}_{\mathrm{nn}})\dot{q}_{\mathrm{norm}}(t^-) \\ &H_{\mathrm{T}}^T(q)M^{-1}(q)\varXi^{-1}(q)\begin{pmatrix} -\mathcal{E}_{\mathrm{nn}}\dot{q}_{\mathrm{norm}}(t^-) \\ \dot{q}_{\tan}(t^+) \end{pmatrix} \\ &\quad + \mathcal{E}_{\mathrm{tt}} H_{\mathrm{T}}^T(q)M^{-1}(q)\varXi^{-1}(q)\begin{pmatrix} \dot{q}_{\mathrm{norm}}(t^-) \\ \dot{q}_{\tan}(t^-) \end{pmatrix} > 0. \end{aligned} \tag{70}$$

This leads us to the following result.

Proposition 10 *Assume that* (*i*) $\mathcal{E}_{\mathrm{nn}}$ *is non-negative*, (*ii*) $G(q) - \mathbf{n}_q^T H_{\mathrm{T}}(q)[\bar{\mu}]$ *is full-rank*, (iii) $(G(q) - \mathbf{n}_q^T H_{\mathrm{T}}(q)[\bar{\mu}])^{-1}$ *is non-negative. Then the mode corresponding to* $v_{\mathrm{t},i}(t^+) + e_{\mathrm{t},i} v_{\mathrm{t},i}(t^-) > 0$ *at each impacting point exists if and only if the inequality*:

$$H_{\mathrm{T}}^T(q) M^{-1}(q) \Xi^{-1}(q) \begin{pmatrix} -\mathcal{E}_{\mathrm{nn}} \dot{q}_{\mathrm{norm}}(t^-) \\ \dot{q}_{\tan}(t^-) + \mathbf{t}_q^T H_{\mathrm{T}}(q)[\bar{\mu}](G(q) - \mathbf{n}_q^T H_{\mathrm{T}}(q)[\bar{\mu}])^{-1}(I + \mathcal{E}_{\mathrm{nn}}) \dot{q}_{\mathrm{norm}}(t^-) \end{pmatrix}$$
$$> -\mathcal{E}_{\mathrm{tt}} H_{\mathrm{T}}^T(q) M^{-1}(q) \Xi^{-1}(q) \begin{pmatrix} \dot{q}_{\mathrm{norm}}(t^-) \\ \dot{q}_{\tan}(t^-) \end{pmatrix} \tag{71}$$

is satisfied.

If $e_{\mathrm{t},i} = 0$ for all i, then the mode corresponds to a sliding mode. Let us recall that all functions are computed at q which is the impact position. Suppose that $\mu_i = \mu$ for all $1 \leqslant i \leqslant p'$, $G(q) > 0$ and that $\mathbf{n}_q^T H_{\mathrm{T}}(q)$ is a symmetric matrix. Then for μ small enough, the matrix $G(q) - \mathbf{n}_q^T H_{\mathrm{T}}(q)[\bar{\mu}]$ is symmetric positive definite (see the proof of Proposition 4) so (ii) is satisfied. If in addition it is a Z and a K-matrix,[11] then (iii) of the proposition is satisfied by [25, Theorem 3.11.10]. Such properties usually yield quite stringent conditions on the mechanical system under study. In fact, they may be over-conservative: examples show that solutions may exist even with relatively large μ. Another point is that not all the elements $\dot{q}_{\mathrm{norm},i}(t^-)$ may be < 0, some of them may be equal to zero. Therefore, the non-negativity condition (iii) for kinetic consistency may also be too conservative. This is the case for the rocking block as Example 3 shows.

A similar system as (70) can be constructed and examined for each mode. In practice, such an enumerative procedure is quite time-consuming and almost impossible for more than few impact points. For a single contact like a falling rod in Sect. 2.3.3, one can derive the conditions of existence of each mode [75]. What is more reasonable is therefore to prove that the generalized equation in (68) possesses at least one solution, and then to use a numerical method to calculate it. In the simplest case of a planar particle hitting a rough ground with $m = 1$ kg, coordinates as in Fig. 1(d), and using (67) (with $G(q) = 1$, $H_{\mathrm{T}} = (1\ 0)^T$, $\mathbf{n}_q = (0\ 1)^T$, $\mathbf{t}_q = (0\ 1)^T$, $\mathbf{n}_q^T H_{\mathrm{T}}(q) = 0$, $\mathbf{t}_q^T H_{\mathrm{T}}(q) = 1$, $\dot{q}_{\mathrm{norm}} = \dot{y} = v_n$, $\dot{q}_{\tan} = \dot{x} = v_{\mathrm{t}}$, $\Xi = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$), then simple manipulations of (68) yield $v_{\mathrm{t}}(t^+) = v_{\mathrm{t}}(t^-) - \mathrm{proj}([-1, 1]; \frac{1+e_t}{\mu(1+e_n)} \frac{v_{\mathrm{t}}(t^-)}{|v_n(t^-)|})$, where proj denotes the orthogonal projection on $[-1, 1]$. One may extend this result to the inclusion (68) to obtain a generalized equation with unknown $\dot{q}_{\tan}(t^+)$, and then deduce general conditions for its well-posedness. Propositions 10 and 11 are a preliminary analysis.

Example 3 Let us consider the system in Sect. 2.3.2, with $\mu_1 = \mu_2 = \mu$. Starting from (33) one can compute that at a double impact with $\theta = 0$ (noticing that $\|\nabla h_i(q)\|_q = \sqrt{f(0)}$ for $i = 1, 2$):

$$G(q) + \mathbf{n}_q^T H_{\mathrm{T}}(q)\mu = \frac{1}{4f(0)} \begin{pmatrix} 4 + L^2 + \mu l L & 4 - L^2 + \mu l L \\ 4 - L^2 - \mu l L & 4 + L^2 - \mu l L \end{pmatrix}$$
$$= \frac{1}{4f(0)} \begin{pmatrix} 4 + L^2 + \mu l L & 4 - L^2 \\ 4 - L^2 & 4 + L^2 - \mu l L \end{pmatrix} + \frac{\mu l L}{4f(0)} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{72}$$

[11] Also called an M-matrix [25, p. 222].

The second matrix in the second line in (72) is skew-symmetric. It is checked that the first matrix is positive definite if and only if $0 \leqslant \mu < \frac{4}{L}$, which implies that the whole matrix is also positive definite. The inverse matrix is given by

$$\big(G(q) + \mathbf{n}_q^T H_{\mathrm{T}}(q)\mu\big)^{-1} = 4f(0)\frac{1}{16L^2}\begin{pmatrix} 4+L^2-\mu lL & -4+L^2-\mu lL \\ -4+L^2+\mu lL & 4+L^2+\mu lL \end{pmatrix} \tag{73}$$

This matrix is non-negative if and only if $\mu \geqslant \frac{4-L^2}{lL}$ and $\mu \leqslant \frac{L^2-4}{lL}$, which implies that $L = 2$ and $\mu = 0$: the frictionless case. Thus, the criterion (iii) of Proposition 10 does not hold. Let us, however, focus on the rocking motion, which implies that $\dot{q}_{\mathrm{norm},2}(t^-) = 0$ (the corner A is in contact at the collision instant at B). Suppose also that we model plastic impacts so that $\mathcal{E}_{\mathrm{nn}} = 0$. Then $\dot{q}_{\mathrm{norm}}(t^-) = \big(\begin{smallmatrix} \dot{q}_{\mathrm{norm},1}(t^-) \\ 0 \end{smallmatrix}\big)$. Thus, only the first column of $(G(q) + \mathbf{n}_q^T H_{\mathrm{T}}(q)\mu)^{-1}$ plays a role and the non-negativity condition yields $4 + L^2 - \mu lL \geqslant 0$ and $L^2 + \mu lL \geqslant 0$, that is $\frac{4-L^2}{lL} \leqslant \mu \leqslant \frac{4+L^2}{lL}$. So conditions (i), (ii), and (iii) of the proposition can be fulfilled for some values of the friction coefficient $0 \leqslant \mu < \min(\frac{4}{l}, \frac{4+L^2}{lL})$.

Another source of conservativeness of the conditions of Proposition 10 comes from the fact that (iii) may not be satisfied, however, there exists some pre-impact velocities such that $\bar{p}_n \geqslant 0$. The next example illustrates this point.

Example 4 Consider now the system in Sect. 2.3.6, with $\mu_1 = \mu_2 = \mu$. One calculates that

$$\big(G(q) + \mathbf{n}_q^T H_{\mathrm{T}}(q)\mu\big)^{-1} = \sqrt{m}\big(1-\mu^2\big)\begin{pmatrix} 1 & -\mu \\ -\mu & 1 \end{pmatrix} \tag{74}$$

so that the non-negativity condition (iii) cannot hold if $\mu > 0$. This means that the analyzed mode does not exist for arbitrary pre-impact condition. However, one has

$$\bar{p}_n = m\big(1-\mu^2\big)\begin{pmatrix} (1+e_{n,1})\dot{y}(t^-) - \mu(1+e_{n,2})\dot{x}(t^-) \\ -\mu(1+e_{n,1})\dot{y}(t^-) + (1+e_{n,2})\dot{x}(t^-) \end{pmatrix} \tag{75}$$

The kinetic consistency holds if $\mu \geqslant \max(\frac{(1+e_{n,1})\dot{y}(t^-)}{(1+e_{n,2})\dot{x}(t^-)}, \frac{(1+e_{n,2})\dot{x}(t^-)}{(1+e_{n,1})\dot{y}(t^-)})$.

Motivated by these examples we can thus reformulate the problem of existence of this mode by stating that it exists only if this condition on the friction and the pre-impact velocity holds true.

Proposition 11 *Let $\mathcal{S}^-$ denote the largest set of pre-impact velocities such that $\bar{p}_n = -(G(q) - \mathbf{n}_q^T H_{\mathrm{T}}(q)[\bar{\mu}])^{\dagger}(I + \mathcal{E}_{\mathrm{nn}})\dot{q}_{\mathrm{norm}}(t^-)$ is componentwise non-negative and solves* (69) (a)–(d). *Then the mode corresponding to $v_{\mathrm{t},i}(t^+) + e_{\mathrm{t},i}v_{\mathrm{t},i}(t^-) > 0$ at each impacting point exists for pre-impact velocities in $\mathcal{S}^-$ if and only if the inequality*:

$$\begin{aligned} &H_{\mathrm{T}}^T(q)M^{-1}(q)\Xi^{-1}(q)\begin{pmatrix} -\mathcal{E}_{\mathrm{nn}}\dot{q}_{\mathrm{norm}}(t^-) \\ \dot{q}_{\mathrm{tan}}(t^-) + \mathbf{t}_q^T H_{\mathrm{T}}(q)[\bar{\mu}](G(q) - \mathbf{n}_q^T H_{\mathrm{T}}(q)[\bar{\mu}])^{\dagger}(I + \mathcal{E}_{\mathrm{nn}})\dot{q}_{\mathrm{norm}}(t^-) \end{pmatrix} \\ &\quad > -\mathcal{E}_{\mathrm{tt}}H_{\mathrm{T}}^T(q)M^{-1}(q)\Xi^{-1}(q)\begin{pmatrix} \dot{q}_{\mathrm{norm}}(t^-) \\ \dot{q}_{\mathrm{tan}}(t^-) \end{pmatrix} \end{aligned} \tag{76}$$

is satisfied.

3.2.5 Energetic consistency (Coulomb's friction)

Besides the existence of solution to systems like in (68) or (69), it is necessary to examine the energetic consistency of the law that couples normal impact restitution with Coulomb's friction at the impulse level. It is known since the celebrated Kane's counter-example that such models may violate the dissipativity [36]. It is therefore of utmost importance to provide some conditions on the parameters that guarantee the dissipativity. The kinetic energy loss is expressed as follows at a time t of impact:

$$\begin{aligned}
T_L(t) &= \frac{1}{2}\dot{q}_{\text{norm}}^T(t^+)D_{\text{n}}(q)\dot{q}_{\text{norm}}(t^+) - \frac{1}{2}\dot{q}_{\text{norm}}^T(t^-)D_{\text{n}}(q)\dot{q}_{\text{norm}}(t^-) \\
&\quad + \frac{1}{2}\dot{q}_{\text{tan}}^T(t^+)D_{\text{t}}(q)\dot{q}_{\text{tan}}(t^+) - \frac{1}{2}\dot{q}_{\text{tan}}^T(t^-)D_{\text{t}}(q)\dot{q}_{\text{tan}}(t^-) \\
&= \frac{1}{2}\left(\dot{q}_{\text{norm}}(t^+) + \dot{q}_{\text{norm}}(t^-)\right)^T D_{\text{n}}(q)\left(\dot{q}_{\text{norm}}(t^+) - \dot{q}_{\text{norm}}(t^-)\right) \\
&\quad + \frac{1}{2}\left(\dot{q}_{\text{tan}}(t^+) + \dot{q}_{\text{tan}}(t^-)\right)^T D_{\text{t}}(q)\left(\dot{q}_{\text{tan}}(t^+) - \dot{q}_{\text{tan}}(t^-)\right) \\
&= \frac{1}{2}\left(\dot{q}_{\text{norm}}(t^+) + \dot{q}_{\text{norm}}(t^-)\right)^T \bar{p}_n + \frac{1}{2}\left(\dot{q}_{\text{norm}}(t^+) + \dot{q}_{\text{norm}}(t^-)\right)^T G(q)\mathbf{n}_q^T H_{\text{T}}(q)p_{\text{t}} \\
&\quad + \frac{1}{2}\left(\dot{q}_{\text{tan}}(t^+) + \dot{q}_{\text{tan}}(t^-)\right)^T D_{\text{t}}(q)\mathbf{t}_q^T H_{\text{T}}(q)p_{\text{t}}
\end{aligned} \tag{77}$$

Let us focus on the mode of Proposition 11. Let us denote $E(q) \stackrel{\Delta}{=} (G(q) - \mathbf{n}_q^T H_{\text{T}}(q)[\bar{\mu}])^\dagger$, and $M(q,\mu) \stackrel{\Delta}{=} E(q) - G(q)\mathbf{n}_q^T H_{\text{T}}(q)[\bar{\mu}]E(q)$.

Lemma 1 *Assume that $G(q)$ is positive definite. Then*:

- (i) *If* $\|G^{-1}(q)\|_2\|\mathbf{n}_q^T H_{\text{T}}(q)\|_2\|[\bar{\mu}]\|_2 < 1$, *one has* $E(q) > 0$.
- (ii) *Let the conditions in* (i) *hold. If* $\|E^{-1}(q)\|_2\|E(q)\|_2\|G(q)\|_2\|\mathbf{n}_q^T H_{\text{T}}(q)\|_2\|[\bar{\mu}]\|_2 < 1$, *then* $M(q,\mu) > 0$.

Proof (i) $G(q)$ is symmetric positive definite. Applying Theorem 3 with $M = G(q)$ and $A = G(q) - \mathbf{n}_q^T H_{\text{T}}(q)[\bar{\mu}]$, one finds that the inequality in (i) guarantees that $G(q) - \mathbf{n}_q^T H_{\text{T}}(q)[\bar{\mu}]$ is positive definite. Then this matrix has a positive definite inverse which is $E(q)$. (ii) The proof follows from Corollary 3, with $M = E(q)$, $B = I - G(q)\mathbf{n}_q^T H_{\text{T}}(q)[\bar{\mu}]$ and $A = M(q,\mu)$. Applying Proposition 9.3.5 in [7] to upper-bound $\|G(q)\mathbf{n}_q^T H_{\text{T}}(q)[\bar{\mu}]\|_2$ by the product of norms, the result follows. □

Given the Delassus matrix, the conditions of (i) can be computed as restrictions on both the normal/tangential couplings and the friction coefficients. Once these conditions hold, the conditions of (ii) can be calculated.

We now suppose that (i) of Lemma 1 holds and that $\bar{p}_n$ can be calculated consistently from the first equation in (68), i.e. $\bar{p}_n = -E^{-1}(q)(I + \mathcal{E}_{\text{nn}})\dot{q}_{\text{norm}}(t^-) \geqslant 0$. Therefore, $p_{\text{t}} = -[\bar{\mu}]\bar{p}_n$ in the considered mode. After some manipulations, $T_L(t)$ in (3.2.5) can be split into three parts (using in particular the second equation in (68) to obtain $\dot{q}_{\text{tan}}(t^+) + \dot{q}_{\text{tan}}(t^-)$):

$$
\begin{aligned}
T_L(t) = &-\frac{1}{2}\dot{q}_{\text{norm}}^T\big(t^-\big)(I-\mathcal{E}_{\text{nn}})^T M(q,\mu)(I+\mathcal{E}_{\text{nn}})\dot{q}_{\text{norm}}\big(t^-\big) \\
&+\frac{1}{2}\dot{q}_{\text{norm}}^T\big(t^-\big)(I+\mathcal{E}_{\text{nn}})^T E^T(q)[\bar{\mu}]H_{\text{T}}^T(q)\mathbf{t}_q D_{\text{t}}(q)\mathbf{t}_q^T H_{\text{T}}(q)[\bar{\mu}] \\
&\times E(q)(I+\mathcal{E}_{\text{nn}})\dot{q}_{\text{norm}}\big(t^-\big) \\
&+\dot{q}_{\tan}^T\big(t^-\big)D_{\text{t}}(q)\mathbf{t}_q^T H_{\text{T}}(q)[\bar{\mu}]E(q)(I+\mathcal{E}_{\text{nn}})\dot{q}_{\text{norm}}\big(t^-\big)
\end{aligned} \tag{78}
$$

The matrix in the second term is always positive semi-definite, so this term is always $\geqslant 0$. Notice that one cannot transform the first quadratic term as was done to pass from (48) to (49) because $M(q,\mu)$ may not be symmetric.

Lemma 2 *Suppose that the conditions of Lemma 1 all hold. Then*:

$$
\begin{aligned}
&\|\mathcal{E}_{\text{nn}}\|_2\big(1+2\|\mathcal{E}_{\text{nn}}\|_2\big) < \frac{1}{\|M(q,\mu)\|_2}\frac{1}{\|(\frac{M(q,\mu)+M^T(q,\mu)}{2})^{-1}\|_2} \\
&\quad\Rightarrow (I-\mathcal{E}_{\text{nn}})^T M(q,\mu)(I+\mathcal{E}_{\text{nn}}) > 0
\end{aligned} \tag{79}
$$

Proof One has $(I-\mathcal{E}_{\text{nn}})^T M(q,\mu)(I+\mathcal{E}_{\text{nn}}) = M(q,\mu)+H(q,\mu,e_{\text{n},i})$, with $H(q,\mu,e_{\text{n},i}) = -\mathcal{E}_{\text{nn}}^T M(q,\mu)\mathcal{E}_{\text{nn}} - \mathcal{E}_{\text{nn}}^T M(q,\mu) + M(q,\mu)\mathcal{E}_{\text{nn}}$. Consider Theorem 3, with $M = M(q,\mu)$ and $A = M(q,\mu) + H(q,\mu,e_{\text{n},i})$. Using Proposition 9.3.5 in [7] and the triangular inequality of norms one finds $\|H(q,\mu,e_{\text{n},i})\|_2 \leqslant \|\mathcal{E}_{\text{nn}}\|_2^2\|M(q,\mu)\|_2 + 2\|\mathcal{E}_{\text{nn}}\|_2\|M(q,\mu)\|_2$. Thus, it suffices that

$$
\left\|\left(\frac{M(q,\mu)+M^T(q,\mu)}{2}\right)^{-1}\right\|_2\big(\|\mathcal{E}_{\text{nn}}\|_2^2\big\|M(q,\mu)\big\|_2 + 2\|\mathcal{E}_{\text{nn}}\|_2\big\|M(q,\mu)\big\|_2\big) < 1
$$

and the result follows. □

Assume that the conditions of Lemmas 1 and 2 hold, and let us denote $\bar{M}(q,\mu,e_{\text{n},i}) \triangleq (I-\mathcal{E}_{\text{nn}})^T M(q,\mu)(I+\mathcal{E}_{\text{nn}}) > 0$ but not necessarily symmetric, and

$$
K(q,\mu,e_{\text{n},i}) \triangleq (I+\mathcal{E}_{\text{nn}})^T E^T(q)[\bar{\mu}]H_{\text{T}}^T(q)\mathbf{t}_q D_{\text{t}}(q)\mathbf{t}_q^T H_{\text{T}}(q)[\bar{\mu}]E(q)(I+\mathcal{E}_{\text{nn}}) \tag{80}
$$

Notice that $\mathcal{E}_{\text{nn}} = I$ implies that $\bar{M}(q,\mu,e_{\text{n},i}) = 0$. One has:

$$
\begin{aligned}
T_L(t) \leqslant &-\frac{1}{2}\dot{q}_{\text{norm}}^T\big(t^-\big)\bar{M}(q,\mu,e_{\text{n},i})\dot{q}_{\text{norm}}\big(t^-\big) \\
&+\frac{1}{2}\dot{q}_{\text{norm}}^T\big(t^-\big)K(q,\mu_i,e_{\text{n},i})\dot{q}_{\text{norm}}\big(t^-\big) \\
&+\dot{q}_{\tan}^T\big(t^-\big)D_{\text{t}}(q)\mathbf{t}_q^T H_{\text{T}}(q)[\bar{\mu}]E(q)(I+\mathcal{E}_{\text{nn}})\dot{q}_{\text{norm}}\big(t^-\big)
\end{aligned} \tag{81}
$$

Theorem 1 *Assume that the conditions of Lemmas 1 and 2 hold. If $\|\dot{q}_{\text{norm}}(t^-)\| \geqslant \epsilon$ for some $\epsilon > 0$, then there exist $\delta_1 > 0$, $\delta_2 > 0$, $\delta_3 > 0$, $\delta_4 > 0$ such that if $\|I+\mathcal{E}_{\text{nn}}\|_2 \leqslant \delta_1$, $\|\dot{q}_{\tan}(t^-)\| \leqslant \delta_2$, $\|D_{\text{t}}(q)\mathbf{t}_q^T H_{\text{T}}(q)\|_2 \leqslant \delta_3$, $\|[\bar{\mu}]\|_2 \leqslant \delta_4$, then $T_L(t) \leqslant 0$.*

Proof From (81), one has

$$\begin{aligned} T_L(t) \leqslant & -\frac{1}{2}\lambda_{\min}\big(\bar{M}(q,\mu,e_{\mathrm{n},i})\big)\big\|\dot{q}_{\mathrm{norm}}\big(t^-\big)\big\|^2 \\ & + \frac{1}{2}\lambda_{\max}\big(K(q,\mu_i,e_{\mathrm{n},i})\big)\big\|\dot{q}_{\mathrm{norm}}\big(t^-\big)\big\|^2 \\ & + \big\|D_{\mathrm{t}}(q)\mathbf{t}_q^T H_{\mathrm{T}}(q)\big\|_2\big\|[\bar{\mu}]\big\|_2\|I+\mathcal{E}_{\mathrm{nn}}\|_2\big\|\dot{q}_{\mathrm{norm}}\big(t^-\big)\big\|\,\big\|\dot{q}_{\mathrm{tan}}\big(t^-\big)\big\| \end{aligned} \tag{82}$$

Eigenvalues are continuous functions of the matrix entries. Choosing $[\mu]=0$ yields $K(q,0,e_{\mathrm{n},i})=0$. Therefore, $\lambda_{\max}(K(q,\mu,e_{\mathrm{n},i}))$ can be made as small as desired by decreasing the coefficients μ_i. On the other hand in view of the definitions of both $E(q)$ and $M(q,\mu)$, $\lambda_{\min}(\bar{M}(q,\mu,e_{\mathrm{n},i}))$ can be made as close as possible to $\lambda_{\min}(G^{-1}(q))>0$ that does not depend on μ_i. The result follows. □

Roughly speaking, Theorem 1 says that if the frictional effects are small enough, then the impact under the considered mode is energetically consistent provided that the quasi-normal pre-impact velocity is sufficiently large while the quasi-tangential pre-impact velocity is sufficiently small. In other words, the impact, viewed in a generalized space, should be close to a "normal" collision. It is noteworthy that this is true even if the normal/tangential coupling term $\mathbf{n}_q^T H_{\mathrm{T}}(q)$ vanishes. The interest of Theorem 1 is that the various bounds can be calculated for a system with a given Delassus' matrix. This result is, however, conservative because it does not take into account the sign of the last term in (78), but only its norm. It is also noteworthy that even if the coupling term $\mathbf{n}_q^T H_{\mathrm{T}}(q)$ vanishes, the energetic consistency is not guaranteed because of the quadratic term with $K(q,\mu_i,e_{\mathrm{n},i})\geqslant 0$. This is nevertheless the case even in the simplest case of a particle hitting a rough plane; see Example 5. Another way to formulate Theorem 1 is as follows:

Theorem 2 *Assume that the conditions of Lemmas 1 and 2 hold. Then provided that*

$$\begin{aligned} &\text{(i)} \quad \lambda_{\min}\big(\bar{M}(q,\mu,e_{\mathrm{n},i})\big) > \lambda_{\max}\big(K(q,\mu_i,e_{\mathrm{n},i})\big) \\ &\text{(ii)} \quad \frac{\|\dot{q}_{\mathrm{norm}}(t^-)\|}{\|\dot{q}_{\mathrm{tan}}(t^-)\|} \geqslant 2\frac{\|D_{\mathrm{t}}(q)\mathbf{t}_q^T H_{\mathrm{T}}(q)\|_2\|[\bar{\mu}]\|_2\|I+\mathcal{E}_{\mathrm{nn}}\|_2}{\lambda_{\min}(\bar{M}(q,\mu,e_{\mathrm{n},i}))-\lambda_{\max}(K(q,\mu_i,e_{\mathrm{n},i}))} \end{aligned} \tag{83}$$

one has $T_L(t)\leqslant 0$.

Proof Follows directly from (82). □

Remark 8 It is clear that one can replace $[\bar{\mu}]$ in (68) by the matrix $[\bar{\bar{\mu}}]=\mathrm{diag}(\mu_i\frac{\xi_i}{\|\nabla h_i(q)\|_{M^{-1}}})$, where $\xi_i\in\mathrm{sgn}(v_{\mathrm{t},i}(t^+)+e_{\mathrm{t},i}v_{\mathrm{t},i}(t^-))$. The analysis of Theorem 1 remains valid, since $\|[\bar{\bar{\mu}}]\|_2\leqslant\|[\bar{\mu}]\|_2$. This means that the energetic consistency criterion is valid for all the modes (whereas existence still has to be checked by a generalized equation like (68)), and this explains its conservativeness.

Example 5 Let us illustrate the above developments on the simplest case of a planar particle hitting a line. The horizontal position is x, the vertical one (normal to the line) is y. One has $\dot{q}_{\mathrm{norm}}=\sqrt{m}\dot{y}$, $\dot{q}_{\mathrm{tan}}=\sqrt{m}\dot{x}$, $\bar{p}_n=\frac{1}{\sqrt{m}}p_n$, $E(q)=1$, $M(q,\mu)=1$, $\bar{M}(q,\mu,e_{\mathrm{n}})=1-e_{\mathrm{n}}^2$, $K(q,\mu_i,e_{\mathrm{n},i})=\mu^2(1+e_{\mathrm{n}})^2$, $D_{\mathrm{t}}=1$, $\mathbf{t}_q^T D_{\mathrm{t}}\mathbf{t}_q=\frac{1}{m}$, $\mathbf{t}_q^T H_{\mathrm{T}}(q)=\frac{1}{\sqrt{m}}$, $G(q)=1$, $\bar{p}_n=$

$-(1+e_{\rm n})\dot{q}_{\rm norm}(t^-)$, $\ddot{q}_{\rm norm}=\bar{p}_n$, $\ddot{q}_{\rm tan}=\frac{1}{\sqrt{m}}p_{\rm t}$, $\dot{q}_{\rm tan}(t^+)-\dot{q}_{\rm tan}(t^-)=\frac{1}{\sqrt{m}}(1+e_{\rm n})\dot{q}_{\rm norm}(t^-)\bar{\mu}\xi$, with $\xi\in\mathrm{sgn}(\dot{x}(t^+)+e_{\rm t}\dot{x}(t^-))$. The conditions of the Theorems imply that $e_{\rm n}\leqslant 1$, while the kinematic admissibility implies that $e_{\rm n}\geqslant 0$. The direct application of Theorem 2 gives:

$$\begin{array}{ll}\text{(i)} & 1-e_{\rm n}>\mu^2(1+e_{\rm n})\\ \text{(ii)} & \dfrac{|\dot{y}(t^-)|}{|\dot{x}(t^-)|}\geqslant\dfrac{2\mu}{1-e_{\rm n}-\mu^2(1+e_{\rm n})}\end{array}\tag{84}$$

Notice that condition (i) implies that $e_{\rm n}<1$. If $e_{\rm n}=0$, then $\mu<1$ and $\frac{|\dot{y}(t^-)|}{|\dot{x}(t^-)|}\geqslant\frac{2\mu}{1-\mu^2}$. If $e_{\rm n}=1$ only, the frictionless case is admitted, because as noted above in that case $\bar{M}(q,\mu,e_{\rm n})=0$, and we have excluded this case from the beginning. For this system, the energy loss is calculated to be:

$$T_L(t)=\frac{1}{2}m(1+e_{\rm n})\big[\big((e_{\rm n}-1)+\mu^2\xi^2(1+e_{\rm n})\big)\dot{y}\big(t^-\big)+2\mu\xi\dot{x}\big(t^-\big)\big]\dot{y}\big(t^-\big)\tag{85}$$

The collision is dissipative if and only if $((e_{\rm n}-1)+\mu^2\xi^2(1+e_{\rm n}))\dot{y}(t^-)+2\mu\xi\dot{x}(t^-)\geqslant 0$. Suppose that $e_{\rm t}=0$, $\dot{x}(t^-)\neq 0$, and the collision does not reverse the tangential velocity, so that $|\xi|=1$ and $\xi\dot{x}(t^-)=|\dot{x}(t^-)|\geqslant 0$. We can rewrite the dissipativity condition as $((1-e_{\rm n})-\mu^2(1+e_{\rm n}))\frac{|\dot{y}(t^-)|}{|\dot{x}(t^-)|}>-2\mu$. Then two cases arise: (a) if $1-e_{\rm n}-\mu^2(1+e_{\rm n})\geqslant 0$ then the inequality is always satisfied; (b) if $1-e_{\rm n}-\mu^2(1+e_{\rm n})<0$ then $\frac{|\dot{y}(t^-)|}{|\dot{x}(t^-)|}<\frac{2\mu}{e_{\rm n}-1+\mu^2(1+e_{\rm n})}$. Case (a) strictly contains (84), because the sign of the third term in (81) has been taken into account.

Glocker [36] analyses the energetic consistency of such kinematic impact laws for various tangential models (isotropic and orthotropic Coulomb friction) in particular cases: orthogonal constraints (all contacts decoupled), restricted coefficients of restitution (small, elastic impacts, plastic impacts). He also formulates friction in a way similar to (67) in [35]. Leine and van de Wouw [55] also derive general criteria, starting from an inclusion formalism for both the normal and the frictional parts. Lubarda [60] derives bounds on the kinematic, kinetic, and energetic restitution coefficients for single collisions between a planar body and a rough fixed surface. Lankarani et al. [52] use Routh's graphical method to analyze single impacts with Poisson's coefficient of restitution in multibody systems. Both articles use Coulomb's friction at the impulse level, as in (67) with $e_{{\rm t},i}=0$.

4 Analysis of the contact LCP (26)

In this section, it is assumed that $p\geqslant 1$ contacts are lasting and are either sticking or sliding. Painlevé paradoxes occur in constrained systems with Coulomb's friction [16]. They represent some kind of singularities in the dynamics, which yield phenomena like non-uniqueness, non-existence, or unboundedness of the contact force [33]. The central tool for the analysis of the Painlevé paradoxes is the complementarity problem (26). One notices that the matrix of the LCP in (26) is the same as the matrix that appears in the first line of (68). The main properties for the analysis of the LCP are the P-property, and co-positivity. Another major difference between Painlevé issues and the impact issues is that q varies in the former, so that the LCP matrix may have changing properties along the system's trajectory. In this section, we just point out some preliminary results that may be used to extend the first step of the analysis made in [33] about the characterization of the LCP matrix. Contrarily to some previous results which state the well-posedness of the contact LCP showing only the existence of small enough friction [63, 85] or for the all-sticking mode only [73], here criteria with explicit bounds are given for general tangential behaviors in the multi-contact case. To the best of the author's knowledge, only [43] has proposed explicit criteria to test the existence and uniqueness of solutions to the contact LCP when $p = 2$, with detailed conditions that guarantee that the contact LCP matrix is a P-matrix.

4.1 Single-contact systems

The analysis starts from the properties of the LCP in (26) to determine the normal multiplier. It essentially relies on the properties of the matrix (here a scalar)

$$1 - \mathbf{n}_q^T H_{\mathrm{T}}(q) \frac{\mu}{\|\nabla h(q)\|_{M^{-1}}} \operatorname{sgn}\big(H_{\mathrm{T}}^T(q)\dot{q}\big), \tag{86}$$

because $G(q) = 1$ when $p = 1$. For instance, in the case of the rod of Sect. 2.3.3, this scalar is equal to $[1 + \frac{3}{1+3\cos^2(\theta)} \mu \cos(\theta)\sin(\theta)\operatorname{sgn}(H_{\mathrm{T}}^T(q)\dot{q})]$. The different modes of the LCP for the sliding rod are easy to obtain in the scalar case, however, the subsequent analysis of the dynamics is rather intricate [33]. Indeed even if the rod is initialized in a well-posed mode, it may evolve towards regions of the $(\theta, \dot{\theta})$ plane where singularities (non-uniqueness or unboundedness of the contact force) occur. This is in contrast with the analysis of impact conditions where q is considered as being constant. A central question in such analysis is: Does there exist a critical value $\mu_c > 0$ such that no singularity occurs for all $0 < \mu < \mu_c$? This is the case for the system of Sect. 2.3.3 where $\mu_c(\theta) = \frac{1+3\cos^2(\theta)}{3\sin(\theta)\cos(\theta)} \geqslant \frac{4}{3}$ [54]. But this is not the case for the system of Sect. 2.3.4, as shown in [54, Appendix A]. The frictional impact oscillator shown in [54, Fig. 4.1] and analyzed therein, also has $\mu_c = \tan(\theta) + \frac{m_1}{m_2}\frac{1}{\tan(\theta)}$ with minimum value at the configuration $\theta = \arctan(\sqrt{m_1/m_2})$ given by $2\sqrt{\frac{m_1}{m_2}}$, where m_1 and m_2 are the lumped masses of a rigid arm connected to a mass-spring-damper, respectively. It is thus possible to design these systems in a way such that singularities occur for arbitrarily small $\mu > 0$ by properly choosing the masses m_1 and m_2. In the scalar case, one nevertheless has the following result:

Proposition 12 *Let $p = 1$. Suppose that $\mathbf{n}_q^T H_{\mathrm{T}}(q) \frac{1}{\|\nabla h(q)\|_{M^{-1}}}$ is a bounded quantity and that $G(q) \geqslant G_{\min} > 0$ for any q in the admissible domain of $\mathcal{C}$. Then there exists $\mu_c > 0$ such that for all $0 \leqslant \mu < \mu_c$ the LCP in (26) has a unique bounded solution for any tangential relative velocity $v_{\mathrm{t}} = H_{\mathrm{T}}^T(q)\dot{q}$.*

Proposition 12 applies to the classical Painlevé example of a rod sliding on a rough surface. Then $\mu_c = \frac{4}{3}$ [33]. Let us examine the system in Sect. 2.3.4. Then we obtain $\mathbf{n}_q^T H_{\mathrm{T}}(q) \frac{1}{\|\nabla h(q)\|_{M^{-1}}} = l \frac{l\sin(\varphi)+r\cos(\varphi)}{k^2+(l\sin(\varphi)+r\cos(\varphi))^2}$. This quantity is bounded for any $k \geqslant 0$ as long as $\varphi \in [0, \frac{\pi}{2}]$. One finds when $k = 0$ that (86) vanishes for $\mu_c = \sin(\varphi) + \frac{r}{l}\cos(\varphi)$ if $\operatorname{sgn}(H_{\mathrm{T}}^T(q)\dot{q}) = +1$ (in [54, Appendix A] the value $\frac{r}{l}$ is found because the system is analyzed at $\varphi = 0$), and that $\mu < \mu_c$ guarantees the positivity of (86).

4.2 Multi-contact systems

The analysis is more intricate in the multi-contact case (i.e. $p \geqslant 2$) for the simple reason that the LCP modes are much less easy to enumerate (we let apart the subsequent complex analysis of the system's trajectories). Roughly speaking, the term $-\mathbf{n}_q^T H_\mathrm{T}(q)\,\mathrm{diag}(\frac{\mu_i \xi_i}{\|\nabla h_i(q)\|_{M^{-1}}})$ in (26) should be a "nice" perturbation of $\mathbb{R}^{p\times p} \ni G(q) > 0$, so that the LCP matrix is a P-matrix (the LCP (26) has a unique solution for any $b(q,\dot{q},t)$), or a strictly co-positive matrix (the LCP has at least a solution for any $b(q,\dot{q},t)$ [25, Theorems 3.3.7 and 3.8.5]).

Example 6 Consider the system of Sect. 2.3.6. One has

$$-\mathbf{n}_q^T H_\mathrm{T}(q)\,\mathrm{diag}\left(\frac{\mu_i \xi_i}{\|\nabla h_i(q)\|_{M^{-1}}}\right) = -\begin{pmatrix} 0 & \mu_2\,\mathrm{sgn}(v_{\mathrm{t},2}) \\ \mu_1\,\mathrm{sgn}(v_{\mathrm{t},1}) & 0 \end{pmatrix} \quad \text{and} \quad G(q) = I.$$

Notice that the perturbation needs not be symmetric, except if $\mu_1 = \mu_2$ and $v_{\mathrm{t},1} = v_{\mathrm{t},2}$, which occurs if $\dot{x} = \dot{y}$. A characterization of a P-matrix is that all its principal minors are positive [25, Definition 3.3.1]. The matrix of interest here is

$$\begin{pmatrix} 1 & -\mu_2\,\mathrm{sgn}(v_{\mathrm{t},2}) \\ -\mu_1\,\mathrm{sgn}(v_{\mathrm{t},1}) & 1 \end{pmatrix}$$

Its first principal minor is 1; the second one is $1 - \mu_1\mu_2\,\mathrm{sgn}(v_{\mathrm{t},2})\,\mathrm{sgn}(v_{\mathrm{t},1})$. So if the two tangential velocities have reversed signs, the matrix is always a P-matrix whatever the friction. If the tangential velocities have the same signs, then it is a P-matrix if and only if $\mu_1\mu_2 < 1$ (the strict inequality is important since otherwise one obtains a P_0-matrix). This allows for a high friction at one contact, provided the other contact has low friction.

Apart from the basic result that if the LCP matrix is a P-matrix then the LCP always has a unique solution, some refined results may be stated, taking into account that $G(q) = \mathbf{n}_q^T M(q)\mathbf{n}_q$ is symmetric positive definite. Let us denote the perturbation matrix as

$$P(q,\mu,v_\mathrm{t}) \triangleq -\mathbf{n}_q^T H_\mathrm{T}(q)\,\mathrm{diag}\left(\frac{\mu_i \xi_i}{\|\nabla h_i(q)\|_{M^{-1}}}\right), \quad \xi_i \in \mathrm{sgn}(v_{\mathrm{t},i}) \tag{87}$$

Proposition 13 *Suppose that* $G(q) > 0$ *and* $P(q,\mu,v_\mathrm{t}) = \epsilon A(q)$ *with* $A(q)$ *symmetric,* ϵ *real. Then there exists* ϵ^* *such that for all* $|\epsilon| < \epsilon^*$ *the LCP matrix* $D(q,\mu,v_\mathrm{t})$ *is positive definite, and the LCP in* (26) *has a unique solution for any* $b(q,\dot{q},t)$.

Proof As for Proposition 4, the result follows from [51, Exercise 8, p. 218]. □

The smallness of ϵ may be a consequence of either small friction, or of small couplings between normal and tangential directions, that is the matrix $\mathbf{n}_q^T H_\mathrm{T}(q)$ may have small entries. One difficulty with the analysis of this problem is that in general, the LCP matrix is not symmetric. Perturbations that preserve the positive definiteness without requiring symmetry can also be characterized as follows.

Proposition 14 *Let* $G(q) > 0$. *Suppose that* $\|\mathbf{n}_q^T H_\mathrm{T}(q)\|_2 (\sum_{i=1}^{p} \frac{\mu_i^2 \xi_i^2}{\|\nabla h_i(q)\|^2_{M^{-1}}})^{\frac{1}{2}} < \frac{1}{\|G^{-1}(q)\|_2}$. *Then the LCP matrix* $D(q,\mu,v_\mathrm{t}) = G(q) + P(q,\mu,v_\mathrm{t})$ *is positive definite, and the LCP in* (26) *has a unique solution for any* $b(q,\dot{q},t)$.

Proof Let $M = G(q)$, $A = G(q) + P(q,\mu,v_\mathrm{t}) = D(q,\mu,v_\mathrm{t})$, so that $M - A = -P(q,\mu,v_\mathrm{t})$. From [23, Theorem 2.11] (see Theorem 3 in Appendix A), the condition $\|G^{-1}(q)\|_2 \times \|P(q,\mu,v_\mathrm{t})\|_2 < 1$ guarantees that $D(q,\mu,v_\mathrm{t}) > 0$. Now from Proposition 9.3.5 in [7] one has $\|P(q,\mu,v_\mathrm{t})\|_2 \leqslant \|\mathbf{n}_q^T H_\mathrm{T}(q)\|_2 \|\,\mathrm{diag}(\frac{\mu_i \xi_i}{\|\nabla h_i(q)\|_{M^{-1}}})\|_2$; the result follows. □

Again the interest of Proposition 14 is that it provides explicit upper-bounds on both the friction and normal/tangential couplings, as a function of the kinetic angles. Another result using the co-positiveness is as follows:

Proposition 15 *Let $G(q) + P(q, \mu, v_{\mathrm{t}})$ be co-positive. If $b(q, \dot{q}, t) \in Q^{*}_{G+P}$, then the LCP in* (26) *has a solution.*

Proof Directly from [25, Theorem 3.8.6]. □

For small dimensions of the contact LCP (say, no more than 3 or 4 contact points), the co-positivity may be checked [39]. Further necessary conditions may be derived from the fact that $G(q)$ has diagonal entries equal to unity, but are not discussed here for the sake of briefness of the paper. The criteria derived in Propositions 6, 7, 9, 14, Lemmas 1, 2, Theorem 2, can be refined when dealing with particular systems.

5 Conclusions

In this paper, it is shown that a specific state-space transformation of the Lagrange dynamics with unilateral constraints and impacts, yields a suitable framework for the analysis of kinematic impact laws, and of the contact linear complementarity problem (closely related to the occurrence of the Painlevé paradoxes). The transformed velocity may also be seen as a quasi-velocity, constructed from projections in the kinetic metric. Kinematic impact laws with and without friction are analyzed, in particular the energetical consistency is carefully studied in all cases. Explicit, computable bounds on the friction coefficients and on the inertial couplings that guarantee the consistency are provided. Preliminary results on the well-posedness of the contact LCP in the multi-contact case are also given. Examples illustrate the developments.

Appendix A: Theorem 2.11 in [23]

We give here just an excerpt of this result, and two easy corollaries of it.

Theorem 3 *Let $M \in \mathbb{R}^{n\times n}$ be a positive definite matrix. Then every matrix*

$$A \in \left\{ A \mid \left\| \left(\frac{M + M^T}{2} \right)^{-1} \right\|_2 \|M - A\|_2 < 1 \right\}$$

is positive definite.

Corollary 2 *Let $D = P + N$, where D, P, and N are $n \times n$ real matrices, and $P > 0$, not necessarily symmetric. If*

$$\|N\|_2 < \frac{1}{\|(\frac{P+P^T}{2})^{-1}\|_2} \tag{88}$$

then $D > 0$.

Proof Follows from Theorem 3 with $A \triangleq D$ and $M \triangleq P$. □

Corollary 3 *Let $M \in \mathbb{R}^{n\times n}$ be a symmetric positive definite matrix. Let $A = BM$ for some matrix B. If*

$$\|M^{-1}\|_2 \|M\|_2 \|I - B\|_2 < 1$$

then A is positive definite.

Proof Since $M = M^T$ applying Theorem 3 gives that $\|M^{-1}\|_2 \|M - BM\|_2 < 1$ guarantees that $A > 0$. Now from [7, Proposition 9.3.5] one has $\|M - BM\|_2 \leqslant \|M\|_2 \|I - B\|_2$: the result follows. □

Appendix B: Theorem 3.8.6 in [25]

Theorem 4 *Let $M \in \mathbb{R}^{n\times n}$ be a co-positive matrix, and let $q \in \mathbb{R}^n$ be given. If the implication*

$$\left[v \geqslant 0, Mv \geqslant 0, v^T M v = 0\right] \quad \Rightarrow \quad \left[v^T q \geqslant 0\right]$$

is valid, then the LCP: $0 \leqslant \lambda \perp M\lambda + q \geqslant 0$ has a solution.

The implication is equivalently rewritten as $q \in Q_M^*$.

Appendix C: Propositions 8.1.2 and 9.3.5 in [7]

Proposition 16 *Let A and B be $n \times n$ symmetric real matrices, and S be an $m \times n$ real matrix. Then: (xi) if $A \leqslant B$ then $SAS^T \leqslant SBS^T$; (xiii) if $SAS^T \leqslant SBS^T$ and rank$(S) = n$, then $A \leqslant B$.*

Proposition 17 *Let A be an $n \times m$ and B an $m \times l$ matrices. If $p \in [1, 2]$, then $\|AB\|_p \leqslant \|A\|_p \|B\|_p$.*

Appendix D: Proof of (59)

Let us start from (58). Since R has full rank and using $P\bar{G}(q) = \Gamma^T$ ($\Leftrightarrow \Gamma R^{-1} = \bar{G}^T R$), this is equivalent to

$$z - Rv(t^-) \in -R^{-1}\Gamma^T N_{W(q)}\left(\Gamma R^{-1}\left(z + R\Lambda v(t^-)\right)\right) \tag{89}$$

Let $y = z + R\Lambda v(t^-)$. In view of the definition of $\bar{W}(q)$, and since $N_{\bar{W}(q)}(\cdot) = \partial\psi_{\bar{W}(q)}(\cdot)$, one has $N_{\bar{W}(q)}(y) = \partial\psi_{\bar{W}(q)}(y) = (\Gamma R^{-1})^T \partial\psi_{W(q)}(\Gamma R^{-1} y)$, where the chain rule of convex analysis has been used. The expression in (60) is then a consequence of the fact that $x - y \in -N_K(x) \Leftrightarrow x = \text{proj}[K; y]$ for any vectors x, y, and convex set K of appropriate dimensions.

References

1. Acary, V., Brogliato, B.: Numerical Methods for Nonsmooth Dynamical Systems: Applications in Mechanics and Electronics. Lecture Notes in Applied and Computational Mechanics, vol. 35. Springer, Berlin (2008)
2. Aghili, F.: A unified approach for inverse and direct dynamics of constrained multibody systems based on linear projection operator: applications to control and simulation. IEEE Trans. Robot. **21**(5), 834–849 (2005)
3. Aghili, F.: Dynamics and control of constrained mechanical systems in terms of reduced quasi-velocities. In: IEEE Int. Conference on Robotics and Automation, Pasaneda, USA, 19–23 May 2008, pp. 1225–1232 (2008)
4. Aghili, F.: Control of redundant mechanical systems under equality and inequality constraints on both input and constraint forces. J. Comput. Nonlinear Dyn. **6**, 031013 (2011)
5. Appel, P.: Traité de Mécanique Rationnelle, Tome 2: Dynamique des Systèmes–Mécanique Analytique, 3rd edn. Gauthier-Villars, Paris (1911)
6. Ballard, P.: The dynamics of discrete mechanical systems with perfect unilateral constraints. Arch. Ration. Mech. Appl. **154**, 199–274 (2000)
7. Bernstein, D.S.: Matrix, Mathematics: Theory, Facts, and Formulas with Application to Linear Systems Theory. Princeton University Press, Princeton (2005)
8. Betsch, P.: The discrete null space method for the energy consistent integration of constrained mechanical systems. Part I. Holonomic constraints. Comput. Methods Appl. Mech. Eng. **194**, 5159+5190 (2005)
9. Blajer, W.: A geometric unification of constrained system dynamics. Multibody Syst. Dyn. **1**, 1–21 (1997)
10. Blajer, W.: A geometrical interpretation and uniform matrix formulation of multibody system dynamics. Z. Angew. Math. Mech. **81**(4), 247–259 (2001)
11. Bowling, A., Montrallo Flickinger, D., Harmeyer, S.: Energetically consistent simulation of simultaneous impacts and contacts in multibody systems with friction. Multibody Syst. Dyn. **22**(1), 27–45 (2009). doi:10.1007/s11044-009-9147-5
12. Brauchli, H.: Mass-orthogonal formulation of equations of motion for multibody systems. Z. Angew. Math. Phys. **42**, 169–182 (1991)
13. Braun, D.J., Goldfarb, M.: Simulation of constrained mechanical systems. Part I. An equation of motion. J. Appl. Mech. **79**, 041017 (2012)
14. Brogliato, B.: Nonsmooth Impact Mechanics: Models, Dynamics and Control. Lecture Notes in Control and Information Sciences, vol. 220. Springer, Berlin (1996)
15. Brogliato, B., Niculescu, S., Orhant, P.: On the control of finite-dimensional mechanical systems with unilateral constraints. IEEE Trans. Autom. Control **42**(2), 200–215 (1997)
16. Brogliato, B.: Nonsmooth Mechanics: Models, Dynamics and Control. Communications and Control Engineering, 2nd edn. Springer, Berlin (1999)
17. Brogliato, B.: Absolute stability and the Lagrange-Dirichlet theorem with monotone multivalued mappings. Systems and Control Letters **51**(5), 343–353 (2004)
18. Brogliato, B., Lozano, R., Maschke, B., Egeland, O.: Dissipative Systems Analysis and Control. Communications and Control Engineering, 2nd edn. Springer, Berlin (2007)
19. Brogliato, B., Zhang, H., Liu, C.: Analysis of a generalized kinematic impact law for multibody-multicontact systems, with application to the planar rocking block and chains of balls. Multibody Syst. Dyn. **27**, 351–382 (2012)
20. Brogliato, B.: Inertial couplings between unilateral and bilateral holonomic constraints in frictionless Lagrangian systems. Multibody Syst. Dyn. **29**(3), 289–325 (2013). doi:10.1007/s11044-012-9317-8
21. Buttazzo, G., Percivale, D.: On the approximation of the elastic bounce problem on Riemannian manifolds. J. Differ. Equ. **47**, 227–275 (1983)
22. Caselli, F., Frémond, M.: Collision of three balls on a plane. Comput. Mech. **43**, 743–754 (2009)
23. Chen, X., Xiang, S.: Perturbation bounds of P-matrix linear complementarity problems. SIAM J. Optim. **18**(4), 1250–1265 (2007)
24. Cholet, C.: Chocs de solides rigides. Ph.D. thesis, University Paris 6, No. 98 PA06 6069 (1998)
25. Cottle, R.W., Pang, J.S., Stone, R.E.: In: The Linear Complementarity Problem. SIAM Classics in Applied Mathematics, vol. 60 (2009)
26. Duindam, V., Stramigioli, S.: Singularity-free dynamic equations of open-chain mechanisms with general holonomic and nonholonomic joints. IEEE Trans. Robot. **24**(3), 517–526 (2008)
27. Dzonou, R., Monteiro Marques, M.D.P., Paoli, L.: A convergence result for a vibro-impact problem with a general inertia operator. Nonlinear Dyn. **58**, 361–384 (2009). doi:10.1007/s11071-009-9484-1
28. Frémond, M.: Rigid bodies collisions. Phys. Lett. A **204**, 33–41 (1995)
29. Frémond, M.: Non-smooth Thermomechanics. Springer, Berlin (2002)

30. From, P.J.: An explicit formulation of singularity-free dynamic equations of mechanical systems in Lagrangian form. Part one. Single rigid bodies. Part two. Multibody systems. Model. Identif. Control **33**(2), 45–68 (2012)
31. Gale, D.: An indeterminate problem in classical mechanics. Am. Math. Mon. **59**(5), 291–295 (1952)
32. Gantmacher, F.: Lectures in Analytical Mechanics. MIR, Moscow (1970)
33. Génot, F., Brogliato, B.: New results on Painlevé paradoxes. Eur. J. Mech. A, Solids **18**, 653–677 (1999)
34. Glocker, C.: Set-Valued Force Laws. Lecture Notes in Applied Mechanics, vol. 1. Springer, Berlin (2001)
35. Glocker, C.: An introduction to impacts. In: Haslinger, J., Stavroulakis, G. (eds.) Nonsmooth Mechanics of Solids. CISM Courses and Lectures, vol. 485, pp. 45–102. Springer, Wien (2006)
36. Glocker, C.: Energetic consistency conditions for standard impacts. Multibody Syst. Dyn. **29**(1), 77–117 (2013). doi:10.1007/s11044-012-9316-9
37. Glocker, C., Pfeiffer, F.: Multiple impacts with friction in rigid multi-body systems. Nonlinear Dyn. **7**, 471–497 (1995)
38. Herman, P., Kozlowski, K.: A survey of equations of motion in terms of inertial quasi-velocities for serial manipulators. Arch. Appl. Mech. **76**, 579–614 (2006)
39. Hiriart Urruty, J.B., Seeger, A.: A variational approach to copositive matrices. SIAM Rev. **52**(4), 593–629 (2010)
40. Hiriart Urruty, J.B., Lemaréchal, C.: Fundamentals of Convex Analysis. Springer, Berlin (2001)
41. Ivanov, A.P.: Impacts in a system with certain unilateral couplings. J. Appl. Math. Mech. **51**(4), 43–442 (1987)
42. Ivanov, A.P.: On multiple impact. J. Appl. Math. Mech. **59**(6), 887–902 (1995)
43. Ivanov, A.P.: Singularities in the dynamics of systems with non-ideal constraints. J. Appl. Math. Mech. **67**(2), 185–192 (2003)
44. Jain, A., Rodriguez, G.: Diagonalized Lagrangian robot dynamics. IEEE Trans. Robot. Autom. **11**(4), 571–584 (1995)
45. Junkins, J.L., Schaub, H.: An instantaneous eigenstructure quasivelocity formulation for nonlinear multibody dynamics. J. Astronaut. Sci. **45**, 279–295 (1997)
46. Kane, T.R., Wang, C.F.: On the derivation of equations of motion. J. Soc. Ind. Appl. Math. **13**(2), 487–492 (1965)
47. Kane, T.R., Levinson, D.A.: The use of Kane's dynamical equations in robotics. Int. J. Robot. Res. **2**(3), 3–21, Fall (1983)
48. Khulief, Y.A.: Modeling of impact in multibody systems: an overview. J. Comput. Nonlinear Dyn. **8**, 021012 (2013)
49. Kovecses, J., Cleghorn, W.L.: Finite and impulsive motion of constrained mechanical systems via Jourdain's principle: discrete and hybrid parameter models. Int. J. Non-Linear Mech. **38**, 935–956 (2003)
50. Kozlov, V.V., Treshchev, D.V.: Billiards: A Genetic Introduction to the Dynamics of Systems with Impacts. American Math. Soc., Providence (1991)
51. Lancaster, P., Tismenetsky, M.: The Theory of Matrices, 2nd edn. Academic Press, San Diego (1985)
52. Lankarani, H.M., Pereira, M.F.O.S.: Treatment of impact with friction in planar multibody mechanical systems. Multibody Syst. Dyn. **6**, 203–227 (2001)
53. Lee, D., Li, P.Y.: Passive decomposition of mechanical systems with coordinate requirements. IEEE Trans. Autom. Control **58**(1), 230–235 (2013). doi:10.1109/TAC.2012.2203062
54. Leine, R.I., Brogliato, B., Nijmeijer, H.: Periodic motion and bifurcations induced by the Painlevé paradox. Eur. J. Mech. A, Solids **21**, 869–896 (2002)
55. Leine, R.I., van de Wouw, N.: Stability and Convergence of Mechanical Systems with Unilateral Constraints. Lecture Notes in Applied and Computational Mechanics, vol. 36. Springer, Berlin (2008)
56. Lewis, A.D.: Simple mechanical control systems with constraints. IEEE Trans. Autom. Control **45**(8), 1420–1436 (2000)
57. Liu, G., Li, Z.: A unified geometric approach to modeling and control of constrained mechanical systems. IEEE Trans. Robot. Autom. **18**(4), 574–587 (2002)
58. Liu, C.Q., Huston, R.L.: Another form of equations of motion for constrained multibody systems. Nonlinear Dyn. **51**, 465–470 (2008)
59. Loduha, T.A., Ravani, B.: On first-order decoupling of equations of motion for constrained dynamical systems. J. Appl. Mech. **62**, 216–222 (1995)
60. Lubarda, V.A.: The bounds on the coefficients of restitution for the frictional impact of rigid pendulum against a fixed surface. J. Appl. Mech. **77**, 011006 (2010)
61. Lurie, A.I.: Analytical Mechanics. Springer, Berlin (2002). (reprint of the 1961 edition)
62. Mabrouk, M.: A unified variational model for the dynamics of perfect unilateral constraints. Eur. J. Mech. A, Solids **17**(5), 819–842 (1998)

63. Matrosov, V.M., Finogenko, I.A.: Right-hand solutions of the differential equations of dynamics for mechanical systems with sliding friction. J. Appl. Math. Mech. **59**(6), 837–844 (1995)
64. McClamroch, N.H., Wang, D.: Feedback stabilization and tracking of constrained robots. IEEE Trans. Autom. Control **33**(5), 419–426 (1988)
65. Modarres Najafabadi, S.A., Kovecses, J., Angeles, J.: Generalization of the energetic coefficient of restitution for contacts in multibody systems. J. Comput. Nonlinear Dyn. **3**, 041008 (2008)
66. Modarres Najafabadi, S.A., Kovecses, J., Angeles, J.: Impacts in multibody systems: modeling and experiments. Multibody Syst. Dyn. **20**, 163–176 (2008)
67. Morarescu, C.I., Brogliato, B.: Trajectory tracking control of multiconstraint complementarity Lagrangian systems. IEEE Trans. Autom. Control **55**(6), 1300–1313 (2010)
68. Moreau, J.J.: Liaisons unilatérales sans frottement et chocs inélastiques. Frictionless unilateral constraints and inelastic shocks. C. R. Séances Acad. Sci., Sér. 2 Méc.-Phys. Chim. Sci. Univers Sci. Terre **296**(19), 1473–1476 (1983)
69. Moreau, J.J.: Unilateral contact and dry friction in finite freedom dynamics. In: Moreau, J.J., Panagiotopoulos, P.D. (eds.) Nonsmooth Mechanics and Applications. CISM Courses and Lectures, vol. 302, pp. 1–82. International Center for Mechanical Sciences, Springer, Berlin (1988)
70. Moreau, J.J.: Some numerical methods in multibody dynamics: application to granular materials. Eur. J. Mech. A, Solids **13**(4), 93–114 (1994)
71. Neimark, J.I., Fufaev, N.A.: Dynamics of Non-holonomic Systems. Am. Math. Soc., Providence (1972)
72. Nguyen, N.S., Brogliato, B.: Multiple Impacts in Dissipative Granular Chains. Lecture Notes in Applied and Computational Mechanics, vol. 72. Springer, Berlin (2013)
73. Pang, J.S., Trinkle, J.C.: Complementarity formulations and existence of solutions of dynamic multi-rigid-body contact problems with Coulomb friction. Math. Program. **73**, 199–226 (1996)
74. Paoli, L.: Continuous dependence on data for vibro-impact problems. Math. Models Methods Appl. Sci. **15**(1), 1–41 (2005)
75. Payr, M., Glocker, C.: Oblique frictional impact of a bar: analysis and comparison of different impact laws. Nonlinear Dyn. **41**, 361–383 (2005)
76. Payr, M.: An Experimental and Theoretical Study of Perfect Multiple Contact Collisions in Linear Chains of Bodies. Ph.D. thesis, ETH Zurich (2008)
77. Pfeiffer, F., Glocker, C.: Multibody Dynamics with Unilateral Contacts. Wiley, New-York (1996)
78. Rockafellar, R.T.: Convex Analysis. Princeton Landmarks in Mathematics (1970)
79. Rockafellar, R.T., Wets, R.J.B.: Variational Analysis. Grundlehren der Mathematischen Wissenschaften, vol. 317. Springer, Berlin (1998)
80. Rodriguez, A., Bowling, A.: Solution to indeterminate multipoint impact with frictional contact using constraints. Multibody Syst. Dyn. **28**(4), 313–330 (2012). doi:10.1007/s11044-012-9307-x
81. Seghete, V., Murphey, T.: Variational solutions to simultaneous collisions between multiple rigid bodies. In: IEEE Int. Conf. on Robotics and Automation, Anchorage Convention District, Anchorage, Alaska, USA, 3–8 May 2010, pp. 2731–2738 (2010)
82. Seghete, V., Murphey, T.: Conditions for uniqueness in simultaneous impact with application to mechanical design. In: IEEE Int. Conf. on Robotics and Automation, RiverCentre, Saint Paul, Minnesota, USA, 14–18 May 2012, pp. 5006–5011 (2012)
83. Sun, Y.L.: The equations of motion of a system under the action of the impulsive constraints. Appl. Math. Mech. **9**(1), 51–60 (1988)
84. Towne, D.H., Hadlock, C.R.: One-dimensional collisions and Chebyshev polynomials. Am. J. Phys. **45**(3), 255–259 (1977)
85. Trinkle, J.C., Pang, J.S., Sudarsky, S., Lo, G.: On dynamic multi-rigid-body contact problems with Coulomb friction. Z. Angew. Math. Mech. **77**(4), 267–279 (1997)
86. Yen, J., Petzold, L.R.: An efficient Newton-type iteration for the numerical solution of highly oscillatory constrained multbody dynamic systems. SIAM J. Sci. Comput. **19**(5), 1513–1534 (1998)
87. Zhang, H., Brogliato, B., Liu, C.: Study of the planar rocking-block dynamics with Coulomb friction: critical kinetic angles. J. Comput. Nonlinear Dyn. **8**, 021002 (2013)