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Sonia Fliss, Patrick Joly

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E-mail: [sonia.fliss@ensta-paristech.fr](mailto:sonia.fliss@ensta-paristech.fr)

# Solutions of the time-harmonic wave equation in periodic waveguides : asymptotic behaviour and radiation condition

Sonia Fliss · Patrick Joly

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**Abstract** In this paper, we give the expression and the asymptotic behaviour of the physical solution of a time harmonic wave equation set in a periodic waveguide. This enables us to define a radiation condition and show well-posedness of the Helmholtz equation set in a periodic waveguide.

## 1 Introduction

Periodic media play a major role in applications, in particular in optics for micro and nano-technology [19, 20, 24, 39]. From the point of view of applications, one of the main interesting features is the possibility offered by such media of selecting ranges of frequencies for which waves can or cannot propagate. Mathematically, this property is linked to the gap structure of the spectrum of the underlying differential operator appearing in the model. For a complete, mathematically oriented presentation, we refer the reader to [24, 25].

This article is a contribution to the theory of time harmonic wave propagation in closed periodic waveguides. We are more specifically interested on the characterization of outgoing solutions via appropriate radiation conditions, the outgoing solutions being defined a priori thanks to the limiting absorption principle. The construction of these radiation conditions relies on the analysis of the behaviour at infinity of the solution constructed by the limiting absorption process. One particular motivation was to provide preparatory material for the development of the so-called factorization method for inverse scattering problems in locally perturbed periodic waveguides [6] which extensively uses the asymptotic behaviour at infinity of the Green's function associated to the unperturbed periodic waveguide. We also expect that this article is a step forward to the theory of the DtN method developed in [21, 13]. This method

provides transparent boundary conditions for reducing numerical computations to a bounded domain, that has been justified in absorbing media. We expect the extension of these absorbing boundary conditions in the case of non dissipative media, already evoked in [21] to be equivalent to the radiation conditions established in this paper.

It is worthwhile that the Russian school [30–32] has extensively contributed to the theory of radiation condition for selfadjoint elliptic problems using Kondrat'ev's weighted spaces. This approach which is a priori quite different from the limiting absorption approach yields to radiation conditions in waveguides [33] and more recently in periodic media [34]. The connection between their approach and ours is most likely possible but not so obvious in our opinion.

The more classical approach to time harmonic solutions of wave propagation problems is the limiting absorption principle that aims at characterizing the physical solution as the limit, when it exists, of the unique solution of finite energy of the corresponding mathematical model with absorption, when this absorption tends to 0. The corresponding mathematical theory has been extensively developed in the literature in various situations such as locally perturbed homogeneous media (see for instance [8, 43, 2]) or locally perturbed stratified media (see for instance [44, 7, 42, 5]). There are much less results in periodic media. In [14], the authors use Mourre's theory [29, 18] to prove the limiting absorption principle for analytically fibered operators, whose differential operators with periodic coefficients are a particular case. In [17], the author uses the same theory to deal with the Laplace-Dirichlet operator in a  $(n + 1)$ -dimensional homogeneous layer with periodically shaped boundary. In [26], a limiting absorption principle for a two-dimensional periodic layer with perturbation is proven. In all these works, the authors are interested mainly in the existence of the limit of the solution with absorption which amounts to study the limit of the resolvent of the corresponding periodic differential operator, and in which sense this limit holds. Their approach is not constructive and therefore cannot be used to study the asymptotic behavior at infinity of the physical solution and construct corresponding radiation conditions at infinity in order to characterize this solution. That is why we consider a more constructive approach, using the Floquet-Bloch theory, that leads to a semi-analytical expression of the limit, the physical solution, as in [16] for the periodic half-waveguide and [37] more general infinite periodic media. In addition, we are able to give a rate of convergence.

The main objective of this work is to use the semi-analytical expression of the solution to deduce, from the analysis of its behavior at infinity, a radiation condition that characterizes this solution, which means that we should be able to show the uniqueness of a solution of the time harmonic wave equation set in a periodic waveguide satisfying this radiation condition. This condition is well-known in the case of a (locally perturbed) homogeneous medium occupying the whole space : this is the so called Sommerfeld radiation condition

that represents the fact that, at infinity, the solution looks like an outgoing spherical wave which decays as  $|x|^{-\frac{d-1}{2}}$  where  $d$  is the space dimension. In the case of a homogeneous closed waveguide, i. e. when the domain of propagation is an infinite cylinder with bounded cross section, the behavior at infinity of the solution is quite different than in the homogeneous whole space : in particular, the solution does not decay any longer at infinity. As a consequence, the radiation condition is of quite different nature and relies on the decomposition of the solution as a infinite sum of evanescent modes and a finite sum of (appropriately chosen) propagative modes : these are the so-called outgoing modes. The situation for a periodic closed waveguide, that is considered in this paper, is similar in nature to the case of the homogeneous waveguide. However, the notion of outgoing modes is much more delicate ( we shall pay a lot of attention to a precise definition and description of such modes) and the analysis relies on quite different mathematical tools (Floquet-Bloch transform, spectral theory of operators depending analytically on a parameter, complex contour integral techniques,...).

## 2 The model problem

We consider in this paper the propagation of a time harmonic scalar wave in a perfect periodic waveguide,  $\Omega \subset \mathbb{R}^{d+1}$  (typically  $d = 1$  or  $2$ ) due to a given source  $f$

$$\begin{cases} -\Delta u - n_p^2 \omega^2 u = f & \text{in } \Omega \\ \nabla u \cdot \mathbf{n} = 0 & \text{on } \partial\Omega \end{cases} \quad (1)$$

$\mathbf{n}$  being the exterior normal of  $\Omega$ .

We shall assume that (see figure 1 for an example)

- The domain of propagation  $\Omega$  is a connected open set of  $\mathbb{R}^{d+1}$  which is unbounded in one direction, let us set  $x_1$ , bounded in the other ones

$$\exists \mathbb{S} \text{ a compact region of } \mathbb{R}^d, \quad \Omega \subset \mathbb{R} \times \mathbb{S},$$

and periodic with period  $L$  in the first direction in the sense that :

$$(x_1, x_s) \in \Omega \quad \Rightarrow \quad (x_1 \pm L, x_s) \in \Omega.$$

The boundary  $\partial\Omega$  of  $\Omega$  is supposed to be smooth enough in order that

$$\{u \in H^1(\Delta, \Omega), \nabla u \cdot \mathbf{n}|_{\partial\Omega} = 0\} \subset H^s(\Omega) \quad \text{for some } s > \frac{3}{2}. \quad (2)$$

where  $H^1(\Delta, \Omega) = \{u \in H^1(\Omega), \Delta u \in L^2(\Omega)\}$ . This is true for instance if  $\partial\Omega$  is piecewise  $C^1$  and globally lipschitz [15]. The unit periodicity cell is defined by

$$\mathcal{C} = \Omega \cap (-L/2, L/2) \times \mathbb{S}$$

and finally

$$\Omega = \bigcup_{p \in \mathbb{Z}} \mathcal{C}_p, \quad \text{with } \mathcal{C}_p = \mathcal{C} + (pL, 0), \quad \forall p \in \mathbb{Z}.$$

Typically  $\Omega$  could be a straight waveguide ( $\Omega = \mathbb{S} \times \mathbb{R}$ ) minus a periodic set of holes (see Figure 1).

- The material properties of the periodic medium, here the refractive index  $n_p$ , is characterized by a  $L^\infty(\Omega)$  function, with  $n_p \geq c > 0$  which is periodic with the same period than the waveguide

$$\forall (x_1, x_s) \in \Omega, \quad n_p(x_1 + L, x_s) = n_p(x_1, x_s).$$

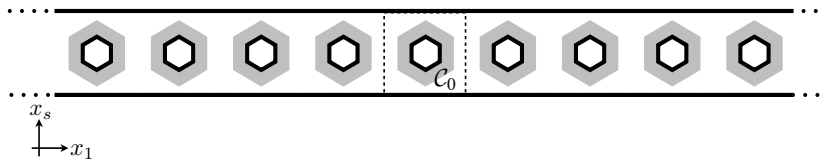
Moreover we assume  $n_p$  positively bounded from below :  $n_p \geq C > 0$ .

- The source term represented by the function  $f$  is supposed to be  $L^2$  with a compact support.

Without loss of generality, we suppose in the following that the period  $L$  is 1.

*Remark 1 (Extensions)* Most of the results of this article can be extended to

- more general symmetric, second order elliptic differential operators with real periodic coefficients (with the same period);
- other boundary conditions as soon as they satisfy the same periodicity properties than the geometry and the coefficients.



**Fig. 1** A periodic domain of propagation : the black lines represents the boundary of  $\Omega$  and typically  $n_p = 1$  in the white region,  $n_p = 2$  in the light grey regions.

The principal difficulty of the problem (1) lies in the definition of the notion of "physical" or "outgoing" solution. Indeed, if the frequency  $\omega$  is such that  $\omega^2$  is in the spectrum of the corresponding periodic differential operator  $-n_p^{-2}\Delta$  in  $\Omega$ , waves can propagate in the periodic waveguide which allows non vanishing solutions at infinity. In this case, the  $H^1$  classical framework is not appropriate anymore : solutions cannot exist in this space. On the other hand, in  $H_{loc}^1 = \{u, \forall \varphi \in \mathcal{D}(\mathbb{R}) \quad \varphi(x_1)u(x_1, x_s) \in H^1(\Omega)\}$ , one can define several solutions to the original problem. Additional conditions are required to define the "good" physical solution of (1). For example, if the waveguide is straight and the refraction index is constant, the classical separation of variables can

be used to show that at infinity any solution is exponentially close to a finite superposition of propagative guided modes. It suffices then to impose that this superposition is only made of the outgoing propagative modes on both sides of the support to make the problem well posed : this corresponds to adding a so-called radiation condition. Let us emphasize that the definition of the outgoing modes is explicit in that case thanks to the separations of variables.

In the general periodic case, a classical and natural idea to define physical solutions is to use the *limiting absorption principle*. This principle consists in adding some absorption to the original problem (the presence of the damping term  $\varepsilon > 0$  guaranteeing the well-posedness of this problem in  $H^1(\Omega)$ )

$$\begin{cases} -\Delta u_\varepsilon - n_p^2(\omega^2 + i\varepsilon) u_\varepsilon = f & \text{in } \Omega \\ \nabla u_\varepsilon \cdot \mathbf{n} = 0 & \text{on } \partial\Omega, \end{cases} \quad (3)$$

and in studying the limit, when  $\varepsilon$  goes to 0, of the family of solutions  $(u_\varepsilon)_{\varepsilon>0}$ . If the limit exists in a certain sense, this limit is defined as the physical or outgoing solution of (1). One says that the limiting absorption principle holds. Applied to the homogeneous waveguide, this approach leads to the outgoing solution as defined in the previous paragraph.

Our main goal in this paper is to characterize the outgoing solution with the help of radiation conditions at infinity, the construction of which relies on the study of the asymptotic behavior of this outgoing solution when  $x_1$  goes to  $\pm\infty$ .

The outline of the remainder of the paper is as follows : in Section 3, we introduce useful mathematical tools and we recall and extend (for our purpose) classical results about differential operators with periodic coefficients. in Section 4, we briefly recall the construction and the semi-analytical expression of the outgoing solution. In Section 5 which is the main section of the paper, we investigate the asymptotic behavior of the outgoing solution when  $x_1$  tends to  $\pm\infty$  and construct the radiation conditions. Finally, in Section 6, we show that this radiation condition ensures the uniqueness of the outgoing solution.

Throughout the remainder of the paper, we will denote for any open set  $\mathcal{O} \subset \Omega$

$$\forall f, g \in L^2(\mathcal{O}), \quad (f, g)_{\mathcal{O}} := \int_{\mathcal{O}} f \bar{g} dx_1 dx_s, \quad (f, g)_{n_p^2, \mathcal{O}} := \int_{\mathcal{O}} f \bar{g} n_p^2 dx_1 dx_s$$

and  $\|\cdot\|_{\mathcal{O}}$ ,  $\|\cdot\|_{n_p^2, \mathcal{O}}$  respectively, the corresponding norms.

### 3 Mathematical tools and results on differential operators with periodic coefficients

#### 3.1 The Floquet Bloch Transformation and properties

We recall the definition and more useful properties of the partial Floquet Bloch Transform in the  $x_1$ -direction, denoted FBT in the following (see [23] for a more complete and general exposition). Most of the proofs of the forthcoming results are straightforward and left to the reader (see also [23]).

**Definition 1** The FBT with period 1 in  $\Omega$  is defined by (see [23])

$$\begin{aligned} \mathcal{F} : \mathcal{D}(\Omega) &\rightarrow L^2(\mathcal{C} \times (-\pi, \pi)) \\ f(x_1, x_s) &\rightarrow \hat{f}(x_1, x_s; k) := \sqrt{\frac{1}{2\pi}} \sum_{p \in \mathbb{Z}} f(x_1 + p, x_s) e^{-ipk}, \end{aligned}$$

where  $\mathcal{D}(\Omega)$  is the set of  $C^\infty$ -functions with compact support (note that the sum in the definition of the FBT is then finite).

The FBT can be extended to  $L^2$  functions thanks to the following Proposition.

**Proposition 1 (Extension of the FBT to  $L^2(\Omega)$ )** *The FBT extends to an isometry between  $L^2(\Omega)$  and  $L^2(\mathcal{C} \times (-\pi, \pi))$  :*

$$\forall f, g \in L^2(\Omega), \quad \int_{\mathcal{C} \times (-\pi, \pi)} \hat{f} \hat{g} dk dx_1 dx_s = \int_{\Omega} f g dx_1 dx_s.$$

The second important formula for us is the inversion formula:

**Proposition 2** *The operator  $\mathcal{F}$  is invertible and its inverse is given by  $\forall \hat{f} \in L^2(\mathcal{C} \times (-\pi, \pi))$ , a.e.  $(x_1, x_s) \in \mathcal{C}$ ,  $\forall p \in \mathbb{Z}$ ,*

$$(\mathcal{F}^{-1} \hat{f})(x_1 + p, x_s) = \sqrt{\frac{1}{2\pi}} \int_{-\pi}^{\pi} \hat{f}(x_1, x_s; k) e^{ipk} dk. \quad (4)$$

We shall use in the sequel the following (obvious) properties of the FBT which make the FBT a privileged tool for the analysis of linear PDEs with periodic coefficients.

**Proposition 3** *The FBT has the following properties :*

1. *it commutes with the differential operators, in the sense that*

$$\forall u \in H^1(\Omega), \forall k \in [-\pi, \pi], \quad \mathcal{F} \left( \frac{\partial u}{\partial x_i} \right) (\cdot; k) = \frac{\partial}{\partial x_i} (\mathcal{F}u(\cdot; k)); \quad (5)$$

2. *it diagonalizes the integer translation operators*

$$\begin{aligned} \forall (x_1, x_s) \in \Omega, \forall p \in \mathbb{Z}, (\tau_p u)(x_1, x_s) &= u(x_1 + p, x_s) \\ \Downarrow & \\ \mathcal{F}(\tau_p u)(x_1, x_s; k) &= e^{-ipk} \mathcal{F}u(x_1, x_s; k), \quad (x_1, x_s; k) \in \mathcal{C} \times (-\pi, \pi); \end{aligned} \quad (6)$$



3. *it commutes with the multiplication by a periodic function: if  $n_p$  is a 1-periodic  $L^\infty$ -function for all  $(x_1, x_s; k) \in \mathcal{C} \times (-\pi, \pi)$*

$$\mathcal{F}(n_p^2 u)(x_1, x_s; k) = n_p^2(x_1, x_s) \mathcal{F}u(x_1, x_s; k). \quad (7)$$

For the sequel, it is essential to know how the Floquet Bloch Transform acts in Sobolev spaces. We recall these results, a proof of which can be found in [12, 13], for instance.

We need first to introduce spaces of functions on the domain  $\mathcal{C}$  of so-called  $k$  quasi-periodic functions ( $k$  being a parameter in  $(-\pi, \pi]$ ). To do so, we need to introduce the  $k$ -quasiperiodic extension operator  $E_k \in \mathcal{L}(L^2(\mathcal{C}), L^2_{loc}(\Omega))$  defined by (note that  $E_k f = f$  in  $\mathcal{C}$ )

$$\forall f \in L^2(\mathcal{C}), \quad \forall p \in \mathbb{Z}, \quad \forall (x_1, x_s) \in \mathcal{C}, \quad E_k f(x_1 + p, x_s) = e^{ipk} f(x_1, x_s). \quad (8)$$

For any  $k \in (-\pi, \pi]$  and for any  $m \in \mathbb{N}$ , let  $H_k^m(\mathcal{C})$  be defined by

$$H_k^m(\mathcal{C}) = \{f \in H^m(\mathcal{C}), \quad E_k f \in H^m_{loc}(\Omega)\}$$

One easily shows that  $H_k^m(\mathcal{C})$  is a closed subspace of  $H^m(\mathcal{C})$ . Consequently,  $H_k^1(\Delta, \mathcal{C}) = \{u \in H_k^1(\mathcal{C}), \Delta u \in L^2_k(\mathcal{C})\}$  is a closed subspace of  $H^1(\Delta, \mathcal{C})$ .

Let us give another characterization of  $H_k^1(\mathcal{C})$  and  $H_k^1(\Delta, \mathcal{C})$  which will be more convenient in the sequel.

**Proposition 4** *For any  $k \in (-\pi, \pi]$ , a function  $u$  belongs to  $H_k^1(\mathcal{C})$  if and only if it belongs to  $H^1(\mathcal{C})$  and satisfies*

$$u|_{x_1=1/2} = e^{ik} u|_{x_1=-1/2} \quad (9)$$

*and it belongs to  $H_k^1(\Delta, \mathcal{C})$  if and only if it belongs to  $H^1(\Delta, \mathcal{C})$ , satisfies (9) and*

$$\partial_{x_1} u|_{x_1=1/2} = e^{ik} \partial_{x_1} u|_{x_1=-1/2} \quad (10)$$

We can now state the main result for FBT in Sobolev spaces.

**Theorem 1** *For any  $m \in \mathbb{N}$ ,  $\mathcal{F}$  is an isometric isomorphism from  $H^m(\Omega)$  into*

$$H^m_{QP}(\mathcal{C} \times (-\pi, \pi)) = \left\{ \hat{u} \in L^2(-\pi, \pi; H^m(\mathcal{C})) / \forall k \in (-\pi, \pi), \hat{u}(\cdot; k) \in H_k^m(\mathcal{C}) \right\},$$

*equipped with the norm of  $L^2(-\pi, \pi; H^m(\mathcal{C}))$ :*

$$\|\hat{u}\|^2 := \int_{-\pi}^{\pi} \|\hat{u}(\cdot; k)\|_{H^m(\mathcal{C})}^2 dk$$

*Remark 2* We can deduce easily that  $\mathcal{F}$  is an isomorphism from  $H^1(\Delta, \Omega)$  into

$$H_{QP}^1(\Delta, \mathcal{C} \times (-\pi, \pi)) = \left\{ \hat{u} \in L^2(-\pi, \pi; H^1(\Delta, \mathcal{C})) / \right. \\ \left. \forall k \in (-\pi, \pi), \hat{u}(\cdot; k) \in H_k^1(\Delta, \mathcal{C}) \right\},$$

equipped with the norm of  $L^2(-\pi, \pi; H^1(\Delta, \mathcal{C}))$ :

$$\|\hat{u}\|^2 := \int_{-\pi}^{\pi} \left[ \|\hat{u}(\cdot; k)\|_{H^1(\mathcal{C})}^2 + \|\Delta \hat{u}(\cdot; k)\|_{L^2(\mathcal{C})}^2 \right] dk.$$

Finally, another important property concerns the relation between the decay at infinity of functions in  $L^2(\Omega)$  with the smoothness with respect to the dual variable of their Floquet transform (it is an equivalent of the Paley-Wiener theorem for the Fourier transform - see [23, Chapter 2]). We first introduce the notion of exponentially decaying in Sobolev spaces as follows.

**Definition 2** Let  $m > 0$ , a function  $u \in H^m(\Omega)$  is exponentially decaying with rate  $\alpha > 0$  in  $H^m(\Omega)$  if

$$\exists C > 0 \text{ such that } \forall p \in \mathbb{Z}, \|u\|_{H^m(\mathcal{C}_p)} \leq C e^{-\alpha|p|} \quad (11)$$

Analogously, we define the exponential decay with rate  $\alpha > 0$  in  $H^1(\Delta, \Omega)$ .

Note that if  $u$  is exponentially decaying in  $H^1(\Delta, \Omega)$  thanks to (2),  $u$  is exponentially decaying in  $H^s(\Omega)$ .

**Proposition 5** Let us consider a function  $u$  of  $H^m(\Omega)$  (resp. of  $H^1(\Delta, \Omega)$ ) which is exponentially decaying with rate  $\alpha > 0$  then its Floquet-Bloch transformation can be extended analytically in a neighborhood of the real axis,  $\{|Im(k)| \leq \alpha\}$ , as a  $2\pi$ -periodic function with values in  $H^m(\mathcal{C})$  (resp. in  $H^1(\Delta, \mathcal{C})$ ).

Conversely, if a function  $\hat{u}$  is in  $H_{QP}^m(\mathcal{C} \times (-\pi, \pi))$  (resp. in  $H_{QP}^1(\Delta, \mathcal{C} \times (-\pi, \pi))$ ) can be extended analytically as a function  $k \rightarrow \hat{u}(\cdot; k) \in H^m(\mathcal{C})$  (resp. in  $H^1(\Delta, \mathcal{C})$ ) in  $\{|Im(k)| \leq \alpha\}$  which is  $2\pi$ -periodic, then its Inverse Floquet Bloch Transform, denoted  $u$  is exponentially decaying with rate  $\alpha > 0$ .

In section 5.2, we shall also use unilateral version of exponential decay

**Definition 3** Let  $m > 0$ ,  $u \in H_{loc}^m(\Omega)$  is exponentially decaying with rate  $\alpha > 0$  in  $H_{loc}^m(\Omega)$  at  $+\infty$  (resp.  $-\infty$ )

$$\exists C > 0 \text{ such that } \|u\|_{H^m(\mathcal{C}_p)} \leq C e^{-\alpha|p|} \quad \forall p > 0 \quad (\text{resp. } \forall p < 0). \quad (12)$$

Analogously, we define the exponential decay with rate  $\alpha > 0$  in  $H^1(\Delta, \Omega)$  at  $\pm\infty$ .

### 3.2 Spectral properties of the operator

Let us introduce the unbounded operator in  $L^2(\Omega)$  defined by

$$\left| \begin{array}{l} A_p = -\frac{1}{n_p^2} \Delta \\ \text{with domain } D(A_p) = \{u \in H^1(\Delta, \Omega), \nabla u \cdot \mathbf{n}|_{\partial\Omega} = 0\} \end{array} \right. \quad (13)$$

One of the main result of the Floquet-Bloch theory is (see [23] for more details) that the spectrum of  $A_p$ ,  $\sigma(A_p)$  is a pure essential spectrum given by

$$\sigma(A_p) = \bigcup_{k \in ]-\pi, \pi]} \sigma(A_p(k))$$

where, for all  $k \in \mathbb{R}$ ,  $A_p(k)$  is the unbounded operator in  $L^2(\mathcal{C}, n_p^2 dx_1 dx_s)$  defined by

$$\left| \begin{array}{l} A_p(k) = -\frac{1}{n_p^2} \Delta \\ \text{with domain } D(A_p(k)) = \{u \in H_k^1(\Delta, \mathcal{C}), \nabla u \cdot \mathbf{n}|_{\partial\Omega \cap \partial\mathcal{C}} = 0\}. \end{array} \right. \quad (14)$$

From the definition of  $H_k^1(\Delta, \mathcal{C})$ , it is obvious that  $k \mapsto A_p(k)$  is  $2\pi$ -periodic. For any  $k \in \mathbb{R}$ , the operator  $A_p(k)$  is a self-adjoint positive operator with compact resolvent so its spectrum is purely discrete. Let us introduce the eigenvalues of the operators  $A_p(k)$  ordered increasingly

$$0 \leq \lambda_0(k) \leq \lambda_1(k) \leq \dots \leq \lambda_n(k) \leq \dots, \quad \text{with } \lim_{n \rightarrow +\infty} \lambda_n(k) = +\infty$$

and a corresponding orthonormal basis of eigenvectors  $(\varphi_n(\cdot; k))_{n \in \mathbb{N}}$ . Implicitly, each eigenvalue  $\lambda_n(k)$  is repeated with its (finite) multiplicity. With this choice, one easily checks that the  $\lambda_n(k)$ 's have the following property

$$\forall k \in \mathbb{R}, \quad \lambda_n(k + 2\pi) = \lambda_n(k) \quad \text{and} \quad \lambda_n(k) = \lambda_n(-k) \quad (15)$$

and the associated eigenvectors can be chosen such that

$$\forall k \in \mathbb{R}, \quad \varphi_n(\cdot; k + 2\pi) = \varphi_n(\cdot; k) \quad \text{and} \quad \varphi_n(\cdot; -k) = \overline{\varphi_n(\cdot; k)}. \quad (16)$$

According to [22], the functions  $k \mapsto \lambda_n(k)$  and  $k \mapsto \varphi_n(\cdot; k)$  are continuous and piecewise differentiable functions, with values in  $\mathbb{R}$  and  $L^2(\mathcal{C})$  respectively. They lose differentiability only at points  $k$  when  $\lambda_n(k)$  is no longer a simple eigenvalue. There is only a finite number of them and the reason is that these functions are in fact piecewise analytic.

To be more precise, one has to adopt a slightly different point of view and to refer to the perturbation theory of differential operators [22], more precisely to chapter 7. Obviously, the definition (14) of the operator  $A_p(k)$  can be extended to  $k \in \mathbb{C}$  and one easily shows that  $A_p(k)$  depends analytically of  $k$  in the sense of the definition of holomorphic non bounded operators, in §1,

Chapter 7 of [22]. The key point is that these operators have compact resolvent for any  $k \in \mathbb{C}$  and are self-adjoint for  $k \in \mathbb{R}$ . As a consequence, we can refer to the Theorem 1.8, Chapter 7 of [22] to claim that it exists a sequence of neighborhoods of the real axis  $D_n := \{|\operatorname{Im}(k)| < \alpha_n\}$ , analytic functions  $\mu_n(k) : D_n \rightarrow \mathbb{C}$  and analytic functions  $\psi_n(\cdot; k) : D_n \rightarrow L^2(\mathcal{C})$  such that for all  $k \in \mathbb{R}$

$$\begin{aligned} \psi_n(\cdot; k) \in D(A_p(k)), \quad A_p(k) \psi_n(\cdot; k) &= \mu_n(k) \psi_n(\cdot; k) \\ \text{and } (\psi_n(\cdot; k), \psi_m(\cdot; k))_{n_p^2, \mathcal{C}} &= \delta_{n,m}. \end{aligned} \quad (17)$$

As a consequence, for real  $k$ , the sequence  $(\mu_n(k))_{n \in \mathbb{N}}$  is simply a rearrangement of the sequence  $(\lambda_n(k))_{n \in \mathbb{N}}$ :

$$\forall k \in \mathbb{R}, \quad \{\mu_n(k), n \in \mathbb{N}\} = \{\lambda_n(k), n \in \mathbb{N}\}.$$

Moreover, the eigenfunctions  $\varphi_n(\cdot; k)$  can be chosen in such way that

$$\forall k \in \mathbb{R}, \quad \{\psi_n(\cdot; k), n \in \mathbb{N}\} = \{\varphi_n(\cdot; k), n \in \mathbb{N}\}.$$

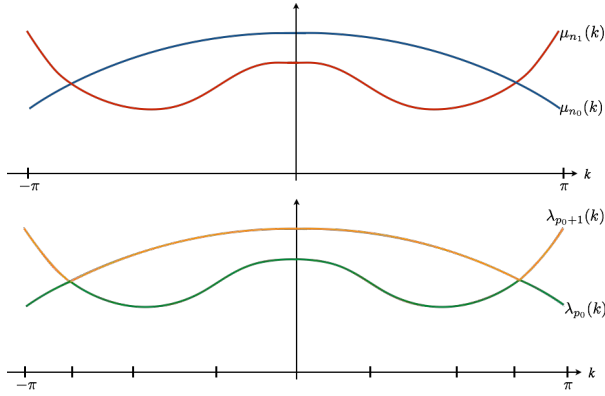
We shall call dispersion curves, the curves  $k \in \mathbb{R} \mapsto \mu_n(k)$ . By analyticity arguments, two dispersion curves either coincide along the real axis or meet only a finite number of times inside  $[-\pi, \pi]$ . As a consequence, going back to the  $\lambda_n(k)$ 's, their multiplicity is constant with respect to  $k$  except maybe for a finite number of points and for each  $n \geq 1$ , the interval  $[-\pi, \pi]$  can be divided into finite numbers of subintervals inside each of which the function  $\lambda_n(k)$  (resp.  $\varphi_n(\cdot; k)$ ) coincides with some functions  $\mu_p(k)$  (resp.  $\psi_p(\cdot; k)$ ), see Figure 2 for an example, which makes precise the definition of piecewise analytic. This also implies that either the multiplicity of  $\mu_n(k)$  is constant, equal to  $M$  for  $k \in [0, 2\pi]$  in which case we repeat in the set  $\{\mu_p(k)\}$ ,  $M$  times the same functions, either multiplicity only increases at each crossing point between different dispersion curves.

We shall use in Section 3.3 and 6, another important property of the functions  $\mu_n(k)$ . By analyticity arguments, we know that either  $\mu_n(k)$  is constant in  $[-\pi, \pi]$  or its derivative  $\mu_n'(k)$  vanishes a finite number of times inside  $[-\pi, \pi]$ . However, a recent non trivial result excludes the first possibility ([27, 40]).

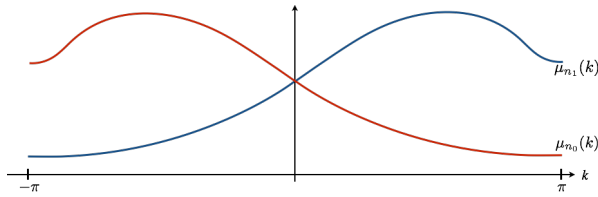
**Proposition 6 ([27, 40])** *None of the constant  $\mu_n(k)$  can be constant along the interval  $[-\pi, \pi]$ .*

In the most part of this paper, we shall work with  $(\mu_n(k), \psi_n(\cdot; k))$  instead of  $(\lambda_n(k), \varphi_n(\cdot; k))$ .

*Remark 3* Let us point out the fact that contrary to the functions  $\lambda_n(k)$ , the functions  $\mu_n(k)$  are not necessarily  $2\pi$ -periodic nor even as illustrated in Figure 3.



**Fig. 2** A picture of two dispersion curves (left) and corresponding  $\lambda_p(k)$  and  $\lambda_{p+1}(k)$  (right)



**Fig. 3** The dispersion curve  $k \mapsto \mu_n(k)$  is not necessarily  $2\pi$ -periodic nor even

### 3.3 The propagating Floquet modes

In order to define the propagating Floquet modes at a given frequency  $\omega$ , let us introduce the set

$$I(\omega) = \{n \in \mathbb{N}, \exists \xi \in ]-\pi, \pi], \mu_n(\xi) = \omega^2\} \quad (18)$$

and for each  $n \in I(\omega)$ , the set

$$\Xi_n(\omega) = \{\xi \in ]-\pi, \pi], \mu_n(\xi) = \omega^2\}. \quad (19)$$

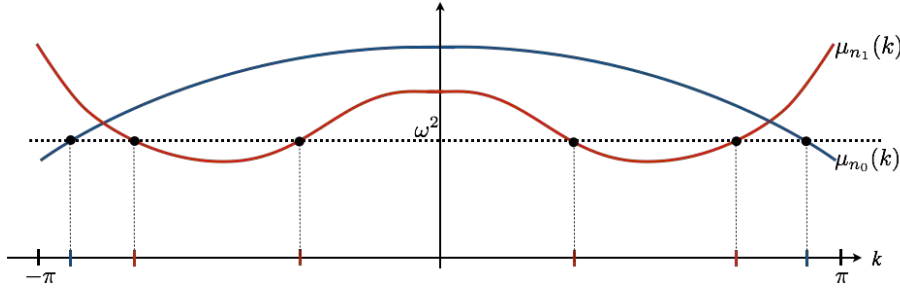
We define the set of propagative waves numbers associated to  $\omega$  as (see Figure 4 for an example)

$$\Xi(\omega) = \bigcup_{n \in I(\omega)} \Xi_n(\omega). \quad (20)$$

where the union is understood in the sense that common elements are repeated in the result, for example

$$\{\xi_1, \xi_2\} \cup \{\xi_1, \xi_3\} = \{\xi_1, \xi_2, \xi_1, \xi_3\}. \quad (21)$$

**Proposition 7** *The sets  $I(\omega)$  and  $\Xi_n(\omega)$  are finite.*



**Fig. 4** In this case,  $I(\omega) = \{n_0, n_1\}$ ,  $\Xi_{n_0}(\omega)$  contains 2 points and  $\Xi_{n_1}(\omega)$  4 points.

*Proof* Using the min-max principle and the Weyl estimation (see [38]), it is easy to show that

$$\forall n, \forall k \in ]-\pi, \pi], \quad C_1 n \leq \lambda_n(k) \leq C_2 n$$

where  $\lambda_n(k)$  are the eigenvalues of  $A(k)$ , ordered increasingly, as defined in Section 3.2 and  $C_1$  and  $C_2$  are related to the lower and the upper bounds of  $n_p$ . Using the relation between the  $\lambda_n(k)$ 's and the  $\mu_n(k)$ 's, we can show by contradiction that

$$\lim_{n \rightarrow +\infty} \min_{k \in [-\pi, \pi]} \mu_n(k) = +\infty$$

and thus that  $I(\omega)$  is finite.

Next, let us consider  $n \in I(\omega)$ . By analyticity arguments, the function  $\mu_n(k)$  is either constant or meets  $\omega^2$  only a finite number of times. However, Proposition 6 excludes the first possibility. Hence, each set  $\Xi_n(\omega)$  is finite.

□

On Figure 4, the sets  $\Xi_n(\omega)$  are symmetric, however this is not necessarily the case in general, see Figure 5 for an example. Nevertheless, this symmetry property is satisfied by the full set  $\Xi(\omega)$ .

**Proposition 8** *The set  $\Xi(\omega)$  is symmetric with respect to the origin in the sense that*

$$\xi \in \Xi(\omega) \quad \Rightarrow \quad -\xi \in \Xi(\omega)$$

*Proof* It is sufficient to use the definition of the set  $\Xi(\omega)$  with the help of the  $\lambda_n$ 's. More precisely, we introduce

$$\tilde{I}(\omega) = \{n \in \mathbb{N}, \exists \xi \in ]-\pi, \pi], \lambda_n(\xi) = \omega^2\} \quad (22)$$

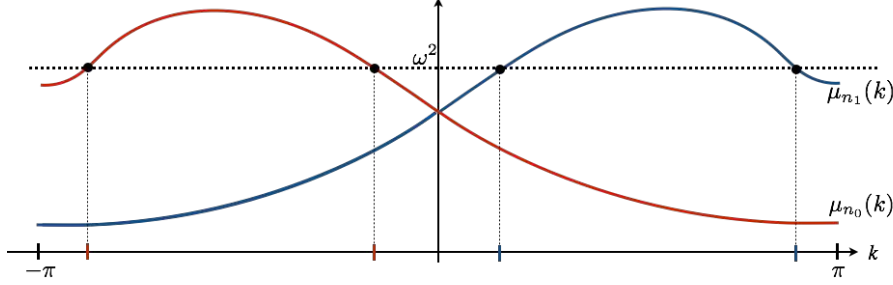
and for all  $n \in \tilde{I}(\omega)$

$$\tilde{\Xi}_n(\omega) = \{\xi \in ]-\pi, \pi], \lambda_n(\xi) = \omega^2\}. \quad (23)$$

These two sets are finite for the same reasons than  $I(\omega)$  and  $\Xi(\omega)$  and we have even

$$\Xi(\omega) = \{\xi \in \tilde{\Xi}_n(\omega), n \in \tilde{I}(\omega)\}.$$

Thanks to (15), we deduce that  $\tilde{\Xi}_n(\omega)$  is symmetric with respect to  $k = 0$ .  $\square$



**Fig. 5** The sets  $\Xi_{n_0}(\omega)$  and  $\Xi_{n_1}(\omega)$  are not symmetric with respect to  $k = 0$  but the set  $\Xi(\omega)$  is.

For all  $n \in I(\omega)$  and  $\xi \in \Xi_n(\omega)$ , let us introduce the corresponding propagating Floquet modes namely the functions in  $H_{loc}^1(\Omega)$  which are the  $\xi$ -quasiperiodic extensions of the eigenvectors  $\psi_n(\cdot; \xi)$ , that we still denote  $\psi_n$  for simplicity

$$\psi_n(\cdot; \xi) := E_\xi \psi_n(\cdot; \xi) \quad (\text{abuse of notation}) \quad (24)$$

where the operator  $E_\xi$  is defined in (8). One easily shows that  $\psi_n(\cdot; \xi)$  is an element of the space

$$\mathcal{V}(\omega) = \{u \in H_{loc}^1(\Omega), \quad -\Delta u - n_p^2 \omega^2 u = 0 \text{ in } \Omega \quad \text{and} \quad \nabla u \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\} \quad (25)$$

Even more,  $\mathcal{V}(\omega)$  is generated by these propagating Floquet modes :

$$\mathcal{V}(\omega) = \text{span} \{\psi_n(\cdot; \xi), \quad \xi \in \Xi_n(\omega), \quad n \in I(\omega)\}.$$

On the space  $\mathcal{V} = \{v \in H_{loc}^1(\Omega), \Delta v \in H_{loc}^1 \text{ and } \nabla v \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}$  which contains  $\mathcal{V}(\omega)$  for any  $\omega$ , we can define, setting  $\Gamma_x = \{x_s, (x, x_s) \in \Omega\}$  and for  $x$  in  $\mathbb{R}$  using trace theorems on  $\Gamma_x$  in  $H_{loc}^s(\Omega)$  with  $s > 3/2$  (see (2))

$$\forall x \in \mathbb{R}, \forall v, w \in \mathcal{V}, \quad q(x; v, w) = \int_{\Gamma_x} \left( \frac{\partial v}{\partial x_1} \bar{w} - \frac{\partial \bar{w}}{\partial x_1} v \right) dx_s. \quad (26)$$

Let  $x < y$  and  $\Omega_{x,y} = \{(x_1, x_s) \in \Omega, x < x_1 < y\}$ , we have by Green's formula

$$\begin{aligned} \int_{\Omega_{x,y}} [(-\Delta - n_p^2 \omega^2) v \bar{w} - (-\Delta - n_p^2 \omega^2) \bar{w} v] dx_1 dx_s \\ = q(x; v, w) - q(y; v, w). \end{aligned} \quad (27)$$

In particular,

$$\forall v, w \in \mathcal{V}(\omega), \forall x \in \mathbb{R}, \quad q(x; v, w) = q(v, w) \quad (\text{independent of } x). \quad (28)$$

Thus  $q(v, w)$  defines a sesquilinear form in  $\mathcal{V}(\omega)$ , already introduced in [32] where the quadratic form  $q(v, v)$  is defined as the energy flux of  $v$ . One main interest of this sesquilinear form is that it "orthogonalizes" the basis  $\{\psi_n(\cdot; \xi), \xi \in \Xi_n(\omega), n \in I(\omega)\}$ . This property will be useful to show the uniqueness result of Theorem 4.

**Proposition 9** *Let  $n, m \in I(\omega)$ ,  $\xi \in \Xi_n(\omega)$ ,  $\xi' \in \Xi_m(\omega)$ , then*

$$\begin{aligned} \text{if } m \neq n \text{ or } \xi \neq \xi' \quad & q(\psi_n(\cdot; \xi), \psi_m(\cdot; \xi')) = 0 \\ \text{otherwise} \quad & q(\psi_n(\cdot; \xi), \psi_n(\cdot; \xi)) = \imath \mu'_n(\xi). \end{aligned}$$

*Proof* Writing that for any  $n, m \in I(\omega)$  and  $(\xi, \xi') \in \Xi_n(\omega) \times \Xi_m(\omega)$ ,  $q(x; \psi_n(\cdot; \xi), \psi_m(\cdot; \xi'))$  is independent of  $x$ , we have in particular

$$\begin{aligned} \int_{\Gamma_{-1/2}} \left( \frac{\partial \psi_n(\cdot; \xi)}{\partial x_1} \overline{\psi_m(\cdot; \xi')} - \frac{\partial \overline{\psi_m(\cdot; \xi')}}{\partial x_1} \psi_n(\cdot; \xi) \right) dx_s \\ - \int_{\Gamma_{1/2}} \left( \frac{\partial \psi_n(\cdot; \xi)}{\partial x_1} \overline{\psi_m(\cdot; \xi')} - \frac{\partial \overline{\psi_m(\cdot; \xi')}}{\partial x_1} \psi_n(\cdot; \xi) \right) dx_s = 0. \end{aligned}$$

Since  $\psi_n(\cdot; \xi)$  (resp.  $\psi_m(\cdot; \xi')$ ) is  $\xi$  (resp.  $\xi'$ ) quasi-periodic, this simply rewrites

$$\left( e^{\imath(\xi - \xi')} - 1 \right) q(\psi_n(\cdot; \xi), \psi_m(\cdot; \xi')) = 0.$$

This gives the result for  $\xi \neq \xi'$ .

We consider now the case  $\xi = \xi'$ . Then let  $n, m \in I(\omega)$  such that  $\mu_n(\xi) = \mu_m(\xi) = \omega^2$ . We shall use the derivative of  $\psi_n(\cdot; k)$  with respect to  $k$ , denoted  $\partial_k \psi_n(\cdot; k)$ . The reader will easily shows that  $\partial_k \psi_n(\cdot; \xi)$  satisfies

$$(-\Delta - \mu_n(\xi) n_p^2) \partial_k \psi_n(\cdot; \xi) = \mu'_n(\xi) n_p^2 \psi_n(\cdot; \xi), \quad \text{in } \Omega \quad (29)$$

$$\partial_k \psi_n(\cdot; \xi)|_{\Gamma_{1/2}} = e^{\imath \xi} \partial_k \psi_n(\cdot; \xi)|_{\Gamma_{-1/2}} + \imath e^{\imath \xi} \psi_n(\cdot; \xi)|_{\Gamma_{-1/2}} \quad (30)$$

$$\partial_{x_1} [\partial_k \psi_n(\cdot; \xi)]|_{\Gamma_{1/2}} = e^{\imath \xi} \partial_{x_1} [\partial_k \psi_n(\cdot; \xi)]|_{\Gamma_{-1/2}} + \imath e^{\imath \xi} \partial_{x_1} \psi_n(\cdot; \xi)|_{\Gamma_{-1/2}} \quad (31)$$

If we use formula (27) with  $x = -1/2$ ,  $y = 1/2$ ,  $v = \partial_k \psi_n(\cdot; \xi) \in \mathcal{V}$  and  $w = \psi_m(\cdot; \xi) \in \mathcal{V}$ , we obtain

$$\begin{aligned} q\left(-\frac{1}{2}, \partial_k \psi_n(\cdot; \xi), \psi_m(\cdot; \xi)\right) - q\left(\frac{1}{2}, \partial_k \psi_n(\cdot; \xi), \psi_m(\cdot; \xi)\right) \\ = \mu'_n(\xi) \int_{\mathcal{C}} \psi_n(\cdot; \xi) \overline{\psi_m(\cdot; \xi)} n_p^2 dx_1 dx_s. \end{aligned}$$



On the other hand, using the boundary conditions (30), (31) and the fact that  $\psi_m(\cdot; \xi)$  is  $\xi$ -quasi-periodic, we obtain after some computations omitted here

$$\begin{aligned} q\left(-\frac{1}{2}, \partial_k \psi_n(\cdot; \xi), \psi_m(\cdot; \xi)\right) - q\left(\frac{1}{2}, \partial_k \psi_n(\cdot; \xi), \psi_m(\cdot; \xi)\right) \\ = -i q(\psi_n(\cdot; \xi), \psi_m(\cdot; \xi)) \end{aligned}$$

One concludes easily by the orthonormality (17) of the  $(\psi_n(\cdot; k))_{n \in \mathbb{N}}$ .  $\square$

*Remark 4 (Physical interpretation)* Introducing for all  $n \in I(\omega)$  and for all  $\xi \in \Xi_n(\omega)$ ,  $\omega_n(\xi) = \sqrt{\mu_n(\xi)}$ , one can reinterpret the Floquet mode as the time harmonic wave obeying the dispersion relation

$$\omega = \omega_n(\xi)$$

and thus define the group velocity by

$$V_n(\xi) := \frac{d\omega_n}{d\xi}(\xi) = \frac{1}{2} \mu_n(\xi)^{-1/2} \mu_n'(\xi).$$

According to the previous Proposition, the sign of the group velocity indicates how the energy propagates in the waveguide.

The propagating Floquet modes can then be divided into those for which  $\mu_n'(\xi) > 0$  (they propagate from the left to the right), those for which  $\mu_n'(\xi) < 0$  (they propagate from the right to the left) and those for which  $\mu_n'(\xi) = 0$ .

We can decompose the set  $\Xi(\omega)$  of propagative wave numbers as

$$\Xi(\omega) = \Xi^0(\omega) \cup \Xi^+(\omega) \cup \Xi^-(\omega)$$

where

$$\Xi^\pm(\omega) = \bigcup_{n \in I(\omega)} \Xi_n^\pm(\omega), \quad \Xi_n^\pm(\omega) = \{\xi \in \Xi_n(\omega), \pm \mu_n'(\xi) > 0\}$$

$$\Xi^0(\omega) = \bigcup_{n \in I(\omega)} \Xi_n^0(\omega), \quad \Xi_n^0(\omega) = \{\xi \in \Xi_n(\omega), \mu_n'(\xi) = 0\}$$

the unions being again defined in the sense of (21). Let us remark that the sets  $\Xi^\pm(\omega)$  and  $\Xi^0(\omega)$  are disjoint but  $\Xi^+(\omega)$  and  $\Xi^-(\omega)$  are not necessarily disjoint in the case of multiple eigenvalues

$$\exists n \neq m, \exists \xi \in [-\pi, \pi], \quad \mu_n(\xi) = \mu_m(\xi) = \omega^2 \quad \text{and} \quad \mu_n'(\xi) \mu_m'(\xi) < 0.$$

**Proposition 10** *The sets  $\Xi^+(\omega)$  and  $\Xi^-(\omega)$  are symmetric the one with respect to the other in the sense that*

$$\xi \in \Xi^+(\omega) \quad \Leftrightarrow \quad -\xi \in \Xi^-(\omega).$$

*Proof* We first show the result for  $\omega \notin \sigma_m$  where  $\sigma_m$  is the set

$$\sigma_m = \{\omega, \exists n \neq p, \exists k \in [-\pi, \pi[, \mu_n(k) = \mu_p(k) = \omega^2\}. \quad (32)$$

Let  $\xi \in \Xi^+(\omega)$  then it exists  $n \in I(\omega)$  such that  $\mu_n(\xi) = \omega^2$  and  $\mu'_n(\xi) > 0$ . Because  $\omega \notin \sigma_m$ , it exists a subinterval of  $(-\pi, \pi)$  containing  $\xi$  in which  $\mu_n(k)$  coincides with some  $\lambda_p(k)$ . Because  $\lambda_p$  is an even function, we have  $\omega^2 = \lambda_p(\xi) = \lambda_p(-\xi)$  and  $\lambda'_p(-\xi) = -\lambda'_p(\xi) < 0$ . Since  $\lambda_p(k)$  coincides with a  $\mu_q(k)$  in a subinterval of  $(-\pi, \pi)$  containing  $-\xi$ , we deduce that  $-\xi \in \Xi_q^-(\omega) \subset \Xi^-(\omega)$ .

For  $\omega \in \sigma_m$ , we want to use the continuity of  $\omega \mapsto \Xi^\pm(\omega)$  which is possible because  $\sigma_m$  is a discrete set for similar reasons than the ones which will be used in the proof on the next proposition.  $\square$

We now introduce a set of frequencies  $\omega$  for which the limiting absorption principle will not hold

$$\begin{aligned} \sigma_0 &= \{\omega \in \mathbb{R}, \exists n, \exists \xi \in (-\pi, \pi], \mu_n(\xi) = \omega^2 \text{ and } \mu'_n(\xi) = 0\} \\ &\equiv \{\omega \in \mathbb{R}, \Xi^0(\omega) \neq \emptyset\}. \end{aligned} \quad (33)$$

Physically, these are the frequencies for which it exists a Floquet modes whose group velocity can vanish. This corresponds for instance to the cut-off frequencies for the homogeneous waveguide for which we know that the limiting absorption principle does not hold [28].

**Proposition 11** *The set  $\sigma_0$  is countable set with at most one point of accumulation  $+\infty$ .*

*Proof* Let us suppose that there exists a non stationary sequence  $(\omega_n)_n$  of  $\sigma_0$ , then

$$\forall n, \exists i_n, \exists \xi_n, \mu_{i_n}(\xi_n) = \omega_n \text{ and } \mu'_{i_n}(\xi_n) = 0$$

which converges towards  $\omega_\infty \in \mathbb{R}^+$ . The sequence  $(\omega_n)_n$  is in particular bounded. Moreover, using that (see the proof of Proposition 7)

$$\lim_{n \rightarrow +\infty} \min_{k \in [-\pi, \pi]} \mu_n(k) = +\infty$$

the set of indices  $(i_n)_n$  is necessarily finite. Leaves to extract a subsequence, we can assume that the sequence  $(i_n)_n$  is constant and equal to  $i_0$  :

$$\forall n, \exists \xi_n, \mu_{i_0}(\xi_n) = \omega_n \text{ and } \mu'_{i_0}(\xi_n) = 0$$

The sequence  $(\omega_n)_n$  is assumed to be non stationary, that implies that  $\mu'_{i_0}(\xi_n) = 0$  for a non stationary sequence  $(\xi_n)_n$ . The function  $\mu_{i_0}$  being analytic, this implies that  $\mu_{i_0}$  is a constant which implies that the sequence  $(\omega_n)_n$  is stationary, which contradicts the initial assumption.  $\square$

*Remark 5* In the Appendix B, we shall show by explicit calculations that, in a straight homogeneous waveguide, the Floquet modes do coincide with the standard guided modes.

#### 4 Limiting absorption principle in the periodic waveguide

There are some contributions in the study of limiting absorption principle for periodic waveguide [26, 13, 16]. We remind in this section the main results and the ideas of the proof of these works : using the Floquet-Bloch Transform, we give a semi-analytical expression of the solution of the time-harmonic scalar wave equation (3) with absorption, show that the family of solutions has a limit in a certain sense when the absorption tends to 0 and finally deduce a semi-analytical expression of the physical solution of (1).

**Proposition 12** *Suppose that  $f \in L^2(\Omega)$ . The solution  $u_\varepsilon$  of (3) is given by*

$$u_\varepsilon = \sum_{n \in \mathbb{N}} u_\varepsilon^n, \quad \text{in } H^1(\Delta, \Omega) \quad (34)$$

where each  $u_\varepsilon^n$  is given by  $\forall (x_1, x_s) \in \mathcal{C}, \forall p \in \mathbb{Z}$ ,

$$u_\varepsilon^n(x_1 + p, x_s) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \frac{\mathbb{P}_n(f)(x_1, x_s; k)}{\mu_n(k) - (\omega^2 + i\varepsilon)} e^{ipk} dk \quad (35)$$

where

$$\mathbb{P}_n(f)(\cdot; k) = (\hat{f}(\cdot; k), \psi_n(\cdot; k))_{n_p^2, \mathcal{C}} \psi_n(\cdot; k). \quad (36)$$

*Proof* Applying the Floquet Bloch Transform to the equation (3), using its properties detailed in Proposition 3, we see that the FBT of  $u_\varepsilon$ , namely  $\hat{u}_\varepsilon(\cdot; k)$ , belongs to  $D(A_p(k))$  and satisfies

$$A_p(k) \hat{u}_\varepsilon(\cdot; k) - (\omega^2 + i\varepsilon) \hat{u}_\varepsilon(\cdot; k) = \frac{1}{n_p^2} \hat{f}(\cdot; k).$$

After diagonalization of the above equation in the basis  $(\psi_n(\cdot; k))_{n \in \mathbb{N}}$ , we obtain

$$\hat{u}_\varepsilon(\cdot; k) = \sum_{n \in \mathbb{N}} \frac{\mathbb{P}_n(f)(\cdot; k)}{\mu_n(k) - (\omega^2 + i\varepsilon)}$$

and finally by applying the inverse Floquet-Bloch Transform

$$\forall (x_1, x_s) \in \mathcal{C}, \forall p \in \mathbb{Z}, \quad u_\varepsilon(x_1 + p, x_s) = \sqrt{\frac{1}{2\pi}} \int_{-\pi}^{\pi} \sum_{n \in \mathbb{N}} \frac{\mathbb{P}_n(f)(x_1, x_s; k)}{\mu_n(k) - (\omega^2 + i\varepsilon)} e^{ipk} dk.$$

To conclude, the only difficulty is to justify the inversion between the integral and the sum. The details use similar arguments than in the Step 1 of Appendix A and are then left to the reader.  $\square$

*Remark 6* Using the same ideas, we can show that the definition of  $\mathbb{P}_n(f)$  can be extended if the source term  $f \in H^r(\Omega)$  for any  $r > -s$  with  $s > \frac{3}{2}$  because  $\psi_n(\cdot; k) \in H^s(\mathcal{C})$  (see assumption (2)). Then the representation of the solution holds in  $H^{r+2}(\mathcal{C})$ . In particular, we can deduce easily from this result the expression of the Green function  $G(\cdot; y)$  for which formally  $f = \delta_y$  with  $y = (y_1 + q, y_s)$ ,  $(y_1, y_s) \in \mathcal{C}$  and  $q \in \mathbb{Z}$ ,  $\forall (x_1, x_s) \in \mathcal{C}$ ,  $\forall p \in \mathbb{Z}$ ,

$$G_\varepsilon(x_1 + p, x_s; y_1 + q, y_s) = \frac{1}{2\pi} \sum_{n \in \mathbb{N}} \int_{-\pi}^{\pi} \frac{\psi_n(x_1, x_s; k) \overline{\psi_n(y_1, y_s; k)}}{\mu_n(k) - (\omega^2 + i\varepsilon)} e^{i(p-q)k} dk.$$

We can now establish the limiting absorption principle which will be proven in Appendix B.

**Theorem 2 (Limiting absorption principle)** *Suppose that  $\omega \notin \sigma_0$  and  $f$  is in  $L^2(\Omega)$  with a compact support. Then*

$$\forall \eta \in (0, 1), \exists C_\eta(\omega), \forall p \in \mathbb{Z}, \quad \|u_\varepsilon - u\|_{H^1(\mathcal{C}_p)} \leq C_\eta(\omega) \varepsilon^{1-\eta} \|f\|_{L^2(\Omega)}.$$

where  $u$  is solution of the Helmholtz equation (1) and is given by

$$u = u_{\text{evan}} + u_{\text{prop}} \quad \text{with} \quad \begin{cases} u_{\text{evan}} = \sum_{n \notin I(\omega)} u^n \in H^1(\Delta, \Omega) \\ u_{\text{prop}} = \sum_{n \in I(\omega)} u^n \in H_{\text{loc}}^1(\Delta, \Omega) \end{cases} \quad (37)$$

where each  $u^n$  is given by  $\forall (x_1, x_s) \in \mathcal{C}$ ,  $\forall p \in \mathbb{Z}$ ,

$$\begin{aligned} n \notin I(\omega), \quad u^n(x_1 + p, x_s) &= \sqrt{\frac{1}{2\pi}} \int_{-\pi}^{\pi} \frac{\mathbb{P}_n(f)(x_1, x_s; k)}{\mu_n(k) - \omega^2} e^{ipk} dk \\ n \in I(\omega), \quad u^n(x_1 + p, x_s) &= \sqrt{\frac{1}{2\pi}} \left[ p.v. \int_{-\pi}^{\pi} \frac{\mathbb{P}_n(f)(x_1, x_s; k)}{\mu_n(k) - \omega^2} e^{ipk} dk \right. \\ &\quad \left. + i\pi \sum_{\xi \in \Xi_n(\omega)} \frac{\mathbb{P}_n(f)(x_1, x_s; \xi)}{|\mu'_n(\xi)|} e^{ip\xi} \right] \end{aligned} \quad (38)$$

and where  $\mathbb{P}_n(f)$  is defined in (36) and

$$p.v. \int_{-\pi}^{\pi} \frac{\mathbb{P}_n(f)(x_1, x_s; k)}{\mu_n(k) - \omega^2} e^{ipk} dk = \lim_{\delta \rightarrow 0^+} \int_{\mathcal{I}_{\omega, n}^\delta} \frac{\mathbb{P}_n(f)(x_1, x_s; k)}{\mu_n(k) - \omega^2} e^{ipk} dk,$$

with  $\mathcal{I}_{\omega, n}^\delta = ]-\pi, \pi[ \setminus \bigcup_{\xi \in \Xi_n(\omega)} [\xi - \delta, \xi + \delta]$ .

*Remark 7* When  $\omega^2 \notin \sigma(A_p)$  (i.e. when  $I(\omega) = \emptyset$ ), we show in Appendix A that  $u \in H^1(\Omega)$  and

$$\exists C(\omega), \forall p \in \mathbb{Z}, \quad \|u_\varepsilon - u\|_{H^1(\mathcal{C}_p)} \leq C(\omega) \varepsilon \|f\|_{L^2(\Omega)}.$$

*Remark 8* Suppose that  $\omega \notin \sigma_0$ ,

$$\forall y \in \Omega, \forall p \in \mathbb{Z}, \quad \lim_{\varepsilon \rightarrow 0} \|G_\varepsilon(\cdot; y) - G(\cdot; y)\|_{L^2 \mathcal{C}_p} = 0$$

where  $G$  is solution of the Helmholtz equation (1) with  $f = \delta_y$ ,  $y = (y_1 + q, y_s)$ ,  $(y_1, y_s) \in \mathcal{C}$  and  $q \in \mathbb{Z}$  and is given by  $\forall (x_1, x_s) \in \mathcal{C}$ ,  $\forall p \in \mathbb{Z}$ ,

$$\begin{aligned} G(x_1 + p, x_s; y_1 + q, y_s) &= \frac{1}{2\pi} \sum_{n \notin I(\omega)} \int_{-\pi}^{\pi} \frac{\psi_n(x_1, x_s; k) \overline{\psi_n(y_1, y_s; k)}}{\mu_n(k) - \omega^2} e^{i(p-q)k} dk \\ &+ \frac{1}{2\pi} \sum_{n \in I(\omega)} \left[ p.v. \int_{-\pi}^{\pi} \frac{\psi_n(x_1, x_s; k) \overline{\psi_n(y_1, y_s; k)}}{\mu_n(k) - \omega^2} e^{i(p-q)k} dk \right. \\ &\quad \left. + i\pi \sum_{\xi \in \Xi_n(\omega)} \frac{\psi_n(x_1, x_s; \xi) \overline{\psi_n(y_1, y_s; \xi)}}{|\mu'_n(\xi)|} e^{i(p-q)\xi} \right]. \quad (39) \end{aligned}$$

## 5 Asymptotic behaviour of the solution when $x_1$ tends to $\pm\infty$

We suppose now that  $\omega \notin \sigma_0$  and  $f$  is in  $L^2(\Omega)$  with a compact support then Theorem 2 holds.

### 5.1 Case where $\omega^2 \notin \sigma(A_p)$

The asymptotic behaviour at  $\infty$  in that case is in fact a well known property of waves in periodic media. We describe this (simple) case just for pedagogical purpose. Suppose that the spectrum of  $A_p$  contains a gap and let  $\omega^2$  lies in this gap. Then, it has been established in [9–11] that the Green's function and then the solution of (1) are exponentially decreasing in the direction of the waveguide with a rate which is proportionnal to the distance between  $\omega^2$  and the essential spectrum of  $A$ . Let us give simple arguments that we shall extend in Section 5.2 which allow us to prove this decay - without estimating the decay rate. These arguments are the analyticity with respect to the dual variable of the FBT of the function and the equivalent of the Paley-Wiener theorem for the FBT, given in Proposition 5.

Indeed, if  $\omega^2 \notin \sigma(A_p)$ , the set  $I(\omega)$  is empty, the problem (1) is well-posed, its unique solution  $u$  given in (37-38) is in  $H^1(\Delta, \Omega)$  and can be written as

$$\forall p \in \mathbb{Z}, \quad u(\cdot + (p, 0)) = \sqrt{\frac{1}{2\pi}} \int_{-\pi}^{\pi} (A(k) - \omega^2)^{-1} \hat{f}(\cdot; k) e^{ipk} dk.$$

It suffices then to note that

- From Section 3.2, we know that  $k \mapsto A_p(k)$  is analytic in  $\mathbb{C}$ . Moreover, for all real  $k$ ,  $\omega^2 \notin \sigma(A_p(k))$ , then  $(A_p(k) - \omega^2)$  is invertible from  $H_k^1(\Delta, \mathcal{C})$  into  $L^2(\mathcal{C})$  with bounded inverse  $(A_p(k) - \omega^2)^{-1}$  in  $L^2(\mathcal{C})$ . Using [22, Chapter VII, Theorem 1.3], the function  $k \mapsto (A_p(k) - \omega^2)^{-1}$  can be extended analytically in  $D_\beta = \{|\operatorname{Im}(k)| \leq \beta\}$  for some  $\beta$ ;
- $f$  is in  $L^2(\Omega)$  with compact support then its FBT  $\hat{f}(\cdot; k)$  is analytic with respect to  $k$  with values in  $L^2(\mathcal{C})$  (see Proposition 5);

Finally, since  $A_p(k)$  and  $\hat{f}(\cdot; k)$  are  $2\pi$ -periodic,  $\hat{u}(\cdot; k)$  is  $2\pi$ -periodic. We deduce that  $u$  is the FB inverse of an analytic function with respect to  $k$  with values in  $H_k^1(\Delta, \mathcal{C})$  so, thanks to Proposition 5,  $u$  is exponentially decreasing in  $H^1(\Delta, \Omega)$  in the sense of Definition 2.

## 5.2 Case where $\omega^2 \in \sigma(A_p)$

Let us remind that each function  $k \mapsto \mu_n(k)$  (resp.  $k \mapsto \psi_n(\cdot; k)$ ) is analytic in  $D_n := \{|\operatorname{Im}(k)| \leq \alpha_n\}$  (see Section 3.2), let us introduce the positive number  $\alpha$  and the domain  $D_\alpha$ , that both depends on  $\omega$ ,

$$\alpha = \min\{\alpha_n, \quad n \in I(\omega)\} > 0 \quad \text{and} \quad D_\alpha := \{|\operatorname{Im}(k)| \leq \alpha\}.$$

For each  $k \in D_\alpha$ , let us introduce  $\mathbb{P}_\omega(k)$  and  $\mathbb{Q}_\omega(k)$  the bounded operators in  $L^2(\mathcal{C})$  defined by

$$\forall g \in L^2(\mathcal{C}), \quad \mathbb{P}_\omega(k)g = \sum_{n \in I(\omega)} (g, \psi_n(\cdot; k))_{n_p^2, \mathcal{C}} \psi_n(\cdot; k) \quad \text{and} \quad \mathbb{Q}_\omega(k) = \mathbb{I} - \mathbb{P}_\omega(k)$$

**Lemma 1** *The operators  $\mathbb{P}_\omega(k)$  and  $\mathbb{Q}_\omega(k)$  have the following properties*

1. for real  $k$ , they are orthogonal projectors in  $L^2(\mathcal{C})$ ;
2. they depend analytically on  $k$  in  $D_\alpha$ ;
3. they are  $2\pi$ -periodic in  $D_\alpha$ .

*Proof* Since the two first points are obvious, only the third point needs clarifications since the  $\psi_n(\cdot; k)$  are not necessarily periodic. Using the same notations as in the proof of Proposition 8, we observe that, for all  $k \in \mathbb{R}$

$$\forall g \in L^2(\mathcal{C}), \quad \mathbb{P}_\omega(k)g = \sum_{n \in \bar{I}(\omega)} (g, \varphi_n(\cdot; k))_{n_p^2, \mathcal{C}} \varphi_n(\cdot; k)$$

From the  $2\pi$ -periodicity of  $k \mapsto \varphi_n(\cdot; k)$  (see (16)), we deduce that  $\mathbb{P}_\omega(k) = \mathbb{P}_\omega(k + 2\pi)$  for any real  $k$ . Finally the function  $k \mapsto \mathbb{P}_\omega(k + 2\pi) - \mathbb{P}_\omega(k)$  is analytic in  $D_\alpha$  and vanishes in the real axis so it vanishes in  $D_\alpha$ .  $\square$

To study the asymptotic behaviour at infinity of  $u$ , we shall analyze the sequence of functions  $p \mapsto u(\cdot + (p, 0)) \in L^2(\mathcal{C})$ . We recall that  $u$  can be written as

$$\forall (x_1, x_s) \in \mathcal{C}, \quad \forall p \in \mathbb{Z}, \quad u(x_1 + p, x_s) = u_{\text{evan}}(x_1 + p, x_s) + u_{\text{prop}}(x_1 + p, x_s)$$

where  $u_{\text{evan}}$  (the evanescent part of the solution) and  $u_{\text{prop}}$  (the propagative part) are given by

$$u_{\text{evan}}(x_1 + p, x_s) = \sqrt{\frac{1}{2\pi}} \sum_{n \notin I(\omega)} \int_{-\pi}^{\pi} \frac{\mathbb{P}_n(k)(f)(x_1, x_s; k)}{\mu_n(k) - \omega^2} e^{ipk} dk \quad (40)$$

$$u_{\text{prop}}(x_1 + p, x_s) = \sqrt{\frac{1}{2\pi}} \sum_{n \in I(\omega)} \left[ p.v. \int_{-\pi}^{\pi} \frac{\mathbb{P}_n(f)(x_1, x_s; k)}{\mu_n(k) - \omega^2} e^{ipk} dk + i\pi \sum_{\xi \in \Xi_n(\omega)} \frac{\mathbb{P}_n(f)(x_1, x_s; \xi)}{|\mu'_n(\xi)|} e^{ip\xi} \right]. \quad (41)$$

The behaviour of the evanescent part can be studied using as in the case where  $\omega^2 \notin \sigma(A)$ , analyticity arguments and Proposition 5:

**Lemma 2** *The function  $u_{\text{evan}}$  is exponentially decaying in  $H^1(\Delta, \Omega)$ .*

*Proof* The idea is to check that the FBT of  $u_{\text{evan}}$ , denoted  $\hat{u}_{\text{evan}}$ , satisfies the assumptions of Proposition 5.

It is obvious from (40) and the formula (4) of the inverse FBT that

$$\hat{u}_{\text{evan}}(\cdot; k) = \sum_{n \notin I(\omega)} \frac{\mathbb{P}_n(f)(x_1, x_s; k)}{\mu_n(k) - \omega^2} \in \text{Im } \mathbb{Q}_\omega(k),$$

and satisfies the equation

$$(A_p(k) - \omega^2) \mathbb{Q}_\omega(k) \hat{u}_{\text{evan}}(\cdot; k) = \mathbb{Q}_\omega(k) \hat{f}(\cdot; k),$$

which is equivalent to

$$(A_p(k) \mathbb{Q}_\omega(k) - \omega^2) \hat{u}_{\text{evan}}(\cdot; k) = \mathbb{Q}_\omega(k) \hat{f}(\cdot; k).$$

Moreover, it is easy to see that the spectrum of the operator  $A_p(k) \mathbb{Q}_\omega(k)$  is purely discrete and

$$\sigma(A_p(k) \mathbb{Q}_\omega(k)) = \{0\} \cup \{\mu_n(k), n \notin I(\omega)\}.$$

We have then

$$\bigcup_{k \in ]-\pi, \pi[} \sigma(A_p(k) \mathbb{Q}_\omega(k)) = \{0\} \cup \sigma(A) \setminus J$$

where  $J$  is an open interval given by

$$J = \bigcup_{n \in I(\omega)} \mu_n(] - \pi, \pi[).$$

From Lemma 1, we know that  $k \mapsto A_p(k) \mathbb{Q}_\omega(k)$  is analytic in  $D_\omega$ . Moreover by definition of  $I(\omega)$ ,  $\omega^2$  belongs to  $J$ . Therefore, for any real  $k$ ,  $(A_p(k) \mathbb{Q}_\omega(k) - \omega^2)$  is invertible from  $H_k^1(\Delta, \Omega)$  into  $L^2(\mathcal{C})$  with bounded inverse  $(A_p(k) \mathbb{Q}_\omega(k) - \omega^2)^{-1}$  in  $L^2(\mathcal{C})$ . Using [22, Chapter VII, Theorem 1.3], the function  $k \mapsto (A_p(k) \mathbb{Q}_\omega(k) - \omega^2)^{-1}$  can be extended analytically in  $D_\beta = \{|\operatorname{Im}(k)| \leq \beta\}$  for some  $\beta \leq \alpha$ . Since  $k \mapsto \hat{f}(\cdot; k)$  is an analytic function in  $\mathbb{C}$  with values in  $L^2(\mathcal{C})$ , the function  $\hat{u}_{\text{evan}}(\cdot; k)$  can then be extended analytically for  $k \in D_\beta$  via the expression

$$\forall k \in D_\beta, \quad \hat{u}_{\text{evan}}(\cdot; k) = (A_p(k) \mathbb{Q}_\omega(k) - \omega^2)^{-1} \mathbb{Q}_\omega(k) \hat{f}(\cdot; k).$$

Finally, since  $A_p(k)$ ,  $\mathbb{Q}_\omega(k)$  and  $\hat{f}(\cdot; k)$  are  $2\pi$ -periodic,  $\hat{u}_{\text{evan}}(\cdot; k)$  is  $2\pi$ -periodic. One then concludes thanks to Proposition 5.  $\square$

We are now going to focus on the first term appearing in the expression (41) of  $u_{\text{prop}}$ , namely  $\forall (x_1, x_s) \in \mathcal{C}$ ,  $\forall p \in \mathbb{Z}$ ,

$$u_{\text{prop}}^*(x_1 + p, x_s) = \sqrt{\frac{1}{2\pi}} \sum_{n \in I(\omega)} p.v. \int_{-\pi}^{\pi} \frac{\mathbb{P}_n(f)(x_1, x_s; k)}{\mu_n(k) - \omega^2} e^{ipk} dk.$$

**Lemma 3**  $\forall (x_1, x_s) \in \mathcal{C}$ ,  $\forall p \in \mathbb{Z}$ ,

$$u_{\text{prop}}^*(x_1 + p, x_s) = \sqrt{\frac{1}{2\pi}} \left[ \pm i\pi \sum_{n \in I(\omega)} \sum_{\xi \in \Xi_n(\omega)} \frac{\mathbb{P}_n(f)(x_1, x_s; \xi)}{\mu'_n(\xi)} e^{ip\xi} \right] + w^\pm(x_1 + p, x_s)$$

where  $w^+$  (resp.  $w^-$ ) is exponentially decaying in  $H_{\text{loc}}^1(\Delta, \Omega)$  at  $+\infty$  (resp.  $-\infty$ ).

*Proof* By definition of the principal value,  $\forall (x_1, x_s) \in \mathcal{C}$ ,  $\forall p \in \mathbb{Z}$

$$\sqrt{2\pi} u_{\text{prop}}^*(x_1 + p, x_s) = \lim_{\delta \rightarrow 0} \sum_{n \in I(\omega)} \int_{\mathcal{I}_\omega^\delta} \frac{\mathbb{P}_n(f)(x_1, x_s; k)}{\mu_n(k) - \omega^2} e^{ipk} dk$$

where

$$\mathcal{I}_\omega^\delta = [-\pi, \pi] \setminus \bigcup_{\xi \in \Xi(\omega)} ]\xi - \delta, \xi + \delta[.$$

when  $k$  belongs to  $\mathcal{I}_\omega^\delta$ ,  $\omega^2$  belongs to the resolvent set of  $A(k)$  and we can write

$$\forall k \in \mathcal{I}_\omega^\delta, \quad \sum_{n \in I(\omega)} \frac{\mathbb{P}_n(f)(x_1, x_s; k)}{\mu_n(k) - \omega^2} = (A_p(k) - \omega^2)^{-1} \mathbb{P}_\omega(k) \hat{f}(\cdot; k). \quad (42)$$

Looking at the left hand side of (42) and using the analyticity property of  $k \mapsto \mu_n(k)$  and  $k \mapsto \psi_n(\cdot; k)$ , we see that the function  $k \mapsto \hat{F}_\omega(\cdot; k) := (A_p(k) - \omega^2)^{-1} \mathbb{P}_\omega(k) \hat{f}(\cdot; k)$  can be extended analytically in  $D_\beta \setminus \Xi(\omega)$ , for some  $\beta > 0$ . Moreover, this function is  $2\pi$ -periodic thanks to Lemma 1.



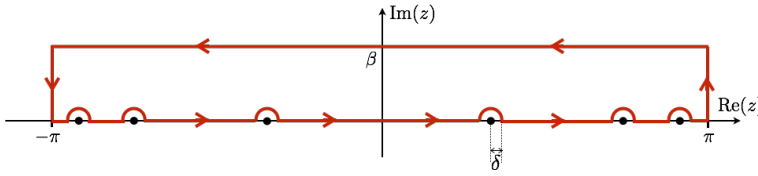
Now we want to analyze the behaviour when  $p$  tends to  $+\infty$ . For this, let us introduce a contour in the complex plane, included in  $D_\beta$  :

$$\Gamma_{\omega,\delta} = \mathcal{I}_\omega^\delta \cup S^\beta \cup S_-^\beta \cup S_+^\beta \cup \bigcup_{\xi \in \Xi(\omega)} \gamma_\delta(\xi)$$

where

$$\begin{cases} S^\beta = ]-\pi, \pi[ + i\beta & \text{horizontal segment} \\ S_\pm^\beta = \pm\pi + i]0, \beta[ & \text{vertical segment} \\ \gamma_\delta(\xi) = \{\xi + \delta e^{i\theta}, \theta \in ]0, \pi[ \} & \text{semi circle of centre } \xi \text{ and radius } \eta \end{cases}$$

are oriented according to the Figure 6. Writing that by Cauchy formula



**Fig. 6** Oriented contour  $\Gamma_{\omega,\delta}$

$$\int_{\Gamma_{\omega,\delta}} \hat{F}_\omega(\cdot; z) e^{ipz} dz = 0$$

and by  $2\pi$ -periodicity that

$$\int_{S_+^\beta \cup S_-^\beta} \hat{F}_\omega(\cdot; k) e^{ipz} dz = 0$$

we obtain

$$\begin{aligned} \sqrt{2\pi} u_{\text{prop}}^*(x_1 + p, x_s) &= e^{-p\beta} \sum_{n \in I(\omega)} \int_{-\pi}^{\pi} \frac{\mathbb{P}_n(f)(x_1, x_s; k + i\beta)}{\mu_n(k + i\beta) - \omega^2} e^{ipk} dk \\ &+ \lim_{\delta \rightarrow 0} \sum_{n \in I(\omega)} \sum_{\xi \in \Xi_n(\omega)} \int_{\gamma_\delta(\xi)} \frac{\mathbb{P}_n(f)(x_1, x_s; z)}{\mu_n(z) - \omega^2} e^{ipz} dz \end{aligned}$$

For the last term, we use a well-known generalization of the Residue Theorem : for any function  $g$  meromorphic in the neighborhood of  $\xi$ , which is supposed to be a simple pole, we have

$$\lim_{\eta \rightarrow 0} \int_{\gamma_\eta(\xi)} g(z) dz = i\pi \lim_{z \rightarrow \xi} (z - \xi) g(z) \quad (43)$$

Since  $\omega \notin \sigma_0$ , for all  $\xi \in \Xi_n(\omega)$ ,  $\mu'_n(\xi) \neq 0$  so  $\xi$  is a simple pole of

$$\frac{\mathbb{P}_n(f)(x_1, x_s; z)}{\mu_n(z) - \omega^2} e^{ipz},$$

and we deduce that

$$\forall \xi \in \Xi_n(\omega), \quad \lim_{\delta \rightarrow 0} \int_{\gamma_\delta(\xi)} \frac{\mathbb{P}_n(f)(x_1, x_s; z)}{\mu_n(z) - \omega^2} e^{ipz} dz = i\pi \frac{\mathbb{P}_n(f)(x_1, x_s; \xi)}{\mu'_n(\xi)} e^{ip\xi}$$

So finally for all  $p \in \mathbb{N}$ , we have found  $\forall (x_1, x_s) \in \mathcal{C}$ ,  $\forall p \in \mathbb{N}$ ,

$$u_{\text{prop}}^*(x_1 + p, x_s) = \sqrt{\frac{1}{2\pi}} \left[ i\pi \sum_{n \in I(\omega)} \sum_{\xi \in \Xi_n(\omega)} \frac{\mathbb{P}_n(f)(x_1, x_s; \xi)}{\mu'_n(\xi)} e^{ip\xi} \right] + w^+(x_1 + p, x_s)$$

where

$$w^+(x_1 + p, x_s) = \sqrt{\frac{1}{2\pi}} e^{-p\beta} \sum_{n \in I(\omega)} \int_{-\pi}^{\pi} \frac{\mathbb{P}_n(f)(x_1, x_s; k + i\beta)}{\mu_n(k + i\beta) - \omega^2} e^{ipk} dk$$

is easily shown to satisfy (the details are omitted)

$$\forall p \in \mathbb{N}, \quad \|w^+\|_{H^1(\Delta, \mathcal{C}_p)} \leq C(\omega) e^{-\beta p} \|f\|_{L^2}.$$

This completes the proof for  $u_{\text{prop}}^*(x_1 + p, x_s)$ . The proof for  $u_{\text{prop}}^*(x_1 - p, x_s)$  is obtained by changing the contour  $\Gamma_{\omega, \eta}$  by its symmetric with respect to the real axis, which in particular changes  $i\pi$  into  $-i\pi$  in the formula (43).  $\square$

Thanks to Lemmas 2 and 3, we deduce the asymptotic behaviour at  $\pm\infty$  of the solution of the Helmholtz equation (1) obtained by limiting absorption principle and given by (37-38).

**Theorem 3 (Asymptotic behaviour of the solution at  $\pm\infty$ )** *For all  $(x_1, x_s) \in \mathcal{C}$  and for all  $p \in \mathbb{Z}$*

$$u(x_1 + p, x_s) = i\sqrt{2\pi} \sum_{n \in I(\omega)} \sum_{\xi \in \Xi_n^\pm(\omega)} \frac{\left( \hat{f}(\cdot; \xi), \psi_n(\cdot; \xi) \right)_{L^2(\mathcal{C})}}{|\mu'_n(\xi)|} \psi_n(x_1, x_s; \xi) e^{ip\xi} + w^\pm(x_1 + p, x_s) \quad (44)$$

where  $w^+$  (resp.  $w^-$ ) is exponentially decaying in  $H_{loc}^1(\Delta, \Omega)$  at  $+\infty$  (resp.  $-\infty$ ).

*Proof* Thanks to Lemmas 2 and 3, the solution of the Helmholtz equation (1) obtained by the limiting absorption principle and given by (37-38) can be written as

$$u(x_1 + p, x_s) \underset{p \rightarrow \pm\infty}{\sim} \sqrt{\frac{1}{2\pi}} \sum_{n \in I(\omega)} \left[ \imath\pi \operatorname{sign}(p) \sum_{\xi \in \Xi_n(\omega)} \frac{\mathbb{P}_n(f)(x_1, x_s; \xi)}{\mu'_n(\xi)} e^{\imath p \xi} + \imath\pi \sum_{\xi \in \Xi_n(\omega)} \frac{\mathbb{P}_n(f)(x_1, x_s; \xi)}{|\mu'_n(\xi)|} e^{\imath p \xi} \right]$$

where  $u$  tends to the finite sum exponentially fast. When  $p$  tends to  $+\infty$ , it remains only the  $\xi \in \Xi_n(\omega)$  such that  $\mu'_n(\xi) > 0$  and when  $p$  tends to  $-\infty$ , it remains only the  $\xi \in \Xi_n(\omega)$  such that  $\mu'_n(\xi) < 0$ .  $\square$

*Remark 9* Suppose that  $\omega \notin \sigma_0$ , for all  $(x_1, x_s) \in \mathcal{C}$ ,  $(y_1, y_s) \in \mathcal{C}$ ,  $p, q \in \mathbb{Z}$

$$G(x_1 + p, x_s; y_1 + q, y_s) = \imath \sum_{n \in I(\omega)} \sum_{\xi \in \Xi_n^\pm(\omega)} \frac{\psi_n(x_1, x_s; \xi) \overline{\psi_n(y_1, y_s; \xi)}}{|\mu'_n(\xi)|} e^{\imath(p-q)\xi} + w^\pm(x_1 + p, x_s) \quad (45)$$

where  $w^+$  (resp.  $w^-$ ) is exponentially decaying in  $H_{\text{loc}}^1(\Delta, \Omega)$  at  $+\infty$  (resp.  $-\infty$ ).

The solution  $u$  behaves when  $x_1$  tends to  $+\infty$  (resp. to  $-\infty$ ) as a linear combination of the Floquet modes  $\psi_n(x, \xi)$ ,  $n \in I(\omega)$ ,  $\xi \in \Xi_n^+(\omega)$  (resp.  $\psi_n(x, \xi)$ ,  $n \in I(\omega)$ ,  $\xi \in \Xi_n^-(\omega)$ ) which propagate to the right (resp. to the left) as  $\mu'_n(\xi) > 0$  (resp.  $\mu'_n(\xi) = \omega^2$  and  $\mu'_n(\xi) < 0$ ). Similarly the solution  $u$  behaves when  $x_1$  tends to  $-\infty$  as a linear combination of the Floquet modes  $\psi_n(x, \xi)$ ,  $n \in I(\omega)$ ,  $\xi \in \Xi_n^-(\omega)$  which propagate to the left as  $\mu'_n(\xi) < 0$ .

For the sequel and in order to simplify the expression of our result, let us introduce a renumbering of the set  $\Xi^+(\omega)$  defined in (20)

$$\Xi^+(\omega) = \bigcup_{n \in I(\omega)} \Xi_n^+(\omega) = \{\xi_1^+ \leq \dots \leq \xi_\ell^+ \leq \dots \leq \xi_{N(\omega)}^+\},$$

where  $N(\omega) = \#\Xi^+(\omega)$  is the number of Floquet modes propagating to the right. Thanks to Proposition 10, we have

$$\Xi^-(\omega) = \bigcup_{n \in I(\omega)} \Xi_n^-(\omega) = \{-\xi_1^+ \geq \dots \geq -\xi_\ell^+ \geq \dots \geq -\xi_{N(\omega)}^+\}$$

We renumber accordingly the set  $\{\psi_n(\cdot; \xi), \xi \in \Xi_n^+(\omega), n \in I(\omega)\}$  and renormalize them for convenience of notation. More precisely we introduce  $\phi_1, \dots, \phi_{N(\omega)}$  respectively  $\phi_{-1}, \dots, \phi_{-N(\omega)}$  defined as

$$\forall n \in I(\omega), \forall \xi \in \Xi_n^\pm(\omega), \quad \xi = \pm \xi_m^+ \Rightarrow \phi_{\pm m} = \frac{\psi_n(\cdot; \xi)}{\sqrt{|\mu'_n(\xi)|}} \quad (46)$$

Then the asymptotic behaviour of Theorem 3 becomes for all  $(x_1, x_s) \in \mathcal{C}$  and for all  $p \in \mathbb{Z}$

$$u(x_1+p, x_s) = i\sqrt{2\pi} \sum_{m=1}^{N(\omega)} \left( \hat{f}(\cdot; \pm\xi_m^\pm), \phi_{\pm m} \right)_{\mathcal{C}} \phi_{\pm m}(x_1, x_s) e^{\pm ip\xi_m^\pm} + w^\pm(x_1+p, x_s)$$

where  $w^+$  (resp.  $w^-$ ) is exponentially decaying in  $H_{\text{loc}}^1(\Delta, \Omega)$  at  $+\infty$  (resp.  $-\infty$ ).

*Remark 10* In the Appendix B, we compute the far field of the fundamental solution in the special case of the straight homogeneous waveguide, that is  $n_p = 1$  in  $\Omega = \mathbb{S} \times \mathbb{R}$ . In particular, we retrieve the expression obtained by using directly the well-known fundamental solution for the homogeneous waveguide.

## 6 Radiation condition and uniqueness of the solution

Thanks to the main result of the last section (see Theorem 3), we can define a radiation condition and establish the well-posedness of the Helmholtz equation set in a periodic waveguide.

**Definition 4 (The outgoing radiation condition)** We say that  $u$  satisfies the outgoing radiation condition if and only if there exist  $(u_n^\pm)_{n \in \llbracket 1, N(\omega) \rrbracket}$  such that

$$\forall (x_1, x_s) \in \mathcal{C}, \quad u(x_1+p, x_s) = \sum_{m=1}^{N(\omega)} u_m^\pm \phi_{\pm m}(x_1, x_s) e^{\pm ip\xi_m^\pm} + w^\pm(x_1+p, x_s) \quad (47)$$

where  $w^+$  (resp.  $w^-$ ) is exponentially decaying in  $H_{\text{loc}}^1(\Delta, \Omega)$  at  $+\infty$  (resp.  $-\infty$ ).

**Theorem 4 (Well-posedness of the problem)** Suppose  $\omega^2 \notin \sigma_0$ . It exists a unique solution of problem (1) which satisfies the outgoing radiation condition.

*Proof* Thanks to theorems 2 and 3, we have shown that the function given by (37-38) is a solution to problem (1) satisfying the outgoing radiation condition. The difficulty is to show the uniqueness of such a solution. Let  $u$  be a solution of

$$\begin{cases} -\Delta u - n_p^2 \omega^2 u = 0 & \text{in } \Omega \\ \nabla u \cdot \mathbf{n} = 0 & \text{on } \partial\Omega \end{cases} \quad (48)$$

which satisfies the outgoing radiation condition, i.e. it exists  $(u_n^\pm)_{n \in \llbracket 1, N(\omega) \rrbracket}$  such that  $\forall (x_1, x_s) \in \mathcal{C}, \forall p \in \mathbb{Z}$ ,

$$u(x_1+p, x_s) = \sum_{m=1}^{N(\omega)} u_m^\pm \phi_{\pm m}(x_1, x_s) e^{\pm ip\xi_m^\pm} + w^\pm(x_1+p, x_s) \quad (RC^\pm)$$

where  $w^+$  (resp.  $w^-$ ) is exponentially decaying in  $H_{\text{loc}}^1(\Delta, \Omega)$  at  $+\infty$  (resp.  $-\infty$ ). Let us show that  $u = 0$ .

We first notice that  $u \in \mathcal{V}(\omega)$  defined in (25). By using the property (28) for  $x = -N$  and  $x = N$  for some  $N \in \mathbb{N}^*$

$$q(-N, u, u) = q(N, u, u). \quad (49)$$

We use now the formula  $(RC^\pm)$  on  $\Gamma_{\pm N} = \{x_s, (\pm N, x_s) \in \Omega\}$

$$\begin{aligned} q(\pm N, u, u) &= \sum_{n,m=1}^{N(\omega)} u_n^\pm \bar{u}_m^\pm q(\phi_{\pm n}, \phi_{\pm m}) \\ &+ \sum_{n=1}^{N(\omega)} \left[ u_n^\pm q(\pm N, \phi_{\pm n}, w^\pm) + \bar{u}_n^\pm q(\pm N, w^\pm, \phi_{\pm n}) \right] + q(\pm N, w^\pm, w^\pm) \\ &= \pm \iota \sum_n^{N(\omega)} |u_n^\pm|^2 \\ &+ \sum_{n=1}^{N(\omega)} \left[ u_n^\pm q(\pm N, \phi_{\pm n}, w^\pm) + \bar{u}_n^\pm q(\pm N, w^\pm, \phi_{\pm n}) \right] + q(\pm N, w^\pm, w^\pm) \end{aligned}$$

thanks to Proposition 9 and Definition (46). Using the exponential decay property of  $w^\pm$ , we know in particular that

$$\|w^\pm\|_{L^2(\Gamma_{\pm N})} + \left\| \frac{\partial w^\pm}{\partial x_1} \right\|_{L^2(\Gamma_{\pm N})} \leq C e^{-\alpha N}$$

for some  $\alpha > 0$  while by quasi-periodicity it is clear that

$$\forall n \in \llbracket 1, N(\omega) \rrbracket, \quad \|\phi_{\pm n}\|_{L^2(\Gamma_{\pm N})} + \left\| \frac{\partial \phi_{\pm n}}{\partial x_1} \right\|_{L^2(\Gamma_{\pm N})} \leq C$$

We deduce that

$$\lim_{N \rightarrow +\infty} q(\pm N, u, u) = \pm \iota \sum_n^{N(\omega)} |u_n^\pm|^2$$

Taking the limit when  $N$  tends to  $+\infty$  in the equality (49), we conclude that

$$\forall n \in \llbracket 1, N(\omega) \rrbracket, \quad u_n^\pm = 0.$$

The function  $u$  is then a solution of (48) which is exponentially decaying and thus belongs to the domain of the operator  $A$ . The spectrum of  $A$  contains no eigenvalue according to [27, 40], in consequence  $u = 0$ .  $\square$

*Remark 11* The reader will note that the existence and uniqueness result of Theorem 4 remains valid under the weaker notion of radiation condition where in the Definition 4 the exponential decay property of  $w^\pm$  is simply replaced by

$$\lim_{N \rightarrow +\infty} \left[ \|w^\pm\|_{L^2(\Gamma_{\pm N})} + \left\| \frac{\partial w^\pm}{\partial x_1} \right\|_{L^2(\Gamma_{\pm N})} \right] = 0$$

The exponential decay of  $w^\pm$  appears then as an additional property of the outgoing solution.

## A Proof of Theorem 2

Let us write  $u_\varepsilon = u_{\text{evan}}^\varepsilon + u_{\text{prop}}^\varepsilon$  where

$$u_{\text{evan}}^\varepsilon = \sum_{n \notin I(\omega)} u_\varepsilon^n \quad \text{and} \quad u_{\text{prop}}^\varepsilon = \sum_{n \in I(\omega)} u_\varepsilon^n.$$

We show separately that  $u_{\text{evan}}^\varepsilon$  tends to  $u_{\text{evan}}$  in  $H^1(\Omega)$  (and implicitly that  $u_{\text{evan}}$  is in  $H^1(\Omega)$ ) and that  $u_{\text{prop}}^\varepsilon$  tends to  $u_{\text{prop}}$  in  $H_{\text{loc}}^1(\Omega)$  when  $\varepsilon$  goes to 0 where  $u_{\text{evan}}$  and  $u_{\text{prop}}$  are defined in (37-38).

**Step 1:**  $\|u_{\text{evan}}^\varepsilon - u_{\text{evan}}\|_{H^1(\Omega)} \leq C(\omega)\varepsilon \|f\|_{L^2(\Omega)}$

We set

$$\forall k \in [-\pi, \pi], \quad d_\varepsilon^n(k) = (\mu_n(k) - (\omega^2 + i\varepsilon))^{-1} - (\mu_n(k) - \omega^2)^{-1}$$

that satisfies

$$\forall k \in [-\pi, \pi], \quad |d_\varepsilon^n(k)| \leq \frac{\varepsilon}{|\mu_n(k) - \omega^2|^2}$$

and by definition of the set  $I(\omega)$ ,

$$\inf_{n \notin I(\omega)} \inf_{k \in [-\pi, \pi]} |\mu_n(k) - \omega^2| > 0$$

from which we deduce that

$$\forall k \in [-\pi, \pi], \quad |d_\varepsilon^n(k)| \leq C_0(\omega)\varepsilon.$$

Using Proposition 1, we have

$$\|u_{\text{evan}}^\varepsilon - u_{\text{evan}}\|_{n_p^2, \Omega}^2 = \int_{-\pi}^{\pi} \|\hat{u}_{\text{evan}}^\varepsilon(\cdot; k) - \hat{u}_{\text{evan}}(\cdot; k)\|_{n_p^2, \mathcal{C}}^2 dk$$

where each  $k \in [-\pi, \pi]$ ,  $\hat{u}_{\text{evan}}^\varepsilon(\cdot; k)$  and  $\hat{u}_{\text{evan}}(\cdot; k)$  are given by

$$\hat{u}_{\text{evan}}^\varepsilon(\cdot; k) = \sum_{n \notin I(\omega)} \frac{\mathbb{P}_n(f)(\cdot, k)}{\mu_n(k) - (\omega^2 + i\varepsilon)}, \quad \hat{u}_{\text{evan}}(\cdot; k) = \sum_{n \notin I(\omega)} \frac{\mathbb{P}_n(f)(\cdot, k)}{\mu_n(k) - \omega^2}$$

From the orthogonality in  $L^2(\mathcal{C}, n_p^2 dx)$  of the functions  $\mathbb{P}_n(f)(\cdot, k)$ , we deduce that

$$\|\hat{u}_{\text{evan}}^\varepsilon(\cdot; k) - \hat{u}_{\text{evan}}(\cdot; k)\|_{n_p^2, \mathcal{C}}^2 = \sum_{n \notin I(\omega)} |d_\varepsilon^n(k)|^2 \left| \langle \hat{f}(\cdot; k), \psi_n(\cdot; k) \rangle_{n_p^2, \mathcal{C}} \right|^2 \leq C_0(\omega)^2 \varepsilon^2 \|f(\cdot; k)\|_{n_p^2, \mathcal{C}}^2$$

that is to say

$$\|u_{\text{evan}}^\varepsilon - u_{\text{evan}}\|_{n_p^2, \Omega} \leq C_0(\omega)\varepsilon \|f\|_{n_p^2, \Omega}$$

In the same way,

$$\|\nabla u_{\text{evan}}^\varepsilon - \nabla u_{\text{evan}}\|_\Omega^2 = \int_{-\pi}^{\pi} \|\nabla \hat{u}_{\text{evan}}^\varepsilon(\cdot; k) - \nabla \hat{u}_{\text{evan}}(\cdot; k)\|_{\mathcal{C}}^2 dk$$

and

$$\|\nabla \hat{u}_{\text{evan}}^\varepsilon(\cdot; k) - \nabla \hat{u}_{\text{evan}}(\cdot; k)\|_{\mathcal{C}}^2 = \sum_{n \notin I(\omega)} |\mu_n(k)| |d_\varepsilon^n(k)|^2 \left| (\hat{f}(\cdot; k), \psi_n(\cdot; k))_{n_p^2, \mathcal{C}} \right|^2$$

One easily concludes after having noticed that

$$\forall k \in [-\pi, \pi], \quad |\mu_n(k)| |d_\varepsilon^n(k)|^2 \leq C_1(\omega)^2 \varepsilon^2.$$

**Step 2:** For all  $n \in I(\omega)$ , for all  $p \in \mathbb{Z}$  and for all  $\eta \in (0, 1)$

$$\|u_n^\varepsilon - u_n\|_{H^1(C_p)} \leq C_\eta(\omega) \varepsilon^{1-\eta} \|f\|_{L^2(\Omega)} \quad (50)$$

To prove the above equality, we shall use the following technical lemma which is obtained by combining Sokhotski-Plemelj theorem [36,41] and the Sobolev continuous embedding  $H^r(-a, b) \hookrightarrow C^{0, r-1/2}(-a, b)$  for  $r \in ]1/2, 3/2[$  [1].

**Lemma 4** *Let  $a, b > 0$ ,  $r \in ]1/2, 3/2[$  and  $X$  a Banach space. For any  $\psi \in H^r(-a, b[, X)$ , we define for any  $\varepsilon > 0$*

$$S_\varepsilon \psi = \int_{-a}^b \frac{\psi(t)}{t - i\varepsilon} dt \in X \quad \text{and} \quad S \psi = p.v. \int_{-a}^b \frac{\psi(t)}{t} dt + \psi(0) \in X$$

then for  $\varepsilon$  small enough, it exists a constant  $C_r$  such that

$$\forall \psi \in H^r(-a, b[, X), \quad \|S_\varepsilon \psi - S \psi\|_X \leq C_r \varepsilon^{r-1/2} \|\psi\|_{H^r(-a, b[, X)}$$

Since  $\omega^2 \notin \sigma_0$ ,  $\mu'_n(\xi) \neq 0$  for all  $\xi \in \Xi_n(\omega)$ . Therefore one can fix a  $\delta > 0$  small enough such that  $\mu_n(k)$  is monotonous in each of the (disjoint) intervals  $]\xi - \delta, \xi + \delta[$ ,  $\xi \in \Xi_n(\omega)$ . Let us set

$$\mathcal{I}_{\omega, n}^\delta = ]-\pi, \pi[ \setminus \bigcup_{\xi \in \Xi_n(\omega)} ]\xi - \delta, \xi + \delta[$$

With this notation, we have  $u_n^\varepsilon = u_n^{\varepsilon, \delta} + \sum_{\xi \in \Xi_n(\omega)} v_n^{\varepsilon, \delta}(\xi; \cdot)$  and  $u_n = u_n^\delta + \sum_{\xi \in \Xi_n(\omega)} v_n^\delta(\xi; \cdot)$

where

$$\begin{aligned} \sqrt{2\pi} u_n^{\varepsilon, \delta}(x_1 + p, x_s) &= \int_{\mathcal{I}_{\omega, n}^\delta} \frac{\mathbb{P}_n(f)(x_1, x_s; k)}{\mu_n(k) - (\omega^2 + i\varepsilon)} e^{ipk} dk \\ \sqrt{2\pi} u_n^\delta(x_1 + p, x_s) &= \int_{\mathcal{I}_{\omega, n}^\delta} \frac{\mathbb{P}_n(f)(x_1, x_s; k)}{\mu_n(k) - \omega^2} e^{ipk} dk \\ \sqrt{2\pi} v_n^{\varepsilon, \delta}(\xi; x_1 + p, x_s) &= \int_{\xi - \delta}^{\xi + \delta} \frac{\mathbb{P}_n(f)(x_1, x_s; k)}{\mu_n(k) - (\omega^2 + i\varepsilon\omega)} e^{ipk} dk \\ \sqrt{2\pi} v_n^\delta(\xi; x_1 + p, x_s) &= p.v. \int_{\xi - \delta}^{\xi + \delta} \frac{\mathbb{P}_n(f)(x_1, x_s; k)}{\mu_n(k) - \omega^2} e^{ipk} dk + i\pi \frac{\mathbb{P}_n(f)(x_1, x_s; \xi)}{|\mu'_n(\xi)|} \end{aligned}$$

Proceeding along the same lines than in Step 1, it is immediate to prove that

$$\left\| u_n^{\varepsilon, \delta} - u_n^\delta \right\|_{H^1(\Omega)} \leq C(\omega, \delta) \varepsilon \|f\|_{L^2(\Omega)}.$$

It remains to estimate  $v_n^{\varepsilon, \delta}(\xi; \cdot) - v_n^\delta(\xi; \cdot)$  for each  $\xi \in \Xi_n(\omega)$ . Thanks to our choice of  $\delta$  and since  $\mu_n(\xi) = \omega^2$ , the function  $k \mapsto \mu_n(k)$  is invertible from  $]\xi - \delta, \xi + \delta[$  onto

$] - a_n(\xi) + \omega^2, b_n(\xi) + \omega^2[$  with  $a_n(\xi), b_n(\xi) > 0$ . We perform the change of variable  $k = \nu_n(t + \omega^2)$  where  $\nu_n = \mu_n^{-1}$  and we obtain

$$\sqrt{2\pi} v_n^{\varepsilon, \delta}(\xi; \cdot) = \int_{-a_n(\xi)}^{b_n(\xi)} \frac{\phi_n(\cdot; t)}{t - i\varepsilon} dt$$

where

$$\forall (x_1, x_s) \in \mathcal{C}, \forall p \in \mathbb{Z}, \quad \phi_n(x_1 + p, x_s; t) = \mathbb{P}_n(f)(x_1, x_s; \nu_n(t + \omega^2)) \frac{e^{ip\nu_n(t + \omega^2)}}{|\mu_n'(\nu_n(t + \omega^2))|}$$

In the same way, we can write

$$\sqrt{2\pi} v_n^{\varepsilon, \delta}(\xi; \cdot) = \text{p.v.} \int_{-a_n(\xi)}^{b_n(\xi)} \frac{\phi_n(\cdot; t)}{t} dt + v\pi \psi_n(\cdot; 0).$$

To conclude, we wish to apply the lemma 4 with  $X = H^1(\mathcal{C})$ . Then it suffices to remark that

$$\phi_n \in H^r([-a_n(\xi), b_n(\xi)], H^1(\mathcal{C})) \quad \text{for any } r > 0.$$

Indeed, since  $f$  is of compact support, we know that

$$\|f\|_{L_r^2(\Omega)}^2 = \int_{\Omega} |f(x_1, x_s)|^2 (1 + x_1^2)^r dx_1 dx_s < +\infty \quad \text{for any } r > 0$$

so using the property of the Floquet Bloch Transform [23, 13], we show that

$$\hat{f} \in H^r([- \pi, \pi], L^2(\mathcal{C})) \quad \text{for any } r > 0.$$

More precisely, exploiting the  $C^\infty$  regularity of  $\nu_n(t + \omega^2)$ ,  $\mu_n'(t)$  and  $\phi_n(\cdot; \nu_n(t + \omega^2))$ , it is not difficult (the details are left to the reader) to show that

$$\exists C(\omega), \quad \|\psi_n\|_{H^r([-a_n(\xi), b_n(\xi)], H^1(\mathcal{C}))} \leq C(\omega) \|f\|_{L_r^2(\Omega)}$$

where

$$\|f\|_{L_r^2(\Omega)} \leq \|f\|_{L^2(\Omega)}$$

because  $f$  is of compact support.

*Remark 12* Let us remark, this is classical in scattering theory, that the result of the theorem does not require that  $f$  has compact support but simply that  $f$  belongs to a  $L_r^2$  for  $r > 1/2$ . If  $r \geq 3/2$ , then the result (50) holds and if  $1/2 < r < 3/2$ , one has to replace  $1 - \eta$  in the estimation (50) by  $r - 1/2$ .

## B The homogeneous waveguide as a particular instance of periodic waveguides

The homogeneous waveguide is obviously a particular case of periodic waveguides for which  $n_p = 1$  in  $\Omega = \mathbb{S} \times \mathbb{R}$ .

Firstly, let us see what are the propagating Floquet modes in that case. For this purpose we introduce the so-called guided modes, which appear when we investigate the solutions of

$$\begin{cases} -\Delta u - \omega^2 u = 0 & \text{in } \Omega \\ \nabla u \cdot \mathbf{n} = 0 & \text{on } \partial\Omega. \end{cases} \quad (51)$$



Indeed, these solutions are the linear combinations of the guided modes given by

$$u_m^\pm(x_1, x_2) = \theta_m(x_s) e^{\pm i\beta_m x_1}, \quad m \geq 1, \quad (52)$$

with

$$\beta_m = \sqrt{\omega^2 - \alpha_m}, \quad \operatorname{Re}(\beta_m), \operatorname{Im}(\beta_m) \geq 0, \quad (53)$$

where  $0 = \alpha_1 \leq \dots \leq \alpha_m \rightarrow +\infty$  are the eigenvalues of the tranverse laplacian  $-\Delta_{\mathbb{S}}$  with Neumann boundary conditions on  $\partial\mathbb{S}$  and  $\theta_1, \dots, \theta_m, \dots$  are associated eigenvectors (forming an orthonormal basis of  $L^2(\mathbb{S})$ ). For example, if  $\mathbb{S} = (0, 1)$

$$\alpha_m = (m-1)^2 \pi^2 \quad (54)$$

and

$$\begin{cases} \theta_1(x_s) = 1 \\ \theta_m(x_s) = \sqrt{2} \cos((m-1)\pi x_s) \quad (m \geq 2). \end{cases} \quad (55)$$

Among the guided modes, the propagating guided modes correspond to the purely real  $\beta_m$ . Because the sequence  $(\alpha_m)_m$  tends to  $+\infty$ , there is only a finite number of propagative modes - for example for  $\mathbb{S} = (0, 1)$ , there are only  $2P$  propagative modes with  $P := C(k/\pi)$ , where  $C$  is the ceiling function. The other guided modes are either increasing or decreasing exponentially with variable  $x_1$ . Let us verify that the two notions, propagating Floquet modes in the homogeneous case and propagating guided modes, do coincide. It is easy to show that the eigenvalues and eigenfunctions of operator  $A_p(k)$  for  $n_p = 1$  are given, for all  $m \geq 1$  and  $l \in \mathbb{Z}$ , by

$$\mu_{m,l}(k) = \alpha_m + (k + 2\pi l)^2, \quad \psi_{m,l}(x; k) = \theta_m(x_s) e^{i(k+2\pi l)x_1}. \quad (56)$$

Hence the  $\lambda_n(k)$  coincide with the  $\mu_{m,l}(k)$ , up to a change of numbering. The propagating Floquet modes are then given, for  $x \in \Omega$ , by

$$u_{m,l}(x; \xi) = \theta_m(x_s) e^{i(\xi+2\pi l)x_1}, \quad (57)$$

for all  $\xi \in (-\pi, \pi]$ ,  $m \geq 1$  and  $l \in \mathbb{Z}$  such that

$$\alpha_m + (\xi + 2\pi l)^2 = \omega^2. \quad (58)$$

Let us consider a propagating Floquet mode  $u_{m,l}(\cdot, \xi)$ . Since (58) is satisfied, then

$$\xi + 2\pi l = \pm \sqrt{\omega^2 - \alpha_m} = \pm \beta_m$$

and therefore  $u_{m,l} = u_m^\pm$  given by (52), that is  $u_{m,l}$  is a propagating guided mode. Conversely, let us consider a propagating guided mode  $u_m^\pm$ . There is a unique pair  $(\xi, l)$  with  $\xi \in (-\pi, \pi]$  and  $l \in \mathbb{Z}$  such that  $\pm\beta_m = \xi + 2\pi l$  (we denote  $\xi = \pm\beta_m \bmod 2\pi$ ). So that  $u_m^\pm$  coincide with the propagating Floquet mode  $u_{m,l}(\cdot, \xi)$  given by (57). This completes the verification.

For some index  $n \in I(\omega)$  and  $\xi \in \Xi_n(\omega)$ , we associate the unique corresponding pair  $(m, l)$  such that the propagating Floquet mode  $u_n(\cdot; \xi) = u_{m,l}(\cdot; \xi)$  coincides with a guided mode  $u_m^+$  or  $u_m^-$ . We have then, for  $n \in I(\omega)$ ,  $\Xi_n(\omega) = \{\pm\beta_m \bmod 2\pi\}$  and by using the expression of  $\mu_{m,l}$  in (56) we obtain that for  $\xi \in \Xi_n(\omega)$ ,  $\lambda_n(\xi) = \pm 2\beta_m$ .

Besides the set  $\sigma_0$  is simply given by

$$\sigma_0 = \{\sqrt{\alpha_m}, m \geq 1\}.$$

*Remark 13* One particular property of periodic waveguides is that there may have band gaps in the spectrum, that is intervals of  $\omega$  for which no propagating Floquet mode exist, in other words  $I(\omega) = \emptyset$ . More generally, the number of propagating Floquet modes does not necessarily increase with  $\omega$ . This is in contrast with the case of the homogeneous waveguide with Neumann boundary conditions, for which at least two propagating modes  $u_1^\pm$  exist whatever  $\omega$  is, and the number of propagating modes increases with  $\omega$ .

Secondly, let us see what becomes the expression (45) of the far field of the fundamental solution when the waveguide is homogeneous. By using the computations above we obtain that such far field is given by

$$G(x_1, x_s; y_1, y_s) \underset{x_1 \rightarrow \pm\infty}{\sim} i \sum_{m=1}^P \frac{u_m^\pm(x_1, x_s) u_m^\mp(y_1, y_s)}{2\beta_m},$$

where the guided modes  $u_m^\pm$  are given by (52) and the  $\beta_m$  are given by (53), while  $P$  is the number of propagating modes in each direction of propagation. Fortunately, we retrieve such expression by considering directly the far field of the fundamental solution of the homogeneous waveguide, that is:

$$G(x, y) = i \sum_{m=1}^{+\infty} \frac{e^{i\beta_m|x_1-y_1|}}{2\beta_m} \theta_m(x_s) \theta_m(y_s),$$

## C Conclusion and Ongoing works

In this paper, we gave a somewhat complete theory for time harmonic wave propagation in a periodic waveguide. Of course, this problem is quite academic but it opens the gate to the analysis of more realistic (from the point of view of applications) situations. The first and simplest extension is a case of a local perturbation of a periodic waveguide. Using compact perturbation theory, it should be straightforward to show, using the same radiation condition as in this paper, an existence and uniqueness result except maybe for a countable set of frequencies located in the resolvent set of the unperturbed periodic operator and that can accumulate only at the edges of the essential spectrum. Another extension, which is slightly similar than the previous is the junction between two different periodic waveguides. A more intricate extension is the junction between an homogeneous and a periodic halfspaces. This last situation seems really interesting for the application in photonics (see for instance [35, 4, 3]).

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