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Thomas Cluzeau*, Alban Quadrat†

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Abstract: The purpose of this paper is to study the connections existing between *Serre's reduction* of linear functional systems – which aims at finding an equivalent system defined by fewer equations and fewer unknowns – and the *decomposition problem* – which aims at finding an equivalent system having a diagonal block structure – in which one of the diagonal blocks is assumed to be the identity matrix. In order to do that, we further develop results on Serre's reduction problem and on the decomposition problem obtained in Boudellioua and Quadrat (2010); Cluzeau and Quadrat (2008). Finally, we show how these techniques can be used to analyze the decomposability problem of standard linear systems of partial differential equations studied in hydrodynamics such as Stokes equations, Oseen equations and the movement of an incompressible fluid rotating with a small velocity around the vertical axis.

Key-words: Mathematical systems theory, linear functional systems, module theory, Serre's reduction, the factorization problem, the decomposition problem, applications to hydrodynamics.

* Université de Limoges ; CNRS ; XLIM UMR 7252, DMI, 123 avenue Albert Thomas, 87060 Limoges Cedex, France. E-mail: thomas.cluzeau@unilim.fr.

† INRIA Saclay -Île-de-France, DISCO project, Supélec, L2S, 3 rue Joliot Curie, 91192 Gif-sur-Yvette, France. E-mail: alban.quadrat@ inria.fr.

RESEARCH CENTRE
SACLAY – ÎLE-DE-FRANCE

Parc Orsay Université
4 rue Jacques Monod
91893 Orsay Cedex

Un nouveau point de vue sur le problème de réduction de Serre

Résumé : Ce papier porte sur l'étude des liens entre la *réduction de Serre* des systèmes fonctionnels linéaires − qui a pour but de trouver un système équivalent défini par moins d'équations et moins d'inconnues − et le *problème de décomposition* − qui a pour but de trouver un système diagonal par blocs équivalent − dans le cas où l'un des blocs diagonaux est une matrice d'identité. Pour cela, nous étendons des résultats obtenus dans Boudellioua and Quadrat (2010); Cluzeau and Quadrat (2008) sur la réduction de Serre et sur le problème de décomposition. Finalement, nous montrons comment ces résultats peuvent être utilisés pour analyser le problème de la décomposabilité des systèmes linéaires d'équations aux dérivées partielles classiquement étudiés en hydrodynamique tels que les équations de Stokes, les équations d'Oseen et le mouvement d'un fluide incompressible en rotation à petite vitesse autour d'un axe vertical.

Mots-clés : Théorie mathématique des systèmes, systèmes fonctionnels linéaires, théorie des modules, réduction de Serre, problème de factorisation, problème de décomposition, applications à l'hydrodynamique.

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1 Introduction

Mathematical systems theory aims at studying general systems defined by mathematical equations. These systems are usually defined by *functional equations*, namely, systems whose unknowns are functions, such as ordinary differential (OD) or partial differential (PD) equations, differential time-delay equations, (partial) difference equations, ... They can be linear, nonlinear, determined, overdetermined or underdetermined. A system can be studied by a broad spectrum of mathematical theories. For instance, mathematical models developed in natural sciences are usually studied by means of techniques coming from mathematical physics, functional analysis, probability and numerical analysis. There are at least two reasons for that. The first one is that it is generally difficult to obtain purely analytical results for such functional systems. The second one is the role that simulations play in nowadays life. Other functional systems coming from mathematical physics, differential geometry, hamiltonian systems, algebraic geometry, ... are usually studied by means of algebraic or differential geometry techniques. More recently, the development of constructive versions of parts of pure mathematical theories (e.g., differential algebra, algebraic geometry, differential geometry, module theory, homological algebra) and their implementations in efficient computer algebra systems allow one to develop a more analytic study of certain functional systems studied, for instance, in control theory and in mathematical physics. The questions raised in this approach are the intrinsic study of these systems, i.e., the study of their built-in properties, their symmetries and their solutions, the computation of particular forms for the systems (e.g., formal integrable forms, Gröbner or Janet bases, block triangular forms, block diagonal forms, equidimensional decomposition), of conservation laws, ... This intrinsic study leads to important information on the system (e.g., dimension of the solution space, invariants, cascade integration, decoupling), the computation of particular solutions (e.g., exponential, hypergeometric, parametrizations), ...

Following the latter approach, the purpose of this paper is to further develop certain results obtained in Cluzeau and Quadrat (2008) and Boudellioua and Quadrat (2010) which study the existence of factorizations of a matrix of functional operators defining a linear functional system, the existence of equivalent block diagonal forms for the system and the existence of equivalent forms defined by fewer unknowns and fewer equations than the original system. More precisely, if $\mathrm{GL}_r(D)$ is the group formed by the $r \times r$ matrices with entries in a ring D which are invertible and $R \in D^{q\times p}$ is a matrix which defines the system equations $R\,\eta = 0$, where $\eta \in \mathcal{F}^p$ is a vector of unknown functions which belong to a functional space $\mathcal{F}$ having a left D-module structure, then the so-called *factorization problem*, *decomposition problem* and *Serre's reduction problem* are respectively defined by:

1. Find $R' \in D^{r\times p}$ and $R'' \in D^{q\times r}$ such that $R = R''\,R'$.
2. Find $V \in \mathrm{GL}_q(D)$ and $W \in \mathrm{GL}_p(D)$ such that

$$V\,R\,W = \begin{pmatrix} \overline{R}_1 & 0 \\ 0 & \overline{R}_2 \end{pmatrix}$$

   for certain matrices $\overline{R}_1 \in D^{s\times t}$ and $\overline{R}_2 \in D^{(q-s)\times(p-t)}$.
3. Find $V \in \mathrm{GL}_q(D)$ and $W \in \mathrm{GL}_p(D)$ such that

$$V\,R\,W = \begin{pmatrix} I_s & 0 \\ 0 & \overline{R}_2 \end{pmatrix}$$

   for a certain matrix $\overline{R}_2 \in D^{(q-s)\times(p-s)}$, where I_s denotes the identity matrix of $\mathrm{GL}_s(D)$.

To do that, we study linear functional systems within the *algebraic analysis approach* (also called *D-module theory*) developed by Malgrange, Bernstein, Sato, Kashiwara ... See Hotta et al. (2008); Kashiwara (1995); Malgrange (1962); Quadrat (2010) and the references therein. In this approach, a linear functional system $R\,\eta = 0$ is studied by means of the left D-module M *finitely presented* by the matrix $R \in D^{q\times p}$ which defines the system equations and whose entries belong to a noncommutative polynomial ring D of functional operators. Using the recent development of *Gröbner or Janet basis techniques* for certain classes of noncommutative polynomial rings of functional operators (Chyzak et al. (2005)), results of algebraic analysis, using module theory and homological algebra, were made algorithmic in Chyzak et al. (2005); Cluzeau and Quadrat (2008); Quadrat (2010) and implemented in the OreModules and OreMorphisms packages (Chyzak et al. (2007); Cluzeau and Quadrat (2009)).

In this paper, we first complete some of the main results obtained in Cluzeau and Quadrat (2008). In particular, we obtain a necessary and sufficient condition for the existence of a non-trivial factorization of the system matrix based on the concept of a *non-generic solution* developed in algebraic analysis. Even if this characterization is not constructive, it generalizes a result obtained in Cluzeau and Quadrat (2008) and gives another explanation to the well-known fact that, for a linear OD operator, the existence of a factorization cannot usually be detected from the knowledge of the associated *eigenring* (see Barkatou (2007); van der Put and Singer (2003) and the references therein). We then consider the decomposition problem and we obtain necessary and sufficient conditions for the existence of a direct decomposition of the module which generalize a result obtained in Cluzeau and Quadrat (2008). We study Serre's reduction problem as a particular case of the decomposition problem, i.e., as the particular case where one of the diagonal block is the identity matrix. We show how to use certain *homotopies* of the trivial idempotents of the left D-module M, namely, of the 0 and identity endomorphisms of M, and the solutions of an *algebraic Riccati equation* (generalized inverses) to obtain necessary and sufficient conditions of Serre's reduction. These conditions are then related to the ones obtained in Boudellioua and Quadrat (2010) following Serre's ideas (see the references of Boudellioua and Quadrat (2010)). In particular, we state a correspondence between these two approaches and show how to explicitly pass from one formulation to the other. Finally, we show how the above results can be used to prove that standard 2-dimensional linear PD systems studied in hydrodynamics (namely, Oseen equations and the movement of an incompressible fluid rotating with a small velocity around the vertical axis) are defined by indecomposable differential modules. These results give a mathematical proof that the matrices of PD operators defining these systems are not equivalent to block diagonal matrices, and thus, that the equations of these systems cannot be uncoupled (which would exhibit independent physical subphenomema). These results are obtained by proving that their endomorphism rings are cyclic differential modules that only admit the two trivial idempotents.

The plan of the paper is the following. In Section 2, we briefly review the main ideas of the algebraic analysis approach to linear systems theory. In Section 3, we first present well-known results on the homomorphisms of finitely presented left modules and then study the multiplicative structure of the endomorphism ring of a finitely presented module over a commutative polynomial ring. In Section 4, we complete the results obtained in Cluzeau and Quadrat (2008) on the factorization problem. In Section 5, we further develop the results of Cluzeau and Quadrat (2008) on the decomposition problem. Serre's reduction problem is first recalled in Section 6 and the main results are reviewed. We then study the deep connections existing between Serre's reduction problem and the decomposition problem in the particular case when one of the diagonal blocks is the identity matrix. We exhibit a non-trivial correspondence between the solutions of Serre's reduction problem and those of the decomposition problem (which are based on the solvability of an algebraic Riccati equation). In Section 7, we illustrate how the techniques developed in this paper can be used to study the endomorphism ring of standard linear PD systems encountered in hydrodynamics and prove that some of these systems are indecomposable, i.e., that they cannot be uncoupled. Finally, the paper ends with Section 8 which is an appendix where the different computations used in Sections 5, 6 and 7, and obtained by means of the OreMorphisms package, are given.

Notation. In this article, D will denote a *left noetherian domain*, namely a ring without zero divisors and which is such that every left ideal of D is finitely generated as a left D-module (see, e.g., Rotman (2009)). When D is a (noncommutative) polynomial ring over a *computational field* k, we shall further assume that *Buchberger's algorithm* terminates for any *admissible term order* and computes a *Gröbner basis* (see Cluzeau and Quadrat (2008) and references therein). Moreover, $D^{q\times p}$ denotes the $D-D$-bimodule formed by the $q \times p$ matrices with entries in D. We simply note $D^{p\times 1}$ by D^p. The group of invertible matrices of $D^{p\times p}$ is denoted by $\mathrm{GL}_p(D)$. If M and N are two left D-modules, $\hom_D(M,N)$ is the abelian group formed by the *left D-homomorphisms* (i.e., left D-linear maps) from M to N. If k is a field and D a k-algebra, then $\hom_D(M,N)$ inherits a k-vector space structure. Two left D-modules M and N are said to be *isomorphic*, which is denoted by $M \cong N$, if there exists an injective and surjective element of $\hom_D(M,N)$. We denote by $M \oplus N$ the *direct sum* of M and N (see, e.g., Rotman (2009)). Finally, $\det(R)$ denotes the determinant of a square matrix R whose entries belong to a commutative ring and $\mathrm{diag}(R_1, R_2)$ is the block diagonal matrix formed by the matrices R_1 and R_2.

2 Algebraic analysis approach to linear functional systems

In this paper, we study *linear systems theory* within the *algebraic analysis framework* (see Chyzak et al. (2005); Cluzeau and Quadrat (2008); Malgrange (1962); Quadrat (2010) and the references therein). Let us briefly state again the main ideas of this approach. Let D be a left noetherian domain, $\mathcal{F}$ a left D-module and $R \in D^{q\times p}$. A *linear system*, also called *behaviour* in control theory, is defined by the following abelian group:

$$\ker_{\mathcal{F}}(R.) := \{\eta \in \mathcal{F}^p \mid R\,\eta = 0\}.$$

If k is a field and D a k-algebra, then $\ker_{\mathcal{F}}(R.)$ inherits a k-vector space structure.

To study the linear system $\ker_{\mathcal{F}}(R.)$, we first introduce the *finitely presented* left D-module M defined as the cokernel of the following left D-homomorphism

$$\begin{aligned} .R\colon D^{1\times q} &\longrightarrow D^{1\times p} \\ \lambda &\longmapsto \lambda\, R, \end{aligned}$$

i.e., with the notation $\mathrm{im}_D(.R) := D^{1\times q}\, R$, defined by the following factor left D-module:

$$M := D^{1\times p}/(D^{1\times q}\, R).$$

Let us explain why this module plays an important role in the algebraic analysis approach to linear systems theory. Let $\pi \in \hom_D(D^{1\times p}, M)$ be the left D-homomorphism sending $\lambda \in D^{1\times p}$ onto its *residue class* $\pi(\lambda) \in M$ (i.e., $\pi(\lambda) = \pi(\lambda')$ if and only if there exists $\mu \in D^{1\times q}$ such that $\lambda = \lambda' + \mu\, R$), $\{f_j\}_{j=1,\ldots,p}$ the *standard basis* of $D^{1\times p}$, i.e., $f_j \in D^{1\times p}$ is the vector formed by 1 at the j^{th} position and 0 elsewhere, and $y_j := \pi(f_j)$ for $j = 1, \ldots, p$. Then, every element $m \in M$ is of the form $m = \pi(\lambda)$ for a certain $\lambda = (\lambda_1 \ \ldots \ \lambda_p) \in D^{1\times p}$, which yields $m = \pi(\sum_{j=1}^{p} \lambda_j\, f_j) = \sum_{j=1}^{p} \lambda_i\, \pi(f_j) = \sum_{j=1}^{p} \lambda_i\, y_j$ and shows that $\{y_j\}_{j=1,\ldots,p}$ is a *family of generators* of M. These generators satisfy the following *left D-linear relations*:

$$\forall\, i = 1, \ldots, q, \quad \sum_{j=1}^{p} R_{ij}\, y_j = \sum_{j=1}^{p} R_{ij}\, \pi(f_j) = \sum_{j=1}^{p} \pi(R_{ij}\, f_j) = \pi((R_{i1} \ \ldots \ R_{ip})) = 0.$$

If we note $y := (y_1 \ \ldots \ y_p)^T$, then we have $R\, y = 0$. For more details, see Chyzak et al. (2005); Cluzeau and Quadrat (2008); Quadrat (2010).

Let M', M and M'' be left D-modules, $f \in \hom_D(M', M)$ and $g \in \hom_D(M, M'')$. If $\ker g = \operatorname{im} f$, then $M' \xrightarrow{f} M \xrightarrow{g} M''$ is called an *exact sequence* at M (see, e.g., Rotman (2009)). If the above sequence is exact at M and if $g = 0$, then $\operatorname{im} f = M$, i.e., f is surjective, or if $f = 0$, then $\ker g = 0$, i.e., g is injective. By definition of M as the cokernel of $.R \in \hom_D(D^{1\times q}, D^{1\times p})$, we have the exact sequence

$$D^{1\times q} \xrightarrow{.R} D^{1\times p} \xrightarrow{\pi} M \longrightarrow 0,$$

which is called a *finite presentation* of M. If we apply the *contravariant left exact functor* $\hom_D(\,\cdot\,, \mathcal{F})$ (see, e.g., Rotman (2009)) to the above exact sequence, we get the following exact sequence

$$\hom_D(D^{1\times q}, \mathcal{F}) \xleftarrow{(.R)^\star} \hom_D(D^{1\times p}, \mathcal{F}) \xleftarrow{\pi^\star} \hom_D(M, \mathcal{F}) \longleftarrow 0,$$

where $(.R)^\star(\phi) = \phi \circ (.R)$ for all $\phi \in \hom_D(D^{1\times p}, \mathcal{F})$ and $\pi^\star(\psi) = \psi \circ \pi$ for all $\psi \in \hom_D(M, \mathcal{F})$. For more details, see, e.g., Rotman (2009). Using the isomorphism $\hom_D(D^{1\times r}, \mathcal{F}) \cong \mathcal{F}^r$ defined by mapping the elements of the standard basis of $D^{1\times r}$ to elements of $\mathcal{F}$, the above exact sequence yields the following exact sequence of abelian groups

$$\mathcal{F}^q \xleftarrow{R.} \mathcal{F}^p \longleftarrow \hom_D(M, \mathcal{F}) \longleftarrow 0,$$

where $(R.)(\eta) = R\,\eta$ for all $\eta \in \mathcal{F}^p$, which finally shows that:

$$\ker_{\mathcal{F}}(R.) = \{\eta \in \mathcal{F}^p \mid R\,\eta = 0\} \cong \hom_D(M, \mathcal{F}). \tag{1}$$

More precisely, we can easily show that $\phi \in \hom_D(M, \mathcal{F})$ yields $\eta := (\phi(y_1) \ \ldots \ \phi(y_p))^T \in \ker_{\mathcal{F}}(R.)$, where $\{y_j\}_{j=1,\ldots,p}$ is the family of generators of M defined as above, and if $\eta \in \ker_{\mathcal{F}}(R.)$, then $\phi_\eta(\pi(\lambda)) := \lambda\,\eta$ for all $\lambda \in D^{1\times p}$ is a left D-homomorphism from M to $\mathcal{F}$. See, e.g., Chyzak et al. (2005). The isomorphism (1) shows that the linear system $\ker_{\mathcal{F}}(R.)$ can be studied by means of M and $\mathcal{F}$ (Malgrange (1962)). The finitely presented left D-module M encodes the algebraic side (i.e., the linear equations) of $\ker_{\mathcal{F}}(R.)$ and the left D-module $\mathcal{F}$ is the (functional) space in which the solutions are sought.

Example 1. A commutative ring A is called a *differential ring* if A is equipped with n *commuting derivations* ∂_i, $i = 1, \ldots, n$, i.e., maps $\partial_i : A \longrightarrow A$ satisfying $\partial_i(a_1 + a_2) = \partial_i\, a_1 + \partial_i\, a_2$, $\partial_i(a_1\, a_2) = (\partial_i\, a_1)\, a_2 + a_1\, \partial_i\, a_2$ (Leibniz rule) for all a_1, $a_2 \in A$, and $\partial_j\, \partial_i\, a = \partial_i\, \partial_j\, a$ for all $a \in A$. A *differential field* K is a field K endowed with a differential ring structure (which yields $\partial_i\, a^{-1} = -a^{-2}\, \partial_i\, a$).

The ring $D := A\langle d_1, \ldots, d_n\rangle$ of PD operators with coefficients in a differential ring $(A, \{\partial_i\}_{i=1,\ldots,n})$ is defined as the noncommutative polynomial ring formed by elements of the form $\sum_{0\le|\mu|\le r} a_\mu\, d^\mu$, where $a_\mu \in A$, $\mu := (\mu_1 \ \ldots \ \mu_n) \in \mathbb{Z}^n_{\ge 0}$, $|\mu| := \mu_1 + \cdots + \mu_n$, $d^\mu := d_1^{\mu_1} \ldots d_n^{\mu_n}$, and the d_i's satisfy the relations:

$$\forall\, a \in A, \quad \forall\, i,\, j = 1, \ldots, n, \quad \begin{cases} d_i\, a = a\, d_i + \partial_i\, a, \\ d_i\, d_j = d_j\, d_i. \end{cases}$$

If $A := k[x_1, \ldots, x_n]$ (resp., $k(x_1, \ldots, x_n)$), then the ring $A\langle d_1, \ldots, d_n\rangle$ is simply denoted by $A_n(k)$ (resp., $B_n(k)$) and is called the *polynomial* (resp., *rational*) *Weyl algebra*. Finally, if $A := k[\![x_1, \ldots, x_n]\!]$ (resp., $A = k\{x_1, \ldots, x_n\}$) is the integral domain of formal (resp., locally convergent) power series in $x_1, \ldots, x_n$ with coefficients in the field k (resp., $k = \mathbb{R}$ or $\mathbb{C}$), and $Q(A)$ its quotient field, i.e., the ring of Laurent formal power series (resp., the ring of Laurent power series), then $Q(A)\langle d_1, \ldots, d_n\rangle$ is simply denoted by $\widehat{\mathcal{D}}_n(k)$ (resp., $\mathcal{D}_n(k)$).

If $D = A\langle d_1, \ldots, d_n\rangle$ is a ring of PD operators with coefficients in a differential ring A, then $R \in D^{q\times p}$ is a $q \times p$ matrix of PD operators. If $\mathcal{F}$ is a left D-module (e.g., $\mathcal{F} = A$), then (1) shows that the solutions $\eta \in \mathcal{F}^p$ of the PD system $R\,\eta = 0$ are in a 1-1 correspondence with the elements of $\hom_D(M, \mathcal{F})$.

See Chyzak et al. (2005); McConnell and Robson (2000) for other noncommutative polynomial algebras of functional operators (e.g., time-delay or shift operators) such as the *Ore extensions* and the *Ore algebras*.

Let us briefly review a part of the classification of *finitely generated* left D-modules, i.e., left D-modules which can be defined by a finite number of generators.

Definition 1 (Lam (1999); McConnell and Robson (2000); Rotman (2009))**.** Let D be a left noetherian domain and M a finitely generated left D-module.

1. M is *free* if there exists $r \in \mathbb{Z}_{\geq 0}$ such that $M \cong D^{1\times r}$. Then, r is called the *rank* of the free left D-module M and is denoted by $\operatorname{rank}_D(M)$.

2. M is *stably free* if there exist r, $s \in \mathbb{Z}_{\geq 0}$ such that $M \oplus D^{1\times s} \cong D^{1\times r}$. Then, $r - s$ is called the *rank* of the stably free left D-module M.

3. M is *projective* if there exist $r \in \mathbb{Z}_{\geq 0}$ and a left D-module N such that $M \oplus N \cong D^{1\times r}$.

4. M is reflexive if the canonical left D-homomorphism $\varepsilon : M \longrightarrow M^{\star\star}$ defined by $\varepsilon(m)(f) = f(m)$ for all $f \in M^\star := \hom_D(M, D)$, where $M^{\star\star} = \hom_D(\hom_D(M, D), D)$, is an isomorphism (which then yields $M \cong M^{\star\star}$).

5. M is *torsion-free* if the *torsion left D-submodule of M*, namely,

$$t(M) := \{m \in M \mid \exists\, d \in D \setminus \{0\} :\ d\,m = 0\},$$

   is reduced to 0, i.e., if $t(M) = 0$. The elements of $t(M)$ are called the *torsion elements* of M.

6. M is *torsion* if $t(M) = M$, i.e., if every element of M is a torsion element.

7. M is *cyclic* if there exists $m \in M$ such that $M = D\,m := \{d\,m \mid d \in D\}$.

8. M is *decomposable* if there exist two proper left D-submodules M_1 and M_2 of M such that:

$$M = M_1 \oplus M_2.$$

   If M is not decomposable, then M is said to be *indecomposable*.

9. A non-zero left D-module M is called *simple* if M has no non-zero proper left D-submodules.

Similar definitions exist for finitely generated right D-modules.

We refer to Chyzak et al. (2005); Fabiańska and Quadrat (2007); Quadrat and Robertz (2007a) for algorithms which test whether or not a finitely presented module M over some classes of noncommutative polynomial rings admits a non-trivial torsion submodule, is torsion-free, projective, stably free or free. These algorithms are implemented in the OreModules (Chyzak et al. (2007)), QuillenSuslin (Fabiańska and Quadrat (2007)) and Stafford (Quadrat and Robertz (2007a)) packages.

A free module is clearly stably free (take $s = 0$ in 2 of Definition 1), a stably free module is projective (take $N = D^{1\times s}$ in 3 of Definition 1) and a projective module is torsion-free (since it can be embedded into a free, and thus into a torsion-free module). More generally, we have the following results.

Theorem 1 (Lam (1999); McConnell and Robson (2000); Rotman (2009)). *Let D be a left noetherian domain. Then, we have the following implications for finitely generated left/right D-modules:*

$$\textit{free} \;\Rightarrow\; \textit{stably free} \;\Rightarrow\; \textit{projective} \;\Rightarrow \textit{reflexive} \;\Rightarrow\; \textit{torsion-free}.$$

The converses of the above results are generally not true. Some of them hold for particular domains playing particular roles in linear systems theory.

Theorem 2 (Lam (1999); McConnell and Robson (2000); Quadrat and Robertz (2014); Rotman (2009)). *We have the following results:*

1. *If D is a* principal ideal domain, *i.e., every left ideal I and every right ideal J of the domain D are* principal, *i.e., are of the form $I = D\,d_1$ and $J = d_2\,D$ for $d_1,\, d_2 \in D$ (e.g., the ring $A\langle\partial\rangle$ of OD operators with coefficients in a differential field A such as the ring $B_1(k)$, $\widehat{\mathcal{D}}_1(k)$, $\mathcal{D}_1(k)$), then every finitely generated torsion-free left or right D-module is free.*
2. *If $D = k[x_1, \ldots, x_n]$ is a commutative polynomial ring with coefficients in a field k, then every finitely generated projective D-module is free (*Quillen-Suslin theorem*).*
3. *If D is the Weyl algebra $A_n(k)$ or $B_n(k)$, where k is a field of characteristic 0 (e.g., $k = \mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$), then every finitely generated projective left/right D-module is stably free and every finitely generated stably free left/right D-module of rank at least 2 is free (*Stafford's theorem*).*
4. *If $D = \widehat{\mathcal{D}}_n(k)$, $\mathcal{D}_n(k)$ or $D = A\langle\partial\rangle$, where $A = k[\![t]\!]$ and k is a field of characteristic 0, or $A = k\{t\}$ and $k = \mathbb{R}$ or $\mathbb{C}$, then every finitely generated projective left/right D-module is stably free and every finitely generated stably free left/right D-module of rank at least 2 is free.*

A matrix $R \in D^{q\times p}$ is said to have *full row rank* if $\ker_D(.R) := \{\mu \in D^{1\times q} \mid \mu\, R = 0\} = 0$, i.e., if the rows of the matrix R are left D-linearly independent. If $R \in D^{q\times p}$ has full row rank, then we have $D^{1\times q} \cong D^{1\times q}\, R \subseteq D^{1\times p}$, which yields $q \leq p$. The next theorem characterizes when a left D-module M, finitely presented by a full row rank matrix R, is a projective or a free module.

Theorem 3 (Fabiańska and Quadrat (2007); Quadrat and Robertz (2007a)). *Let $M = D^{1\times p}/(D^{1\times q}\, R)$ be a left D-module finitely presented by a full row rank matrix $R \in D^{q\times p}$. Then, we have:*

1. *M is a projective left D-module if and only if M is a stably free left D-module.*
2. *M is a stably free left D-module of rank $p - q$ if and only if R admits a* right inverse, *i.e., if and only if there exists a matrix $S \in D^{p\times q}$ such that $R\,S = I_q$.*
3. *M is a free left D-module of rank $p - q$ if and only if there exists $U \in \mathrm{GL}_p(D)$ such that:*

$$R\,U = (I_q \quad 0).$$

If $U := (S \quad Q)$, where $S \in D^{p\times q}$ and $Q \in D^{p\times(p-q)}$, then we have the following isomorphisms

$$\begin{array}{rcl} \psi : M & \longrightarrow & D^{1\times(p-q)} \\ \pi(\lambda) & \longmapsto & \lambda\, Q, \end{array} \qquad \begin{array}{rcl} \psi^{-1} : D^{1\times(p-q)} & \longrightarrow & M \\ \mu & \longmapsto & \pi(\mu\, T), \end{array}$$

where the matrix $T \in D^{(p-q)\times p}$ is defined by:

$$U^{-1} := \left(\begin{array}{c} R \\ T \end{array}\right) \in D^{p\times p}.$$

In particular, we have $M \cong D^{1\times p}\, Q = D^{1\times(p-q)}$. The matrix Q is called an injective parametrization *of M. If $T_{i\bullet}$ denotes the i^{th} row of T, then $\{\pi(T_{i\bullet})\}_{i=1,\ldots,p-q}$ is a basis of the free left D-module M of rank $p - q$.*

The Quillen-Suslin theorem (resp., Stafford's theorem) is implemented in the QuillenSuslin package (Fabiańska and Quadrat (2007)) (resp., Stafford package (Quadrat and Robertz (2007a))). Hence, for $D = k[x_1, \ldots, x_n]$ and $k = \mathbb{Q}$, $A_n(\mathbb{Q})$ or $B_n(\mathbb{Q})$, bases and injective parametrizations of finitely generated free left D-modules can be computed.

3 Homomorphisms of finitely presented left D-modules

In this section, we first briefly review the characterization of a left D-homomorphism of two finitely presented left D-modules. For more details, see Cluzeau and Quadrat (2008); Rotman (2009).

Lemma 1 (Cluzeau and Quadrat (2008)). *Let $M = D^{1\times p}/(D^{1\times q}\, R)$ and $M' = D^{1\times p'}/(D^{1\times q'}\, R')$ be two finitely presented left D-modules and $\pi : D^{1\times p} \longrightarrow M$ and $\pi' : D^{1\times p'} \longrightarrow M'$ the canonical projections onto M and M'.*

1. *The existence of $f \in \hom_D(M, M')$ is equivalent to the existence of a pair of matrices $P \in D^{p\times p'}$ and $Q \in D^{q\times q'}$ satisfying the following relation:*

$$R\,P = Q\,R'. \tag{2}$$

Hence, we have the following commutative exact diagram

$$\begin{tikzcd} D^{1\times q} \arrow[r, ".R"] \arrow[d, ".Q"] & D^{1\times p} \arrow[r, "\pi"] \arrow[d, ".P"] & M \arrow[r] \arrow[d, "f"] & 0 \\ D^{1\times q'} \arrow[r, ".R'"] & D^{1\times p'} \arrow[r, "\pi'"] & M' \arrow[r] & 0, \end{tikzcd}$$

i.e., (2) holds and $f \circ \pi = \pi' \circ (.P)$, where $f \in \hom_D(M, M')$ is defined by:

$$\forall\, \lambda \in D^{1\times p}, \quad f(\pi(\lambda)) = \pi'(\lambda\, P). \tag{3}$$

2. *Let $P \in D^{p\times p'}$ and $Q \in D^{q\times q'}$ satisfy (2) and $R'_2 \in D^{q'_2\times q'}$ be such that $\ker_D(.R') = \mathrm{im}_D(.R'_2)$. Then, the matrices defined by*

$$\begin{cases} \overline{P} := P + Z\,R', \\ \overline{Q} := Q + R\,Z + Z_2\,R'_2, \end{cases}$$

for all $Z \in D^{p\times q'}$ and $Z_2 \in D^{q\times q'_2}$, satisfy the identity $R\,\overline{P} = \overline{Q}\,R'$ and we have:

$$\forall\, \lambda \in D^{1\times p}, \quad f(\pi(\lambda)) = \pi'(\lambda\, P) = \pi'(\lambda\, \overline{P}).$$

For two finitely presented left D-modules M and M', the problem of characterizing elements of $\hom_D(M, M')$ is considered in Cluzeau and Quadrat (2008) for certain classes of noncommutative polynomial rings and algorithms are given (see Algorithms 2.1 and 2.2 of Cluzeau and Quadrat (2008)). An implementation is available in the OreMorphisms package (Cluzeau and Quadrat (2009)).

If $f \in \hom_D(M, M')$, then we can define the following finitely generated left D-modules:

$$\begin{cases} \ker f := \{m \in M \mid f(m) = 0\}, \\ \mathrm{im}\, f := \{m' \in M' \mid \exists\, m \in M :\ m' = f(m)\}, \\ \mathrm{coim}\, f := M/\ker f, \\ \mathrm{coker}\, f := M'/\mathrm{im}\, f. \end{cases}$$

For two finitely presented left D-modules M and M', let us explicitly characterize the kernel, image, coimage and cokernel of $f \in \hom_D(M, M')$.

Lemma 2 (Cluzeau and Quadrat (2008)). *Let $M = D^{1\times p}/(D^{1\times q}\, R)$ (resp., $M' = D^{1\times p'}/(D^{1\times q'}\, R')$) be a left D-module finitely presented by $R \in D^{q\times p}$ (resp., $R' \in D^{q'\times p'}$) and $f \in \hom_D(M, M')$ defined by (3), where $P \in D^{p\times p'}$ satisfies (2) for a certain matrix $Q \in D^{q\times q'}$.*

1. *Let $S \in D^{r\times p}$ and $T \in D^{r\times q'}$ be two matrices such that*

$$\ker_D\left(.\begin{pmatrix} P^T & R'^T \end{pmatrix}^T\right) = \mathrm{im}_D(.(S \quad -T)), \tag{4}$$

$L \in D^{q\times r}$ a matrix satisfying $R = L\, S$ and $Q = L\, T$, and $S_2 \in D^{r_2 \times r}$ a matrix such that $\ker_D(.S) = \mathrm{im}_D(.S_2)$. Then, we have:

$$\ker f = (D^{1\times r}\, S)/(D^{1\times q}\, R) \cong D^{1\times r} / \left(D^{1\times (q+r_2)} \begin{pmatrix} L^T & S_2^T \end{pmatrix}^T \right). \tag{5}$$

Hence, f is injective if and only if the matrix $(L^T \quad S_2^T)^T$ admits a left inverse, i.e., if and only if there exists $X = (X_1 \quad X_2) \in D^{r\times(q+r_2)}$ such that $X_1\, L + X_2\, S_2 = I_r$.

2. *With the above notations, we have:*

$$\mathrm{coim}\, f = D^{1\times p}/(D^{1\times r}\, S) \cong \mathrm{im}\, f = \left(D^{1\times(p+q')} \begin{pmatrix} P^T & R'^T \end{pmatrix}^T \right) / (D^{1\times q'}\, R').$$

Moreover, we have the following commutative exact diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & \\
 & & \ker f \arrow[d] & \\
D^{1\times q} \arrow[r, ".R"] \arrow[d, ".L"] & D^{1\times p} \arrow[r, "\pi"] \arrow[d, equal] & M \arrow[r] \arrow[d, "\rho"] & 0 \\
D^{1\times r} \arrow[r, ".S"] & D^{1\times p} \arrow[r, "\kappa"] & \mathrm{coim}\, f \arrow[r] \arrow[d] & 0, \\
 & & 0 &
\end{tikzcd}$$

where $\rho : M \longrightarrow \mathrm{coim}\, f = M/\ker f$ is the canonical projection.

3. *We have* $\mathrm{coker}\, f = D^{1\times p'} / \left(D^{1\times(p+q')} \begin{pmatrix} P^T & R'^T \end{pmatrix}^T \right)$ *and the following long exact sequence*

$$D^{1\times r} \xrightarrow{.(S \quad -T)} D^{1\times(p+q')} \xrightarrow{.\left(P^T \quad R'^T\right)^T} D^{1\times p'} \xrightarrow{\epsilon} \mathrm{coker}\, f \longrightarrow 0$$

defining the beginning of a finite free resolution *of* $\mathrm{coker}\, f$. *Hence, f is surjective if and only if the matrix $(P^T \quad R'^T)^T$ admits a left inverse, i.e., if and only if there exists $Y = (Y_1 \quad Y_2) \in D^{p'\times(p+q')}$ such that $Y_1\, P + Y_2\, R' = I_{p'}$.*

4. *We have the following commutative exact diagram*

$$\begin{tikzcd}
 & & 0 \arrow[d] & \\
D^{1\times r} \arrow[r, ".S"] \arrow[d, ".T"] & D^{1\times p} \arrow[r, "\kappa"] \arrow[d, ".P"] & \mathrm{coim}\, f \arrow[r] \arrow[d, "f^\sharp"] & 0 \\
D^{1\times q'} \arrow[r, ".R'"] & D^{1\times p'} \arrow[r, "\pi'"] & M' \arrow[r] \arrow[d] & 0, \\
 & & \mathrm{coker}\, f \arrow[d] & \\
 & & 0 &
\end{tikzcd}$$

where $f^\sharp \in \hom_D(\mathrm{coim}\, f, M')$ is defined by $f^\sharp(\kappa(\lambda)) = \pi'(\lambda\, P)$ for all $\lambda \in D^{1\times p}$.

To study the *decomposition problem*, namely the problem of recognizing whether or not a finitely presented left D-module M is decomposable (see of 8 of Definition 1), we shall focus on the case $M' = M$, i.e., on the study of the *endomorphism ring* $\mathrm{end}_D(M) := \hom_D(M, M)$ of M.

In many standard examples coming from linear systems theory and mathematical physics (see, e.g., the examples considered in Section 7), D is a commutative polynomial ring. In this particular case, $\hom_D(M, M')$ inherits a D-module structure (which is usually not the case for a noncommutative ring D) and an explicit description of the D-module $\hom_D(M, M')$ in terms of generators and relations can be given. For more details and explicit algorithms, we refer to Cluzeau and Quadrat (2008).

Till the end of this section, we assume that D is a commutative ring. From Lemma 1, it follows that the ring $\mathrm{end}_D(M)$ can be written as the factor of two D-modules, i.e., we have

$$\mathrm{end}_D(M) \cong \mathcal{B} := \mathcal{A}/(D^{p\times q}\, R),$$

where $\mathcal{A} := \{P \in D^{p\times p} \,|\, \exists\, Q \in D^{q\times q} : R\,P = Q\,R\}$ is a ring called *eigenring*. Indeed, we clearly have $0 \in \mathcal{A}$, $I_p \in \mathcal{A}$ and if P_1, $P_2 \in \mathcal{A}$, i.e., $R\,P_1 = Q_1\,R$ and $R\,P_2 = Q_2\,R$ for some matrices Q_1, $Q_2 \in D^{q\times q}$, then we have $R\,(P_1 + P_2) = (Q_1 + Q_2)\,R$ and $R\,(P_1\,P_2) = (Q_1\,Q_2)\,R$ so that $P_1 + P_2 \in \mathcal{A}$ and $P_1\,P_2 \in \mathcal{A}$. The other properties of a ring can easily be checked. The ring $\mathcal{A}$ is a noncommutative ring since $P_1\,P_2$ is usually different from $P_2\,P_1$. Moreover, $D^{p\times q}\,R$ is a two-sided ideal of $\mathcal{A}$. Indeed, if P_1, $P_2 \in \mathcal{A}$ and $Z_1\,R$, $Z_2\,R \in D^{p\times q}\,R$, where $Z_i \in D^{p\times q}$ for $i = 1, 2$, then we have:

$$\begin{cases} P_1\,(Z_1\,R) + P_2\,(Z_2\,R) = (P_1\,Z_1 + P_2\,Z_2)\,R, \\ (Z_1\,R)\,P_1 + (Z_2\,R)\,P_2 = (Z_1\,Q_1 + Z_2\,Q_2)\,R. \end{cases}$$

Thus, $\mathcal{B} = \mathcal{A}/(D^{p\times q}\,R)$ is a noncommutative ring and $\kappa := \mathrm{id}_p \otimes \pi : \mathcal{A} \longrightarrow \mathcal{B}$ is the canonical projection onto $\mathcal{B}$. In particular, the product of $\mathcal{B}$ is defined by:

$$\forall\, P_1,\, P_2 \in \mathcal{A}, \quad \kappa(P_1)\,\kappa(P_2) = \kappa(P_1\,P_2).$$

We call *opposite ring* of $\mathcal{B}$, denoted by $\mathcal{B}^{\mathrm{op}}$, the ring defined by $\mathcal{B}$ as an abelian group but equipped with the *opposite multiplication* $\bullet$ defined by:

$$\forall\, b_1,\, b_2 \in \mathcal{B}, \quad b_1 \bullet b_2 := b_2\, b_1.$$

If $\phi : \mathcal{B} \longrightarrow \mathrm{end}_D(M)$ is the abelian group isomorphism mapping $\kappa(P)$ to $\phi(\kappa(P))$ defined by

$$\forall\, \lambda \in D^{1\times p}, \quad \phi(\kappa(P))(\pi(\lambda)) = \pi(\lambda\, P),$$

then we have

$$\forall\, \lambda \in D^{1\times p}, \quad (\phi(\kappa(P_2)) \circ \phi(\kappa(P_1)))(\pi(\lambda)) = \pi(\lambda\, P_1\, P_2) = \phi(\kappa(P_1\, P_2))(\pi(\lambda)) = \phi(\kappa(P_1)\,\kappa(P_2))(\pi(\lambda)),$$

i.e., using the opposite ring $\mathcal{B}^{\mathrm{op}}$, we obtain:

$$\phi(\kappa(P_2) \bullet \kappa(P_1)) = \phi(\kappa(P_1)\,\kappa(P_2)) = \phi(\kappa(P_2)) \circ \phi(\kappa(P_1)).$$

Since $\phi(\kappa(I_p)) = \mathrm{id}_M$, ϕ is a ring isomorphism, i.e.:

$$\mathrm{end}_D(M) \cong \mathcal{B}^{\mathrm{op}}.$$

Algorithm 2.1 in Cluzeau and Quadrat (2008) computes a family of generators $\{f_i\}_{i=1,\ldots,s}$ of the finitely generated D-module $\mathrm{end}_D(M)$. The f_i's are given by means of two matrices $P_i \in D^{p\times p}$ and $Q_i \in D^{q\times q}$ satisfying $R\,P_i = Q_i\,R$, i.e., $f_i(\pi(\lambda)) = \pi(\lambda\,P_i)$ for all $\lambda \in D^{1\times p}$ and $i = 1, \ldots, s$ (see Lemma 1).

Let us now explain how to obtain a finite family of D-linear relations among these generators, i.e., $X\,F = 0$, where $F = (f_1 \ \ldots \ f_s)^T$ and $X \in D^{t\times s}$. A D-linear relation $\sum_{j=1}^{s} d_j\, f_j = 0$ between the f_i's is equivalent to the existence of $Z \in D^{p\times q}$ satisfying:

$$\sum_{j=1}^{s} d_j\, P_j = Z\, R. \tag{6}$$

To solve (6), let us introduce a few definitions and a standard result which holds for matrices with entries in a commutative ring D. If $F \in D^{q \times p}$, then $\mathrm{row}(F) \in D^{1 \times q\, p}$ denotes the row vector obtained by concatenating the rows of F. If $F \in D^{q \times p}$ and $F' \in D^{q' \times p'}$, then $K := F \otimes F'$ stands for the *Kronecker product* of F and F', namely, the matrix $K \in D^{q\, q' \times p\, p'}$ defined by $(F_{ij}\, F')_{1 \leq i \leq q,\, 1 \leq j \leq p}$. If $F \in D^{q \times p}$, $G \in D^{r \times q}$ and $H \in D^{s \times r}$, then a standard result on Kronecker products states that we have:

$$\mathrm{row}(H\, G\, F) = \mathrm{row}(G)\, (H^T \otimes F).$$

Applying the above identity to (6), we get:

$$\sum_{j=1}^{s} d_j \,\mathrm{row}(P_j) - \mathrm{row}(Z)\, (I_p \otimes R) = 0 \quad \Longleftrightarrow \quad (d_1 \; \ldots \; d_s \; - \mathrm{row}(Z)) \begin{pmatrix} \mathrm{row}(P_1) \\ \vdots \\ \mathrm{row}(P_s) \\ I_p \otimes R \end{pmatrix} = 0.$$

If we introduce the matrices

$$\left\{ \begin{array}{l} U := \left(\mathrm{row}(P_1)^T \; \ldots \; \mathrm{row}(P_s)^T\right)^T \in D^{s \times p^2}, \\ V := I_p \otimes R \in D^{p\, q \times p^2}, \\ W := (U^T \quad V^T)^T \in D^{(s + p\, q) \times p^2}, \end{array} \right. \tag{7}$$

then there exist $X \in D^{t \times s}$ and $Y \in D^{t \times p\, q}$ satisfying $\ker_D(.W) = D^{1 \times t}\, (X \quad - Y)$. If $Y_{i,j}$ denotes the $i \times j$ entry of the matrix Y and for $i = 1, \ldots, t$,

$$Z_i = \begin{pmatrix} Y_{i,1} & \ldots & Y_{i,q} \\ Y_{i,(q+1)} & \ldots & Y_{i,2\, q} \\ \vdots & & \vdots \\ Y_{i,(p-1)\, q+1} & \ldots & Y_{i,p\, q} \end{pmatrix} \in D^{p \times q},$$

then $\sum_{j=1}^{s} X_{ij}\, P_j = Z_i\, R$, and thus the f_i's satisfy the following D-linear relations:

$$\forall\, i, = 1, \ldots, t, \quad \sum_{j=1}^{s} X_{ij}\, f_j = 0. \tag{8}$$

Hence, we get $\mathrm{end}_D(M) \cong D^{1 \times s}/(D^{1 \times t}\, X)$, i.e., $\mathrm{end}_D(M)$ is finitely presented by the matrix $X \in D^{t \times s}$.

Now, the ring structure of $\mathrm{end}_D(M)$ is characterized by the expressions of the $f_i \circ f_j$'s in terms of the generators f_k's of the D-module $\mathrm{end}_D(M)$, i.e.:

$$\forall\, i, j = 1, \ldots, s, \quad f_i \circ f_j = \sum_{k=1}^{s} \gamma_{ijk}\, f_k, \quad \gamma_{ijk} \in D. \tag{9}$$

The γ_{ijk}'s look like the *structure constants* appearing in the theory of finite-dimensional algebras. The matrix Γ formed by the γ_{ijk} satisfies $F \otimes F = \Gamma\, F$. Γ is called a *multiplication table* in group theory. If $D\langle f_1, \ldots f_s\rangle$ denotes the free associative D-algebra generated by the f_i's and

$$J = \left\langle \sum_{j=1}^{s} X_{ij}\, f_j, \; i = 1, \ldots, t, \; f_i \circ f_j - \sum_{k=1}^{s} \gamma_{ijk}\, f_k, \; i, j = 1, \ldots, s \right\rangle$$

is the two-sided ideal of $D\langle f_1, \ldots f_s\rangle$ generated by the relations (8) and (9), then the noncommutative ring $\mathrm{end}_D(M)$ is defined by $\mathrm{end}_D(M) = D\langle f_1, \ldots f_s\rangle / J$, which shows that $\mathrm{end}_D(M)$ can be defined as the quotient of a free associative algebra by a two-sided ideal generated by linear and quadratic relations over D.

Using (7), the structure constants γ_{ijk}'s can be computed as follows. The computation of the normal form of the rows $\mathrm{row}(P_i\, P_j)$ with respect to a Gröbner basis of the D-module $D^{1\times(s+p\,q)}\, W$ for $i, j = 1, \ldots, s$ yields a matrix $(\Gamma_1 \quad \Gamma_2) \in D^{s^2\times(s+p\,q)}$, where $\Gamma_1 \in D^{s^2\times s}$ and $\Gamma_2 \in D^{s^2\times p\,q}$. Then, the matrix Γ_1 defines the multiplication table of the family of generators $\{f_i\}_{i=1,\ldots,s}$ of $\mathrm{end}_D(M)$. The computation of the endomorphism ring $\mathrm{end}_D(M)$ (i.e., generators, relations and multiplication table) for a finitely presented module over a commutative polynomial ring D is implemented in the OreMorphisms package (Cluzeau and Quadrat (2009)).

Example 2. Let us consider the motion of a fluid in a one-dimensional tank studied in Dubois et al. (1999) and defined by the following linear system of OD time-delay equations

$$\begin{cases} y_1(t-2\,h) + y_2(t) - 2\,\dot{u}(t-h) = 0, \\ y_1(t) + y_2(t-2\,h) - 2\,\dot{u}(t-h) = 0, \end{cases} \tag{10}$$

where h a positive real number. Let $D = \mathbb{Q}(\alpha)[\partial, \delta]$ be the commutative polynomial ring of OD time-delay operators with rational constant coefficients, i.e., $\partial\, y(t) = \dot{y}(t)$, $\delta\, y(t) = y(t-h)$ and $\partial\,\delta = \delta\,\partial$,

$$R = \begin{pmatrix} \delta^2 & 1 & -2\,\partial\,\delta \\ 1 & \delta^2 & -2\,\partial\,\delta \end{pmatrix} \in D^{2\times 3} \tag{11}$$

the matrix defining (10), and the D-module $M = D^{1\times 3}/(D^{1\times 2}\, R)$ finitely presented by R. Applying Algorithm 2.1 of Cluzeau and Quadrat (2008) to R, the D-module structure of $\mathrm{end}_D(M)$ is generated by $f_{e_1}, f_{e_2}, f_{e_3}, f_{e_4} \in \mathrm{end}_D(M)$ defined by $f_\alpha(\pi(\lambda)) = \pi(\lambda\, P_\alpha)$ for all $\lambda \in D^{1\times 3}$, where $\alpha = (\alpha_1\ \alpha_2\ \alpha_3\ \alpha_4) \in D^{1\times 4}$, $\{e_i\}_{i=1,\ldots,4}$ is the standard basis of $D^{1\times 4}$ and:

$$P_\alpha = \begin{pmatrix} \alpha_1 & \alpha_2 & 2\,\alpha_3\,\partial\,\delta \\ \alpha_2 + 2\,\alpha_4\,\partial & \alpha_1 - 2\,\alpha_4\,\partial & 2\,\alpha_3\,\partial\,\delta \\ \alpha_4\,\delta & -\alpha_4\,\delta & \alpha_1 + \alpha_2 + \alpha_3\,(\delta^2+1) \end{pmatrix},$$

$$Q_\alpha = \begin{pmatrix} \alpha_1 - 2\,\alpha_4\,\partial & \alpha_2 + 2\,\alpha_4\,\partial \\ \alpha_2 & \alpha_1 \end{pmatrix}.$$

Let us simply set $f_i := f_{e_i}$. We can check that the generators $\{f_i\}_{i=1,\ldots,4}$ of the D-module structure of $\mathrm{end}_D(M)$ satisfy the following D-linear relations:

$$(\delta^2 - 1)\, f_4 = 0, \quad \delta^2\, f_1 + f_2 - f_3 = 0, \quad f_1 + \delta^2\, f_2 - f_3 = 0. \tag{12}$$

A complete description of the noncommutative ring $\mathrm{end}_D(M)$ is given by the knowledge of the expressions of the compositions $f_i \circ f_j$ in the family of generators $\{f_k\}_{k=1,\ldots,4}$ for $i, j = 1, \ldots, 4$:

$$\begin{cases} f_1 \circ f_i = f_i \circ f_1 = f_i, \quad i = 1, \ldots, 4, \\ f_2 \circ f_2 = f_1, \\ f_2 \circ f_3 = f_3 \circ f_2 = f_3, \\ f_2 \circ f_4 = 2\,\partial\, f_1 - 2\,\partial\, f_2 + f_4, \\ f_4 \circ f_2 = -f_4, \end{cases} \qquad \begin{cases} f_3 \circ f_3 = (\delta^2+1)\, f_3, \\ f_3 \circ f_4 = 2\,\partial\, f_1 - 2\,\partial\, f_2 + 2\, f_4, \\ f_4 \circ f_3 = 0, \\ f_4 \circ f_4 = -2\,\partial\, f_4. \end{cases} \tag{13}$$

Denoting by $f_c \circ f_r$ the composition of an element f_c in the first column by an element f_r in the first row of the table below, we can write (13) in the form of the following multiplication table:

$f_c \circ f_r$	f_1	f_2	f_3	f_4
f_1	f_1	f_2	f_3	f_4
f_2	f_2	f_1	f_3	$2\,\partial\, f_1 - 2\,\partial\, f_2 + f_4$
f_3	f_3	f_3	$(\delta^2+1)\, f_3$	$2\,\partial\, f_1 - 2\,\partial\, f_2 + 2\, f_4$
f_4	f_4	$-f_4$	0	$-2\,\partial\, f_4$

We finally obtain $\mathrm{end}_D(M) = D\langle f_1, f_2, f_3, f_4\rangle / J$, where

$$J = \langle (\delta^2 - 1)\, f_4, \delta^2\, f_1 + f_2 - f_3, f_1 + \delta^2\, f_2 - f_3, f_1 \circ f_1 - f_1, \ldots, f_4 \circ f_4 + 2\,\partial\, f_4\rangle$$

is the two-sided ideal of the free D-algebra $D\langle f_1, f_2, f_3, f_4\rangle$ generated by the polynomials defined by the identities (12) and (13).

4 Factorization problem

In this section, we complete results of Cluzeau and Quadrat (2008) to obtain a necessary and sufficient condition for the existence of a *strict factorization* of a linear functional system.

Let us first give a necessary and sufficient condition for the existence of a factorization of $R \in D^{q\times p}$.

Lemma 3. *If $R \in D^{q\times p}$, then the following assertions are equivalent:*

1. *There exist two matrices $L \in D^{q\times r}$ and $S \in D^{r\times p}$ such that:*
$$R = L\, S. \tag{14}$$
2. *There exist a finitely presented left D-module M' and $f \in \hom_D(M, M')$, where $M = D^{1\times p}/(D^{1\times q}\, R)$, such that:*
$$\mathrm{coim}\, f = D^{1\times p}/(D^{1\times r}\, S), \quad \ker f = (D^{1\times r}\, S)/(D^{1\times q}\, R). \tag{15}$$

Proof. $1 \Rightarrow 2$. Let $M' := D^{1\times p}/(D^{1\times r}\, S)$ be the left D-module finitely presented by S. The relation $R = L\, S$ induces the commutative exact diagram

$$\begin{array}{ccccccc} D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow & 0 \\ \downarrow{.L} & & \| & & & & \\ D^{1\times r} & \xrightarrow{.S} & D^{1\times p} & \xrightarrow{\kappa} & M' & \longrightarrow & 0, \end{array}$$

which defines $f \in \hom_D(M, M')$ by $f(\pi(\lambda)) = \kappa(\lambda)$ for all $\lambda \in D^{1\times p}$. Indeed, if $\pi(\lambda) = \pi(\lambda')$ for some $\lambda' \in D^{1\times p}$, then there exists $\mu \in D^{1\times q}$ such that $\lambda = \lambda' + \mu\, R$, which yields:

$$f(\pi(\lambda)) = \kappa(\lambda) = \kappa(\lambda') + \kappa(\mu\, R) = \kappa(\lambda') + \kappa((\mu\, L)\, S) = \kappa(\lambda') = f(\pi(\lambda')).$$

Using $\ker_D(.(I_p^T \quad S^T)^T) = D^{1\times r}\,(S \quad -I_r)$, 1 and 2 of Lemma 2 yield (15).

$2 \Rightarrow 1$ is proved in Theorem 3.1 of Cluzeau and Quadrat (2008). For the sake of completeness, we repeat the proof here. Let $M' := D^{1\times p'}/(D^{1\times q'}\, R')$ and $f \in \hom_D(M, M')$ satisfy (15). From Lemma 1, f is defined by (3) where $P \in D^{p\times p'}$ satisfies (2) for a certain matrix $Q \in D^{q\times q'}$. Using (2) and (4) of Lemma 2, we get $\mathrm{im}_D(.(R \quad -Q)) \subseteq \ker_D\left(.\left(P^T \quad R'^T\right)^T\right) = \mathrm{im}_D(.(S \quad -T))$, which shows that there exists a matrix $L \in D^{q\times r}$ such that $R = L\, S$ and $Q = L\, T$.

□

Using Gröbner basis techniques for a polynomial ring D, the factorization (14) can be computed (see, e.g., Chyzak et al. (2005, 2007)).

Definition 2. A factorization $R = L\, S$, where $R \in D^{q\times p}$, $L \in D^{q\times s}$ and $S \in D^{s\times p}$, is called *strict* if:

$$\mathrm{im}_D(.R) \subsetneq \mathrm{im}_D(.S).$$

If $\mathcal{F}$ is a left D-module and $R = L\, S$, then $\ker_{\mathcal{F}}(S.) \subseteq \ker_{\mathcal{F}}(R.)$, i.e., every $\mathcal{F}$-solution of $S\,\eta = 0$ is a $\mathcal{F}$-solution of $R\,\eta = 0$. Hence, finding solutions of a linear functional system is an application of the problem of factoring matrices of functional operators.

Proposition 1. *If $R = L\, S$ is not a strict factorization, then we have $\ker_{\mathcal{F}}(R.) = \ker_{\mathcal{F}}(S.)$ for all left D-modules $\mathcal{F}$.*

Proof. Since, by definition of S, we have $D^{1\times q}\,R \subseteq D^{1\times r}\,S$, $D^{1\times r}\,S = D^{1\times q}\,R$ is equivalent to the existence of $F \in D^{r\times q}$ such that $S = F\,R$. Combining this identity with $R = L\,S$, we get $(I_q - L\,F)\,R = 0$ and $(I_r - F\,L)\,S = 0$, and thus there exist two matrices $X \in D^{r\times q_2}$ and $Y \in D^{r\times r_2}$ such that

$$\begin{cases} L\,F = I_q + X\,R_2, \\ F\,L = I_r + Y\,S_2, \end{cases} \tag{16}$$

where $R_2 \in D^{q_2\times q}$ (resp., $S_2 \in D^{r_2\times r}$) satisfies $\ker_D(.R) = \operatorname{im}_D(.R_2)$ (resp., $\ker_D(.S) = \operatorname{im}_D(.S_2)$). Then, using (16), we can easily check that we have

$$R\,\eta = L\,(S\,\eta) = 0 \;\Rightarrow\; \begin{cases} L\,\theta = 0, \\ S\,\eta = \theta, \end{cases} \;\Rightarrow\; S\,\eta = 0,$$

since $\theta \in \mathcal{F}^r$ satisfies $S_2\,\theta = 0$, and thus $\theta = F\,(L\,\theta) - Y\,(S_2\,\theta) = 0$ by (16), and conversely

$$S\,\eta = F\,(R\,\eta) = 0 \;\Rightarrow\; \begin{cases} F\,\zeta = 0, \\ R\,\eta = \zeta, \end{cases} \;\Rightarrow\; R\,\eta = 0,$$

since $\zeta \in \mathcal{F}^q$ satisfies $R_2\,\zeta = 0$, and thus $\zeta = L\,(F\,\zeta) - X\,(R_2\,\zeta) = 0$ by (16), i.e., $\ker_{\mathcal{F}}(R.) = \ker_{\mathcal{F}}(S.)$. $\square$

Remark 1. Homological algebra techniques can be used to give another proof of Proposition 1. Indeed, by $1 \Rightarrow 2$ of Lemma 3, the factorization $R = L\,S$ defines $f \in \hom_D(M, M')$, where $M' = D^{1\times p}/(D^{1\times r}\,S)$, such that we have (15). Then, applying the contravariant left exact functor $\hom_D(\,\cdot\,, \mathcal{F})$ to the canonical short exact sequence $0 \longrightarrow \ker f \longrightarrow M \longrightarrow \operatorname{coim} f \longrightarrow 0$ and using (1) and (15), we get the following long exact sequence

$$\begin{aligned} 0 \longrightarrow \ker_{\mathcal{F}}(S.) &\longrightarrow \ker_{\mathcal{F}}(R.) \longrightarrow \hom_D(\ker f, \mathcal{F}) \\ \longrightarrow \operatorname{ext}^1_D(\operatorname{coim} f, \mathcal{F}) &\longrightarrow \operatorname{ext}^1_D(M, \mathcal{F}) \longrightarrow \operatorname{ext}^1_D(\ker f, \mathcal{F}) \\ \longrightarrow \operatorname{ext}^2_D(\operatorname{coim} f, \mathcal{F}) &\longrightarrow \operatorname{ext}^2_D(M, \mathcal{F}) \longrightarrow \cdots, \end{aligned}$$

where the $\operatorname{ext}^i_D(M, \mathcal{F})$'s are the so-called *extension abelian groups* (see, e.g., Rotman (2009)). Hence, if $R = L\,S$ is not a strict factorization, i.e., $D^{1\times r}\,S = D^{1\times q}\,R$, or equivalently $\ker f = 0$, then $\hom_D(\ker f, \mathcal{F}) = 0$, and thus $\ker_{\mathcal{F}}(S.) = \ker_{\mathcal{F}}(R.)$.

Now, if $\mathcal{F}$ is a so-called *injective* left D-module, i.e. if we have $\operatorname{ext}^i_D(P, \mathcal{F}) = 0$ for all left D-modules P and for $i \geq 1$ (see, e.g., Rotman (2009)), then the above long exact sequence reduces to the following short exact sequence:

$$0 \longrightarrow \ker_{\mathcal{F}}(S.) \longrightarrow \ker_{\mathcal{F}}(R.) \longrightarrow \hom_D(\ker f, \mathcal{F}) \longrightarrow 0.$$

We then get $\ker_{\mathcal{F}}(R.)/\ker_{\mathcal{F}}(S.) \cong \hom_D(\ker f, \mathcal{F}) \cong \ker_{\mathcal{F}}((L^T \quad S_2^T)^T.)$ by (1) and (5). In particular, if S has full row rank, i.e., $S_2 = 0$, then we finally obtain:

$$\ker_{\mathcal{F}}(R.)/\ker_{\mathcal{F}}(S.) \cong \ker_{\mathcal{F}}(L.).$$

In particular, this result holds for $\mathcal{F} = C^\infty(\mathbb{R}^n)$ and $D = \mathbb{R}\langle d_1, \ldots, d_n\rangle = \mathbb{R}[d_1, \ldots, d_n]$.

Let us now introduce the concept of a *generic solution* of the linear system $\ker_{\mathcal{F}}(R.)$.

Definition 3. Let $\mathcal{F}$ be a left D-module, $M = D^{1\times p}/(D^{1\times q}R)$ a finitely presented left D-module, and $\pi : D^{1\times p} \longrightarrow M$ the canonical projection onto M. Then, $\eta \in \ker_{\mathcal{F}}(R.)$ is called *a generic solution* if $\phi_\eta \in \hom_D(M, \mathcal{F})$, defined by $\phi_\eta(\pi(\lambda)) = \lambda\,\eta$ for all $\lambda \in D^{1\times p}$, is injective.

For instance, with the notations of Section 2, $y = (y_1 \;\ldots\; y_p)^T$ is a generic solution of $\ker_M(R.) \cong \operatorname{end}_D(M)$ corresponding to id_M.

The next result is a reformulation of the concept of a strict factorization in terms of homomorphisms.

Theorem 4. *If $R \in D^{q \times p}$, then the following assertions are equivalent:*

1. *The matrix R admits a strict factorization, i.e., there exist $L \in D^{q \times r}$ and $S \in D^{r \times q}$ such that $R = L\,S$ with $\mathrm{im}_D(.R) \subsetneq \mathrm{im}_D(.S)$.*

2. *There exist a finitely presented left D-module $\mathcal{F}$ and $f \in \hom_D(M, \mathcal{F})$ such that $\ker f \neq 0$.*

3. *There exists a finitely presented left D-module $\mathcal{F}$ such that the linear system $\ker_{\mathcal{F}}(R.)$ admits a non-generic solution in the sense of Definition 3.*

Proof. By Lemma 3, the existence of a factorization $R = L\,S$ is equivalent to the existence of a finitely presented left D-module $\mathcal{F}$ and $f \in \hom_D(M, \mathcal{F})$ such that $\mathrm{coim}\, f = D^{1 \times p}/(D^{1 \times r}\,S)$ and $\ker f = (D^{1 \times r}\,S)/(D^{1 \times q}\,R)$. Moreover, the factorization is strict if and only if $\ker f \neq 0$, i.e., if and only if the linear system $\ker_{\mathcal{F}}(R.)$ admits a non-generic solution.

□

Theorem 4 shows that the factorization problem cannot simply be solved by studying the ring $\mathrm{end}_D(M)$ since the factorizations of R correspond to finitely presented left D-modules $\mathcal{F}$ which are usually not equal to M.

Example 3. We illustrate the known fact that an operator $R \in D = B_1(\mathbb{Q})$ can admit a strict factorization $R = L\,S$ even if $\mathrm{end}_D(M)$ is reduced to $k\,\mathrm{id}_M$ (see van der Put and Singer (2003); Barkatou (2007)). Let us consider the OD operator $R = d^2 + t\,d \in D$. An element of $\mathrm{end}_D(M)$ can be defined by $P = a\,d + b$, where $a,\, b \in \mathbb{Q}(t)$, which satisfies $R\,P = Q\,R$ for a certain $Q \in D$. We have:

$$R\,P = (d^2 + t\,d)\,(a\,d + b) = a\,d^3 + (2\,\dot{a} + t\,a + b)\,d^2 + (\ddot{a} + t\,(\dot{a} + b) + 2\,\dot{b})\,d + \ddot{b} + t\,\dot{b}.$$

Hence, Q has the form $Q = a\,d + c$, where $c \in \mathbb{Q}(t)$, which yields

$$Q\,R = (a\,d + c)\,(d^2 + t\,d) = a\,d^3 + (t\,a + c)\,d^2 + (a + t\,c)\,d,$$

and thus $R\,P = Q\,R$ is equivalent to the following linear OD system:

$$\begin{cases} 2\,\dot{a} + b - c = 0, \\ \ddot{a} + t\,(\dot{a} + b - c) + 2\,\dot{b} - a = 0, \\ \ddot{b} + t\,\dot{b} = 0. \end{cases} \tag{17}$$

If we note $u := \dot{b}$, then the last equation of (17) gives $\dot{u} + t\,u = 0$, i.e., $u = c_1\,e^{-t^2/2}$, and thus we have $b = c_1 \int_0^t e^{-s^2/2}\,ds + c_2$, where c_1 and c_2 are two arbitrary constants, i.e., $c_1, c_2 \in \mathbb{Q}$. Since $b \in \mathbb{Q}(t)$, we get $c_1 = 0$ and $b = c_2$ and the above system becomes:

$$\begin{cases} \ddot{a} - t\,\dot{a} - a = \frac{d}{dt}\,(\dot{a} - t\,a) = 0, \\ b = c_2, \\ c = 2\,\dot{a} + c_2. \end{cases}$$

The integration of the first equation gives $\dot{a} - t\,a = c_3$ so that we get $a = (c_4 + c_3 \int_0^t e^{-s^2/2}\,ds)\,e^{t^2/2}$, where c_3 and c_4 are two arbitrary constants, i.e., $c_3, c_4 \in \mathbb{Q}$. Since $a \in \mathbb{Q}(t)$, we must have $c_3 = c_4 = 0$, i.e., $a = 0$ and $b = c = c_2$. Hence, we obtain $P = Q = c_2$, i.e., every element of $\mathrm{end}_D(M)$ has the form of $f = c_2\,\mathrm{id}_M$, where $c_2 \in \mathbb{Q}$, and thus $\ker\, f = 0$ if $c_2 \neq 0$. An algorithm for computing rational solutions of linear OD systems can be found in Barkatou (1999). See also Barkatou (2007); van der Put and Singer (2003) and references therein for the computation of the eigenring of a linear OD operator and a first order linear OD system.

Theorem 4 asserts that R admits a strict factorization if and only if there exists a finitely presented left D-module $\mathcal{F}$ and $f \in \hom_D(M, \mathcal{F})$ such that $\ker f \neq 0$. If we take $\mathcal{F} = D/(D\,d) \cong \mathbb{Q}(t)$ and $f \in \hom_D(M, \mathcal{F})$ defined by $f(\pi(\lambda)) = \kappa(\lambda)$ for all $\lambda \in D$, where $\kappa : D \longrightarrow \mathcal{F}$ is the canonical projection onto $\mathcal{F}$, then we get $\ker f = (D\,d)/(D\,R) \neq 0$, which shows that the OD equation $\ddot{\eta} + t\,\dot{\eta} = 0$ admits the non-generic solution $\eta = 1$ and yields the strict factorization $R = L\,S$, where $L = d + t$ and $S = d$.

We refer the reader to the AlgebraicAnalysis package (Cluzeau et al. (2013)) which computes general homomorphisms of two finitely presented differential modules by integrating linear PD systems in the unknown coefficients of a fixed order ansatz for P. For instance, for the above example, the AlgebraicAnalysis package integrates (17) to get that the general endomorphism of M is defined by:

$$P = \left(c_4 + c_3 \int_0^t e^{-s^2/2}\, ds\right) e^{t^2/2}\, d + c_1 \int_0^t e^{-s^2/2}\, ds + c_2, \quad c_1, \ldots, c_4 \in \mathbb{Q}.$$

If M is a simple left D-module (see 9 of Definition 1) and $f \in \hom_D(M, \mathcal{F}) \setminus \{0\}$, then we have $\ker f = 0$, which shows that f is injective. If M is a left D-module finitely presented by $R \in D^{q \times p}$, then R does not admit a strict factorization by Theorem 4. Moreover, if $\mathcal{F} = M$, then $\operatorname{im} f = M$ since $\operatorname{im} f$ is a non-trivial left D-submodule of M, which shows that a non-trivial $f \in \operatorname{end}_D(M)$ is an automorphism, i.e., $f \in \operatorname{aut}_D(M)$. This result is the so-called *Schur's lemma* stating that the endomorphism ring $\operatorname{end}_D(M)$ of a simple left D-module M is a *division ring* (see, e.g., McConnell and Robson (2000)).

Example 4. Let us show that $M = D/(D\, d_1 + D\, d_2) \cong k[x_1, x_2]$ is a simple left $D = A_2(\mathbb{Q})$-module. If L is a non-trivial left D-submodule of M and $z := d\, y \in L$, where $d \in D \setminus \{0\}$, $y = \pi(1)$ is the generator of M and $\pi : D \longrightarrow M$ the canonical projection onto M, then we can assume without loss of generality that $d \in k[x_1, x_2]$ since y satisfies the following relations:

$$\begin{cases} d_1\, y = 0, \\ d_2\, y = 0. \end{cases} \tag{18}$$

Using (18), we get $d_i\, z = d_i\, (d\, y) = d\, d_i\, y + \frac{\partial\, d}{\partial x_i}\, y = \frac{\partial\, d}{\partial x_i}\, y = 0$ for $i = 1,\, 2$. Thus, there exists $d' \in D$ such that $y = d'\, z \in L$ for a certain $d' \in D \setminus \{0\}$, i.e., $L = M$, which proves that M is a simple left D-module. Using Proposition 2.5 of Cluzeau and Quadrat (2008), we can easily prove that $\operatorname{aut}_D(M) = k \setminus \{0\}$.

5 Decomposition problem

5.1 General results

The existence of a non-trivial decomposition $M = M_1 \oplus M_2$ of a left D-module M is known to be equivalent to the existence of a non-trivial *idempotent element* $f \in \operatorname{end}_D(M)$, i.e., $f^2 = f$, where f is neither id_M nor 0. See, e.g., McConnell and Robson (2000); Cluzeau and Quadrat (2008).

Let us state a characterization of an idempotent element of $\operatorname{end}_D(M)$.

Lemma 4 (Cluzeau and Quadrat (2008))**.** *Let $M = D^{1 \times p}/(D^{1 \times q}\, R)$ be the left D-module finitely presented by $R \in D^{q \times p}$ and $R_2 \in D^{r \times q}$ a matrix such that $\ker_D(.R) = \operatorname{im}_D(.R_2)$. Then, $f \in \operatorname{end}_D(M)$, defined by a matrix $P \in D^{p \times p}$ satisfying $R\, P = Q\, R$ for a certain $Q \in D^{q \times q}$, is an idempotent element of $\operatorname{end}_D(M)$, i.e., $f^2 = f$, if and only if there exists $Z \in D^{p \times q}$ such that:*

$$P^2 = P + Z\, R. \tag{19}$$

Then, there exists a matrix $Z' \in D^{q \times r}$ such that:

$$Q^2 = Q + R\, Z + Z'\, R_2.$$

In particular, if $R \in D^{q \times p}$ has full row rank, then we have $Q^2 = Q + R\, Z$.

An algorithm for the computation of idempotents of $\operatorname{end}_D(M)$ is given in Algorithm 4.1 of Cluzeau and Quadrat (2008).

If $f^2 = f \in \operatorname{end}_D(M)$, then we have $M = \ker f \oplus \operatorname{im} f$. Indeed, we have $m = f(m) + (m - f(m))$ for all $m \in M$, where $m - f(m) \in \ker f$.

Let us now generalize Lemma 4.4 of Cluzeau and Quadrat (2008).

Lemma 5. *Let* $R \in D^{q\times p}$, $\ker_D(.R) = \operatorname{im}_D(.R_2)$, $\ker_D(.R_2) = \operatorname{im}_D(.R_3)$, $M = D^{1\times p}/(D^{1\times q}\, R)$ *and* $f \in \operatorname{end}_D(M)$ *an idempotent defined by* $P \in D^{p\times p}$ *satisfying* $P^2 = P + Z\, R$ *and* $R\, P = Q\, R$ *for a certain matrix* $Z \in D^{p\times q}$ *and a matrix* Q *necessarily of the form* $Q^2 = Q + R\, Z + Z'\, R_2$ *for a certain matrix* $Z' \in D^{q\times r}$*. Moreover, let* $S \in D^{r\times r}$ *be such that* $R_2\, Q = S\, R_2$*. If there exist* $\Delta \in D^{p\times q}$, $\Delta_2 \in D^{q\times r}$, $U \in D^{p\times r}$ *and* $V \in D^{q\times s}$ *such that*

$$\begin{cases} \Delta\, R\, \Delta + (P - I_p)\, \Delta + \Delta\, Q + Z = U\, R_2, \\ \Delta_2\, R_2\, \Delta_2 + (Q - I_q + R\, \Delta)\, \Delta_2 + \Delta_2\, S + R\, U + Z' = V\, R_3, \end{cases} \tag{20}$$

then the matrices defined by $\overline{P} := P + \Delta\, R$ *and* $\overline{Q} := Q + R\, \Delta + \Delta_2\, R_2$ *satisfy* $R\, \overline{P} = \overline{Q}\, R$, $\overline{P}^2 = \overline{P}$, $\overline{Q}^2 = \overline{Q}$ *and* $f(\pi(\lambda)) = \pi(\lambda\, \overline{P})$ *for all* $\lambda \in D^{1\times p}$*.*

If R *has full row rank, then (20) reduces to the following* algebraic Riccati equation*:*

$$\Delta\, R\, \Delta + (P - I_p)\, \Delta + \Delta\, Q + Z = 0. \tag{21}$$

Proof. Considering $\overline{P} := P + \Delta\, R$, we can check that

$$\overline{P}^2 - \overline{P} = (\Delta\, R\, \Delta + (P - I_p)\, \Delta + \Delta\, Q + Z)\, R,$$

which shows that $\overline{P}^2 = \overline{P}$ if and only if the first equation of (20) holds for a certain $U \in D^{p\times r}$. Now, using the first equation of (20), we can check that the matrix $\overline{Q} := Q + R\, \Delta + \Delta_2\, R_2$ satisfies

$$\begin{aligned} \overline{Q}^2 - \overline{Q} &= R\, (\Delta\, R\, \Delta + (P - I_p)\, \Delta + \Delta\, Q + Z) + (\Delta_2\, R_2\, \Delta_2 + (Q - I_q + R\, \Delta)\, \Delta_2 + \Delta_2\, S + Z')\, R_2 \\ &= (\Delta_2\, R_2\, \Delta_2 + (Q - I_q + R\, \Delta)\, \Delta_2 + \Delta_2\, S + R\, U + Z')\, R_2, \end{aligned}$$

and thus $\overline{Q}^2 = \overline{Q}$ if and only the second equation of (20) holds for a certain $V \in D^{q\times s}$. Finally, (20) reduces to (21) when R has full row rank. □

Remark 2. If D is a polynomial ring over a computational field k, then a solution $\Delta \in D^{p\times q}$ of the first equation of (20) can be obtained by considering an ansatz for Δ for a fixed total degree and by solving the quadratic equations in the parameters of the ansatz so that all the normal forms of the rows of $\Delta\, R\, \Delta + (P - I_p)\, \Delta + \Delta\, Q + Z$ with respect of a Gröbner basis of the D-module $D^{1\times r}\, R_2$ reduce to zero. In this way, we can obtain a solution Δ of the first equation of (20) for a certain $U \in D^{p\times r}$. Then, the second equation of (20) can be solved by considering an ansatz for Δ_2 for a fixed total degree and by solving the quadratic equations in the parameters of the ansatz so that all the normal forms of the rows of $\Delta_2\, R_2\, \Delta_2 + (Q - I_q + R\, \Delta)\, \Delta_2 + \Delta_2\, S + R\, U + Z'$ with respect of a Gröbner basis of the D-module $D^{1\times s}\, R_3$ reduce to zero. We can get a solution Δ_2 of the second equation of (20) for a certain $V \in D^{q\times s}$.

The interest of defining an idempotent f of $\operatorname{end}_D(M)$ by two idempotent matrices $\overline{P} \in D^{p\times p}$ and $\overline{Q} \in D^{q\times q}$ (i.e., two *projectors*) is that the left D-modules $\ker_D(.\overline{P})$, $\operatorname{im}_D(.\overline{P})$, $\ker_D(.\overline{Q})$ and $\operatorname{im}_D(.\overline{Q})$ then satisfy

$$\begin{cases} D^{1\times p} = \ker_D(.\overline{P}) \oplus \operatorname{im}_D(.\overline{P}), \\ D^{1\times q} = \ker_D(.\overline{Q}) \oplus \operatorname{im}_D(.\overline{Q}), \end{cases} \tag{22}$$

which shows that $\ker_D(.\overline{P})$, $\operatorname{im}_D(.\overline{P})$, $\ker_D(.\overline{Q})$ and $\operatorname{im}_D(.\overline{Q})$ are finitely generated projective left D-modules (see 3 of Definition 1). In this case, we also have $\ker_D(.\overline{P}) = \operatorname{im}_D(.(I_p - \overline{P}))$ and $\operatorname{im}_D(.\overline{P}) = \ker_D(.(I_p - \overline{P}))$ and similarly with $\overline{Q}$.

Let us state again two standard results of homological algebra that will be used in what follows.

Proposition 2 (Rotman (2009))**.** *Let* $0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$ *be a short exact sequence. Then, the following assertions are equivalent:*

1. *There exists* $u \in \hom_D(M'', M)$ *such that* $g \circ u = \operatorname{id}_{M''}$*.*
2. *There exists* $v \in \hom_D(M, M')$ *such that* $v \circ f = \operatorname{id}_{M'}$*.*

3. *There exist* $u \in \hom_D(M'', M)$ *and* $v \in \hom_D(M, M')$ *such that* $f \circ v + u \circ g = \mathrm{id}_M$.
4. *We have* $M \cong M' \oplus M''$, *where the isomorphism is defined by* $(f \quad u) : M' \oplus M'' \longrightarrow M$ *and* $(v^T \quad g^T)^T : M \longrightarrow M' \oplus M''$, *with* u *and* v *defined as above, i.e.:*

$$(f \quad u) \circ \begin{pmatrix} v \\ g \end{pmatrix} = \mathrm{id}_M, \quad \begin{pmatrix} v \\ g \end{pmatrix} \circ (f \quad u) = \mathrm{id}_{M' \oplus M''}. \tag{23}$$

The short exact sequence is then said to split *or is a* split short exact sequence, *which is denoted by:*

$$0 \longrightarrow M' \underset{v}{\overset{f}{\rightleftarrows}} M \underset{u}{\overset{g}{\rightleftarrows}} M'' \longrightarrow 0. \tag{24}$$

Proposition 3 (Rotman (2009)). *If* $0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$ *is a short exact sequence and* M'' *is a projective left* D*-module, then the exact sequence splits, i.e.* $M \cong M' \oplus M''$.

Example 5. Let us suppose that the left D-module $M = D^{1\times p}/(D^{1\times q}\, R)$ is free of rank m. Then, composing the left D-isomorphism $\iota : M \longrightarrow D^{1\times m}$ with $\pi \in \hom_D(D^{1\times p}, M)$ defined by the finite presentation $0 \longrightarrow \mathrm{im}_D(.R) \xrightarrow{i} D^{1\times p} \xrightarrow{\pi} M \longrightarrow 0$ of M, we obtain the short exact sequence $0 \longrightarrow \mathrm{im}_D(.R) \xrightarrow{i} D^{1\times p} \xrightarrow{.Q} D^{1\times m} \longrightarrow 0$, where the matrix $Q := (Q_{1\bullet}^T \ \ldots \ Q_{p\bullet}^T)^T \in D^{p\times m}$ is defined by $Q_{j\bullet} = (\iota \circ \pi)(f_j) \in D^{1\times m}$ for $j = 1, \ldots, p$ and $\{f_j\}_{j=1,\ldots,p}$ is the standard basis of $D^{1\times p}$, i.e., $\iota \circ \pi = .Q$. Using Proposition 3, the above exact sequence splits, and thus there exists $T \in D^{m\times p}$ such that $T\, Q = I_m$. Hence, if M is free of rank m, then there exist $Q \in D^{p\times m}$ and $T \in D^{m\times p}$ such that $\ker_D(.Q) = \mathrm{im}_D(.R)$ and $T\, Q = I_m$. Conversely, if such matrices exist, then the above short exact sequence holds, which shows that $M = \operatorname{coker} i \cong D^{1\times m}$, i.e., that M is a free left D-module of rank m. For more details, see Chyzak et al. (2005); Fabiańska and Quadrat (2007); Quadrat and Robertz (2007a).

We now recall Theorem 4.2 of Cluzeau and Quadrat (2008).

Theorem 5 (Cluzeau and Quadrat (2008)). *Let* $M = D^{1\times p}/(D^{1\times q}\, R)$ *be the left* D*-module finitely presented by* $R \in D^{q\times p}$ *and* $f \in \mathrm{end}_D(M)$ *an idempotent, i.e.,* $f^2 = f$, *defined by two idempotent matrices* $P \in D^{p\times p}$ *and* $Q \in D^{q\times q}$ *satisfying the relations* $R\,P = Q\,R$, $P^2 = P$ *and* $Q^2 = Q$. *If the finitely generated projective left* D*-modules* $\ker_D(.P)$, $\mathrm{im}_D(.P)$, $\ker_D(.Q)$ *and* $\mathrm{im}_D(.Q)$ *are free of rank* m, $p - m = \mathrm{trace}(P)$, l, $q - l = \mathrm{trace}(Q)$, *then there exist four matrices* $U_1 \in D^{m\times p}$, $U_2 \in D^{(p-m)\times p}$, $V_1 \in D^{l\times q}$ *and* $V_2 \in D^{(q-l)\times q}$ *satisfying*

1. $U := (U_1^T \quad U_2^T)^T \in \mathrm{GL}_p(D)$, $V := (V_1^T \quad V_2^T)^T \in \mathrm{GL}_q(D)$,
2. $\overline{R} := V\,R\,U^{-1} = \begin{pmatrix} V_1\,R\,W_1 & 0 \\ 0 & V_2\,R\,W_2 \end{pmatrix} \in D^{q\times p}$, *where* $U^{-1} := (W_1 \quad W_2)$, $W_1 \in D^{p\times m}$ *and* $W_2 \in D^{p\times(p-m)}$.

In particular, the full row rank matrix U_1 *(resp.,* U_2, V_1, V_2*) defines a basis of the free left* D*-module* $\ker_D(.P)$ *(resp.,* $\mathrm{im}_D(.P)$, $\ker_D(.Q)$, $\mathrm{im}_D(.Q)$*) of rank* m *(resp.,* $p - m$, l, $q - l$*), i.e., we have:*

$$\begin{cases} \ker_D(.P) = \mathrm{im}_D(.U_1), \\ \mathrm{im}_D(.P) = \mathrm{im}_D(.U_2), \\ \ker_D(.Q) = \mathrm{im}_D(.V_1), \\ \mathrm{im}_D(.Q) = \mathrm{im}_D(.V_2). \end{cases} \tag{25}$$

If $V^{-1} := (X_1 \quad X_2)$, *where* $X_1 \in D^{q\times l}$ *and* $X_2 \in D^{(q-l)\times q}$, *then we have the following diagram formed by horizontal split exact sequences, vertical exact sequences and whose squares commute in both directions:*

$$\begin{tikzcd}
0 \arrow[r] & D^{1\times l} \arrow[r, shift left, ".V_1"] \arrow[d, ".V_1\,R\,W_1"'] & D^{1\times q} \arrow[l, shift left, ".X_1"] \arrow[r, shift left, ".X_2"] \arrow[d, ".R"'] & D^{1\times (q-l)} \arrow[l, shift left, ".V_2"] \arrow[r] \arrow[d, ".V_2\,R\,W_2"] & 0 \\
0 \arrow[r] & D^{1\times m} \arrow[r, shift left, ".U_1"] \arrow[d, "\tau"'] & D^{1\times p} \arrow[l, shift left, ".W_1"] \arrow[r, shift left, ".W_2"] \arrow[d, "\pi"'] & D^{1\times (p-m)} \arrow[l, shift left, ".U_2"] \arrow[r] \arrow[d, "\sigma"] & 0 \\
0 \arrow[r] & \ker f \arrow[r, shift left, "i"] \arrow[d] & M \arrow[l, shift left, "\mathrm{id}_M - f"] \arrow[r, shift left, "f"] \arrow[d] & \operatorname{im} f \arrow[l, shift left, "f_{|\operatorname{im} f}"] \arrow[r] \arrow[d] & 0. \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{26}$$

In particular, we have $M = \ker f \oplus \operatorname{im} f$, *where*

$$\begin{cases} \ker f \cong D^{1\times m}/(D^{1\times l}\,(V_1\,R\,W_1)), \\ \operatorname{im} f \cong D^{1\times (p-m)}/(D^{1\times (q-l)}\,(V_2\,R\,W_2)), \end{cases}$$

i.e., the first (resp., second) diagonal block of $\overline{R}$ *corresponds to* $\ker f$ *(resp.,* $\operatorname{im} f$*) up to an isomorphism.*

For rings D and modules satisfying the conditions of Theorem 2, the projective left D-modules $\ker_D(.P)$, $\operatorname{im}_D(.P)$, $\ker_D(.Q)$ and $\operatorname{im}_D(.Q)$ satisfying (22) are free of finite rank.

We now prove the converse of Theorem 5.

Theorem 6. *A matrix* $R \in D^{q\times p}$ *is equivalent to a block-diagonal matrix* $\overline{R} \in D^{q\times p}$*, i.e., there exist* $U \in \mathrm{GL}_p(D)$ *and* $V \in \mathrm{GL}_q(D)$ *such that*

$$\overline{R} := V\,R\,U^{-1} = \begin{pmatrix} \overline{R}_1 & 0 \\ 0 & \overline{R}_2 \end{pmatrix}, \quad \overline{R}_1 \in D^{l\times m}, \quad \overline{R}_2 \in D^{(q-l)\times(p-m)}, \tag{27}$$

if and only if there exist two idempotent matrices $P \in D^{p\times p}$ *and* $Q \in D^{q\times q}$*, i.e.,* $P^2 = P$*,* $Q^2 = Q$*, satisfying* $R\,P = Q\,R$ *and such that the projective left* D*-modules* $\ker_D(.P)$*,* $\operatorname{im}_D(.P)$*,* $\ker_D(.Q)$ *and* $\operatorname{im}_D(.Q)$ *are free of rank respectively* m*,* $p-m$*,* l*, and* $q-l$*. We then have* $\ker_D(.P) = D^{1\times m}\,U_1$*,* $\operatorname{im}_D(.P) = D^{1\times(p-m)}\,U_2$*,* $\ker_D(.Q) = D^{1\times l}\,V_1$ *and* $\operatorname{im}_D(.Q) = D^{1\times(q-l)}\,V_2$*, where* $U := (U_1^T \quad U_2^T)^T$*,* $U_1 \in D^{m\times p}$ *and* $U_2 \in D^{(p-m)\times p}$*, and* $V = (V_1^T \quad V_2^T)^T$*,* $V_1 \in D^{l\times q}$ *and* $V_2 \in D^{(q-l)\times q}$*.*

Proof. Let us suppose that R is equivalent to the matrix $\overline{R}$ defined by (27). We can check that the matrices $\overline{P} \in D^{p\times p}$ and $\overline{Q} \in D^{q\times q}$ defined by

$$\overline{P} := \begin{pmatrix} 0 & 0 \\ 0 & I_{p-m} \end{pmatrix}, \quad \overline{Q} := \begin{pmatrix} 0 & 0 \\ 0 & I_{q-l} \end{pmatrix}$$

satisfy $\overline{P}^2 = \overline{P}$, $\overline{Q}^2 = \overline{Q}$, and $\overline{R}\,\overline{P} = \overline{Q}\,\overline{R}$. Now, if $\lambda = (\lambda_1 \quad \lambda_2) \in \ker_D(.\overline{P})$, where $\lambda_1 \in D^{1\times m}$, $\lambda_2 \in D^{1\times(p-m)}$, then we get $\lambda_2 = 0$, i.e., $(\lambda_1 \quad 0) \in \ker_D(.\overline{P})$, which proves that $\ker_D(.\overline{P}) = D^{1\times m}\,(I_m \quad 0)$ and, since $(I_m \quad 0)$ has full row rank, $\ker_D(.\overline{P})$ is a free left D-module of rank m. Similarly, $\ker_D(.\overline{Q})$ (resp., $\ker_D(.(I_p - \overline{P}))$, $\ker_D(.(I_q - \overline{Q}))$) is a free left D-module of rank l (resp., $p-m$, $q-l$). Now, if we set $P := U^{-1}\,\overline{P}\,U$ and $Q := V^{-1}\,\overline{Q}\,V$, then we can easily check that $R\,P = Q\,R$, $P^2 = P$ and $Q^2 = Q$. Moreover, we have $\ker_D(.P) = \ker_D(.\overline{P})\,U$, $\operatorname{im}_D(.P) = \operatorname{im}_D(.\overline{P})\,U$, $\ker_D(.Q) = \ker_D(.\overline{Q})\,V$ and $\operatorname{im}_D(.Q) = \operatorname{im}_D(.\overline{Q})\,V$, and since $U \in \mathrm{GL}_p(D)$ and $V \in \mathrm{GL}_q(D)$, we obtain that $\ker_D(.P)$, $\operatorname{im}_D(.P)$, $\ker_D(.Q)$ and $\operatorname{im}_D(.Q)$ are free left D-modules of rank respectively m, $p-m$, l, and $q-l$. Finally, if we note $U := (U_1^T \quad U_2^T)^T$, where $U_1 \in D^{m\times p}$ and $U_2 \in D^{(p-m)\times p}$, and $V := (V_1^T \quad V_2^T)^T$, where $V_1 \in D^{l\times q}$ and $V_2 \in D^{(q-l)\times q}$, then we get $\ker_D(.P) = D^{1\times m}\,(I_m \quad 0)\,U = D^{1\times m}\,U_1$, $\operatorname{im}_D(.P) = D^{1\times(p-m)}\,(0 \quad I_{p-m})\,U = D^{1\times(p-m)}\,U_2$, $\ker_D(.Q) = D^{1\times l}(I_l \quad 0)\,V = D^{1\times l}\,V_1$ and $\operatorname{im}_D(.Q) = D^{1\times(q-l)}\,(0 \quad I_{q-l})\,V = D^{1\times(q-l)}\,V_2$. □

Finally, we show that Theorems 5 and 6 are particular instances of the following more general result which proves that the diagram (26) is a particular instance of the diagram (28) below.

Theorem 7. *The following results are equivalent:*

1. *The following diagram, formed by horizontal split exact sequences and vertical exact sequences, commutes in both directions:*

$$\begin{tikzcd} 0 \arrow[r] & D^{1\times l} \arrow[r, shift left, ".V_1"] \arrow[d, ".R'"'] & D^{1\times q} \arrow[l, shift left, ".X_1"] \arrow[r, shift left, ".X_2"] \arrow[d, ".R"'] & D^{1\times (q-l)} \arrow[l, shift left, ".V_2"] \arrow[r] \arrow[d, ".R''"] & 0 \\ 0 \arrow[r] & D^{1\times m} \arrow[r, shift left, ".U_1"] \arrow[d, "\pi'"'] & D^{1\times p} \arrow[l, shift left, ".W_1"] \arrow[r, shift left, ".W_2"] \arrow[d, "\pi"'] & D^{1\times (p-m)} \arrow[l, shift left, ".U_2"] \arrow[r] \arrow[d, "\pi''"] & 0 \\ 0 \arrow[r] & M' \arrow[r, shift left, "f"] \arrow[d] & M \arrow[l, shift left, "v"] \arrow[r, shift left, "g"] \arrow[d] & M'' \arrow[l, shift left, "u"] \arrow[r] \arrow[d] & 0. \\ & 0 & 0 & 0 & \end{tikzcd} \tag{28}$$

2. *There exist matrices* $V \in \mathrm{GL}_q(D)$ *and* $W \in \mathrm{GL}_p(D)$ *such that:*

$$V\,R\,W = \begin{pmatrix} R' & 0 \\ 0 & R'' \end{pmatrix}. \tag{29}$$

With the notations (28) and (29), the unimodular matrices V *and* W *can be defined by*

$$V := (V_1^T \quad V_2^T)^T, \quad V^{-1} = X := (X_1 \quad X_2), \quad W := (W_1 \quad W_2), \quad W^{-1} = U := (U_1^T \quad U_2^T)^T \tag{30}$$

and we have $R' = V_1\,R\,W_1$ *and* $R'' = V_2\,R\,W_2$.

Proof. 1 $\Rightarrow$ 2. Since the first and the second horizontal exact sequences of (28) split, using (23), the matrices defined by $V := (V_1^T \quad V_2^T)^T \in D^{q\times q}$, $X := (X_1 \quad X_2) \in D^{q\times q}$, $U := (U_1^T \quad U_2^T) \in D^{p\times p}$ and $W := (W_1 \quad W_2) \in D^{p\times p}$ satisfy $V\,X = X\,V = I_q$ and $U\,W = W\,U = I_p$, which shows that $V \in \mathrm{GL}_q(D)$, $X = V^{-1}$, $U \in \mathrm{GL}_p(D)$ and $W = U^{-1}$. Moreover, we have:

$$V\,R\,W = \begin{pmatrix} V_1\,R\,W_1 & V_1\,R\,W_2 \\ V_2\,R\,W_1 & V_2\,R\,W_2 \end{pmatrix}. \tag{31}$$

Using the commutativity of (28) in the both directions, we have the relations $V_1\,R = R'\,U_1$, $V_2\,R = R''\,U_2$, $X_1\,R' = R\,W_1$ and $X_2\,R'' = R\,W_2$. Using the identities $U_1\,W_1 = I_m$, $U_2\,W_2 = I_{p-m}$, $U_1\,W_2 = 0$ and $U_2\,W_1 = 0$, we obtain

$$\left\{ \begin{array}{l} V_1\,R\,W_1 = R'\,U_1\,W_1 = R', \\ V_2\,R\,W_2 = R''\,U_2\,W_2 = R'', \\ V_1\,R\,W_2 = R'\,U_1\,W_2 = 0, \\ V_2\,R\,W_1 = R''\,U_2\,W_1 = 0, \end{array} \right.$$

which finally proves 2.

2 $\Rightarrow$ 1. Using (23) and the notations $V = (V_1^T \quad V_2^T)^T$, where $V_1 \in D^{l\times q}$, $V_2 \in D^{(q-l)\times q}$, and $W = (W_1 \quad W_2)$, where $W_1 \in D^{p\times m}$ and $W_2 \in D^{p\times (p-m)}$, the facts that $V \in \mathrm{GL}_q(D)$ and $W \in \mathrm{GL}_p(D)$ are equivalent to the first two horizontal splits exact sequences of (28), where $X := V^{-1} = (X_1 \quad X_2)$, $U := W^{-1} = (U_1^T \quad U_2^T)^T$, $X_1 \in D^{q\times l}$, $X_2 \in D^{q\times (q-l)}$, $U_1 \in D^{m\times p}$ and $U_2 \in D^{(p-m)\times p}$. Using (29) and (31), we get $R' = V_1\,R\,W_1$, $R'' = V_2\,R\,W_2$, $V_1\,R\,W_2 = 0$ and $V_2\,R\,W_1 = 0$. Using the identity $W_1\,U_1 + W_2\,U_2 = I_p$, we get $R'\,U_1 = V_1\,R\,(W_1\,U_1) = V_1\,R\,(I_p - W_2\,U_2) = V_1\,R$. Similarly, we have $R''\,U_2 = V_2\,R\,(W_2\,U_2) = V_2\,R\,(I_p - W_1\,U_1) = V_2\,R$. Now, $V_1\,R\,W_2 = 0$ yields $X_1\,V_1\,R\,W_2 = 0$ which, combines

with the identity $X_1 V_1 + X_2 V_2 = I_q$, gives $R W_2 = X_2 (V_2 R W_2) = X_2 R''$. Similarly, $V_2 R W_1 = 0$ yields $X_2 V_2 R W_1 = 0$ which, combines with the identity $X_1 V_1 + X_2 V_2 = I_q$, gives $R W_1 = X_1 (V_1 R W_1) = X_1 R'$, which shows that the following diagram

$$\begin{tikzcd} 0 \arrow[r] & D^{1\times l} \arrow[r, shift left, ".V_1"] \arrow[d, ".R'"'] & D^{1\times q} \arrow[l, shift left, ".X_1"] \arrow[r, shift left, ".X_2"] \arrow[d, ".R"'] & D^{1\times (q-l)} \arrow[l, shift left, ".V_2"] \arrow[r] \arrow[d, ".R''"] & 0 \\ 0 \arrow[r] & D^{1\times m} \arrow[r, shift left, ".U_1"] & D^{1\times p} \arrow[l, shift left, ".W_1"] \arrow[r, shift left, ".W_2"] & D^{1\times (p-m)} \arrow[l, shift left, ".U_2"] \arrow[r] & 0 \end{tikzcd}$$

is formed by horizontal splits exact sequences and commutes in both direction.

Finally, let us define $M' := D^{1\times m}/(D^{1\times l} R')$, $M := D^{1\times p}/(D^{1\times q} R)$, $M'' := D^{1\times (p-m)}/(D^{1\times (q-l)} R'')$, $\pi' : D^{1\times m} \longrightarrow M'$, $\pi : D^{1\times p} \longrightarrow M$ and $\pi'' : D^{1\times (p-m)} \longrightarrow M''$ the corresponding canonical projections, and the following well-defined homomorphisms:

$$\begin{array}{llllllllllll} f: M' & \longrightarrow & M & g: M & \longrightarrow & M'' & u: M'' & \longrightarrow & M & v: M & \longrightarrow & M' \\ \pi'(\lambda') & \longmapsto & \pi(\lambda' U_1), & \pi(\lambda) & \longmapsto & \pi''(\lambda W_2), & \pi''(\lambda'') & \longmapsto & \pi(\lambda'' U_2), & \pi(\lambda) & \longmapsto & \pi'(\lambda W_1). \end{array}$$

Using the identities $U_1 W_2 = 0$, $U_2 W_1 = 0$, $U_1 W_1 = I_m$, $U_2 W_2 = I_{p-m}$ and $W_1 U_1 + W_2 U_2 = I_p$, we get

$$\left\{ \begin{array}{l} (g \circ f)(\pi'(\lambda')) = \pi''(\lambda' (U_1 W_2)) = 0, \\ (v \circ u)(\pi''(\lambda'')) = \pi'(\lambda'' (U_2 W_1)) = 0, \\ (v \circ f)(\pi'(\lambda')) = \pi'(\lambda' (U_1 W_1)) = \pi'(\lambda'), \\ (g \circ u)(\pi''(\lambda'')) = \pi''(\lambda'' (U_2 W_2)) = \pi''(\lambda''), \\ (\mathrm{id}_M - f \circ v - u \circ g)(\pi(\lambda)) = \pi(\lambda (I_p - W_1 U_1 - W_2 U_2)) = 0, \end{array} \right.$$

i.e., $g \circ f = 0$, $v \circ u = 0$, $v \circ f = \mathrm{id}_{M'}$, $g \circ u = \mathrm{id}_{M''}$ and $f \circ v + u \circ g = \mathrm{id}_M$, which shows that f and u are injective, g and v are surjective, $\mathrm{im}\, f \subseteq \ker g$ and $\mathrm{im}\, u \subseteq \ker v$. If $m \in \ker g$ (resp., $m \in \ker v$), using the identity $f \circ v + u \circ g = \mathrm{id}_M$, we then get $m = f(v(m)) \in \mathrm{im}\, f$ (resp., $m = u(g(m)) \in \mathrm{im}\, u$), which shows that $\ker g = \mathrm{im}\, f$ and $\ker v = \mathrm{im}\, u$, and thus that (24) is a split exact sequence, which proves 1. □

Remark 3. If D is a commutative ring and M a decomposable D-module, then so is the D-module $\mathrm{end}_D(M)$. Indeed, if D is a commutative ring and M a decomposable D-module, i.e., $M = M_1 \oplus M_2$ where M_1 and M_2 are two non-trivial D-modules, i.e., $M_i \neq 0$ for $i = 1, 2$, then we get

$$\begin{aligned} \mathrm{end}_D(M) &= \mathrm{hom}_D(M_1 \oplus M_2, M_1 \oplus M_2), \\ &= \mathrm{end}_D(M_1) \oplus \mathrm{end}_D(M_2) \oplus \mathrm{hom}_D(M_1, M_2) \oplus \mathrm{hom}_D(M_2, M_1), \end{aligned}$$

by the additivity property of the $\mathrm{hom}_D(\cdot\,,\,\cdot)$ bifunctor for the category of D-modules (Rotman (2009)). Since the rings $\mathrm{end}_D(M_1)$ and $\mathrm{end}_D(M_2)$ respectively contain id_{M_1} and id_{M_2}, they are non-trivial, which shows that the D-module $\mathrm{end}_D(M)$ is decomposable. In other words, if $\mathrm{end}_D(M)$ is indecomposable as a D-module, then so is M. This result can sometimes be used to prove that a finitely generated module over a commutative ring D is indecomposable.

5.2 The decomposition problem with an identity diagonal block

We now consider the case where one of the diagonal blocks of (27) is the identity matrix, say, for instance, $\overline{R}_1$. In this case, the linear system $\ker_{\mathcal{F}}(R.)$ is then equivalent to the linear system $\ker_{\mathcal{F}}(\overline{R}_2.)$ defined by fewer unknowns and fewer equations. Such a reduction is called *Serre's reduction* of $\ker_{\mathcal{F}}(R.)$ (Boudellioua and Quadrat (2010)). This problem will also be studied in Section 6 by means of different techniques than the ones developed in this section and we shall compare them.

Given a matrix $R \in D^{q\times p}$ and the left D-module M finitely presented by R, the endomorphism ring $\mathrm{end}_D(M)$ always contains the two trivial idempotents, namely:

1. $f = \mathrm{id}_M$ defined by $P = I_p$ and $Q = I_q$,

2. $f = 0_M$ defined by $P = 0_p$ and $Q = 0_q$.

Applying Lemma 5 to these trivial idempotents with $Z = 0$ and $Z' = 0$ and considering $U = 0$, $V = 0$ and $\Delta_2 = 0$, we obtain the following corollary.

Corollary 1. *We have the following results.*

1. *If $\Delta \in D^{p\times q}$ is a solution of $\Delta\, R\, \Delta = -\Delta$, then $\overline{P} := I_p + \Delta\, R$ and $\overline{Q} := I_q + R\, \Delta$ satisfy $R\, \overline{P} = \overline{Q}\, R$, $\overline{P}^2 = \overline{P}$, $\overline{Q}^2 = \overline{Q}$, and $f(\pi(\lambda)) = \pi(\lambda\, \overline{P}) = \pi(\lambda)$ for all $\lambda \in D^{1\times p}$, i.e., $f = \mathrm{id}_M$.*

2. *If $\Delta \in D^{p\times q}$ is a solution of $\Delta\, R\, \Delta = \Delta$, then $\overline{P} := \Delta\, R$ and $\overline{Q} := R\, \Delta$ satisfy $R\, \overline{P} = \overline{Q}\, R$, $\overline{P}^2 = \overline{P}$, $\overline{Q}^2 = \overline{Q}$, and $f(\pi(\lambda)) = 0$ for all $\lambda \in D^{1\times p}$, i.e., $f = 0_M$.*

Remark 4. If $\Delta \in D^{p\times q}$ satisfies $\Delta\, R\, \Delta = \Delta$, then $\Theta := -\Delta$ satisfies $\Theta\, R\, \Theta = -\Theta$ and conversely. Thus, the idempotent matrices $\overline{P}_1 := \Delta\, R$ and $\overline{Q}_1 := R\, \Delta$ define the idempotent endomorphism 0_M if and only if the idempotent matrices $\overline{P}_2 := I_p - \Delta\, R$ and $\overline{Q}_2 := I_q - R\, \Delta$ define the idempotent id_M. Moreover, we have $\overline{P}_1 + \overline{P}_2 = I_p$ and $\overline{Q}_1 + \overline{Q}_2 = I_q$.

The next two remarks will play an important role in what follows.

Remark 5. Let $\Delta \in D^{p\times q}$ satisfy $\Delta\, R\, \Delta = -\Delta$, i.e., $\Delta\, (-R)\, \Delta = \Delta$ and let us consider the following short exact sequence

$$0 \longrightarrow \mathrm{im}_D(.\Delta) \xrightarrow{i} D^{1\times q} \xrightarrow{\gamma} \mathrm{coker}_D(.\Delta) \longrightarrow 0, \tag{32}$$

where i is the canonical inclusion and γ the canonical projection. If $v \in \hom_D(D^{1\times q}, \mathrm{im}_D(.\Delta))$ is defined by $v(\mu) = -\mu\, R\, \Delta$ for all $\mu \in D^{1\times q}$, then we get $(v \circ i)(\nu\, \Delta) = -\nu\, \Delta\, R\, \Delta = \nu\, \Delta$, i.e., $v \circ i = \mathrm{id}_{\mathrm{im}_D(.\Delta)}$, which shows that the short exact sequence (32) splits by 2 of Proposition 2. By 4 of Proposition 2, this yields $D^{1\times q} \cong \mathrm{im}_D(.\Delta) \oplus \mathrm{coker}_D(.\Delta)$, which proves that $\mathrm{coker}_D(.\Delta) = D^{1\times q}/\mathrm{im}_D(.\Delta)$ and $\mathrm{im}_D(.\Delta)$ are two finitely generated projective left D-modules by 3 of Definition 1.

By 3 of Proposition 2, there exists $u \in \hom_D(\mathrm{coker}_D(.\Delta), D^{1\times q})$ such that $\mathrm{id}_{D^{1\times q}} = i \circ v + u \circ \gamma$, which yields $(u \circ \gamma)(\mu) = (\mathrm{id}_{D^{1\times q}} - i \circ v)(\mu) = \mu\, (I_q + R\, \Delta) = \mu\, \overline{Q}$, where $\overline{Q} := I_q + R\, \Delta$. Using the fact that $\gamma \circ u = \mathrm{id}_{\mathrm{coker}_D(.\Delta)}$ (see 1 of Proposition 2), $e := u \circ \gamma$ satisfies $e^2 = u \circ (\gamma \circ u) \circ \gamma = e$, i.e., e is an idempotent of $\mathrm{end}_D(D^{1\times q}) \cong D^{q\times q}$, i.e., $\overline{Q}^2 = \overline{Q}$.

Applying Proposition 3 to the following canonical short exact sequence

$$0 \longrightarrow \ker_D(.\Delta) \xrightarrow{j} D^{1\times p} \xrightarrow{.\Delta} \mathrm{im}_D(.\Delta) \longrightarrow 0, \tag{33}$$

where $\mathrm{im}_D(.\Delta)$ is a projective left D-module, we get $D^{1\times p} \cong \ker_D(.\Delta) \oplus \mathrm{im}_D(.\Delta)$, which shows that $\ker_D(.\Delta)$ is also a finitely generated projective left D-module by 3 of Definition 1. Let us explicitly describe the splitting of the exact sequence (33). If $\alpha \in \hom_D(\mathrm{im}_D(.\Delta), D^{1\times p})$ is defined by $\alpha(\nu\, \Delta) = -(\nu\, \Delta)\, R$ for all $\nu \in D^{1\times p}$, then we can check that $((.\Delta) \circ \alpha)(\nu\, \Delta) = -\nu\, \Delta\, R\, \Delta = \nu\, \Delta$, i.e., $(.\Delta) \circ \alpha = \mathrm{id}_{\mathrm{im}_D(.\Delta)}$. By 3 of Proposition 2, there exists $\beta \in \hom_D(D^{1\times p}, \ker_D(.\Delta))$ such that $\mathrm{id}_{D^{1\times p}} = j \circ \beta + \alpha \circ (.\Delta)$, which yields $(j \circ \beta)(\lambda) = (\mathrm{id}_{D^{1\times p}} - \alpha \circ (.\Delta))(\lambda) = \lambda\, (I_p + \Delta\, R) = \lambda\, \overline{P}$, where $\overline{P} := I_p + \Delta\, R$. Using the fact that $\beta \circ j = \mathrm{id}_{\ker_D(.\Delta)}$ (see 2 of Proposition 2), $e' := j \circ \beta$ satisfies $(e')^2 = j \circ (\beta \circ j) \circ \beta = e'$, i.e., e' is an idempotent of $\mathrm{end}_D(D^{1\times p}) \cong D^{p\times p}$, i.e., $\overline{P}^2 = \overline{P}$.

Remark 6. Let $\Delta \in D^{p\times q}$ satisfy $\Delta\, R\, \Delta = \Delta$. Using the short exact sequences (32) and (33), if $v \in \hom_D(D^{1\times q}, \mathrm{im}_D(.\Delta))$ and $\alpha \in \hom_D(\mathrm{im}_D(.\Delta), D^{1\times p})$ are respectively defined by $v(\mu) = \mu\, R\, \Delta$ for all $\mu \in D^{1\times q}$ and by $\alpha(\nu\, \Delta) = \nu\, \Delta\, R$ for all $\nu \in D^{1\times p}$, then we can check that $v \circ i = \mathrm{id}_{\mathrm{im}_D(.\Delta)}$ and $((.\Delta) \circ \alpha)(\nu\, \Delta) = \mathrm{id}_{\mathrm{im}_D(.\Delta)}$. Then, we get that the projectors $\overline{P} := \Delta\, R$ and $\overline{Q} := R\, \Delta$ are such that $i \circ v = .\overline{Q}$ and $\alpha \circ (.\Delta) = .\overline{P}$. Finally, the short exact sequences (32) and (33) split, which shows that $\mathrm{coker}_D(.\Delta)$, $\mathrm{im}_D(.\Delta)$ and $\ker_D(.\Delta)$ are three finitely generated projective left D-modules.

Now, let us assume that $\overline{P}$ and $\overline{Q}$ define the endomorphism id_M (resp., 0_M) and the projective left D-modules $\ker_D(.\overline{P})$, $\mathrm{im}_D(.\overline{P})$, $\ker_D(.\overline{Q})$ and $\mathrm{im}_D(.\overline{Q})$ are free of rank respectively m, $p-m$, l, and $q-l$ with $1 \leq m \leq p$ and $1 \leq l \leq q$. Then, Theorem 5 holds with $\ker f = 0$ (resp., $\mathrm{im}\, f = 0$). Moreover, let us assume that R has full row rank. With the notations of Theorem 5, since R has full row rank, so are $V\,R\,U^{-1}$, $V_1\,R\,W_1$ and $V_2\,R\,W_2$, i.e., $\ker_D(.(V_1\,R\,W_1)) = 0$ and $\ker_D(.(V_2\,R\,W_2)) = 0$. Thus, the commutative split exact diagram (26) provides one of the following two results:

1. If $f = \mathrm{id}_M$, then we have the short exact sequence

$$0 \longrightarrow D^{1\times l} \xrightarrow{.V_1\,R\,W_1} D^{1\times m} \longrightarrow 0,$$

   which yields $m = l$ and $V_1\,R\,W_1 \in \mathrm{GL}_m(D)$.

2. If $f = 0_M$, then we have the short exact sequence

$$0 \longrightarrow D^{1\times (q-l)} \xrightarrow{.V_2\,R\,W_2} D^{1\times (p-m)} \longrightarrow 0,$$

   which yields $p-m = q-l$ and $V_2\,R\,W_2 \in \mathrm{GL}_{p-m}(D)$.

We thus obtain the following corollary of Theorem 5.

Corollary 2. *Let $R \in D^{q\times p}$ be a full row rank matrix.*

1. *Let $\Delta \in D^{p\times q}$ satisfy $\Delta\,R\,\Delta = -\Delta$ and let us note $\overline{P} := I_p + \Delta\,R$ and $\overline{Q} := I_q + R\,\Delta$. If the projective left D-modules $\ker_D(.\overline{P})$, $\mathrm{im}_D(.\overline{P})$, $\ker_D(.\overline{Q})$ and $\mathrm{im}_D(.\overline{Q})$ are free of rank respectively m, $p-m$, l and $q-l$, with $1 \leq m \leq p$ and $1 \leq l \leq q$, then we have $m = l$ and there exist $V \in \mathrm{GL}_q(D)$, $W \in \mathrm{GL}_p(D)$ and $\overline{R}_2 \in D^{(q-l)\times(p-l)}$ such that:*

$$V\,R\,W = \begin{pmatrix} I_l & 0 \\ 0 & \overline{R}_2 \end{pmatrix}.$$

2. *Let $\Delta \in D^{p\times q}$ satisfy $\Delta\,R\,\Delta = \Delta$ and let us note $\overline{P} := \Delta\,R$ and $\overline{Q} := R\,\Delta$. If the projective left D-modules $\ker_D(.\overline{P})$, $\mathrm{im}_D(.\overline{P})$, $\ker_D(.\overline{Q})$ and $\mathrm{im}_D(.\overline{Q})$ are free of rank respectively m, $p-m$, l and $q-l$, with $1 \leq m \leq p$ and $1 \leq l \leq q$, then we have $p-m = q-l$ and there exist $V \in \mathrm{GL}_q(D)$, $W \in \mathrm{GL}_p(D)$ and $\overline{R}_1 \in D^{l\times m}$ such that:*

$$V\,R\,W = \begin{pmatrix} \overline{R}_1 & 0 \\ 0 & I_{q-l} \end{pmatrix}.$$

Theorem 2 shows that Corollary 2 holds for the different rings D of functional operators interesting for mathematical systems theory and for modules satisfying the possible rank conditions of Theorem 2.

In order to refine Corollary 2, let us now study the links between the left D-modules defined by the left kernels and the left images of the matrices Δ, $\overline{P}$ and $\overline{Q}$.

Proposition 4. *Let $\Delta \in D^{p\times q}$ satisfy $\Delta\,R\,\Delta = -\Delta$ and let us note $\overline{P} := I_p + \Delta\,R$ and $\overline{Q} := I_q + R\,\Delta$. Then, we have:*

$$\begin{cases} \mathrm{im}_D(.\overline{P}) = \ker_D(.\Delta), \\ \ker_D(.\overline{Q}) = \mathrm{im}_D(.\Delta), \\ \ker_D(.\overline{P}) = \mathrm{im}_D(.(\Delta\,R)) \cong \ker_D(.\overline{Q}) = \mathrm{im}_D(.\Delta), \\ \mathrm{im}_D(.\overline{Q}) = \ker_D(.(R\,\Delta)) \cong \mathrm{coker}_D(.\Delta). \end{cases}$$

Hence, $\ker_D(.\overline{P})$ (resp., $\mathrm{im}_D(.\overline{P})$, $\ker_D(.\overline{Q})$, $\mathrm{im}_D(.\overline{Q})$) is a free left D-module if and only if so is $\mathrm{im}_D(.\Delta)$ (resp., $\ker_D(.\Delta)$, $\mathrm{im}_D(.\Delta)$, $\mathrm{coker}_D(.\Delta)$).

If $\ker_D(.\Delta)$ *is a free left* D*-module of rank* $p-l$*, i.e., if there exists a full row rank matrix* $U_2 \in D^{(p-l)\times p}$ *such that* $\ker_D(.\Delta) = \operatorname{im}_D(.U_2)$*, then we have* $\operatorname{im}_D(.\overline{P}) = \ker_D(.\Delta) = \operatorname{im}_D(.U_2)$*. In particular, there exists a unique matrix* $Z \in D^{p\times(p-l)}$ *such that:*

$$\overline{P} = Z\, U_2. \tag{34}$$

If $\operatorname{im}_D(.\Delta)$ *is a free left* D*-module of rank* l*, i.e., if there exists a full row rank matrix* $V_1 \in D^{l\times q}$ *such that* $\operatorname{im}_D(.\Delta) = \operatorname{im}_D(.V_1)$*, then there exists a unique matrix* $Y \in D^{p\times l}$ *such that*

$$\Delta = Y\, V_1, \tag{35}$$

and $V_1\, R \in D^{l\times p}$ *defines a basis of* $\ker_D(.\overline{P})$*, i.e.,* $\ker_D(.\overline{P}) = \operatorname{im}_D(.(V_1\, R))$*.*

If $\operatorname{coker}_D(.\Delta)$ *is a free left* D*-module of rank* $q-l$ *and* $\Phi \in D^{(q-l)\times q}$ *is a full row rank matrix such that* $\{\gamma(\Phi_{i\bullet})\}_{i=1,\ldots,q-l}$ *is a basis of* $\operatorname{coker}_D(.\Delta)$*, then* $\operatorname{im}_D(.\overline{Q}) = \operatorname{im}_D(.(\Phi\,\overline{Q}))$*, i.e.,* $\Phi\,\overline{Q} \in D^{(q-l)\times q}$ *defines a basis of* $\operatorname{im}_D(.\overline{Q})$*. Finally, there exists* $\Psi \in D^{q\times(q-l)}$ *such that*

$$\overline{Q} = \Psi\, \Phi\, \overline{Q}, \tag{36}$$

and Ψ *can be chosen to be an injective parametrization of* $\operatorname{coker}_D(.\Delta)$*, namely:*

$$\ker_D(.\Psi) = \operatorname{im}_D(.\Delta), \quad \Phi\,\Psi = I_{q-l}.$$

Proof. By Remark 5, we have $.\overline{P} = j \circ \beta$ and $.\overline{Q} = u \circ \gamma$. Thus, we get $\ker_D(.\overline{Q}) = \ker(u \circ \gamma) = \ker \gamma = \operatorname{im} i = \operatorname{im}_D(.\Delta)$ since u is injective and i is the canonical inclusion. We also have $\ker_D(.\overline{P}) = \ker(j \circ \beta) = \ker \beta = \operatorname{im} \alpha = \operatorname{im}_D(.(\Delta\, R))$ since j is injective. Using the fact that α is injective, we have $\operatorname{im} \alpha \cong \operatorname{im}_D(.\Delta) = \ker_D(.\overline{Q})$, which shows that $\ker_D(.\overline{P}) \cong \ker_D(.\overline{Q})$, where the isomorphism $\phi : \ker_D(.\overline{Q}) \longrightarrow \ker_D(.\overline{P})$ is defined by $\phi(\theta) = -\theta\, R$ and $\phi^{-1} : \ker_D(.\overline{P}) \longrightarrow \ker_D(.\overline{Q})$ is defined by $\phi^{-1}(\lambda) = \lambda\, \Delta$. Moreover, we have $\operatorname{im}_D(.\overline{P}) = \operatorname{im}(j \circ \beta) = \operatorname{im} j = \ker_D(.\Delta)$ since β is surjective and j is the canonical inclusion. We also have $\operatorname{im}_D(.\overline{Q}) = \operatorname{im}(u \circ \gamma) = \operatorname{im} u = \ker v = \ker_D(.(R\,\Delta))$ since γ is surjective. Using the fact that u is injective, we get $\operatorname{im} u \cong \operatorname{coker}_D(.\Delta)$, and thus we obtain $\operatorname{im}_D(.\overline{Q}) \cong \operatorname{coker}_D(.\Delta)$, where the isomorphism $u : \operatorname{coker}_D(.\Delta) \longrightarrow \operatorname{im}_D(.\overline{Q})$ is defined by $u(\gamma(\mu)) = \mu\,\overline{Q}$ for all $\mu \in D^{1\times q}$ (which is well-defined since $\gamma(\mu) = \gamma(\mu')$ yields $\mu = \mu' + \lambda\,\Delta$ for a certain $\lambda \in D^{1\times p}$ and thus $\mu\,\overline{Q} = \mu'\,\overline{Q} + \lambda\,\Delta\,\overline{Q} = \mu\,\overline{Q}$ since $\Delta\,\overline{Q} = 0$) and $u^{-1} : \operatorname{im}_D(.\overline{Q}) \longrightarrow \operatorname{coker}_D(.\Delta)$ is defined by $u^{-1}(\mu\,\overline{Q}) = \gamma(\mu)$ for all $\mu \in D^{1\times q}$.

The second point is just a straightforward consequence of the above results.

If $\ker_D(.\Delta)$ is a free left D-module of rank $p-l$, i.e., if there exists a full row rank matrix $U_2 \in D^{(p-l)\times p}$ such that $\ker_D(.\Delta) = \operatorname{im}_D(.U_2)$, then we have $\operatorname{im}_D(.\overline{P}) = \ker_D(.\Delta) = \operatorname{im}_D(.U_2)$, which shows that there exists a matrix $Z \in D^{p\times(p-l)}$ such that (34). The matrix Z is unique since U_2 has full row rank.

If $\operatorname{im}_D(.\Delta)$ is a free left D-module of rank l, i.e., if there exists a full row rank matrix $V_1 \in D^{l\times q}$ such that $\operatorname{im}_D(.\Delta) = \operatorname{im}_D(.V_1)$, then there exists a matrix $Y \in D^{p\times l}$ such that (35). The matrix Y is unique since V_1 has full row rank. Moreover, using ϕ, we obtain that $V_1\, R \in D^{l\times p}$ is a basis of $\ker_D(.\overline{P})$.

If $\operatorname{coker}_D(.\Delta)$ is a free left D-module of rank $q-l$, then there exist $\Psi \in D^{q\times(q-l)}$ and $\Phi \in D^{(q-l)\times q}$ such that $\ker_D(.\Psi) = \operatorname{im}_D(.\Delta)$ and $\Phi\,\Psi = I_{q-l}$ (see Example 5), i.e., such that we have the following commutative exact sequence

$$\begin{tikzcd}
D^{1\times p} \arrow[r, ".\Delta"] \arrow[d, equal] & D^{1\times q} \arrow[r, "\gamma"] \arrow[d, equal] & \operatorname{coker}_D(.\Delta) \arrow[r] \arrow[d, "\iota"] & 0 \\
D^{1\times p} \arrow[r, ".\Delta"] & D^{1\times q} \arrow[r, shift left, ".\Psi"] & D^{1\times(q-l)} \arrow[l, shift left, ".\Phi"] \arrow[r] & 0,
\end{tikzcd} \tag{37}$$

where the isomorphism $\iota \in \hom_D(\operatorname{coker}_D(.\Delta), D^{1\times(q-l)})$ is defined by $\iota(\gamma(\mu)) = \mu\,\Psi$ for all $\mu \in D^{1\times q}$ and $\iota^{-1}(\theta) = \gamma(\theta\,\Phi)$ for all $\theta \in D^{1\times(q-l)}$. The isomorphism $u \circ \iota^{-1} : D^{1\times(q-l)} \longrightarrow \operatorname{im}_D(.\overline{Q})$ is then defined by $(u \circ \iota^{-1})(\theta) = \theta\,(\Phi\,\overline{Q})$, which yields $\operatorname{im}_D(.\overline{Q}) = \operatorname{im}_D(.(\Phi\,\overline{Q}))$, i.e., $\Phi\,\overline{Q}$ defines a basis of $\operatorname{im}_D(.\overline{Q})$. Finally, using $\Phi\,\Psi = I_{q-l}$ (see (37)), we have $(I_q - \Psi\,\Phi)\,\Psi = 0$, which yields $\operatorname{im}_D(.(I_q - \Psi\,\Phi)) \subseteq \ker_D(.\Psi) = \operatorname{im}_D(.\Delta)$, and thus there exists $\Omega \in D^{q\times p}$ such that $I_q - \Psi\,\Phi = \Omega\,\Delta$, i.e., $\Psi\,\Phi + \Omega\,\Delta = I_q$. Finally, combining this identity with the fact that $\Delta\,\overline{Q} = \Delta + \Delta\, R\,\Delta = 0$, we get $\overline{Q} = (\Psi\,\Phi + \Omega\,\Delta)\,\overline{Q} = \Psi\,\Phi\,\overline{Q}$. $\square$

Similarly, we have the following results.

Proposition 5. *Let $\Delta \in D^{p\times q}$ satisfy $\Delta\, R\, \Delta = \Delta$ and let us note $\overline{P} := \Delta\, R$ and $\overline{Q} := R\, \Delta$. Then, we have:*

$$\left\{\begin{array}{l} \ker_D(.\overline{P}) = \ker_D(.\Delta), \\ \operatorname{im}_D(.\overline{Q}) = \operatorname{im}_D(.\Delta), \\ \operatorname{im}_D(.\overline{P}) = \operatorname{im}_D(.(\Delta\, R)) \cong \operatorname{im}_D(.\overline{Q}) = \operatorname{im}_D(.\Delta), \\ \ker_D(.\overline{Q}) = \ker_D(.(R\, \Delta)) \cong \operatorname{coker}_D(.\Delta). \end{array}\right.$$

Hence, $\ker_D(.\overline{P})$ (resp., $\operatorname{im}_D(.\overline{P})$, $\ker_D(.\overline{Q})$, $\operatorname{im}_D(.\overline{Q})$) is a free left D-module if and only if so is $\ker_D(.\Delta)$ (resp., $\operatorname{im}_D(.\Delta)$, $\operatorname{coker}_D(.\Delta)$, $\operatorname{im}_D(.\Delta)$).

If $\ker_D(.\Delta)$ is a free left D-module of rank $m = p - q + l$, i.e., if there exists a full row rank matrix $U_1 \in D^{m\times p}$ such that $\ker_D(.\Delta) = \operatorname{im}_D(.U_1)$, there exists exists a unique matrix $Z \in D^{p\times m}$ such that:

$$I_p - \overline{P} = Z\, U_1.$$

If $\operatorname{im}_D(.\Delta)$ is a free left D-module of rank $q-l$, i.e., if there exists a full row rank matrix $V_2 \in D^{(q-l)\times q}$ such that $\operatorname{im}_D(.\Delta) = \operatorname{im}_D(.V_2)$, then there exists a unique matrix $Y \in D$ such that

$$\Delta = Y\, V_2,$$

and $V_2\, R \in D^{(q-l)\times p}$ defines a basis of $\operatorname{im}_D(.\overline{P})$, i.e., $\operatorname{im}_D(.\overline{P}) = \operatorname{im}_D(.(V_2\, R))$.

If $\operatorname{coker}_D(.\Delta)$ is a free left D-module of rank l and $\Phi \in D^{l\times q}$ is a full row rank matrix such that $\{\gamma(\Phi_{i\bullet})\}_{i=1,\ldots,l}$ is a basis of $\operatorname{coker}_D(.\Delta)$, then $\ker_D(.\overline{Q}) = \operatorname{im}_D(.(\Phi\,(I_q - \overline{Q})))$, i.e., $\Phi\,(I_q - \overline{Q})$ defines a basis of $\ker_D(.\overline{Q})$. Finally, there exists $\Psi \in D^{q\times l}$ such that

$$I_q - \overline{Q} = \Psi\, \Phi\, (I_q - \overline{Q}),$$

and Ψ can be chosen to be an injective parametrization of $\operatorname{coker}_D(.\Delta)$, namely:

$$\ker_D(.\Psi) = \operatorname{im}_D(.\Delta), \quad \Phi\, \Psi = I_l.$$

Proof. By Remark 6, we have $.\overline{P} = \alpha \circ .\Delta$ and $.\overline{Q} = i \circ v$. Thus, we get $\ker_D(.\overline{P}) = \ker(\alpha \circ .\Delta) = \ker_D(.\Delta)$ since α is injective and v is surjective. Moreover, we have $\operatorname{im}_D(.\overline{Q}) = \operatorname{im}(i \circ v) = \operatorname{im} v = \operatorname{im}_D(.\Delta)$ since i is an inclusion. We also have $\operatorname{im}_D(.\overline{P}) = \operatorname{im}(\alpha \circ .\Delta) = \operatorname{im} \alpha = \operatorname{im}_D(.(\Delta\, R))$ since $.\Delta$ is surjective. Using the fact that α is injective, we get $\operatorname{im} \alpha \cong \operatorname{im}_D(.\Delta) = \operatorname{im}_D(.\overline{Q})$, which shows that $\operatorname{im}_D(.\overline{P}) \cong \operatorname{im}_D(.\overline{Q})$, where the isomorphism $\alpha : \operatorname{im}_D(.\overline{Q}) \longrightarrow \operatorname{im}_D(.\overline{P})$ is defined by $\alpha(\mu\, R\, \Delta) = (\mu\, R\, \Delta)\, R = (\mu\, R)\, \overline{P}$ and $\alpha^{-1} : \operatorname{im}_D(.\overline{P}) \longrightarrow \operatorname{im}_D(.\overline{Q})$ is defined by $\alpha^{-1}(\lambda\, \Delta\, R) = (\lambda\, \Delta\, R)\, \Delta = (\lambda\, \Delta)\, \overline{Q}$. We also have $\ker_D(.\overline{Q}) = \ker(i \circ v) = \ker v = \ker_D(.(R\, \Delta))$. Using $\ker v = \operatorname{im} u \cong \operatorname{coker}_D(.\Delta)$ since u is injective, we get that $\ker_D(.\overline{Q}) \cong \operatorname{coker}_D(.\Delta)$, where the isomorphism $u : \operatorname{coker}_D(.\Delta) \longrightarrow \ker_D(.\overline{Q})$ is defined by $u(\gamma(\mu)) = \mu\,(I_q - \overline{Q})$ for all $\mu \in D^{1\times q}$, and $u^{-1} : \ker_D(.\overline{Q}) \longrightarrow \operatorname{coker}_D(.\Delta)$ is defined by $u^{-1}(\mu) = \gamma(\mu)$ for all $\mu \in D^{1\times q}$.

The second point is a straightforward consequence of the above results. Finally, the last points can be proved as in Proposition 4 where the matrix $\overline{P}$ (resp., $\overline{Q}$) is replaced by $I_p - \overline{P}$ (resp., $I_q - \overline{Q}$). □

Remark 7. One interesting application of the results stated in Propositions 4 and 5 is the following. If we want to compute a presentation matrix of the left D-module $M = D^{1\times p}/(D^{1\times q}\, R)$ of minimal size, using the equality $m = l$ (resp., $p - m = q - l$) of 1 (resp., 2) of Corollary 2, we then have to seek for the solutions $\Delta \in D^{p\times q}$ of the equation $\Delta\, R\, \Delta = -\Delta$ (resp., $\Delta\, R\, \Delta = \Delta$) which are such that the projective left D-modules $\operatorname{im}_D(.\Delta)$ (resp., $\ker_D(.\Delta)$) are free with maximal (resp., minimal) rank.

Using Proposition 4, we then obtain the following theorem which is a refinement of 1 of Corollary 2.

Theorem 8. *Let $R \in D^{q\times p}$ have full row rank. If there exists a matrix $\Delta \in D^{p\times q}$ such that:*

1. $\Delta\, R\, \Delta = -\Delta$.

2. *The projective left D-modules* $\ker_D(.\Delta)$, $\operatorname{im}_D(.\Delta)$ *and* $\operatorname{coker}_D(.\Delta)$ *are free of rank respectively* $p-l$, l *and* $q-l$.

Then, there exist $V \in \mathrm{GL}_q(D)$, $W \in \mathrm{GL}_p(D)$ *and* $\overline{R}_2 \in D^{(q-l)\times(p-l)}$ *such that:*

$$V\,R\,W = \begin{pmatrix} I_l & 0 \\ 0 & \overline{R}_2 \end{pmatrix}. \tag{38}$$

The matrices $V := (V_1^T \quad V_2^T)^T$, $V_1 \in D^{l\times q}$, $V_2 \in D^{(q-l)\times q}$, *and* $W^{-1} = U := (U_1^T \quad U_2^T)^T$, $U_1 \in D^{l\times p}$, $U_2 \in D^{(p-l)\times p}$, *can be chosen such as*

$$\begin{cases} \ker_D(.\Delta) = \operatorname{im}_D(.U_2), \\ \operatorname{im}_D(.\Delta) = \operatorname{im}_D(.V_1), \\ U_1 := V_1\,R, \\ V_2 := \Phi\,(I_q + R\,\Delta), \end{cases} \tag{39}$$

where the full row rank matrices U_2, V_1 *and* Φ *define a basis of* $\ker_D(.\Delta)$, $\operatorname{im}_D(.\Delta)$ *and* $\operatorname{coker}_D(.\Delta)$.

Moreover, if $Y \in D^{p\times l}$ *and* $Z \in D^{p\times(p-l)}$ *are the unique matrices defined by (35) and (34), and* $\Psi \in D^{p\times(p-l)}$ *is an injective parametrization of* $\operatorname{coker}_D(.\Delta)$, *i.e., we have* $\ker_D(.\Psi) = \operatorname{im}_D(.\Delta)$ *and* $\Phi\,\Psi = I_{p-l}$ *(see (37)), then the matrices* $X := V^{-1} = (X_1 \quad X_2)$, $X_1 \in D^{q\times l}$, $X_2 \in D^{q\times(q-l)}$, *and* $W := (W_1 \quad W_2)$, $W_1 \in D^{p\times l}$ *and* $W_2 \in D^{p\times(p-l)}$, *are defined by:*

$$\begin{cases} X_1 := -R\,Y, \\ X_2 := \Psi, \\ W_1 := -Y, \\ W_2 := Z. \end{cases} \tag{40}$$

Finally, we have $\overline{R}_2 = \Phi\,(I_q + R\,\Delta)\,R\,Z = \Phi\,R\,Z$.

Proof. (38) is a direct consequence of 1 of Corollary 2 and of Proposition 4.

Now, using (39), let us consider the matrices $U := (U_1^T \quad U_2^T)^T$ and $V := (V_1^T \quad V_2^T)^T$. If we note $\overline{P} := I_p + \Delta\,R$ and $\overline{Q} := I_q + R\,\Delta$, then using the identities $\overline{Q}\,R = R\,\overline{P}$ and $\overline{P} = Z\,U_2$ (see Proposition 4), we obtain:

$$V\,R = \begin{pmatrix} V_1 \\ \Phi\,\overline{Q} \end{pmatrix} R = \begin{pmatrix} V_1\,R \\ \Phi\,\overline{Q}\,R \end{pmatrix} = \begin{pmatrix} V_1\,R \\ \Phi\,R\,Z\,U_2 \end{pmatrix} = \begin{pmatrix} I_l & 0 \\ 0 & \Phi\,R\,Z \end{pmatrix} \begin{pmatrix} V_1\,R \\ U_2 \end{pmatrix} = \begin{pmatrix} I_l & 0 \\ 0 & \Phi\,R\,Z \end{pmatrix} U.$$

By Theorem 5, we have $U \in \mathrm{GL}_p(D)$ and $V \in \mathrm{GL}_q(D)$, which proves the second result.

Finally, let us now compute V^{-1} and W^{-1}. Combining the short exact sequence

$$0 \longrightarrow D^{1\times q} \xrightarrow{.R} D^{1\times p} \xrightarrow{\pi} M \longrightarrow 0$$

with (33) and (32), and using the notations of Remark 5, we get the following commutative exact diagram

$$
\begin{tikzcd}
 & & 0 \arrow[d] & & \\
0 \arrow[r] & \mathrm{im}_D(.\Delta) \arrow[r, shift left, "i"] \arrow[d, equal] & D^{1\times q} \arrow[l, shift left, "v"] \arrow[r, shift left, "\gamma"] \arrow[d, ".-R"'] & \mathrm{coker}_D(.\Delta) \arrow[l, shift left, "u"] \arrow[r] \arrow[d, "\delta"] & 0 \\
0 \arrow[r] & \mathrm{im}_D(.\Delta) \arrow[r, shift left, "\alpha"] & D^{1\times p} \arrow[l, shift left, ".\Delta"] \arrow[r, shift left, "\beta"] \arrow[d, "\pi"'] & \mathrm{ker}_D(.\Delta) \arrow[l, shift left, "j"] \arrow[r] & 0, \\
 & & M \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}
\tag{41}
$$

where $\delta \in \hom_D(\mathrm{coker}_D(.\Delta), \ker_D(.\Delta))$ is defined by $\delta(\gamma(\mu)) = \beta(-\mu\, R) = -\mu\, R\, \overline{P}$ for all $\mu \in D^{1\times q}$, since $\beta \in \hom_D(D^{1\times p}, \ker_D(.\Delta))$ is defined by $\beta(\lambda) = \lambda\, \overline{P}$. By (35), we have $\Delta = Y\, V_1$ for a unique $Y \in D^{p\times l}$. Combining these results with (37) and using the notations of the proof of Proposition 4, we get the following exact diagram:

$$
\begin{tikzcd}
 & 0 & & 0 & \\
0 \arrow[r] & \mathrm{im}_D(.\Delta) \arrow[u] \arrow[r, shift left, "i"] & D^{1\times q} \arrow[l, shift left, "v"] \arrow[r, shift left, "\gamma"] & \mathrm{coker}_D(.\Delta) \arrow[u] \arrow[l, shift left, "u"] \arrow[r] & 0. \\
 & D^{1\times l} \arrow[u, ".V_1"] & & D^{1\times (q-l)} \arrow[u, "\iota^{-1}"'] & \\
 & 0 \arrow[u] & & 0 \arrow[u] &
\end{tikzcd}
$$

The left D-homomorphism $i \circ (.V_1) : D^{1\times l} \longrightarrow D^{1\times q}$ is defined by $i \circ (.V_1)(\theta) = \theta\, V_1$ for all $\theta \in D^{1\times l}$ and $\iota \circ \gamma : D^{1\times q} \longrightarrow D^{1\times (q-l)}$ is defined by $(\iota \circ \gamma)(\mu) = \mu\, \Psi$ for all $\mu \in D^{1\times q}$. Moreover, using $\Delta = Y\, V_1$, we have $v(\mu) = -\mu\, R\, \Delta = (-\mu\, R\, Y)\, V_1$ for all $\mu \in D^{1\times q}$, which shows that the left D-homomorphism $\rho : D^{1\times q} \longrightarrow D^{1\times l}$ defined by $\rho(\mu) = -\mu\, R\, Y$ for all $\mu \in D^{1\times q}$ is such that $v = .V_1 \circ \rho$. Now the left D-homomorphism $u \circ \iota^{-1} : D^{1\times (q-l)} \longrightarrow D^{1\times q}$ is defined by $(u \circ \iota^{-1})(\xi) = \xi\, \Phi\, \overline{Q}$ for all $\xi \in D^{1\times (q-l)}$. Hence, we obtain the following diagram which commutes in both directions and is formed by horizontal split exact sequences:

$$
\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & \mathrm{im}_D(.\Delta) \arrow[u] \arrow[r, shift left, "i"] & D^{1\times q} \arrow[u] \arrow[l, shift left, "v"] \arrow[r, shift left, "\gamma"] & \mathrm{coker}_D(.\Delta) \arrow[u] \arrow[l, shift left, "u"] \arrow[r] & 0 \\
0 \arrow[r] & D^{1\times l} \arrow[u, ".V_1"] \arrow[r, shift left, ".V_1"] & D^{1\times q} \arrow[u, ".I_q"] \arrow[l, shift left, ".-R\,Y"] \arrow[r, shift left, ".\Psi"] & D^{1\times (q-l)} \arrow[u, "\iota^{-1}"'] \arrow[l, shift left, ".\Phi\,\overline{Q}"] \arrow[r] & 0. \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}
\tag{42}
$$

Similarly, using Remark 5, $\mathrm{im}_D(.\Delta) = \mathrm{im}_D(.V_1)$ and $\ker_D(.\Delta) = \mathrm{im}_D(.U_2)$, where V_1 and U_2 are two full row rank matrices, we get the following exact diagram:

$$\begin{tikzcd}
 & 0 & & 0 & \\
0 \arrow[r] & \mathrm{im}_D(.\Delta) \arrow[u] \arrow[r, shift left, "\alpha"] & D^{1\times p} \arrow[l, shift left, ".\Delta"] \arrow[r, shift left, "\beta"] & \ker_D(.\Delta) \arrow[u] \arrow[l, shift left, "j"] \arrow[r] & 0. \\
 & D^{1\times l} \arrow[u, ".V_1"] & & D^{1\times (p-l)} \arrow[u, ".U_2"'] & \\
 & 0 \arrow[u] & & 0 \arrow[u] &
\end{tikzcd}$$

The left D-homomorphism $\alpha \circ .V_1 : D^{1\times l} \longrightarrow D^{1\times p}$ is defined by $(\alpha \circ .V_1)(\theta) = -\theta\, V_1\, R$ for all $\theta \in D^{1\times l}$, and the identity $\Delta = Y\, V_1$ implies that the left D-homomorphism $.Y : D^{1\times p} \longrightarrow D^{1\times l}$ is such that $.\Delta = .V_1 \circ .Y$. By (34), we have $\overline{P} = Z\, U_2$ for a unique $Z \in D^{p\times (p-l)}$. The left D-homomorphism $.Z : D^{1\times p} \longrightarrow D^{1\times (p-l)}$ is such that $\beta = .\overline{P} = .U_2 \circ .Z$. Hence, we obtain the following diagram which commutes in both directions and is formed by horizontal split exact sequences:

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & \mathrm{im}_D(.\Delta) \arrow[u] \arrow[r, shift left, "\alpha"] & D^{1\times p} \arrow[u] \arrow[l, shift left, ".\Delta"] \arrow[r, shift left, "\beta"] & \ker_D(.\Delta) \arrow[u] \arrow[l, shift left, "j"] \arrow[r] & 0 \\
0 \arrow[r] & D^{1\times l} \arrow[u, ".V_1"] \arrow[r, shift left, ".-V_1\, R"] & D^{1\times p} \arrow[u, ".I_p"] \arrow[l, shift left, ".Y"] \arrow[r, shift left, ".Z"] & D^{1\times (p-l)} \arrow[u, ".U_2"] \arrow[l, shift left, ".U_2"] \arrow[r] & 0. \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd} \tag{43}$$

Let $\overline{R}_2 := \Phi\, \overline{Q}\, R\, Z \in D^{(q-l)\times (p-l)}$. The identities $\overline{Q}\, R = R\, \overline{P}$, $\overline{P} = Z\, U_2$ and $U_2\, Z = I_{p-l}$ yield:

$$\overline{R}_2 = \Phi\, (\overline{Q}\, R)\, Z = \Phi\, (R\, \overline{P})\, Z = \Phi\, R\, Z\, (U_2\, Z) = \Phi\, R\, Z.$$

Then, combining this identity with $\Phi\, \Psi = I_{q-l}$, we get:

$$\Psi\, \overline{R}_2 = (\Psi\, \Phi)\, R\, Z = R\, Z. \tag{44}$$

Moreover, the identities $\overline{P} = Z\, U_2$, $R\, \overline{P} = \overline{Q}\, R$ and $\overline{Q}^2 = \overline{Q}$ yield:

$$\overline{R}_2\, U_2 = \Phi\, \overline{Q}\, R\, Z\, U_2 = \Phi\, \overline{Q}\, (R\, \overline{P}) = \Phi\, \overline{Q}^2\, R = \Phi\, \overline{Q}\, R. \tag{45}$$

Hence, combining (41), (42), (43), (44) and (45), we obtain the following diagram which commutes in both directions and is formed by horizontal split exact sequences:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & D^{1\times l} \arrow[r, shift left, ".V_1"] \arrow[d, ".I_l"'] & D^{1\times q} \arrow[l, shift left, ".-R\,Y"] \arrow[r, shift left, ".\Psi"] \arrow[d, ".-R"'] & D^{1\times (q-l)} \arrow[l, shift left, ".\Phi\,\overline{Q}"] \arrow[r] \arrow[d, ".-\overline{R}_2"] & 0 \\
0 \arrow[r] & D^{1\times l} \arrow[r, shift left, ".-V_1\,R"] \arrow[d] & D^{1\times p} \arrow[l, shift left, ".Y"] \arrow[r, shift left, ".Z"] \arrow[d, "\pi"'] & D^{1\times (p-l)} \arrow[l, shift left, ".U_2"] \arrow[r] \arrow[d, "\kappa"] & 0 \\
 & 0 \arrow[r] & M \arrow[r, "\varphi"] \arrow[d] & L \arrow[r] \arrow[d] & 0, \\
 & & 0 & 0 &
\end{tikzcd}$$

where $L := D^{1\times(p-l)}/(D^{1\times(q-l)}\,\overline{R}_2)$ is the left D-module finitely presented by $\overline{R}_2$ and the left D-isomorphism $\varphi \in \hom_D(M, L)$ is defined by $\varphi(\pi(\lambda)) = \kappa(\lambda\, Z)$ for all $\lambda \in D^{1\times p}$. Now, changing signs in the above diagram, we get the following one

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & D^{1\times l} \arrow[r, shift left, ".V_1"] \arrow[d, ".I_l"'] & D^{1\times q} \arrow[l, shift left, ".-R\,Y"] \arrow[r, shift left, ".\Psi"] \arrow[d, ".R"'] & D^{1\times (q-l)} \arrow[l, shift left, ".\Phi\,\overline{Q}"] \arrow[r] \arrow[d, ".\overline{R}_2"] & 0 \\
0 \arrow[r] & D^{1\times l} \arrow[r, shift left, ".V_1\,R"] \arrow[d] & D^{1\times p} \arrow[l, shift left, ".-Y"] \arrow[r, shift left, ".Z"] \arrow[d, "\pi"'] & D^{1\times (p-l)} \arrow[l, shift left, ".U_2"] \arrow[r] \arrow[d, "\kappa"] & 0 \\
 & 0 \arrow[r] & M \arrow[r, "\varphi"] \arrow[d] & L \arrow[r] \arrow[d] & 0, \\
 & & 0 & 0 &
\end{tikzcd} \tag{46}$$

which finally proves the result (see also Theorem 7). □

Remark 8. Using matrix computations, let us check again (46).

Using $\Phi\,\Psi = I_{q-l}$ (see (37)), we have $(I_q - \Psi\,\Phi)\,\Psi = 0$, which yields $\mathrm{im}_D(.(I_q - \Psi\,\Phi)) \subseteq \ker_D(.\Psi) = \mathrm{im}_D(.\Delta)$, and thus there exists $\Omega \in D^{q\times p}$ such that $I_q - \Psi\,\Phi = \Omega\,\Delta$, i.e., $\Psi\,\Phi + \Omega\,\Delta = I_q$. Now, using $\Delta = Y\,V_1$, we get $I_q - \overline{Q} = -R\,\Delta = -R\,Y\,V_1$, which combined with (36) yields:

$$\Psi\,(\Phi\,\overline{Q}) + (-R\,Y)\,V_1 = I_q. \tag{47}$$

Now, multiplying (47) by V_1 and using the fact that $V_1\,\Psi = 0$ since $\ker_D(.\Psi) = \mathrm{im}_D(.\Delta) = \mathrm{im}_D(.V_1)$, we get $(I_l + V_1\,R\,Y)\,V_1 = 0$, which yields $-V_1\,R\,Y = I_l$ since V_1 has full row rank. In particular, we get that the projector $I_q - \overline{Q}$ of $D^{q\times q}$ is such that:

$$I_q - \overline{Q} = (-R\,Y)\,V_1, \quad V_1\,(-R\,Y) = I_l.$$

Now, using $\Delta\,\Psi = 0$ and $\Phi\,\Psi = I_{q-l}$, we obtain $\Phi\,\overline{Q}\,\Psi = \Phi\,(I_q + R\,\Delta)\,\Psi = \Phi\,\Psi = I_{q-l}$. Multiplying (47) by $\Phi\,\overline{Q}$ and using $\Phi\,\overline{Q}\,\Psi = I_{q-l}$, we get $\Phi\,\overline{Q}\,R\,Y\,V_1 = 0$, and thus we obtain $\Phi\,\overline{Q}\,R\,Y = 0$ since V_1 has full row rank, which shows that $\mathrm{im}_D(.(\Phi\,\overline{Q})) \subseteq \ker_D(.(R\,Y))$. Finally, if $\mu \in \ker_D(.(R\,Y))$, using (47), we have $\mu = (\mu\,\Psi)\,(\Phi\,\overline{Q}) \in \mathrm{im}_D(.(\Phi\,\overline{Q}))$, which shows that $\ker_D(.(R\,Y)) = \mathrm{im}_D(.(\Phi\,\overline{Q}))$ and finally proves that the first horizontal sequence of (46) is a split short exact sequence.

Now, using $\overline{P} = I_p + \Delta\,R = Z\,U_2$ and $\Delta = Y\,V_1$, we first get the identity:

$$Z\,U_2 + (-Y)\,(V_1\,R) = I_p. \tag{48}$$

Using $\mathrm{im}_D(.\overline{P}) = \mathrm{im}_D(.U_2)$, there exists $Z' \in D^{(p-l)\times p}$ such that $U_2 = Z'\,\overline{P}$. Combining this identity with $\overline{P} = Z\,U_2$, we obtain $(I_{p-l} - Z'\,Z)\,U_2 = 0$, which yields $Z'\,Z = I_{p-l}$ since the matrix U_2 has full row rank. In particular, the left D-homomorphim $.Z$ is surjective. Using $\overline{P} = Z\,U_2$ and $\overline{P}^2 = \overline{P}$, we get $(Z\,U_2\,Z - Z)\,U_2 = 0$, which yields $Z\,U_2\,Z = Z$ since U_2 has full row rank. Thus, we obtain $Z\,(I_{p-l} - U_2\,Z) = 0$, which yields $U_2\,Z = I_{p-l}$ since $.Z$ is surjective. The projector $\overline{P}$ of $D^{p\times p}$ satisfies:

$$\overline{P} = Z\,U_2, \quad U_2\,Z = I_{p-l}.$$

Multiplying (48) by Z and using $Z\,U_2\,Z = Z$, we get $Y\,V_1\,R\,Z = 0$, and thus $V_1\,R\,Z = 0$ since $.Y$ is surjective. Thus, we have $\mathrm{im}_D(.(-V_1\,R)) \subseteq \ker_D(.Z)$. If $\lambda \in \ker_D(.Z)$, then using (48), we obtain $\lambda = (\lambda\,Y)(-V_1\,R) \in \mathrm{im}_D(.(-V_1\,R))$, which shows that $\ker_D(.Z) = \mathrm{im}_D(.(-V_1\,R))$ and proves the exactness of the second horizontal sequence of (46). Multiplying (48) by U_2 and using $U_2\,Z = I_{p-l}$, we get $(U_2\,Y)\,(-V_1\,R) = 0$, which yields $U_2\,Y = 0$ since $-V_1\,R$ has full row rank due to the identity $-V_1\,R\,Y = I_l$, which shows that $\mathrm{im}_D(.U_2) \subseteq \ker_D(.Y)$. If $\lambda \in \ker_D(.Y)$, then (48) yields $\lambda = (\lambda\,Z)\,U_2 \in \mathrm{im}_D(.U_2)$, which shows that $\ker_D(.Y) = \mathrm{im}_D(.U_2)$ and proves that the second horizontal short exact sequence splits.

Finally, the commutativity of (46) in both directions is proved in (44) and (45).

Similarly, using Proposition 5, we have the following refinement of 2 of Corollary 2.

Theorem 9. *Let $R \in D^{q\times p}$ have full row rank. If there exists a matrix $\Delta \in D^{p\times q}$ such that:*

1. $\Delta\,R\,\Delta = \Delta.$
2. *The projective left D-modules $\ker_D(.\Delta)$, $\mathrm{im}_D(.\Delta)$ and $\mathrm{coker}_D(.\Delta)$ are free of rank respectively $m = p - q + l$, $q - l$ and l.*

Then, there exist $V \in \mathrm{GL}_q(D)$, $W \in \mathrm{GL}_p(D)$ and $\overline{R}_1 \in D^{l\times m}$ such that:

$$V\,R\,W = \begin{pmatrix} \overline{R}_1 & 0 \\ 0 & I_{q-l} \end{pmatrix}.$$

The matrices $V := (V_1^T \quad V_2^T)^T \in \mathrm{GL}_q(D)$, $V_1 \in D^{l\times q}$, $V_2 \in D^{(q-l)\times q}$, and $W^{-1} = U := (U_1^T \quad U_2^T)^T$, $U_1 \in D^{m\times p}$, $U_2 \in D^{(q-l)\times p}$, can be chosen such as

$$\begin{cases} \ker_D(.\Delta) = \mathrm{im}_D(.U_1), \\ \mathrm{im}_D(.\Delta) = \mathrm{im}_D(.V_2), \\ U_2 := V_2\,R, \\ V_1 := \Phi\,(I_q - R\,\Delta), \end{cases}$$

where the full row rank matrices U_1, V_2 and Φ define a basis of $\ker_D(.\Delta)$, $\mathrm{im}_D(.\Delta)$ and $\mathrm{coker}_D(.\Delta)$.

Moreover, if $Y \in D^{p\times(p-l)}$ and $Z \in D^{p\times m}$ are the unique matrices defined by

$$\begin{cases} \Delta = Y\,V_2, \\ I_p - \overline{P} = Z\,U_1, \end{cases}$$

and $\Psi \in D^{q\times l}$ is an injective parametrization of $\mathrm{coker}_D(.\Delta)$, i.e., we have $\ker_D(.\Psi) = \mathrm{im}_D(.\Delta)$ and $\Phi\,\Psi = I_l$ (see (37)), then the matrices $X := V^{-1} = (X_1 \quad X_2)$, $X_1 \in D^{q\times l}$, $X_2 \in D^{(q-l)\times q}$, and $W := (W_1 \quad W_2)$, $W_1 \in D^{p\times m}$ and $W_2 \in D^{p\times(p-l)}$, are defined by:

$$\begin{cases} X_1 := \Psi, \\ X_2 := R\,Y, \\ W_1 := Z, \\ W_2 := Y. \end{cases} \tag{49}$$

Finally, we have $\overline{R}_1 = \Phi\,(I_q - R\,\Delta)\,R\,Z = \Phi\,R\,Z$ and the following diagram which commutes in both directions and is formed by horizontal split exact sequences:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & D^{1\times l} \arrow[r, shift left, ".\Phi\,(I_q-\overline{Q})"] \arrow[d, ".\overline{R}_1"'] & D^{1\times q} \arrow[l, shift left, ".\Psi"] \arrow[r, shift left, ".R\,Y"] \arrow[d, ".R"'] & D^{1\times (q-l)} \arrow[l, shift left, ".V_2"] \arrow[r] \arrow[d, ".I_{q-l}"] & 0 \\
0 \arrow[r] & D^{1\times (p-q+l)} \arrow[r, shift left, ".U_1"] \arrow[d, "\kappa"] & D^{1\times p} \arrow[l, shift left, ".Z"] \arrow[r, shift left, ".Y"] \arrow[d, "\pi"'] & D^{1\times (q-l)} \arrow[l, shift left, ".V_2\,R"] \arrow[r] \arrow[d] & 0. \\
0 \arrow[r] & L \arrow[r, "\varphi^{-1}"] \arrow[d] & M \arrow[r] \arrow[d] & 0 & \\
 & 0 & 0 & &
\end{tikzcd}$$

In Section 6.2, we shall prove that the converse of Theorem 8 holds, i.e., if there exist $V \in \mathrm{GL}_q(D)$, $W \in \mathrm{GL}_p(D)$ and $\overline{R}_2 \in D^{(q-l)\times(p-l)}$ satisfying (38), then there exists $\Delta \in D^{p\times q}$ satisfying the conditions 1 and 2 of Theorem 8. A similar result holds for Theorem 9.

Example 6. Let us consider the wind tunnel model studied in Manitius (1984) described by a linear OD time-delay system defined by the following matrix of functional operators

$$R := \begin{pmatrix} d+a & k\,a\,\delta & 0 & 0 \\ 0 & d & -1 & 0 \\ 0 & \omega^2 & d+2\,\zeta\,\omega & -\omega^2 \end{pmatrix},$$

where $d\,y(t) = \dot{y}(t)$ is the OD operator, $\delta\,y(t) = y(t-1)$ is the time-delay operator and ζ, k, ω and a are constant parameters of the system. If $D = \mathbb{Q}(\zeta, k, \omega, a)[d, \delta]$ is the commutative polynomial ring of OD time-delay operators with coefficients in the field $\mathbb{Q}(\zeta, k, \omega, a)$, using Algorithm 4.1 of Cluzeau and Quadrat (2008), we obtain that the following matrix

$$\Delta := \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ \omega^{-2} & 2\,\zeta\,\omega^{-1} - a\,\omega^{-2} & \omega^{-2} \end{pmatrix} \in D^{4\times 3} \tag{50}$$

satisfies the algebraic Riccati equation $\Delta\,R\,\Delta = -\Delta$. Then, $\overline{P} := I_4 + \Delta\,R$ and $\overline{Q} := I_3 + R\,\Delta$ defined by

$$\overline{P} := \begin{pmatrix} 1 & -d & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & d & 0 & 0 \\ \omega^{-2}\,(d+a) & \omega^{-2}\,((2\,\zeta\,\omega - a)\,d + k\,a\,\delta) + 1 & \omega^{-2}\,(d+a) & 0 \end{pmatrix},$$

$$\overline{Q} := \begin{pmatrix} 1 & -d-a & 0 \\ 0 & 0 & 0 \\ -1 & d+a & 0 \end{pmatrix},$$

satisfy $R\,\overline{P} = \overline{Q}\,R$, $\overline{P}^2 = \overline{P}$ and $\overline{Q}^2 = \overline{Q}$, i.e., they define $f = \mathrm{id}_M$, where $M = D^{1\times 4}/(D^{1\times 3}\,R)$. Since the entries of Δ belong to the field $\mathbb{Q}(\zeta, k, \omega, a)$, the D-modules $\ker_D(.\Delta)$, $\mathrm{im}_D(.\Delta)$ and $\mathrm{coker}_D(.\Delta)$ are free. Hence, by Theorem 8, the matrix R is equivalent to a block diagonal matrix with a first identity block. Let us compute the unimodular matrices V, W, U and X. We have $\ker_D(.\Delta) = \mathrm{im}_D(.U_2)$, where

$$U_2 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

and $\mathrm{im}_D(.\Delta) = \mathrm{im}_D(.V_1)$, where $V_1 = T_2\,\Delta$ and T_2 is a left inverse of a minimal parametrization Q_2 of $\mathrm{coker}_D(.U_2) \cong \mathrm{im}_D(.\Delta)$ (see Chyzak et al. (2005)), i.e.:

$$Q_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad V_1 = \begin{pmatrix} 0 & -1 & 0 \\ \omega^{-2} & \omega^{-2}\,(2\,\zeta\,\omega - a) & \omega^{-2} \end{pmatrix}.$$

Computing a minimal parametrization of $\mathrm{coker}_D(.\Delta) = \mathrm{coker}_D(.V_1)$, we obtain that $\ker_D(.\Psi) = \mathrm{im}_D(.\Delta)$ and $\Phi \in D^{1\times 3}$ such that $\Phi\,\Psi = 1$, where:

$$\Psi = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \Phi = (0 \quad 0 \quad -1)\,.$$

Then, we obtain:

$$\begin{array}{l} U_1 := V_1\,R = \begin{pmatrix} 0 & -d & 1 & 0 \\ \omega^{-2}\,(d+a) & \omega^{-2}\,((2\,\zeta\,\omega - a)\,d + k\,a\,\delta + \omega^2) & \omega^{-2}\,(d+a) & -1 \end{pmatrix}, \\ V_2 := \Phi\,\overline{Q} = (1 \quad -d-a \quad 0)\,. \end{array}$$

Let $Y \in D^{4\times 2}$ and $Z \in D^{4\times 2}$ be the unique matrices defined by $\Delta = Y\,V_1$ and $\overline{P} = Z\,U_2$, i.e.,

$$Y = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & -d \\ 0 & 1 \\ 0 & d \\ \omega^{-2}\,(d+a) & \omega^{-2}\,((2\,\zeta\,\omega - a)\,d + k\,a\,\delta + \omega^2) \end{pmatrix},$$

and let us define the matrices $X_2 = \Psi$, $W_1 = -Y$, $W_2 = Z$ and:

$$X_1 := -R\,Y = \begin{pmatrix} -d-a & 0 \\ -1 & 0 \\ d + 2\,\zeta\,\omega & \omega^2 \end{pmatrix}.$$

Then, we have $V := (V_1^T \quad V_2^T)^T \in \mathrm{GL}_3(D)$, $U := (U_1^T \quad U_2^T)^T \in \mathrm{GL}_4(D)$, $U^{-1} := (W_1 \quad W_2)$ and $V^{-1} := (X_1 \quad X_2)$. Finally, we obtain $\overline{R} := V\,R\,U^{-1} = \mathrm{diag}(I_2, \overline{R}_2)$, where:

$$\overline{R}_2 := \Phi\,R\,Z = (d + a \quad -(d^2 + a\,d - k\,a\,\delta)).$$

Similarly, if we consider the following first-order solution of the Riccati equation $\Delta\,R\,\Delta = -\Delta$

$$\Delta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \omega^{-2}\,(d + 2\,\zeta\,\omega) & \omega^{-2} \end{pmatrix}, \tag{51}$$

then we obtain

$$\overline{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & \omega^{-2}\,(d^2 + 2\,\zeta\,\omega\,d + \omega^2) & 0 & 0 \end{pmatrix}, \quad \overline{Q} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$U_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$V_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & \omega^{-2}\,(d + 2\,\zeta\,\omega) & \omega^{-2} \end{pmatrix}, \quad U_1 = V_1\,R = \begin{pmatrix} 0 & d & -1 & 0 \\ 0 & \omega^{-2}\,(d^2 + 2\,\zeta\,\omega\,d + \omega^2) & 0 & -1 \end{pmatrix},$$

$$\Psi = X_2 = (1 \quad 0 \quad 0)^T, \quad \Phi = (1 \quad 0 \quad 0), \quad V_2 = \Phi\,\overline{Q} = (1 \quad 0 \quad 0),$$

$$X_1 = -R\,Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ -d - 2\,\zeta\,\omega & \omega^2 \end{pmatrix}, \quad W_1 = -Y = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ -1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$Z = W_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & d \\ 0 & \omega^{-2}\,(d^2 + 2\,\zeta\,\omega\,d + \omega^2) \end{pmatrix},$$

which shows that $V := (V_1^T \quad V_2^T)^T \in \mathrm{GL}_3(D)$, $W := (W_1 \quad W_2) \in \mathrm{GL}_4(D)$ and $V\,R\,W = \mathrm{diag}(I_2, \overline{R}_2)$, where $\overline{R}_2 = \Phi\,R\,Z = (d + a \quad k\,a\,\delta)$.

6 Serre's reduction problem as a particular decomposition problem

We now study *Serre's reduction* of linear functional systems, i.e., the possibility to define an equivalent system by fewer equations and fewer unknowns. This problem can be seen as a particular decomposition problem where one of the two diagonal blocs of the matrix $\overline{R}$ defined by (27) is the identity matrix.

6.1 Serre's reduction

Let us first state again the main theorems of Boudellioua and Quadrat (2010) concerning Serre's reduction of linear functional systems.

Theorem 10 (Boudellioua and Quadrat (2010))**.** *Let $M = D^{1\times p}/(D^{1\times q}\,R)$ be the left D-module finitely presented by a full row rank matrix $R \in D^{q\times p}$. For $0 \le r \le q-1$, let $\Lambda \in D^{q\times(q-r)}$, $P = (R \quad -\Lambda) \in D^{q\times(p+q-r)}$ and $E = D^{1\times(p+q-r)}/(D^{1\times q}\,P)$ be the left D-module finitely presented by P. Then, the following results are equivalent:*

1. *The left D-module E is stably free of rank $p-r$.*
2. *The matrix P admits a right inverse, i.e., there exists $S \in D^{(p+q-r)\times q}$ such that $P\,S = I_q$.*
3. $\mathrm{ext}^1_D(E, D) := D^q/\left(P\,D^{(p+q-r)}\right) = 0$.
4. *$\{\tau(\Lambda_{\bullet i})\}_{i=1,\ldots,q-r}$ generates the right D-module $\mathrm{ext}^1_D(M, D) := D^q/(R\,D^p)$, where the right D-homomorphism $\tau : D^q \longrightarrow D^q/(R\,D^p)$ is the canonical projection onto $\mathrm{ext}^1_D(M, D)$ and $\Lambda_{\bullet i}$ denotes the i^{th} column of the matrix Λ.*

The above results depend only on the residue class $\rho(\Lambda)$ of $\Lambda \in D^{q\times(q-r)}$ in the right D-module

$$\mathrm{ext}^1_D\left(M, D^{1\times(q-r)}\right) := D^{q\times(q-r)}/\left(R\, D^{p\times(q-r)}\right), \tag{52}$$

i.e., they depend only on $(\tau(\Lambda_{\bullet 1}) \ \ldots \ \tau(\Lambda_{\bullet (q-r)})) \in \mathrm{ext}^1_D(M, D)^{1\times(q-r)}$.

If the conditions of Theorem 10 are satisfied, then the existence of a Serre's reduction for M, i.e., the possibility to define M by fewer than p generators and fewer than q relations, relies on the fact that the stably free left D-module E is free. Using Theorem 3, we obtain the following result.

Theorem 11 (Boudellioua and Quadrat (2010))**.** *Let $R \in D^{q\times p}$ be a full row rank matrix, $0 \le r \le q-1$ and $\Lambda \in D^{q\times(q-r)}$ such that there exists $Z \in \mathrm{GL}_{p+q-r}(D)$ satisfying*

$$(R \quad -\Lambda)\, Z = (I_q \quad 0), \tag{53}$$

i.e., such that the left D-module $E = D^{1\times(p+q-r)}/(D^{1\times q}\,(R \quad -\Lambda))$ is free of rank $p-r$. If we note

$$Z := \begin{pmatrix} S_1 & Q_1 \\ S_2 & Q_2 \end{pmatrix}, \tag{54}$$

where $S_1 \in D^{p\times q}$, $S_2 \in D^{(q-r)\times q}$, $Q_1 \in D^{p\times(p-r)}$ and $Q_2 \in D^{(q-r)\times(p-r)}$, then we have:

$$M = D^{1\times p}/(D^{1\times q}\, R) \;\cong\; L := D^{1\times(p-r)}/(D^{1\times(q-r)}\, Q_2). \tag{55}$$

Conversely, if M is isomorphic to the finitely presented left D-module $L := D^{1\times(p-r)}/(D^{1\times(q-r)}\, Q_2)$, then there exist $\Lambda \in D^{q\times(q-r)}$ and $Z \in \mathrm{GL}_{p+q-r}(D)$ such that (53) holds, i.e., such that the left D-module $E = D^{1\times(p+q-r)}/(D^{1\times q}\,(R \quad -\Lambda))$ is free of rank $p-r$.

Corollary 3 (Boudellioua and Quadrat (2010))**.** *With the notations of Theorem 11, the isomorphism (55) given in Theorem 11 is defined by*

$$\begin{array}{rcl} M = D^{1\times p}/(D^{1\times q}\, R) & \xrightarrow{\varphi} & L = D^{1\times(p-r)}/(D^{1\times(q-r)}\, Q_2) \\ \pi(\lambda) & \longmapsto & \kappa(\lambda\, Q_1). \end{array}$$

and its inverse $\varphi^{-1} : L \longrightarrow M$ is defined by $\varphi^{-1}(\kappa(\mu)) = \pi(\mu\, T_1)$, where T_1 is defined by:

$$Z^{-1} = \begin{pmatrix} R & -\Lambda \\ T_1 & -T_2 \end{pmatrix} \in \mathrm{GL}_{p+q-r}(D), \quad T_1 \in D^{(p-r)\times p}, \quad T_2 \in D^{(p-r)\times(q-r)}. \tag{56}$$

The above results depend only on the residue class $\rho(\Lambda)$ of $\Lambda \in D^{q\times(q-r)}$ in the right D-module $\mathrm{ext}^1_D\left(M, D^{1\times(q-r)}\right)$ defined by (52).

The next theorem gives a necessary and sufficient condition for Serre's reduction problem to be equivalent to a decomposition problem where one of the diagonal blocks is equal to an identity matrix. It is a slight reformulation of a result obtained in Boudellioua and Quadrat (2010) which is given in Cluzeau and Quadrat (2013).

Theorem 12 (Cluzeau and Quadrat (2013))**.** *If $R \in D^{q\times p}$ has full row rank and $0 \le r \le q-1$, then the following assertions are equivalent:*

1. *There exist $V \in \mathrm{GL}_q(D)$, $W \in \mathrm{GL}_p(D)$ and $Q_2 \in D^{(q-r)\times(p-r)}$ such that:*

$$\overline{R} := V\, R\, W = \begin{pmatrix} I_r & 0 \\ 0 & Q_2 \end{pmatrix}. \tag{57}$$

2. *There exists a matrix $\Lambda \in D^{q\times(q-r)}$ such that:*

(a) *A matrix* $\Gamma \in D^{(q-r)\times q}$ *exists and satisfies* $\Gamma\,\Lambda = I_{q-r}$.

(b) *The stably free left* D*-module* $\ker_D(.\Lambda)$ *is free of rank* r.

(c) *The left* D*-module* $E = D^{1\times(p+q-r)}/(D^{1\times q}\,(R \quad -\Lambda))$ *is free of rank* $p-r$.

Proof. In Corollaries 4.10 and 4.14 of Boudellioua and Quadrat (2010), it is proved that 1 is equivalent to 2.a, 2.c and to the condition 2.b' defined by the fact that the stably free left D-module $\ker_D(.Q_1)$ is free of rank r, where $Q_1 \in D^{p\times(p-r)}$ is a matrix defined by (54) and $Z \in \mathrm{GL}_{p+q-r}(D)$ satisfies the condition (53) (which is equivalent to 2.c (see 3 of Theorem 3)). Let us now show that 2.a, 2.b and 2.c are equivalent to 2.a, 2.b' and 2.c. To do that, we show that $\ker_D(.\Lambda) \cong \ker_D(.Q_1)$. Using 2.c and 3 of Theorem 3, there exists $Z \in \mathrm{GL}_{p+q-r}(D)$ which satisfies (53) so that Theorem 11 and Corollary 3 hold. Using (54), we get the relation $R\,Q_1 = \Lambda\,Q_2$ which yields the following commutative exact diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \ker_D(.\Lambda) \arrow[d] & \ker_D(.Q_1) \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & D^{1\times q} \arrow[r, ".R"] \arrow[d, ".\Lambda"'] & D^{1\times p} \arrow[r, "\pi"] \arrow[d, ".Q_1"'] & M \arrow[r] \arrow[d, "\varphi"] & 0 \\
0 \arrow[r] & D^{1\times(q-r)} \arrow[r, ".Q_2"] \arrow[d] & D^{1\times(p-r)} \arrow[r, "\kappa"] \arrow[d] & L \arrow[r] \arrow[d] & 0. \\
 & \mathrm{coker}_D(.\Lambda) \arrow[d] & \mathrm{coker}_D(.Q_1) \arrow[d] & 0 & \\
 & 0 & 0 & &
\end{tikzcd} \tag{58}$$

Since φ is an isomorphism, the standard *snake lemma* in homological algebra (see, e.g., Rotman (2009)) then shows that $\ker_D(.\Lambda) \cong \ker_D(.Q_1)$ (and $\mathrm{coker}_D(.\Lambda) \cong \mathrm{coker}_D(.Q_1)$), which proves the result. $\square$

Remark 9. Let $\Lambda \in D^{q\times(q-r)}$ be such that $\Gamma\,\Lambda = I_{q-r}$ for a certain $\Gamma \in D^{(q-r)\times q}$. If D is the ring defined in 1 or 2 of Theorem 2 or in 3 and 4 of Theorem 2 and $r \geq 2$, then the stably free left D-module $\ker_D(.\Lambda)$ of rank r is free.

In Theorem 12, the conditions 2.a and 2.b on Λ mean that we are searching for a particular splitting $D^{1\times q} \cong D^{1\times r} \oplus D^{1\times(q-r)}$ of $D^{1\times q}$, i.e., for a splitting short exact sequence of the form:

$$0 \longrightarrow D^{1\times r} \underset{.\Xi}{\overset{.\Theta}{\rightleftarrows}} D^{1\times q} \underset{.\Gamma}{\overset{.\Lambda}{\rightleftarrows}} D^{1\times(q-r)} \longrightarrow 0. \tag{59}$$

In other words, the conditions 2.a and 2.b are equivalent to the existence of $V := \left(\Theta^T \quad \Gamma^T\right)^T \in \mathrm{GL}_q(D)$ with $X := V^{-1} = (\Xi \quad \Lambda) \in \mathrm{GL}_q(D)$ (see (23)). Condition 2.c asserts that we have (55) and thus (58) holds.

Remark 10. We note that the condition 2.c is also equivalent to the fact that the *pushout* of the two left D-homomorphisms $.\Lambda : D^{1\times q} \longrightarrow D^{1\times(q-r)}$ and $.R : D^{1\times q} \longrightarrow D^{1\times p}$ (see, e.g., Rotman (2009)), i.e., the finitely presented left D-module $E = D^{1\times(p+q-r)}/(D^{1\times q}\,(R \quad -\Lambda))$ is free of rank $p-r$. A property of the pushout (see, e.g., Rotman (2009)) shows that we have the following commutative exact diagram

$$\begin{tikzcd}
0 \arrow[r] & D^{1\times q} \arrow[r, ".R"] \arrow[d, ".\Lambda"] & D^{1\times p} \arrow[r, "\pi"] \arrow[d, "\delta\circ i_1"] & M \arrow[r] \arrow[d, "\mathrm{id}_M"] & 0 \\
0 \arrow[r] & D^{1\times(q-r)} \arrow[r, "\delta\circ i_2"] & E \arrow[r, "\beta"] & M \arrow[r] & 0,
\end{tikzcd}$$

where $\delta : D^{1\times(p+q-r)} \longrightarrow E$ denotes the canonical projection onto E and $i_1 : D^{1\times p} \longrightarrow D^{1\times(p+q-r)}$ and $i_2 : D^{1\times(q-r)} \longrightarrow D^{1\times(p+q-r)}$ are the two canonical injections. Hence, if $E \cong D^{1\times(p-r)}$, then we can easily show that the above commutative exact diagram becomes (58) (Boudellioua and Quadrat (2010)).

Let us now develop the links between Theorem 12 and Theorem 7. The exact sequence (59) shows that $\ker_D(.\Lambda) = \operatorname{im}_D(.\Theta)$ and $\operatorname{coker}_D(.\Lambda) = 0$ since $.\Lambda$ is a surjective left D-homomorphism. Then, (58) implies that $\ker_D(.Q_1) = \ker_D(.\Lambda)\, R = \operatorname{im}_D(.\Theta)\, R = \operatorname{im}_D(.(\Theta\, R))$ and $\operatorname{coker}_D(.Q_1) \cong \operatorname{coker}_D(.\Lambda) = 0$. Since both R and Θ have full row rank so has $U_1 := \Theta\, R \in D^{r\times p}$. Thus, (58) yields the following commutative exact diagram:

$$\begin{tikzcd}
0 \arrow[r] & D^{1\times r} \arrow[r, ".\Theta"] \arrow[d, equal] & D^{1\times q} \arrow[r, ".\Lambda"] \arrow[d, ".R"] & D^{1\times (q-r)} \arrow[r] \arrow[d, ".Q_2"] & 0 \\
0 \arrow[r] & D^{1\times r} \arrow[r, ".U_1"] & D^{1\times p} \arrow[r, ".Q_1"] & D^{1\times (p-r)} \arrow[r] & 0.
\end{tikzcd} \tag{60}$$

Hence, the second horizontal exact sequence of (60) splits (see Proposition 3), i.e.

$$\begin{tikzcd}
0 \arrow[r] & D^{1\times r} \arrow[r, shift left, ".U_1"] & D^{1\times p} \arrow[l, shift left, ".W_1"] \arrow[r, shift left, ".Q_1"] & D^{1\times (p-r)} \arrow[l, shift left, ".U_2"] \arrow[r] & 0,
\end{tikzcd} \tag{61}$$

which yields $D^{1\times p} = \operatorname{im}_D(.U_1) \oplus \operatorname{im}_D(.U_2) \cong D^{1\times r} \oplus D^{1\times (p-r)}$. Equivalently, using (23), we obtain $U := (U_1^T \quad U_2^T)^T \in \operatorname{GL}_p(D)$ and $W := U^{-1} = (W_1 \quad Q_1) \in \operatorname{GL}_p(D)$. We point out that $\Gamma\, R$ is not necessarily equal to $Q_2\, U_2$, and thus we cannot yet conclude that 2 of Theorem 7 does necessarily hold.

From the identity $U\, W = I_p$, we get $U_1\, (W_1 \quad Q_1) = (I_r \quad 0)$ which combined with $U_1 = \Theta\, R$ yields

$$\begin{pmatrix} \Theta \\ \Gamma \end{pmatrix} R\, (W_1 \quad Q_1) = \begin{pmatrix} \Theta\, R \\ \Gamma\, R \end{pmatrix} (W_1 \quad Q_1) = \begin{pmatrix} I_r & 0 \\ \Gamma\, R\, W_1 & \Gamma\, R\, Q_1 \end{pmatrix},$$

and thus we get:

$$\begin{pmatrix} I_r & 0 \\ -\Gamma\, R\, W_1 & I_{q-r} \end{pmatrix} \begin{pmatrix} \Theta \\ \Gamma \end{pmatrix} R\, (W_1 \quad Q_1) = \begin{pmatrix} I_r & 0 \\ 0 & \Gamma\, R\, Q_1 \end{pmatrix}.$$

Hence, if we consider $V' := (\Theta^T \quad (\Gamma\, (I_q - R\, W_1\, \Theta))^T)^T \in \operatorname{GL}_q(D)$, whose inverse is defined by

$$\begin{aligned} V'^{-1} &:= (\Xi \quad \Lambda) \begin{pmatrix} I_r & 0 \\ \Gamma\, R\, W_1 & I_{q-r} \end{pmatrix} = (\Xi + \Lambda\, \Gamma\, R\, W_1 \quad \Lambda) \\ &= (\Xi + (I_q - \Xi\, \Theta)\, R\, W_1 \quad \Lambda) = (\Xi\, (I_r - \Theta\, R\, W_1) + R\, W_1 \quad \Lambda) = (R\, W_1 \quad \Lambda), \end{aligned}$$

since $U_1\, W_1 = \Theta\, R\, W_1 = I_r$, then we get:

$$V'\, R\, W = \operatorname{diag}(I_r, \Gamma\, R\, Q_1).$$

Finally, pre-multiplying $R\, Q_1 = \Lambda\, Q_2$ by Γ and using $\Gamma\, \Lambda = I_{q-r}$, we obtain $Q_2 = \Gamma\, R\, Q_1$, which shows that $V'\, R\, W = \operatorname{diag}(I_r, Q_2)$, where $V' \in \operatorname{GL}_q(D)$ and $W \in \operatorname{GL}_p(D)$.

Let us note $\Gamma' := \Gamma\, (I_q - R\, W_1\, \Theta)$ and $\Xi' := R\, W_1$. Using the identities $W_1\, U_1 + Q_1\, U_2 = I_p$, $U_1 = \Theta\, R$ and $U_1\, W_1 = I_r$, we get $\Gamma'\, R = \Gamma\, (I_q - R\, W_1\, \Theta)\, R = \Gamma\, R\, (I_p - W_1\, U_1) = (\Gamma\, R\, Q_1)\, U_2 = Q_2\, U_2$, which shows that following diagram is formed by horizontal split exact sequences and commutes in both directions:

$$\begin{tikzcd}
0 \arrow[r] & D^{1\times r} \arrow[r, shift left, ".\Theta"] \arrow[d, equal] & D^{1\times q} \arrow[l, shift left, ".\Xi'"] \arrow[r, shift left, ".\Lambda"] \arrow[d, ".R"] & D^{1\times (q-r)} \arrow[l, shift left, ".\Gamma'"] \arrow[r] \arrow[d, ".Q_2"] & 0 \\
0 \arrow[r] & D^{1\times r} \arrow[r, shift left, ".U_1"] & D^{1\times p} \arrow[l, shift left, ".W_1"] \arrow[r, shift left, ".Q_1"] & D^{1\times (p-r)} \arrow[l, shift left, ".U_2"] \arrow[r] & 0.
\end{tikzcd}$$

Note that if $\Gamma\, R = Q_2\, U_2$, then we get $\Gamma\, R\, W_1\, \Theta = Q_2\, U_2\, W_1\, \Theta = 0$ since $U_2\, W_1 = 0$, which then yields $\Gamma' := \Gamma\, (I_q - R\, W_1\, \Theta) = \Gamma$.

In particular, Theorem 7 holds with the following matrices

$$\left\{\begin{array}{l} V := \left(\Theta^T \quad (\Gamma\,(I_q - R\,W_1\,\Theta))^T\right)^T \in \mathrm{GL}_q(D), \\ X := V^{-1} = (R\,W_1 \quad \Lambda), \\ U := ((\Theta\,R)^T \quad U_2^T)^T \in \mathrm{GL}_p(D), \\ W := U^{-1} = (W_1 \quad Q_1), \\ R' = I_r, \\ R'' = Q_2 = \Gamma\,R\,Q_1, \end{array}\right. \tag{62}$$

where the matrices Θ and Γ are defined by the split exact sequence (59) and the matrices W_1 and U_2 are defined by the split exact sequence (61). We have the following commutative exact diagram:

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & D^{1\times r} \arrow[r, shift left, ".\Theta"] \arrow[d, ".I_r"'] & D^{1\times q} \arrow[l, shift left, ".\Xi'"] \arrow[r, shift left, ".\Lambda"] \arrow[d, ".R"'] & D^{1\times (q-r)} \arrow[l, shift left, ".\Gamma'"] \arrow[r] \arrow[d, ".Q_2"] & 0 \\ 0 \arrow[r] & D^{1\times r} \arrow[r, shift left, ".U_1"] \arrow[d] & D^{1\times p} \arrow[l, shift left, ".W_1"] \arrow[r, shift left, ".Q_1"] \arrow[d, "\pi"'] & D^{1\times (p-r)} \arrow[l, shift left, ".U_2"] \arrow[r] \arrow[d, "\kappa"] & 0 \\ & 0 \arrow[r] & M \arrow[r, "\varphi"] \arrow[d] & L \arrow[r] \arrow[d] & 0. \\ & & 0 & 0 & \end{tikzcd} \tag{63}$$

To compute the matrices defined in (62), we first compute a basis of the free left D-module $\ker_D(.\Lambda)$ to obtain a full row rank matrix $\Theta \in D^{r\times q}$ such that $\ker_D(.\Lambda) = \mathrm{im}_D(.\Theta)$. Then, we can define the matrix $U_1 := \Theta\,R$ and compute a right inverse $W_1 \in D^{p\times r}$ of U_1. Finally, we can define the matrices $X := (R\,W_1 \quad \Lambda)$ and $W := (W_1 \quad Q_1)$ and finally compute $V := X^{-1}$ (and $U = W^{-1}$).

Let us write (62) in a different form. The identities $U_1 = \Theta\,R$, $R\,S_1 - \Lambda\,S_2 = I_q$ and $\Theta\,\Lambda = 0$ yield:

$$U_1\,S_1 = \Theta\,R\,S_1 = \Theta\,(I_q + \Lambda\,S_2) = \Theta.$$

Using this identity, $W_1\,U_1 + Q_1\,U_2 = I_p$, $R\,S_1 - \Lambda\,S_2 = I_q$, $Q_2 = \Gamma\,R\,Q_1$ and $\Gamma\,\Lambda = I_{q-r}$, we get:

$$\begin{aligned} \Gamma' = \Gamma\,(I_q - R\,W_1\,\Theta) &= \Gamma\,(I_q - R\,W_1\,U_1\,S_1) = \Gamma\,(I_q - R\,(I_p - Q_1\,U_2)\,S_1) \\ &= \Gamma\,(I_q - R\,S_1 + R\,Q_1\,U_2\,S_1) = -\Gamma\,\Lambda\,S_2 + (\Gamma\,R\,Q_1)\,U_2\,S_1 = -S_2 + Q_2\,U_2\,S_1. \end{aligned}$$

Hence, we can rewrite (62) in a form where Θ is replaced by U_1:

$$\left\{\begin{array}{l} V := \left((U_1\,S_1)^T \quad (-S_2 + Q_2\,U_2\,S_1)^T\right)^T \in \mathrm{GL}_q(D), \\ X := V^{-1} = (R\,W_1 \quad \Lambda), \\ U := (U_1 \quad U_2^T)^T \in \mathrm{GL}_p(D), \\ W := U^{-1} = (W_1 \quad Q_1), \\ R' = I_r, \\ R'' = Q_2 = \Gamma\,R\,Q_1. \end{array}\right. \tag{64}$$

The above expressions are the ones obtained in Boudellioua and Quadrat (2010) in which the condition that $\ker_D(.Q_1)$ is a free left D-module of rank r (i.e., $\ker_D(.Q_1) = \mathrm{im}_D(.U_1)$ for a full row rank matrix $U_1 \in D^{r\times p}$) is used instead of 2.b of Theorem 12 (i.e., $\ker_D(.\Lambda) = \mathrm{im}_D(.\Theta)$ for a full row rank $\Theta \in D^{r\times q}$).

Let check again the result. Using $V\,X = I_q$, where $X = (\Xi' \quad \Lambda) = (R\,W_1 \quad \Lambda)$, we get $V\,(R\,W_1) = (I_r^T \quad 0^T)^T$. Using this identity, $R\,Q_1 = \Lambda\,Q_2$, $\Theta\,\Lambda = 0$, $\Gamma' := \Gamma\,(I_q - R\,W_1\,\Theta)$ and $Q_2 = \Gamma\,R\,Q_1$, we get:

$$V\,R\,W = \begin{pmatrix} \Theta \\ \Gamma' \end{pmatrix} R\,(W_1 \quad Q_1) = \begin{pmatrix} \Theta \\ \Gamma' \end{pmatrix} (R\,W_1 \quad R\,Q_1) = \begin{pmatrix} I_r & \Theta\,R\,Q_1 \\ 0 & \Gamma'\,R\,Q_1 \end{pmatrix}$$
$$= \begin{pmatrix} I_r & \Theta\,\Lambda\,Q_2 \\ 0 & \Gamma\,R\,Q_1 - \Gamma\,R\,W_1\,\Theta\,\Lambda\,Q_2 \end{pmatrix} = \begin{pmatrix} I_r & 0 \\ 0 & \Gamma\,R\,Q_1 \end{pmatrix} = \begin{pmatrix} I_r & 0 \\ 0 & Q_2 \end{pmatrix}.$$

Finally, the matrices V and W can be obtained as follows. We first compute a basis of the free left D-module $\ker_D(.Q_1)$ to get a full row rank matrix $U_1 \in D^{r\times p}$ such that $\ker_D(.Q_1) = \mathrm{im}_D(.U_1)$. Then, we compute a right inverse $W_1 \in D^{p\times r}$ of U_1 and define the matrices $W := (W_1 \quad Q_1)$ and $X := (R\,W_1 \quad \Lambda)$ and finally compute $V = X^{-1}$ (and $U = W^{-1}$). See Boudellioua and Quadrat (2010).

We summarize the above results in the following corollary.

Corollary 4. *We have the following results:*

1. *If 2 of Theorem 12 holds, then with the notations (54) and (56), the matrices*

$$\begin{cases} V = (V_1^T \quad V_2^T)^T, & V_1 \in D^{r\times q}, & V_2 \in D^{(q-r)\times q}, \\ V^{-1} = (X_1 \quad X_2), & X_1 \in D^{q\times r}, & X_2 \in D^{q\times (q-r)}, \\ W = (W_1 \quad W_2), & W_1 \in D^{p\times r}, & W_2 \in D^{p\times (p-r)}, \\ W^{-1} = (U_1^T \quad U_2^T)^T, & U_1 \in D^{r\times p}, & U_2 \in D^{(p-r)\times p}, \end{cases} \tag{65}$$

defined in 1 of Theorem 12 can be chosen as follows

$$\begin{cases} V_1 = \Theta, \\ V_2 = \Gamma\,(I_q - R\,W_1\,\Theta), \\ W_2 = Q_1, \end{cases} \qquad \begin{cases} X_1 = R\,W_1, \\ X_2 = \Lambda, \\ U_1 = \Theta\,R, \end{cases} \tag{66}$$

where $\Theta \in D^{r\times q}$ is a full row rank matrix such that $\ker_D(.\Lambda) = \mathrm{im}_D(.\Theta)$, *$W_1 \in D^{p\times r}$ is a right inverse of $U_1 = \Theta\,R$, i.e., $U_1\,W_1 = I_r$, $U_2 \in D^{(p-r)\times p}$ satisfies*

$$I_p - W_1\,U_1 = Q_1\,U_2, \tag{67}$$

and $Q_2 = \Gamma\,R\,Q_1$, where $\Gamma \in D^{(q-r)\times q}$ is a left inverse of Λ, i.e., $\Gamma\,\Lambda = I_{q-r}$.

Equivalently, the matrices defined in 1 of Theorem 12 can be defined by

$$\begin{cases} V_1 = U_1\,S_1, \\ V_2 = -S_2 + Q_2\,U_2\,S_1, \\ W_2 = Q_1, \end{cases} \qquad \begin{cases} X_1 = R\,W_1, \\ X_2 = \Lambda, \end{cases} \tag{68}$$

where $U_1 \in D^{r\times p}$ is a full row rank matrix such that $\ker_D(.Q_1) = \mathrm{im}_D(.U_1)$, *$W_1 \in D^{p\times r}$ is a right inverse of U_1, i.e., $U_1\,W_1 = I_r$, $U_2 \in D^{(p-r)\times p}$ satisfies (67) and $Q_2 = \Gamma\,R\,Q_1$, where $\Gamma \in D^{(q-r)\times q}$ is a left inverse of Λ, i.e., $\Gamma\,\Lambda = I_{q-r}$.*

2. *If 1 of Theorem 12 holds, then, with the notations of (65), the matrices of 2 of Theorem 12 can be chosen as follows:*

$$\Lambda = X_2, \quad \Gamma = V_2, \quad \Theta = V_1, \quad Z = \begin{pmatrix} W_1\,V_1 & W_2 \\ -V_2 & Q_2 \end{pmatrix}, \quad Z^{-1} = \begin{pmatrix} R & -\Lambda \\ U_2 & 0 \end{pmatrix}.$$

Proof. 1 has been proved above and 2 is proved in Corollary 4.14 of Boudellioua and Quadrat (2010) by using the isomorphism $\ker_D(.\Lambda) \cong \ker_D(.Q_1)$ given by (58).

Let us give a proof of 2 based on the results obtained above. By Theorem 7, 2 is equivalent to (28) with $R' = I_r$, $R'' = Q_2$ and $l = m = r$. Comparing (28) with (63), and considering $\Lambda := X_2$, $\Theta := V_1$, $\Gamma' := V_2$, $\Xi' := X_1$ and $Q_1 := W_2$, the relations $X_1\, V_1 + X_2\, V_2 = I_q$ and $X_1 = R\, W_1$ yield $R\,(W_1\, V_1) - X_2\,(-\Lambda) = I_q$. Using the relations $R\, W_2 - X_2\, Q_2 = 0$, $U_2\, W_2 = I_{q-r}$ and $U_2\, W_1 = 0$, we get:

$$\begin{pmatrix} R & -\Lambda \\ U_2 & 0 \end{pmatrix} \begin{pmatrix} W_1\, V_1 & W_2 \\ -V_2 & Q_2 \end{pmatrix} = I_{p+q-r}.$$

The relations $W_1\, U_1 + W_2\, U_2 = I_p$, $U_1 = V_1\, R$, $V_2\, R = Q_2\, U_2$, $V_1\, X_2 = 0$ and $V_2\, X_2 = I_{q-r}$ yield

$$\begin{pmatrix} W_1\, V_1 & W_2 \\ -V_2 & Q_2 \end{pmatrix} \begin{pmatrix} R & -\Lambda \\ U_2 & 0 \end{pmatrix} = I_{p+q-r},$$

which proves 2.c of Theorem 12. Finally, $V_2\, X_2 = I_{q-r}$ yields 2.b of Theorem 12 and the fact that $\ker_D(.X_2) = \mathrm{im}_D(.V_1) \cong D^{1\times r}$ since V_1 has full row rank gives 2.a of Theorem 12. □

Example 7. Let us consider again the wind tunnel model studied in Example 6. Let $\mathrm{ext}^1_D(M, D) := D^3/(R\, D^4)$ be the $D = \mathbb{Q}(\zeta, k, \omega, a)[d, \delta]$-module defined by (52) with $r = q - 1 = 2$. Computing a Gröbner basis of $\mathrm{ext}^1_D(M, D)$, we obtain that this D-module is a finite-dimensional $\mathbb{Q}(\zeta, k, \omega, a)$-vector space of dimension 1 defined by the basis $\tau(\Lambda)$, where $\Lambda := (1 \quad 0 \quad 0)^T$ and $\tau : D^3 \longrightarrow \mathrm{ext}^1_D(M, D)$ is the canonical projection. We can check that the matrix $P := (R \quad -\Lambda)$ admits the right inverse $S = (S_1^T \quad S_2^T)^T$, where

$$S_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & -\omega^{-2}\,(d + 2\,\zeta\,\omega) & -\omega^{-2} \end{pmatrix}, \quad S_2 = (-1 \quad 0 \quad 0),$$

which shows that the D-module $E := \mathrm{coker}_D(.P)$ is stably free (see Theorem 10), and thus free of rank 2 by the Quillen-Suslin theorem (see 2 of Theorem 2 or Remark 9). Computing a basis of the free D-module E (see Fabiańska and Quadrat (2007)), we obtain that the following matrices

$$Q_1 = \begin{pmatrix} 1 & 0 \\ 0 & \omega^2 \\ 0 & \omega^2\, d \\ 0 & d^2 + 2\,\zeta\,\omega\, d + \omega^2 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} d + a & k\, a\, \omega^2\, \delta \end{pmatrix},$$

$$T_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \omega^{-2} & 0 & 0 \end{pmatrix}, \qquad T_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

are such that the matrix Z defined by (54) satisfies $Z \in \mathrm{GL}_5(D)$ and $P\, Z = (I_3 \quad 0)$. By Theorem 11, we obtain that $M = D^{1\times 4}/(D^{1\times 3}\, R) \cong L := D^{1\times 2}/(D\, Q_2)$, i.e., the wind tunnel model can be defined by a single OD time-delay equation in two unknown functions, i.e., $\dot{z}(t) + a\, z(t) + k\, a\, \omega^2\, v(t-1) = 0$.

We note that $\Gamma := (1 \quad 0 \quad 0)$ is a left inverse of Λ. Hence, by 2 of Theorem 12, R is equivalent to the diagonal matrix $\mathrm{diag}(I_2, Q_2)$. Using 1 of Corollary 4, let us compute the matrices $V \in \mathrm{GL}_3(D)$ and $W \in \mathrm{GL}_4(D)$ such that $V\, R\, W = \mathrm{diag}(I_2, Q_2)$ as well as $X = V^{-1}$ and $U = W^{-1}$. Computing first a basis of $\ker_D(.\Lambda)$, we obtain that $\ker_D(.\Lambda) = \mathrm{im}_D(.V_1)$, where the full row rank matrix V_1 is defined by:

$$V_1 = \Theta := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Using (66), we obtain that the following matrices

$$U_1 := \Theta\, R = \begin{pmatrix} 0 & d & -1 & 0 \\ 0 & \omega^2 & d + 2\,\zeta\,\omega & -\omega^2 \end{pmatrix}, \quad W_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ -1 & 0 \\ -\omega^{-2}\,(d + 2\,\zeta\,\omega) & -\omega^{-2} \end{pmatrix},$$

$$U_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \omega^{-2} & 0 & 0 \end{pmatrix}, \qquad X_1 = R\,W_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$V_2 = \Gamma\,(I_3 - R\,W_1\,\Theta) = (1 \quad 0 \quad 0), \qquad X_2 = \Lambda = (1 \quad 0 \quad 0)^T,$$

$$W_2 = Q_1 = \begin{pmatrix} 1 & 0 \\ 0 & \omega^2 \\ 0 & \omega^2\, d \\ 0 & d^2 + 2\,\zeta\,\omega\, d + \omega^2 \end{pmatrix}, \qquad Q_2 = \Gamma\, R\, Q_1 = \begin{pmatrix} d + a & k\, a\, \omega^2\, \delta \end{pmatrix},$$

where W_1 is a right inverse of U_1, are such that $V := (V_1^T \quad V_2^T)^T \in \mathrm{GL}_3(D)$, $W = (W_1 \quad W_2) \in \mathrm{GL}_4(D)$, $X = V^{-1}$, $U = W^{-1}$ and $V\,R\,W = \mathrm{diag}(I_2, Q_2)$. Equivalently, these matrices can also be obtained by (68), where the matrix $U_1 \in D^{2\times 4}$, which defines a basis of $\ker_D(.Q_1)$, can be taken as above.

In the forthcoming Example 8, we shall illustrate Serre's reduction techniques with an interesting class of examples. To do that, we first need to introduce the concept of the *Fitting ideals* of a finitely presented module over a commutative ring and state a few standard results. For more details, see, e.g., Eisenbud (1995).

Definition 4 (Eisenbud (1995))**.** Let D be a commutative ring, $R \in D^{q\times p}$ and $M = D^{1\times p}/(D^{1\times q}\,R)$. Then, the i^{th} *Fitting ideal* $\mathrm{Fitt}_i(M)$ of M is the ideal of D generated by the $(p-i)\times(p-i)$ minors of R, with the conventions that $\mathrm{Fitt}_i(M) = 0$ if $p - i > q$, i.e., if $i < p - q$, and $\mathrm{Fitt}_i(M) = D$ for $i \geq p$.

Theorem 13 (Eisenbud (1995))**.** *Let D be a commutative ring, $R \in D^{q\times p}$ and $M = D^{1\times p}/(D^{1\times q}\,R)$. The Fitting ideals $\mathrm{Fitt}_i(M)$'s depend only on M and not on the presentation matrix R of M.*

If the D-module $M = D^{1\times p}/(D^{1\times q}\,R)$ can be generated by r elements with $r \leq p$, i.e., M admits a finite presentation of the form $D^{1\times s} \xrightarrow{.Q} D^{1\times r} \xrightarrow{\sigma} M \longrightarrow 0$, then using Theorem 13 and Definition 4, we obtain that $\mathrm{Fitt}_r(M) = D$. Hence, we have the following corollary.

Corollary 5 (Eisenbud (1995))**.** *Let D be a commutative ring, $R \in D^{q\times p}$ and $M = D^{1\times p}/(D^{1\times q}\,R)$. If M can be generated by r elements, then we have $\mathrm{Fitt}_r(M) = D$.*

The above corollary will be used in Section 7. Let us now state two results that will be used below.

Proposition 6 (Eisenbud (1995))**.** *Let D be a commutative ring, $R \in D^{q\times p}$ and $M = D^{1\times p}/(D^{1\times q}\,R)$. Then, M is a projective D-module of rank r if and only if $\mathrm{Fitt}_i(M) = 0$ for $i = 0, \ldots, r-1$, and $\mathrm{Fitt}_r(M) = D$.*

Proposition 7 (Eisenbud (1995))**.** *Let D be a commutative ring, $R \in D^{q\times p}$, $M = D^{1\times p}/(D^{1\times q}\,R)$ and E a commutative ring which is a D-module. Then, we have $\mathrm{Fitt}_i(E \otimes_D M) \cong E \otimes_D \mathrm{Fitt}_i(M)$ for $i \geq 0$.*

Example 8. If $D = k[x_1, \ldots, x_n]$ is a commutative polynomial ring over a field k, in Lin et al. (2006), it is shown that every matrix $R \in D^{p\times p}$ whose determinant $\det(R)$ is of the form $x_1 - f(x_2, \ldots, x_n)$, where $f \in k[x_2, \ldots, x_n]$, admits Serre's reduction. Let us prove this using the above results. We first characterize $\mathrm{Fitt}_0(M)$ and $\mathrm{Fitt}_1(M)$. We clearly have $\mathrm{Fitt}_0(M) = (\det(R))$ and

$$1 = \frac{\partial \det(R)}{\partial x_1} = \sum_{i=1}^{p} \det\left(R_{\bullet 1} \; \ldots \; \frac{\partial R_{\bullet i}}{\partial x_1} \; \ldots \; R_{\bullet p}\right),$$

where $R_{\bullet i}$ denotes the i^{th} column of R. If we develop each determinant in the last above sum with respect to the column which is differentiated, then we get that the corresponding determinant is a polynomial combination of $(p-1)\times(p-1)$ minors of R, i.e., is an element of $\text{Fitt}_1(M)$. This proves that $\text{Fitt}_1(M) = D$.

Now, if $E := D/\text{Fitt}_0(M) \cong A := k[x_2, \ldots, x_n]$, then applying the *covariant right exact functor* $E \otimes_D \cdot$ (see, e.g., Rotman (2009)) to the finite presentation of the D-module $N := D^p/(R\, D^p)$, i.e., to the following short exact sequence of D-modules

$$0 \longleftarrow N \xleftarrow{\kappa} D^p \xleftarrow{R.} D^p \longleftarrow 0,$$

we get the following exact sequence of E-modules

$$0 \longleftarrow E \otimes_D N \longleftarrow E^p \xleftarrow{\overline{R}.} E^p \longleftarrow \text{tor}_1^D(N, E) \longleftarrow 0, \tag{69}$$

where $\overline{R} := R(f(x_2, \ldots, x_n), x_2, \ldots, x_n)$ and $\text{tor}_1^D(N, E) \cong \ker_E(\overline{R}.)$. For more details, see, e.g., Rotman (2009). Similarly as before with R^T instead of R, we get $\text{Fitt}_0(E \otimes_D N) = 0$ and $\text{Fitt}_1(E \otimes_D N) = E$, which shows that $E \otimes_D N$ is a projective E-module by Proposition 6, and thus a free E-module of rank 1 by the Quillen-Suslin theorem (see 2 of Theorem 2). Thus, the above long exact sequence splits by Proposition 3, which shows that $\text{im}_E(\overline{R}.)$ (resp., $\ker_E(\overline{R}.)$) is a stably free, i.e., a free E-module of rank $p-1$ (resp., 1) by the Quillen-Suslin theorem. Computing a basis of $\ker_E(\overline{R}.)$, we get $\overline{R}_2 \in E^p$ such that $\ker_E(\overline{R}.) = \overline{R}_2\, E$ and $\overline{S}_2\, \overline{R}_2 = 1$ for a certain $\overline{S}_2 \in E^{1\times p}$. See Fabiańska and Quadrat (2007) for a way to compute these two matrices based only on the fact that $E \otimes_D N$ is free of rank 1 over a commutative ring E. Now, computing a basis of the free E-module $\text{coker}_E(\overline{R}_2.) \cong \text{im}_E(\overline{R}.)$ of rank $p-1$, we obtain two matrices $\overline{Q}_2 \in E^{(p-1)\times p}$ and $\overline{T}_2 \in E^{p\times(p-1)}$ such that

$$(\overline{T}_2 \quad \overline{R}_2) \begin{pmatrix} \overline{Q}_2 \\ \overline{S}_2 \end{pmatrix} = I_p, \quad \begin{pmatrix} \overline{Q}_2 \\ \overline{S}_2 \end{pmatrix} (\overline{T}_2 \quad \overline{R}_2) = I_p, \tag{70}$$

i.e., $W := (\overline{T}_2 \quad \overline{R}_2) \in \text{GL}_p(E)$. For more details, see Fabiańska and Quadrat (2007); Quadrat and Robertz (2007a). Since the entries of the matrices $\overline{T}_2$ and $\overline{R}_2$ can be chosen in A, we have $W \in \text{GL}_p(A)$. Then, the identity $\overline{R}\,\overline{R}_2 = 0$ yields $R\,\overline{R}_2 = \Lambda \det(R)$ for a certain $\Lambda \in D^p$, and thus we have:

$$R\,W = R\,(\overline{T}_2 \quad \overline{R}_2) = (R\,\overline{T}_2 \quad \Lambda \det(R)) = (R\,\overline{T}_2 \quad \Lambda) \begin{pmatrix} I_{p-1} & 0 \\ 0 & \det(R) \end{pmatrix}.$$

Since $\det(W)$ can be chosen to be 1, the above identity yields $\det(R\,\overline{T}_2 \quad \Lambda) = 1$, i.e., $X := (R\,\overline{T}_2 \quad \Lambda) \in \text{GL}_p(D)$, and we obtain that $V\,R\,W = \text{diag}(I_{p-1}, \det(R))$, where $V := X^{-1} \in \text{GL}_p(D)$.

Finally, let us add a few more comments. Since $E \otimes_D N$ is a free E-module of rank 1, (69) yields the following split long exact sequence

$$\begin{tikzcd} 0 & E \arrow[l] \arrow[r, shift left, "\overline{S}_0."] & E^p \arrow[l, shift left, "\overline{R}_0."] \arrow[r, shift left, "\overline{S}_1."] & E^p \arrow[l, shift left, "\overline{R}_1."] \arrow[r, shift left, "\overline{S}_2."] & E \arrow[l, shift left, "\overline{R}_2."] & 0 \arrow[l] \end{tikzcd},$$

with the notation $\overline{R}_1 = \overline{R}$, which can be rewritting as follows:

$$\begin{tikzcd} E^p \arrow[r, shift left, "\overline{S}_2."] \arrow[d, shift left, "\overline{R}_1."] & E \arrow[l, shift left, "\overline{R}_2."] \arrow[d, "0"] & 0 \arrow[l] \\ E^p \arrow[u, shift left, "\overline{S}_1."] \arrow[r, shift left, "\overline{R}_0."] & E \arrow[l, shift left, "\overline{S}_0."] \arrow[r] & 0. \end{tikzcd}$$

Now, (70) is equivalent to the following split short exact sequence:

$$\begin{tikzcd} 0 \arrow[r] & E^{p-1} \arrow[r, shift left, "\overline{T}_2."] & E^p \arrow[l, shift left, "\overline{Q}_2."] \arrow[r, shift left, "\overline{S}_2."] & E \arrow[l, shift left, "\overline{R}_2."] & 0. \arrow[l] \end{tikzcd}$$

Using the fact that $\ker_E(\overline{R}_0.) = \operatorname{im}_E(\overline{R}_1.)$, we get $\operatorname{im}_E((\overline{R}_1\,\overline{T}_2).) \subseteq \ker_E(\overline{R}_0.)$. If $\lambda \in \ker_E(\overline{R}_0.)$, there exists $\mu \in E^p$ such that $\lambda = \overline{R}_1\,\mu$. Now, using $\overline{T}_2\,\overline{Q}_2 + \overline{R}_2\,\overline{S}_2 = I_p$ and $\ker_E(\overline{R}_1.) = \operatorname{im}_E(\overline{R}_2.)$, we obtain

$$\lambda = (\overline{R}_1\,\overline{T}_2)\,(\overline{Q}_2\,\mu) + (\overline{R}_1\,\overline{R}_2)\,(\overline{S}_2\,\mu) = (\overline{R}_1\,\overline{T}_2)\,(\overline{Q}_2\,\mu) \in \operatorname{im}_E((\overline{R}_1\,\overline{T}_2).)$$

which shows that $\ker_E(\overline{R}_0.) = \operatorname{im}_E((\overline{R}_1\,\overline{T}_2).)$. Let us now compute $\ker_E((\overline{R}_1\,\overline{T}_2).)$. If $\nu \in \ker_E((\overline{R}_1\,\overline{T}_2).)$, then there exists $\theta \in E$ such that $\overline{T}_2\,\nu = \overline{R}_2\,\theta$, i.e., $\nu = \overline{Q}_2\,\overline{R}_2\,\theta = 0$ since $\overline{Q}_2\,\overline{T}_2 = I_{p-1}$ and $\overline{Q}_2\,\overline{R}_2 = 0$. Hence, the matrix $\overline{R}_1\,\overline{T}_2$ has full column rank, and thus it defines a basis of $\ker_E(\overline{R}_0.)$, i.e., $\ker_E(\overline{R}_0.) = \operatorname{im}_E(\overline{Q}_0.)$, with the notation $\overline{Q}_0 = \overline{R}_1\,\overline{T}_2$. In particular, there exists a matrix $\overline{T}_0 \in E^{(p-1)\times p}$ such that $\overline{T}_0\,\overline{Q}_0 = I_{p-1}$, i.e., $\overline{T}_0\,\overline{R}_1\,\overline{T}_2 = I_{p-1}$.

If we note $\overline{Q}_2' = \overline{T}_0\,\overline{R}_1$, then we have $\overline{Q}_2'\,\overline{T}_2 = \overline{T}_0\,\overline{R}_1\,\overline{T}_2 = I_{p-1}$. If $\overline{R}_2' \in E^p$ is a matrix such that $I_p - \overline{T}_2\,\overline{Q}_2' = \overline{R}_2'\,\overline{S}_2$ and $\overline{S}_0' \in E^p$ is such that $I_p - \overline{Q}_0\,\overline{T}_0 = \overline{S}_0'\,\overline{R}_0$, then we obtain the following diagram which is formed by horizontal split short exact sequences and commutes in both directions:

$$\begin{tikzcd}
0 \arrow[r] & E^{p-1} \arrow[r, shift left, "\overline{T}_2."] \arrow[d, equal] & E^p \arrow[l, shift left, "\overline{Q}_2'."] \arrow[r, shift left, "\overline{S}_2."] \arrow[d, "\overline{R}_1."] & E \arrow[l, shift left, "\overline{R}_2'."] \arrow[d, "0"] & 0 \arrow[l] \\
0 \arrow[r] & E^{p-1} \arrow[r, shift left, "\overline{Q}_0."] & E^p \arrow[l, shift left, "\overline{T}_0."] \arrow[r, shift left, "\overline{R}_0."] & E \arrow[l, shift left, "\overline{S}_0'."] \arrow[r] & 0.
\end{tikzcd}$$

Then, Theorem 7 shows that $\overline{R}_1$ is equivalent to $\operatorname{diag}(I_{p-1}, 0)$, which proves again that $\operatorname{coker}_E(\overline{R}_1.) \cong E$. More precisely, if $Y = (\overline{T}_2 \quad \overline{R}_2') \in \mathrm{GL}_p(E)$ and $Z = \left(\overline{T}_0^T \quad \overline{R}_0^T\right)^T \in \mathrm{GL}_p(E)$, then we have

$$Z\,\overline{R}_1\,Y = \begin{pmatrix} I_{p-1} & 0 \\ 0 & 0 \end{pmatrix},$$

where $Y^{-1} = \left(\overline{Q}_2'^T \quad \overline{S}_2^T\right)^T$ and $Z^{-1} = (\overline{Q}_0 \quad \overline{S}_0')$.

6.2 From Serre's reduction problem to the decomposition problem and vice versa

The purpose of this section is to show how to go from Theorem 12 (which deals with Serre's reduction problem) to Theorem 8 or Corollary 2 (which deals with the decomposition problem for the trivial idempotents id_M and 0_M) and vice versa.

We have the following first result.

Theorem 14. *Let $R \in D^{q\times p}$ be a full row rank matrix and $\Lambda \in D^{q\times(q-r)}$ satisfy the conditions 2 of Theorem 12. With the notations (54), (56) and (66) or (68), we have:*

1. *$\Delta := -W_1\,V_1 \in D^{p\times q}$ satisfies $\Delta\,R\,\Delta = -\Delta$.*
2. *The matrices $\overline{P} := Q_1\,U_2 = I_p + \Delta\,R \in D^{p\times p}$ and $\overline{Q} := X_2\,V_2 = I_q + R\,\Delta \in D^{q\times q}$ are two idempotents, i.e., $\overline{P}^2 = \overline{P}$ and $\overline{Q}^2 = \overline{Q}$, and they satisfy $R\,\overline{P} = \overline{Q}\,R$.*
3. *The left D-modules $\ker_D(.\overline{P})$, $\operatorname{im}_D(.\overline{P})$, $\ker_D(.\overline{Q})$ and $\operatorname{im}_D(.\overline{Q})$ are defined by $\ker_D(.\overline{P}) = \operatorname{im}_D(.U_1)$, $\operatorname{im}_D(.\overline{P}) = \operatorname{im}_D(.U_2)$, $\ker_D(.\overline{Q}) = \operatorname{im}_D(.V_1)$ and $\operatorname{im}_D(.\overline{Q}) = \operatorname{im}_D(.V_2)$ i.e., they are free of rank respectively r, $p-r$, r and $q-r$.*

Hence, the hypotheses of 1 of Corollary 2 are satisfied, i.e., 1 of Corollary 2 holds.

Proof. 1. Using $U_1 = V_1\,R$ and $U_1\,W_1 = I_r$, we get $\Delta\,R\,\Delta = W_1\,V_1\,R\,W_1\,V_1 = W_1\,U_1\,W_1\,V_1 = W_1\,V_1 = -\Delta$.

2. Since $W_1\,U_1 + Q_1\,U_2 = I_p$ and $U_1 = V_1\,R$, we get $\overline{P} := Q_1\,U_2 = I_p - W_1\,U_1 = I_p - W_1\,V_1\,R$, i.e., $\overline{P} = I_p + \Delta\,R$. Using (66), we have $X_1\,V_1 + X_2\,V_2 = I_q$, where $X_1 = R\,W_1$. Thus, we get

$\overline{Q} := X_2\, V_2 = I_q - X_1\, V_1 = I_q - R\, W_1\, V_1 = I_q + R\, \Delta$. Finally, we have $\overline{P}^2 = \overline{P}$, $\overline{Q}^2 = \overline{Q}$ and $R\, \overline{P} = \overline{Q}\, R$ since $\Delta\, R\, \Delta = -\Delta$.

3. $U\, W = I_p$ yields $\ker_D(.Q_1) = \operatorname{im}_D(.U_1)$, where $U_1 = V_1\, R$, $U_1\, W_1 = I_r$ and $U_2\, Q_1 = I_{p-r}$. The identity $U_1\, W_1 = I_r$ shows that $.U_1$ is injective, i.e., that U_1 has full row rank. The identity $U_2\, Q_1 = I_{p-r}$ shows that $.Q_1$ is surjective and $.U_2$ is injective, i.e., that U_2 has full row rank. Thus, we obtain

$$\begin{cases} \ker_D(.\overline{P}) = \ker_D(.(Q_1\, U_2)) = \ker_D(.Q_1) = \operatorname{im}_D(.U_1) \cong D^{1\times r}, \\ \operatorname{im}_D(.\overline{P}) = \operatorname{im}_D(.(Q_1\, U_2)) = \operatorname{im}_D(.U_2) \cong D^{1\times (p-r)}, \end{cases}$$

which proves that $\ker_D(.\overline{P})$ and $\operatorname{im}_D(.\overline{P})$ are free left D-modules of rank respectively r and $p-r$.

The identity $V\, X = I_q$ yields $\ker_D(.X_2) = \operatorname{im}_D(.V_1)$, $V_1\, X_1 = I_r$ and $V_2\, X_2 = I_{q-r}$. The identity $V_1\, X_1 = I_r$ shows that $.V_1$ is injective, i.e., that V_1 has full row rank. The identity $V_2\, X_2 = I_{q-r}$ shows that $.X_2$ is surjective and $.V_2$ is injective, i.e., V_2 has full row rank. Using $\overline{Q} = X_2\, V_2$, we then obtain

$$\begin{cases} \ker_D(.\overline{Q}) = \ker_D(.(X_2\, V_2)) = \ker_D(.X_2) = \operatorname{im}_D(.V_1) \cong D^{1\times r}, \\ \operatorname{im}_D(.\overline{Q}) = \operatorname{im}_D(.(X_2\, V_2)) = \operatorname{im}_D(.V_2) \cong D^{1\times (q-r)}, \end{cases}$$

which proves that $\ker_D(.\overline{Q})$ and $\operatorname{im}_D(.\overline{Q})$ are free left D-modules of rank respectively r and $q-r$. □

Example 9. We consider again Example 7. According to 1 of Theorem 14, we can check again that

$$\Delta := -W_1\, V_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \omega^{-2}\,(d + 2\,\zeta\,\omega) & \omega^{-2} \end{pmatrix}$$

satisfies the equation $\Delta\, R\, \Delta = -\Delta$. With the notations of Example 7, we can check that

$$\overline{P} := Q_1\, U_2 = I_p + \Delta\, R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & \omega^{-2}\,(d^2 + 2\,\zeta\,\omega\, d + \omega^2) & 0 & 0 \end{pmatrix},$$

$$\overline{Q} := X_2\, V_2 = I_q + R\, \Delta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

are two idempotents, i.e., $\overline{P}^2 = \overline{P}$ and $\overline{Q}^2 = \overline{Q}$, satisfying $R\, \overline{P} = \overline{Q}\, R$. Finally, we can check that $\ker_D(.\overline{P}) = \operatorname{im}_D(.U_1)$ and $\operatorname{im}_D(.\overline{P}) = \ker_D(.(I_4 - \overline{P})) = \operatorname{im}_D(.U_2)$, $\ker_D(.\overline{Q}) = \operatorname{im}_D(.V_1)$ and $\operatorname{im}_D(.\overline{Q}) = \ker_D(.(I_3 - \overline{Q})) = \operatorname{im}_D(.V_2)$, where the full row rank matrices U_1, U_2, V_1 and V_2 are defined in Example 7, i.e., $\ker_D(.\overline{P})$ (resp., $\operatorname{im}_D(.\overline{P})$, $\ker_D(.\overline{Q})$, $\operatorname{im}_D(.\overline{Q})$) is a free D-module of rank 2 (resp., 2, 2, 1), which shows that the hypotheses of 1 of Corollary 2 are satisfied.

Conversely, we have the following result.

Theorem 15. *Let $R \in D^{q\times p}$ be a full row rank matrix, $\Delta \in D^{p\times q}$ a matrix satisfying $\Delta\, R\, \Delta = -\Delta$ and which is such that the projective left D-modules $\ker_D(.\Delta)$, $\operatorname{im}_D(.\Delta)$ and $\operatorname{coker}_D(.\Delta)$ are free of rank respectively $p-l$, l and $q-l$. Let the full row rank matrix $U_2 \in D^{(p-l)\times p}$ (resp., $V_1 \in D^{l\times q}$ and $\Phi \in D^{(q-l)\times q}$) define a basis of $\ker_D(.\Delta)$ (resp., $\operatorname{im}_D(.\Delta)$ and $\operatorname{coker}_D(.\Delta)$), i.e.,*

$$\begin{cases} \ker_D(.\Delta) = \operatorname{im}_D(.U_2), \\ \operatorname{im}_D(.\Delta) = \operatorname{im}_D(.V_1), \end{cases}$$

and $\{\gamma(\Phi_{i\bullet})\}_{i=1,\ldots,q-l}$ *is a basis of* $\operatorname{coker}_D(.\Delta)$*, where* $\gamma : D^{1\times q} \longrightarrow \operatorname{coker}_D(.\Delta) = D^{1\times q}/\operatorname{im}_D(.\Delta)$ *is the canonical projection. Then, the following full row rank matrices*

$$U_1 := V_1\, R \in D^{l\times p}, \quad V_2 := \Phi\,(I_q + R\,\Delta) \in D^{(q-l)\times q}$$

are such that $U := (U_1^T \quad U_2^T)^T \in \operatorname{GL}_p(D)$ *and* $V := (V_1^T \quad V_2^T)^T \in \operatorname{GL}_q(D)$*.*

Moreover, with the notations (40), we have

$$\begin{pmatrix} R & -X_2 \\ U_2 & 0 \end{pmatrix} \begin{pmatrix} W_1\, V_1 & W_2 \\ -V_2 & V_2\, R\, W_2 \end{pmatrix} = I_{q+p-l}, \tag{71}$$

which shows that Theorem 12 holds with the matrix $\Lambda := X_2 \in D^{q\times(q-l)}$ *which admits the left inverse* $V_2 \in D^{(q-l)\times q}$ *and* $\ker_D(.X_2) = \operatorname{im}_D(.V_1) \cong D^{1\times l}$ *is a free left D-module of rank l.*

Proof. The fact that $U := (U_1^T \quad U_2^T)^T \in \operatorname{GL}_p(D)$ and $V := (V_1^T \quad V_2^T)^T \in \operatorname{GL}_q(D)$ is proved in Theorem 8 (see also Remark 8 or Proposition 4.3 of Cluzeau and Quadrat (2008)). Theorem 8 yields

$$V\,R\,W = \begin{pmatrix} I_l & 0 \\ 0 & V_2\, R\, W_2 \end{pmatrix} \quad \Longleftrightarrow \quad R\,W = X \begin{pmatrix} I_l & 0 \\ 0 & V_2\, R\, W_2 \end{pmatrix},$$

which is equivalent to:

$$\begin{cases} R\,W_1 = X_1, \\ R\,W_2 = X_2\,(V_2\, R\, W_2). \end{cases} \tag{72}$$

Combining $X_1 = R\,W_1$ with $X_1\, V_1 + X_2\, V_2 = I_q$, we obtain:

$$R\,(W_1\, V_1) - X_2\,(-V_2) = I_q. \tag{73}$$

Moreover, the identity $U\,W = I_p$ yields:

$$U_2\, W_1 = 0, \quad U_2\, W_2 = I_{p-l}. \tag{74}$$

Combining (73), the second identity of (72) and (74), we obtain (71). Since D is a noetherian domain, it is *stably finite*, namely, for any $r \in \mathbb{Z}_{\geq 0}$ and for all $A, B \in D^{r\times r}$ satisfying $A\,B = I_r$, we have $B\,A = I_r$, i.e., $A \in \operatorname{GL}_r(D)$ and $B = A^{-1}$ (see, e.g., Lam (1999)). Hence, both square matrices in the left-hand side of (71) belong to $\operatorname{GL}_{q+p-l}(D)$. Let us check again this result by direct computation. Using the identities $V_1\, R\, W_2 = 0$, $W_1\, U_1 + W_2\, U_2 = I_p$ and $V_1\, R\, W_1 = I_l$, we first get $V_1\, R = V_1\, R\,(W_1\, U_1 + W_2\, U_2) = U_1$. Combining this identity with $V_1\, X_2 = 0$, $V_2\, X_2 = I_{q-l}$, $W_1\, U_1 + W_2\, U_2 = I_p$ and $V_2\, R\, W_1 = 0$, we obtain:

$$\begin{pmatrix} W_1\, V_1 & W_2 \\ -V_2 & V_2\, R\, W_2 \end{pmatrix} \begin{pmatrix} R & -X_2 \\ U_2 & 0 \end{pmatrix} = \begin{pmatrix} W_1\,(V_1\, R) + W_2\, U_2 & -W_1\,(V_1\, X_2) \\ V_2\, R\,(W_2\, U_2 - I_p) & V_2\, X_2 \end{pmatrix} = I_{p+q-l}.$$

Using $V\,X = I_q$, we get $V_2\, X_2 = V_2\, \Lambda = I_{q-l}$. Moreover, we have $\ker_D(.X_2) = \operatorname{im}_D(.V_1) \cong D^{1\times l}$ since the matrix V_1 has full row rank. Hence, Theorem 12 holds. □

Remark 11. We note that the matrix $\Lambda := X_2$ of Theorem 15 is an injective parametrization of the free left D-module $\operatorname{coker}_D(.\Delta)$ and the residue classes of the rows of V_2 defines a basis of $\operatorname{coker}_D(.\Delta)$.

Example 10. We consider again Example 6. First considering the matrix Δ defined by (50), we can check again that (71) holds, i.e.:

$$\begin{pmatrix} d+a & k\,a\,\delta & 0 & 0 & -1 \\ 0 & d & -1 & 0 & 0 \\ 0 & \omega^2 & d+2\,\zeta\,\omega & -\omega^2 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$
$$\begin{pmatrix} 0 & 1 & 0 & 1 & -d \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & d \\ -\omega^{-2} & -\omega^{-2}\,(2\,\zeta\,\omega-a) & -\omega^{-2} & \omega^{-2}\,(d+a) & \omega^{-2}\,((2\,\zeta\,\omega-a)\,d+k\,a\,\delta+\omega^2) \\ -1 & d+a & 0 & d+a & -d^2-a\,d+k\,a\,\delta \end{pmatrix} = I_5.$$

Moreover, we have $\ker_D(.X_2) = \mathrm{im}_D(.V_1)$, where V_1 is the full row rank matrix defined in Example 6, i.e., $\ker_D(.X_2)$ is a free D-module of rank 2. Finally, X_2 admits the left inverse V_2, where V_2 is defined in Example 6, which shows that Theorem 12 holds with $\Lambda = X_2 = (1 \quad 0 \quad -1)^T$.

Now, if we consider the matrix Δ defined by (51), we can check again that (71) holds, i.e.:

$$\begin{pmatrix} d+a & k\,a\,\delta & 0 & 0 & -1 \\ 0 & d & -1 & 0 & 0 \\ 0 & \omega^2 & d+2\,\zeta\,\omega & -\omega^2 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$
$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & d \\ 0 & -\omega^{-2}\,(d+2\,\zeta\,\omega) & -\omega^{-2} & 0 & \omega^{-2}\,(d^2+2\,\zeta\,\omega\,d+\omega^2) \\ -1 & 0 & 0 & d+a & k\,a\,\delta \end{pmatrix} = I_5.$$

Moreover, we have $\ker_D(.X_2) = \mathrm{im}_D(.V_1)$, where V_1 is the full row rank matrix defined at the end of Example 6, i.e., $\ker_D(.X_2)$ is a free D-module of rank 2. Finally, X_2 admits the left inverse V_2, where V_2 is defined at the end of Example 6, which shows that Theorem 12 holds with $\Lambda = X_2 = (1 \quad 0 \quad 0)^T$.

Similarly, we have the following two results.

Theorem 16. *Let $R \in D^{q\times p}$ be a full row rank matrix and $\Lambda \in D^{q\times(q-r)}$ satisfy the conditions 2 of Theorem 12. With the notations (54), (56) and (66), we have:*

1. *$\Delta := W_1\,V_1 \in D^{p\times q}$ satisfies $\Delta\,R\,\Delta = \Delta$.*
2. *The matrices $\overline{P} := W_1\,U_1 = \Delta\,R \in D^{p\times p}$ and $\overline{Q} := X_1\,V_1 = R\,\Delta \in D^{q\times q}$ are two idempotents, i.e., $\overline{P}^2 = \overline{P}$ and $\overline{Q}^2 = \overline{Q}$, and they satisfy $R\,\overline{P} = \overline{Q}\,R$.*
3. *The left D-modules $\ker_D(.\overline{P})$, $\mathrm{im}_D(.\overline{P})$, $\ker_D(.\overline{Q})$ and $\mathrm{im}_D(.\overline{Q})$ are defined by $\ker_D(.\overline{P}) = \mathrm{im}_D(.U_2)$, $\mathrm{im}_D(.\overline{P}) = \mathrm{im}_D(.U_1)$, $\ker_D(.\overline{Q}) = \mathrm{im}_D(.V_2)$ and $\mathrm{im}_D(.\overline{Q}) = \mathrm{im}_D(.V_1)$ i.e., they are free of rank respectively $p-r$, r, $q-r$ and r.*

Hence, the hypotheses of 1 of Corollary 2 are satisfied, i.e., 2 of Corollary 2 holds.

Theorem 17. *Let $R \in D^{q\times p}$ be a full row rank matrix, $\Delta \in D^{p\times q}$ a matrix satisfying $\Delta\,R\,\Delta = \Delta$ and which is such that the projective left D-modules $\ker_D(.\Delta)$, $\mathrm{im}_D(.\Delta)$ and $\mathrm{coker}_D(.\Delta)$ are free of rank* *respectively* $p-q+l$, $q-l$ *and* l. *Let the full row rank matrix* $U_1 \in D^{(p-q+l)\times p}$ *(resp.,* $V_2 \in D^{(q-l)\times q}$ *and* $\Phi \in D^{l\times q}$*) define a basis of* $\ker_D(.\Delta)$ *(resp.,* $\mathrm{im}_D(.\Delta)$ *and* $\mathrm{coker}_D(.\Delta)$*), i.e.,*

$$\begin{cases} \ker_D(.\Delta) = \mathrm{im}_D(.U_1), \\ \mathrm{im}_D(.\Delta) = \mathrm{im}_D(.V_2), \end{cases}$$

and $\{\gamma(\Phi_{i\bullet})\}_{i=1,\ldots,l}$ *is a basis of* $\mathrm{coker}_D(.\Delta)$*, where* $\gamma : D^{1\times q} \longrightarrow \mathrm{coker}_D(.\Delta) = D^{1\times q}/\mathrm{im}_D(.\Delta)$ *is the canonical projection. Then, the following full row rank matrices*

$$U_2 := V_2\, R \in D^{(q-l)\times p}, \quad V_1 := \Phi\,(I_q - R\,\Delta) \in D^{l\times q}$$

are such that $U := (U_1^T \quad U_2^T)^T \in \mathrm{GL}_p(D)$ *and* $V := (V_1^T \quad V_2^T)^T \in \mathrm{GL}_q(D)$*.*

Moreover, with the notations (49), we have

$$\begin{pmatrix} R & -X_1 \\ U_1 & 0 \end{pmatrix} \begin{pmatrix} W_2\, V_2 & W_1 \\ -V_1 & V_1\, R\, W_1 \end{pmatrix} = I_{q+p-l},$$

which shows that Theorem 12 holds with the matrix $\Lambda := X_1 \in D^{q\times l}$ *which admits the left inverse* $V_1 \in D^{l\times q}$ *and* $\ker_D(.X_1) = \mathrm{im}_D(.V_2) \cong D^{1\times(q-l)}$ *is a free left* D*-module of rank* $q-l$*.*

7 Applications to linear PD systems studied in hydrodynamics

In this last section, we illustrate how Serre's reduction techniques can be applied to study the decomposability of standard linear PD systems.

7.1 Oseen equations

Let us consider the *Oseen equations* in $\mathbb{R}^2$ defined by

$$\begin{cases} d_t\,\vec{u} - \nu\,\Delta\,\vec{u} + (\vec{b}\,.\,\vec{\nabla})\,\vec{u} + \vec{\nabla}\,p = 0, \\ \vec{\nabla}\,.\,\vec{u} = 0, \end{cases} \tag{75}$$

where $\vec{u}$ is the velocity, p the pressure, ν the viscosity, $\vec{b} = (b_1 \quad b_2)^T$ a steady velocity, $\vec{\nabla} = (d_x \quad d_y)^T$ the gradient operator in $\mathbb{R}^2$ and $\Delta = d_x^2 + d_y^2$ the Laplacian operator in $\mathbb{R}^2$. The Oseen equations describe the flow of a viscous and incompressible fluid at small Reynolds numbers (linearization of the incompressible Navier-Stokes equations at a steady state). See, e.g., Dolean et al. (2005). Let $D = \mathbb{Q}(\nu, b_1, b_2)[d_t, d_x, d_y]$ be the commutative polynomial ring of PD operators with coefficients in the field $\mathbb{Q}(\nu, b_1, b_2)$,

$$R = \begin{pmatrix} d_t + \vec{b}\,.\,\vec{\nabla} - \nu\,\Delta & 0 & d_x \\ 0 & d_t + \vec{b}\,.\,\vec{\nabla} - \nu\,\Delta & d_y \\ d_x & d_y & 0 \end{pmatrix} \in D^{3\times 3}, \tag{76}$$

$M = D^{1\times 3}/(D^{1\times 3}\,R)$ the D-module finitely presented by R and $\pi : D^{1\times 3} \longrightarrow M$ the canonical projection. Using Algorithm 2.1 of Cluzeau and Quadrat (2008) and its implementation in the OreMorphisms package (Cluzeau and Quadrat (2009)), we find that the endomorphism ring $\mathrm{end}_D(M)$ of M is defined by the family of generators $\{f_i\}_{i=1,\ldots,5}$, where $f_i(\pi(\lambda)) = \pi(\lambda\,P_i)$ for all $\lambda \in D^{1\times 3}$, and:

$$P_1 = I_3, \quad P_2 = \begin{pmatrix} 0 & -d_y & 0 \\ 0 & d_x & 0 \\ 0 & 0 & d_x \end{pmatrix}, \quad P_3 = \begin{pmatrix} 0 & 0 & d_x \\ 0 & 0 & d_y \\ 0 & 0 & -(d_t + b_1\,d_x + b_2\,d_y) \end{pmatrix},$$

$$P_4 = \begin{pmatrix} 0 & \nu\,d_x\,d_y & 0 \\ 0 & -(d_t + b_1\,d_x + b_2\,d_y - \nu\,d_y^2) & -d_y \\ 0 & 0 & \nu\,d_y^2 \end{pmatrix}, \quad P_5 = \begin{pmatrix} 0 & d_y\,(d_t + b_2\,d_y - \nu\,d_y^2) & d_y^2 \\ 0 & -d_x\,(d_t + b_2\,d_y - \nu\,d_y^2) & -d_x\,d_y \\ 0 & 0 & d_y^2\,(\nu\,d_x - b_1) \end{pmatrix}.$$

For more details, see Section 8.3. If $(u_1\ u_2\ p)^T$ is a solution of $R\,(u_1\ u_2\ p)^T = 0$ then so is $P_i\,(u_1\ u_2\ p)^T$ for $i = 1, \ldots, 5$, i.e., the P_i's send a solution of $R\,(u_1\ u_2\ p)^T = 0$ to another solution of the same system. The generators f_i's of the D-module $\mathrm{end}_D(M)$ satisfy D-linear relations. Using the results explained at the end of Section 3, we obtain that a generating set of D-linear relations among the generators f_i's of $\mathrm{end}_D(M)$ is defined by $L\,(f_1\ \ldots\ f_5)^T = 0$ where:

$$L = \begin{pmatrix} d_x & -1 & 0 & 0 & 0 \\ -d_t - b_2\,d_y + \nu\,d_y^2 & -b_1 & -1 & -1 & 0 \\ 0 & -\nu\,d_x & 0 & -1 & 0 \\ 0 & -\nu\,(d_t + b_1\,d_x + b_2\,d_y - \nu\,d_y^2) & -\nu\,d_x & -b_1 & -\nu \\ 0 & -\nu^2\,d_x\,d_y^2 & -\nu\,d_y^2 & -(d_t + b_2\,d_y) & \nu\,d_x \\ 0 & 0 & 0 & \nu\,d_x - b_1 & -\nu \end{pmatrix} \in D^{6\times 5}.$$

For more details, see Section 8.3.

Using Serre's reduction, let us state a first result.

Proposition 8. *The D-module $\mathrm{end}_D(M) = D^{1\times 5}/(D^{1\times 6}\,L)$, finitely presented by the matrix L defined above, is cyclic and is generated by id_M.*

Proof. If $\Lambda = (1 \quad 0 \quad 0 \quad 0 \quad 0) \in D^{1\times 5}$ and $P = (L^T \quad \Lambda^T)^T \in D^{7\times 5}$, then we can check that P admits a left inverse $X = (X_1 \quad X_2) \in D^{5\times 7}$ where:

$$X_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ -b_1\,\nu\,d_x & -1 & 1 & 0 & 0 & 0 \\ \nu\,d_x & 0 & -1 & 0 & 0 & 0 \\ -d_x\,(b_1 - \nu\,d_x) & 0 & \nu^{-1}\,(b_1 - \nu\,d_x) & 0 & 0 & -\nu^{-1} \end{pmatrix},$$

$$X_2 = \begin{pmatrix} 1 & d_x & -\,(d_t + b_1\,d_x + b_2\,d_y - \nu\,\Delta) & -\,\nu\,d_x^2 & -\,d_x^2\,(\nu\,d_x - b_1) \end{pmatrix}^T.$$

We get $D^{1\times 5} = D^{1\times 7}\,P$, which yields $\mathrm{end}_D(M) = D^{1\times 5}/(D^{1\times 6}\,L) = (D^{1\times 7}\,P)/(D^{1\times 6}\,L)$ and shows that the D-module $\mathrm{end}_D(M) = D^{1\times 5}/(D^{1\times 6}\,L)$ is cyclic and is generated by the residue class of Λ in $\mathrm{end}_D(M)$. Moreover, we have $X_1\,L + X_2\,\Lambda = I_5$, $L\,f = 0$, where $f = (f_1\ \ldots\ f_5)^T$, and $\Lambda\,f = f_1 = \mathrm{id}_M$, which yields:

$$f = X_1\,(L\,f) + X_2\,(\Lambda\,f) = X_2\,f_1 \ \Leftrightarrow\ \begin{cases} f_1 = f_1, \\ f_2 = d_x\,f_1, \\ f_3 = -(d_t + b_1\,d_x + b_2\,d_y - \nu\,\Delta)\,f_1, \\ f_4 = -\nu\,d_x^2\,f_1, \\ f_5 = -d_x^2\,(\nu\,d_x - b_1)\,f_1. \end{cases}$$

Hence, for every $f \in \mathrm{end}_D(M)$, there exist $d_1, \ldots, d_5 \in D$ such that $f = \sum_{i=1}^5 d_i\,f_i = \left(\sum_{i=1}^5 d_i\,X_{2i}\right)\,f_1$, where X_{2i} is the i^{th} entry of the column vector X_2, which shows that $\mathrm{end}_D(M)$ is generated by $f_1 = \mathrm{id}_M$ as a D-module, i.e., $\mathrm{end}_D(M) = D\,f_1$ is a cyclic D-module (see Definition 1). □

Theorem 18. *The D-module M finitely presented by the matrix R defined by (76), i.e., defined by the Oseen equations (75), is indecomposable.*

Proof. Let us determine the annihilator of $f_1 = \mathrm{id}_M \in \mathrm{end}_D(M)$, i.e., $\mathrm{ann}_D(f_1) := \{d \in D \mid d\,f_1 = 0\}$. Using Gröbner basis techniques, we can compute $\ker_D(.P)$, where P is the matrix given in the proof of Proposition 8, and we obtain $\ker_D(.P) = D^{1\times 2}\,(T_1 \quad T_2)$, where $T_1 \in D^{2\times 6}$ is a certain matrix and $T_2 = \left(0 \quad -\,\nu^2\,\Delta\,(d_t + \vec{b}\,.\,\vec{\nabla} - \nu\,\Delta)\right)^T \in D^2$. Moreover, we have $L = (I_6 \quad 0)\,P$. Using Lemma 3.1 of Cluzeau and Quadrat (2008), we then obtain

$$\begin{aligned} \mathrm{end}_D(M) &= D^{1\times 5}/(D^{1\times 6}\,L) = (D^{1\times 7}\,P)/(D^{1\times 6}\,L) \cong D^{1\times 7}/\left(D^{1\times 8}\begin{pmatrix} T_1 & T_2 \\ I_6 & 0 \end{pmatrix}\right) \\ &\cong D/(D^{1\times 2}\,T_2) = D/(D\,(\Delta\,(d_t + \vec{b}\,.\,\vec{\nabla} - \nu\,\Delta))), \end{aligned}$$

i.e., $\mathrm{end}_D(M) = D\, f_1 \cong D/\mathrm{ann}_D(f_1)$ and $\mathrm{ann}_D(f_1) = D\,(\Delta\,(d_t + \vec{b}\,.\,\vec{\nabla} - \nu\,\Delta))$. We recall that the decomposability of M is equivalent to the existence of non-trivial idempotent in $\mathrm{end}_D(M)$. Hence, to study whether or not the D-module M is decomposable, let us search for non-trivial idempotents of $\mathrm{end}_D(M) = D\,\mathrm{id}_M$. If $\alpha \in D$, then $e = \alpha\,\mathrm{id}_M$ is an idempotent of $\mathrm{end}_D(M)$ if and only if $e^2 - e = (\alpha^2 - \alpha)\,\mathrm{id}_M = 0$, i.e., if and only if there exists $\beta \in D$ such that:

$$\alpha\,(\alpha - 1) = \beta\,\Delta\,(d_t + \vec{b}\,.\,\vec{\nabla} - \nu\,\Delta). \tag{77}$$

We first show that the two simple solutions of (77) lead to the trivial idempotents 0 and id_M of $\mathrm{end}_D(M)$. If $\Delta\,(d_t + \vec{b}\,.\,\vec{\nabla} - \nu\,\Delta)$ divides α, i.e., if there exists $\gamma \in D$ such that $\alpha = \gamma\,\Delta\,(d_t + \vec{b}\,.\,\vec{\nabla} - \nu\,\Delta)$, then $e = \alpha\,\mathrm{id}_M = 0$. If $\Delta\,(d_t + \vec{b}\,.\,\vec{\nabla} - \nu\,\Delta)$ divides $\alpha - 1$, i.e., $\alpha = 1 + \gamma\,\Delta\,(d_t + \vec{b}\,.\,\vec{\nabla} - \nu\,\Delta)$ for a certain $\gamma \in D$, then $e = \alpha\,\mathrm{id}_M = \mathrm{id}_M$.

We can check that Δ and $d_t + \vec{b}\,.\,\vec{\nabla} - \nu\,\Delta$ are two irreducible polynomials over the field $\mathbb{Q}(\nu, b_1, b_2)$, their greatest common divisor is 1 and α and $\alpha - 1$ are coprime. Hence, the only two remaining possibilities for (77) to hold are:

1. Δ divides α and $d_t + \vec{b}\,.\,\vec{\nabla} - \nu\,\Delta$ divides $\alpha - 1$, i.e., $\alpha = \gamma\,\Delta$ and $\alpha = 1 + \gamma'\,(d_t + \vec{b}\,.\,\vec{\nabla} - \nu\,\Delta)$ for certain γ, $\gamma' \in D$. This then leads to $\gamma\,\Delta - \gamma'\,(d_t + \vec{b}\,.\,\vec{\nabla} - \nu\,\Delta) = 1$ which is clearly impossible since $1 \notin (\Delta, d_t + \vec{b}\,.\,\vec{\nabla} - \nu\,\Delta) = (\Delta, d_t + \vec{b}\,.\,\vec{\nabla})$.
2. Δ divides $\alpha - 1$ and $d_t + \vec{b}\,.\,\vec{\nabla} - \nu\,\Delta$ divides α, i.e., $\alpha = 1 + \gamma\,\Delta$ and $\alpha = \gamma'\,(d_t + \vec{b}\,.\,\vec{\nabla} - \nu\,\Delta)$ for certain γ, $\gamma' \in D$. This then leads to $\gamma'\,(d_t + \vec{b}\,.\,\vec{\nabla} - \nu\,\Delta) - \gamma\,\Delta = 1$ which is also impossible as shown above.

Thus, $\mathrm{end}_D(M)$ does not admit any non-trivial idempotent element so that M is an indecomposable D-module. □

Finally, from the above computations, the endomorphisms of M defined by $g_1 = \Delta\,\mathrm{id}_M$ and $g_2 = (d_t + \vec{b}\,.\,\vec{\nabla} - \nu\,\Delta)\,\mathrm{id}_M$ are not injective since we have $(d_t + \vec{b}\,.\,\vec{\nabla} - \nu\,\Delta)\,g_1 = 0$ and $\Delta\,g_2 = 0$ and thus $g_1((d_t + \vec{b}\,.\,\vec{\nabla} - \nu\,\Delta)\,m) = 0$ and $g_2(\Delta\,m) = 0$ for all $m \in M$. Then, R admits the two strict factorizations $R = L_1\,S_1$ and $R = L_2\,S_2$ defined by:

$$L_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad S_1 = \begin{pmatrix} \nu\,d_x & \nu\,d_y & -1 \\ d_x & d_y & 0 \\ d_t + \vec{b}\,.\,\vec{\nabla} - \nu\,\Delta & 0 & d_x \\ 0 & d_t + \vec{b}\,.\,\vec{\nabla} - \nu\,\Delta & d_y \end{pmatrix},$$

$$L_2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ b_1 - \nu\,d_x & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} -d_y & d_x & 0 \\ (\nu\,d_x - b_1)\,d_y & -b_2\,d_y - d_t + \nu\,d_y^2 & -d_y \\ -d_x & -d_y & 0 \\ \nu\,\Delta - d_t - \vec{b}\,.\,\vec{\nabla} & 0 & -d_x \end{pmatrix}.$$

Using the factorization $R = L_i\,S_i$ for $i = 1, 2$, we get $\ker_{\mathcal{F}}(S_i.) \subseteq \ker_{\mathcal{F}}(R.)$ for $i = 1, 2$, i.e., the solutions of the PD systems defined by S_1 and S_2 are particular solutions of (75). Computing a Gröbner basis of the D-module $D^{1\times 4}\,S_i$ for $i = 1, 2$, we obtain that:

$$S_1\,\eta = 0 \quad \Longleftrightarrow \quad \left\{ \begin{array}{l} p = 0, \\ d_t\,\vec{u} - \nu\,\Delta\,\vec{u} + (\vec{b}\,.\,\vec{\nabla})\,\vec{u} = 0, \\ \vec{\nabla}\,.\,\vec{u} = 0, \end{array} \right.$$

$$S_2\,\eta = 0 \quad \Longleftrightarrow \quad \left\{ \begin{array}{l} d_x\,u_1 + d_y\,u_2 = 0, \\ d_y\,u_1 - d_x\,u_2 = 0, \\ (d_t + b_1\,d_x)\,u_1 + b_2\,d_x\,u_2 + d_x\,p = 0, \\ b_1\,d_y\,u_1 + (d_t + b_2\,d_y)\,u_2 + d_y\,p = 0. \end{array} \right.$$

We note that the last PD system yields $\Delta\,u_i = 0$ for $i = 1, 2$ and $\Delta\,p = 0$, i.e., the components of the velocity $\vec{u}$ and the pressure p are harmonic functions.

7.2 Implicit scheme for the Oseen equations and Stokes equations

Within implicit schemes of the time dependent Oseen equations, the term $d_t\,\vec{u}$ in (75) is replaced by $c\,\vec{u}$, where the constant c corresponds to the inverse of the time step. Let $E = \mathbb{Q}(\nu, b_1, b_2, c)[d_x, d_y]$ and N be the E-module finitely presented by the matrix obtained by replacing d_t by c in (76). Then we can redo the computations of the previous section and prove that the endomorphism ring $\mathrm{end}_E(N)$ is a cyclic E-module generated by id_N and $\mathrm{end}_E(N) = E\,\mathrm{id}_N \cong E/\mathrm{ann}_E(\mathrm{id}_N) \cong E/(\Delta\,(\nu\,\Delta - \vec{b}\,.\,\vec{\nabla} - c))$. In particular, this result also holds when $b_1 = 0$ or $b_2 = 0$.

Proposition 9. *If $\vec{b} \neq \vec{0}$, then the E-module N is indecomposable.*

Proof. As in the proof of Theorem 18, $e = \alpha\,\mathrm{id}_N \in \mathrm{end}_E(N)$, where $\alpha \in E$, is an idempotent of $\mathrm{end}_E(N)$ if and only if there exists $\beta \in D$ such that:

$$\alpha\,(\alpha - 1) = \beta\,\Delta\,(\nu\,\Delta - \vec{b}\,.\,\vec{\nabla} - c). \tag{78}$$

The two trivial solutions $\alpha = 0$ or $\alpha = 1$ and $\beta = 0$ of (78) lead to the trivial idempotents 0 and id_N of $\mathrm{end}_E(N)$. Now, since Δ and $\nu\,\Delta - \vec{b}\,.\,\vec{\nabla} - c$ are irreducible over $\mathbb{Q}(\nu, b_1, b_2, c)$, (78) can hold only if:

1. Δ divides α and $\nu\,\Delta - \vec{b}\,.\,\vec{\nabla} - c$ divides $\alpha - 1$, i.e., $\alpha = \gamma\,\Delta$ and $\alpha = 1 + \gamma'\,(\nu\,\Delta - \vec{b}\,.\,\vec{\nabla} - c)$ for certain $\gamma,\, \gamma' \in E$. We then get $\gamma\,\Delta = 1 + \gamma'\,(\nu\,\Delta - \vec{b}\,.\,\vec{\nabla} - c)$. In particular, we must have $\deg\gamma = \deg\gamma'$ and $(\gamma - \nu\,\gamma')\,\Delta + \gamma'\,\vec{b}\,.\,\vec{\nabla} + \gamma'\,c - 1 = 0$. Moreover, γ' must be a constant as if $\deg\gamma' > 0$, then the constant 1 cannot be cancelled. Then, we obtain $\gamma = \nu\,\gamma'$, $\gamma'\,b_1 = 0$, $\gamma'\,b_2 = 0$ and $\gamma'\,c = 1$, i.e., $\gamma' = 1/c$ which yields $\gamma'\,b_i = b_i/c = 0$, i.e., $b_i = 0$, for $i = 1,\,2$.

2. Δ divides $\alpha - 1$ and $\nu\,\Delta - \vec{b}\,.\,\vec{\nabla} - c$ divides α, i.e., $\alpha = 1 + \gamma\,\Delta$ and $\alpha = \gamma'\,(\nu\,\Delta - \vec{b}\,.\,\vec{\nabla} - c)$ for certain $\gamma,\, \gamma' \in E$. We then get $1 + \gamma\,\Delta = \gamma'\,(\nu\,\Delta - \vec{b}\,.\,\vec{\nabla} - c)$. In particular, we must have $\deg\gamma = \deg\gamma'$ and $(\gamma - \nu\,\gamma')\,\Delta + \gamma'\,\vec{b}\,.\,\vec{\nabla} + \gamma'\,c + 1 = 0$, and thus $\deg\gamma' = 0$ and $\gamma'\,c = -1$, $\gamma = \nu\,\gamma'$, $\gamma'\,b_1 = 0$ and $\gamma'\,b_2 = 0$, i.e., $\gamma' = -1/c$, which yields $\gamma'\,b_i = -b_i/c = 0$, i.e., $b_1 = b_2 = 0$.

$\square$

Let us now consider the case $\vec{b} = \vec{0}$ which corresponds to an implicit scheme of the time dependent Stokes equations, namely:

$$\left\{\begin{array}{l} c\,u - \nu\,\Delta\,\vec{u} + \vec{\nabla}\,p = 0, \\ \vec{\nabla}\,.\,\vec{u} = 0. \end{array}\right. \tag{79}$$

The matrix of PD operators associated with (79) is then defined by:

$$R' = \begin{pmatrix} c - \nu\,\Delta & 0 & d_x \\ 0 & c - \nu\,\Delta & d_y \\ d_x & d_y & 0 \end{pmatrix} \in E^{3\times 3}. \tag{80}$$

Let $M' = E^{1\times 3}/(E^{1\times 3}\,R')$ be the E-module finitely presented by R'.

From the proof of Proposition 9, we obtain the following result.

Corollary 6. *The E-module M' finitely presented by R' is decomposable.*

Proof. From the proof of Proposition 9, in Case 1, we get that $\alpha = (\nu/c)\,\Delta$ is a non-trivial solution of (78) and thus $e = (\nu/c)\,\Delta\,\mathrm{id}_M$ is a non-trivial idempotent of $\mathrm{end}_E(M')$. Similarly, in Case 2, $\alpha = 1 - (\nu/c)\,\Delta$ is a non-trivial solution of (78) and thus $e = \alpha\,\mathrm{id}_M$ is a non-trivial idempotent of $\mathrm{end}_E(M')$. We have then found non-trivial idempotents of $\mathrm{end}_E(M')$ which proves that the E-module M' is decomposable. $\square$

This result is used to compute a parametrization of the solutions of (79). From the latter proof, we know that the matrices $P = Q = \left(1 - \frac{\nu}{c}\,\Delta\right)\,I_3 \in E^{3\times 3}$ yield an endomorphism $f \in \mathrm{end}_E(M')$ defined by $f(\pi(\lambda)) = \pi(\lambda\, P)$ for all $\lambda \in E^{1\times 3}$, where $\pi : E^{1\times 3} \longrightarrow M'$ is the canonical projection. Then, using Lemma 2, we get $\operatorname{coim} f = E^{1\times 3}/(E^{1\times 4}\, S)$, where

$$S = \begin{pmatrix} -c\, d_y & c\, d_x & 0 \\ \nu\, c\, d_x\, d_y & c\,(\nu\, d_y^2 - c) & -c\, d_y \\ -c\, d_x & -c\, d_y & 0 \\ c\,(\nu\,\Delta - c) & 0 & -c\, d_x \end{pmatrix} \in E^{4\times 3},$$

and $\ker f \cong E^{1\times 4}/(E^{1\times 4}\,(L^T \quad S_2^T)^T)$, where the matrix L is defined by

$$L = -\frac{1}{c}\begin{pmatrix} 0 & 0 & 0 & 1 \\ \nu\, d_x & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in E^{3\times 4},$$

and $S_2 = (\nu\, d_y^2 - c \quad -d_x \quad 0 \quad d_y) \in E^{1\times 4}$. Using a Gröbner basis computation, we obtain:

$$S\,\zeta = 0 \quad\Longleftrightarrow\quad \begin{cases} \Delta\,\zeta_3 = 0, \\ c\,\zeta_2 + d_y\,\zeta_3 = 0, \\ c\,\zeta_1 + d_x\,\zeta_3 = 0, \end{cases} \quad\Longleftrightarrow\quad \begin{cases} \zeta_1 = -\dfrac{1}{c}\, d_x\,\zeta_3, \\ \zeta_2 = -\dfrac{1}{c}\, d_y\,\zeta_3, \\ \zeta_3 = \zeta_3, \\ \Delta\,\zeta_3 = 0. \end{cases} \tag{81}$$

Moreover, we have:

$$\begin{pmatrix} L \\ S_2 \end{pmatrix} \tau = 0 \quad\Longleftrightarrow\quad \begin{cases} \tau_2 = -\nu\, d_x\,\tau_1, \\ \tau_3 = 0, \\ \tau_4 = 0, \\ (\nu\,\Delta - c)\,\tau_1 = 0. \end{cases} \tag{82}$$

Using the factorization $R' = L\, S$ of R' and the notation $\eta := (u_1 \quad u_2 \quad p)^T$, we then get:

$$R'\,\eta = 0 \quad\Longleftrightarrow\quad \begin{cases} S\,\eta = \tau, \\ L\,\tau = 0, \\ S_2\,\tau = 0. \end{cases} \tag{83}$$

Since f is an idempotent, we have $M' \cong \ker f \oplus \operatorname{coim} f$, which shows that the short exact sequence $0 \longrightarrow \ker f \xrightarrow{i} M' \xrightarrow{\rho} \operatorname{coim} f \longrightarrow 0$ splits by 4 of Proposition 2. By Quadrat and Robertz (2007b), there exist matrices $U_1 \in E^{4\times 3}$, $U_2 \in E^4$ and $V \in E^{3\times 4}$ such that $U_1\, L + U_2\, S_2 + S\, V = I_4$. We can take:

$$U_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ d_x & d_y & -c \\ 0 & 0 & 0 \end{pmatrix}, \quad U_2 = \frac{1}{c}\begin{pmatrix} -1 \\ 0 \\ 0 \\ \nu\, d_y \end{pmatrix}, \quad V = -\frac{1}{c^2}\begin{pmatrix} \nu\, d_y & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ \nu^2\, d_x\, d_y & \nu\, d_y & 0 & \nu\, d_x \end{pmatrix}.$$

Using $U_1\, L + U_2\, S_2 + S\, V = I_4$, we get that $\eta^\star := V\,\tau$ is a particular solution of $S\,\eta = \tau$, where τ satisfies (82). The general solution of $S\,\eta = \tau$ is then of the form $\eta = \eta^\star + \zeta$, where ζ satisfies $S\,\zeta = 0$, i.e., ζ is defined by (81), and we obtain the following parametrization of the solutions of (79)

$$\begin{cases} u_1 = -\dfrac{1}{c}\left(d_x\,\zeta_3 + \dfrac{\nu}{c}\, d_y\,\tau_1\right), \\ u_2 = \dfrac{1}{c}\left(-d_y\,\zeta_3 + \dfrac{\nu}{c}\, d_x\,\tau_1\right), \\ p = \zeta_3, \end{cases}$$

where ζ_3 (resp., τ_1) satisfies the PD equation $\Delta\,\zeta_3 = 0$ (resp., $(\nu\,\Delta - c)\,\tau_1 = 0$).

Let us now compute a decomposition of the matrix R' defined by (80). The matrices $P = Q = (1 - \frac{\nu}{c}\,\Delta)\,I_3$ considered above define an idempotent $e \in \mathrm{end}_E(M')$. Moreover, we have $P^2 = P + Z\,R$, where $Z \in E^{3\times 3}$ is defined by:

$$Z = \frac{\nu}{c^2}\left(\begin{array}{ccc} -d_y^2 & d_x\,d_y & d_x\,(\nu\,\Delta - c) \\ d_x\,d_y & -d_x^2 & d_y\,(\nu\,\Delta - c) \\ d_x\,(\nu\,\Delta - c) & d_y\,(\nu\,\Delta - c) & (\nu\,\Delta - c)^2 \end{array}\right).$$

From Lemma 5, we can consider the algebraic Riccati equation $\Lambda\,R'\,\Lambda + (P - I_3)\,\Lambda + \Lambda\,Q + Z = 0$. For instance, we can check that we have the following solution

$$\Lambda = \frac{1}{c}\left(\begin{array}{ccc} -1 & 0 & 0 \\ 0 & -1 & 0 \\ \nu\,d_x & \nu\,d_y & \nu\,(\nu\,\Delta - c) \end{array}\right),$$

which yields the following two idempotent matrices

$$\overline{P} := P + \Lambda\,R' = -\frac{1}{c}\left(\begin{array}{ccc} 0 & 0 & d_x \\ 0 & 0 & d_y \\ 0 & 0 & -c \end{array}\right),$$
$$\overline{Q} := Q + R'\,\Lambda = \frac{\nu}{c}\left(\begin{array}{ccc} d_x^2 & d_x\,d_y & d_x\,(\nu\,\Delta - c) \\ d_x\,d_y & d_y^2 & d_y\,(\nu\,\Delta - c) \\ -d_x & -d_y & -\nu\,\Delta + c \end{array}\right),$$

i.e., $\overline{P}^2 = \overline{P}$ and $\overline{Q}^2 = \overline{Q}$, which satisfy the relation $R'\,\overline{P} = \overline{Q}\,R'$. Thus, the idempotent e can be defined by means of the idempotent matrices $\overline{P}$ and $\overline{Q}$. Since $\overline{P}^2 = \overline{P}$ and $\overline{Q}^2 = \overline{Q}$, $\ker_E(.\overline{P})$, $\mathrm{im}_E(.\overline{P})$, $\ker_E(.\overline{Q})$ and $\mathrm{im}_E(.\overline{Q})$ are then finitely generated projective E-modules (see (22)), and thus free by the Quillen-Suslin theorem (see 2 of Theorem 2). Syzygy module computations yield $\ker_E(.\overline{P}) = \mathrm{im}_E(.X)$ and $\ker_E(.\overline{Q}) = \mathrm{im}_E(.Y)$, where the matrices X and Y are defined by:

$$X = \left(\begin{array}{ccc} c & 0 & d_x \\ -d_y & d_x & 0 \\ 0 & c & d_y \end{array}\right),\; Y = \left(\begin{array}{ccc} 1 & 0 & \nu\,d_x \\ -d_y & d_x & 0 \\ 0 & 1 & \nu\,d_y \end{array}\right).$$

Moreover, we have $\mathrm{im}_E(.\overline{P}) = \ker_E(.(I_3 - \overline{P})) = E\,(0 \quad 0 \quad 1)$ and $\mathrm{im}_E(.\overline{Q}) = \ker_E(.(I_3 - \overline{Q})) = E\,(d_x \quad d_y \quad \nu\,\Delta - c)$. The matrix X does not define a basis of $\ker_E(.\overline{P})$ since $\mathrm{rank}_E(\ker_E(.\overline{P})) \leq 2$ and X has three rows. A similar comment holds for the matrix Y and $\ker_E(.\overline{Q})$. Thus, the rows of X and Y are E-linearly dependent, i.e.:

$$\left\{\begin{array}{l} \ker_E(.X) = E\,(-d_y \quad -c \quad d_x), \\ \ker_E(.Y) = E\,(-d_y \quad -1 \quad d_x). \end{array}\right.$$

If $X_{i\bullet}$ denotes the i^{th} row of X, then we have:

$$\left\{\begin{array}{l} c\,X_{2\bullet} = -d_y\,X_{1\bullet} + d_x\,X_{3\bullet}, \\ Y_{2\bullet} = -d_y\,Y_{1\bullet} + d_x\,Y_{3\bullet}. \end{array}\right.$$

Consequently, a basis of $\ker_E(.\overline{P})$ (resp., $\ker_E(.\overline{Q})$) is defined by the first and third rows of X (resp., Y), i.e., $\ker_E(.\overline{P}) = \mathrm{im}_E(.U_1)$ and $\ker_E(.\overline{Q}) = \mathrm{im}_E(.V_1)$, where the matrices U_1 and V_1 are defined by:

$$U_1 = \left(\begin{array}{ccc} c & 0 & d_x \\ 0 & c & d_y \end{array}\right), \quad V_1 = \left(\begin{array}{ccc} 1 & 0 & \nu\,d_x \\ 0 & 1 & \nu\,d_y \end{array}\right).$$

These results can directly be obtained by means of a constructive version of the Quillen-Suslin theorem implemented in the QuillenSuslin package (Fabiańska and Quadrat (2007)).

Now, from Theorem 5, if we define the following unimodular matrices

$$U = \begin{pmatrix} c & 0 & d_x \\ 0 & c & d_y \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}_3(E), \quad V = \begin{pmatrix} 1 & 0 & \nu\, d_x \\ 0 & 1 & \nu\, d_y \\ d_x & d_y & \nu\,\Delta - c \end{pmatrix} \in \mathrm{GL}_3(E),$$

then the matrix R' defined by (80) is equivalent to the following block diagonal matrix:

$$\overline{R} := V\, R'\, U^{-1} = \begin{pmatrix} -\frac{\nu}{c}\, d_y^2 + 1 & \frac{\nu}{c}\, d_x\, d_y & 0 \\ \frac{\nu}{c}\, d_x\, d_y & -\frac{\nu}{c}\, d_x^2 + 1 & 0 \\ 0 & 0 & \Delta \end{pmatrix}.$$

Let us finally prove that the E-module $O := E^{1\times 2}/(E^{1\times 2}\, T)$ finitely presented by the first 2×2 diagonal block T of $\overline{R}$ is indecomposable. Applying Algorithm 2.1 of Cluzeau and Quadrat (2008), we obtain that $\mathrm{end}_E(O)$ is finitely generated by $\{g_i\}_{i=1,\ldots,4}$, where the g_i's are defined by $g_i(\kappa(\lambda)) = \kappa(\lambda\, P_i)$ for all $\lambda \in E^{1\times 2}$, the matrices P_i's are defined by

$$P_1 = \begin{pmatrix} 0 & \nu\, d_x\, d_y \\ 0 & \nu\, d_y^2 - c \end{pmatrix}, \quad P_2 = I_2, \quad P_3 = \begin{pmatrix} 0 & -\nu\, d_y^2 \\ 0 & \nu\, d_x\, d_y \end{pmatrix}, \quad P_4 = \begin{pmatrix} 0 & c\, d_y \\ 0 & -c\, d_x \end{pmatrix},$$

and $\kappa : E^{1\times 2} \longrightarrow O$ is the canonical projection onto N. The g_i's satisfy the following E-linear relations:

$$\begin{pmatrix} -1 & \nu\, d_y^2 - c & 0 & 0 \\ -c & 0 & 0 & \nu\, d_x \\ -d_x & 0 & d_y & 1 \\ 0 & c\, d_x & 0 & 1 \\ 0 & 0 & c & \nu\, d_y \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \end{pmatrix} = 0.$$

These E-linear relations yield $g_1 = (\nu\, d_y^2 - c)\, g_2$, $g_4 = -c\, d_x\, g_2$ and $g_3 = -\frac{\nu}{c}\, g_4 = \nu\, d_x\, d_y\, g_2$, where g_2 satisfies $(\nu\,\Delta - c)\, g_2 = 0$, which shows that $\mathrm{end}_E(O)$ is a cyclic E-module generated by $g_2 = \mathrm{id}_O$ and:

$$\mathrm{end}_E(O) = E\, g_2 \cong E/(\nu\,\Delta - c).$$

Now, $e = \alpha\, g_2$ is an idempotent of $\mathrm{end}_E(O)$, where $\alpha \in E$, if and only if $e^2 = e$, i.e., if and only if there exists $\beta \in E$ such that $\alpha\,(\alpha - 1) = \beta\,(\nu\,\Delta - c)$. Since the polynomial $\nu\,\Delta - c$ is irreducible over $\mathbb{Q}(\nu, c)$, then $\nu\,\Delta - c$ either divides α or $\alpha - 1$, i.e., $\alpha = \gamma\,(\nu\,\Delta - c)$ or $\alpha = 1 + \gamma'\,(\nu\,\Delta - c)$ for certain $\gamma,\, \gamma' \in E$, which shows that we either have $e = \gamma\,(\nu\,\Delta - c)\, g_2 = 0$ or $e = (1 + \gamma'\,(\nu\,\Delta - c))\, g_2 = g_2 = \mathrm{id}_O$. Therefore, $\mathrm{end}_E(O)$ admits only the trivial idempotents 0 and id_O, which proves that O is an indecomposable E-module. Consequently, T is not equivalent to a diagonal matrix over E.

7.3 Fluid dynamics

In Sections 7.1 and 7.2, Serre's reduction techniques were used to prove the (in)decomposability of finitely presented differential modules associated with 2-dimensional linear PD systems studied in hydrodynamics. The approach is based on the fact that $\mathrm{end}_D(M)$ can be proved to be a cyclic D-module. Unfortunately, for the 3-dimensional case, we are not able to prove that $\mathrm{Fitt}_1(\mathrm{end}_D(M)) = D$ so that Corollary 5 cannot be used to conclude that the endomorphism ring of the corresponding linear PD systems is cyclic.

In this section, we develop a slightly different approach using Serre's reduction to prove the indecomposabilty of a 3-dimensional linear PD system also studied in fluid dynamics.

The movement of an incompressible fluid rotating with a small velocity around the axis lying along the x_3 axis can be defined by

$$\left\{\begin{array}{l} \rho_0 \dfrac{\partial u_1}{\partial t} - 2\,\rho_0\,\Omega_0\,u_2 + \dfrac{\partial p}{\partial x_1} = 0, \\ \rho_0 \dfrac{\partial u_2}{\partial t} + 2\,\rho_0\,\Omega_0\,u_1 + \dfrac{\partial p}{\partial x_2} = 0, \\ \rho_0 \dfrac{\partial u_3}{\partial t} + \dfrac{\partial p}{\partial x_3} = 0, \\ \dfrac{\partial u_1}{\partial x_1} + \dfrac{\partial u_2}{\partial x_2} + \dfrac{\partial u_3}{\partial x_3} = 0, \end{array}\right. \tag{84}$$

where $\vec{u} = (u_1 \quad u_2 \quad u_3)^T$ is the local rate of velocity, p the pressure, ρ_0 the constant fluid density and Ω_0 the constant angle speed (Landau and Lifschitz (1989)). Let $D = \mathbb{Q}(\rho_0, \Omega_0)[d_t, d_1, d_2, d_3]$ be the commutative polynomial ring of PD operators with coefficients in the field $\mathbb{Q}(\rho_0, \Omega_0)$,

$$R = \begin{pmatrix} \rho_0\,d_t & -2\,\rho_0\,\Omega_0 & 0 & d_1 \\ 2\,\rho_0\,\Omega_0 & \rho_0\,d_t & 0 & d_2 \\ 0 & 0 & \rho_0\,d_t & d_3 \\ d_1 & d_2 & d_3 & 0 \end{pmatrix} \in D^{4\times 4}, \tag{85}$$

the presentation matrix of (84), and the D-module $M = D^{1\times 4}/(D^{1\times 4}\,R)$ associated with (84). Let us study the decomposability of the D-module M. Using Algorithm 2.1 of Cluzeau and Quadrat (2008) and its implementation in the OreMorphisms package (Cluzeau and Quadrat (2009)), we find that $\mathrm{end}_D(M)$ is defined by the family of generators $\{f_i\}_{i=1,2}$, where $f_i(\pi(\lambda)) = \pi(\lambda\,P_i)$ for all $\lambda \in D^{1\times 4}$, and:

$$P_1 = I_4, \quad P_2 = \begin{pmatrix} 0 & d_3 & -d_2 & 0 \\ -d_3 & 0 & d_1 & 0 \\ d_2 & -d_1 & 0 & 0 \\ 0 & 0 & 2\,\rho_0\,\Omega_0 & 0 \end{pmatrix}.$$

For more details, see Section 8.5. Using the results explained at the end of Section 3, a generating set of D-linear relations among the generators f_i's of $\mathrm{end}_D(M)$ is defined by $L\,(f_1 \quad f_2)^T = 0$, where the matrix L is defined by:

$$L = \begin{pmatrix} 2\,\rho_0\,\Omega_0\,d_3 & \rho_0\,d_t \\ \rho_0\,d_t\,\Delta & -2\,\rho_0\,\Omega_0\,d_3 \end{pmatrix}, \quad \Delta := d_1^2 + d_2^2 + d_3^2.$$

The endomorphism ring $\mathrm{end}_D(M)$ of M is isomorphic to $N = D^{1\times 2}/(D^{1\times 2}\,L)$. The first Fitting ideal $\mathrm{Fitt}_1(\mathrm{end}_D(M)) = (d_t, d_3, d_t\,\Delta) = (d_t, d_3)$ of $\mathrm{end}_D(M)$ formed by the entries of L is not equal to D, which shows that the D-module $\mathrm{end}_D(M)$ is not cyclic by Corollary 5. Using again Serre's reduction techniques, we can prove the following result.

Proposition 10. *If $D_t = \mathbb{Q}(\rho_0, \Omega_0, d_t)[d_1, d_2, d_3]$ and $M_t = D_t \otimes_D M$, then the D_t-module $\mathrm{end}_{D_t}(M_t) \cong D_t^{1\times 2}/(D_t^{1\times 2}\,L)$ is cyclic and indecomposable, and thus the D_t-module M_t is indecomposable.*

Proof. If $\Lambda = (1 \quad 0) \in D_t^{1\times 2}$ and $P = (\Lambda^T \quad L^T)^T \in D_t^{3\times 2}$, then P admits the following left inverse:

$$S = \begin{pmatrix} 1 & 0 & 0 \\ -2\,\dfrac{\Omega_0\,d_3}{d_t} & -\dfrac{1}{\rho_0\,d_t} & 0 \end{pmatrix} \in D_t^{2\times 3}.$$

If we note $S = (S_1 \quad S_2)$, where $S_1 \in D_t^2$ and $S_2 \in D_t^{2\times 2}$, then $S_1\,\Lambda + S_2\,L = I_2$. Using $L\,f = 0$, where $f = (f_1 \quad f_2)^T$, and $\Lambda\,f = f_1$, we then obtain $f = S_1\,f_1$, i.e., $f_1 = f_1$ and $f_2 = (-2\,\Omega_0\,d_3/d_t)\,f_1$. Thus, the D_t-module $\mathrm{end}_{D_t}(M_t)$ is a cyclic D_t-module generated by $f_1 = \mathrm{id}_{M_t}$. Using Lemma 3.1 of Cluzeau and Quadrat (2008), we get

$$\mathrm{end}_{D_t}(M_t) = D_t\,f_1 = (D_t^{1\times 3}\,P)/(D_t^{1\times 2}\,L) \cong D_t^{1\times 3}/D_t^{1\times 3}\,((F^T \quad P_2^T)^T) \cong D_t/(\rho_0\,(d_t^2\,\Delta + 4\,\Omega_0^2\,d_3^2)),$$

where $F = (0 \quad I_2) \in D_t^{2\times 3}$ is such that $L = F\,P$ and $P_2 = (\rho_0\,(d_t^2\,\Delta + 4\,\Omega_0^2\,d_3^2) \quad -2\,\Omega_0\,d_3 \quad -d_t)$ is such that $\ker_{D_t}(.P) = D_t\,P_2$. We note that $\rho_0\,(d_t^2\,\Delta + 4\,\Omega_0^2\,d_3^2) = \det(R)$. An idempotent e of $\mathrm{end}_{D_t}(M_t) \cong D_t/(\det(R))$ is then of the form $e = \alpha\,\mathrm{id}_{M_t}$, where $\alpha \in D_t$ satisfies $\alpha\,(\alpha - 1) = \gamma\,\det(R)$ for a certain $\gamma \in D_t$. Since $\det(R)$ is irreducible over D_t, the only idempotents of $\mathrm{end}_{D_t}(M_t)$ are then 0 and id_{M_t}, which shows that M_t is an indecomposable D_t-module. □

To deduce the indecomposability of the D-module M, we shall need the next lemma.

Lemma 6. *Let D be a commutative polynomial ring over a field k, $R \in D^{p\times p}$ a square full row rank matrix, i.e., $\det(R) \neq 0$, $M = D^{1\times p}/(D^{1\times p}\,R)$ the D-module finitely presented by R and $\pi : D^{1\times p} \longrightarrow M$ the canonical projection onto M. Then, we have:*

1. $\det(R) \in \mathrm{ann}_D(M) := \{d \in D \mid \forall\, m \in M : \; d\,m = 0\}$.
2. *If $\det(R)$ is irreducible over k, then every element $d \in D$ satisfying $d\,m = 0$ for some $m \in M \setminus \{0\}$ is a multiple of $\det(R)$.*

Proof. 1. Let us consider $m = \pi(\lambda) \in M$, where $\lambda \in D^{1\times p}$. We have $\det(R)\,m = \pi(\lambda\,\det(R))$. If $\mathrm{Adj}(R)$ denotes the *adjugate matrix* of R, namely the transpose of the matrix of *cofactors* of R, then using the identity $\det(R)\,I_p = \mathrm{Adj}(R)\,R = R\,\mathrm{Adj}(R)$, we get $\det(R)\,m = \pi(\lambda\,\mathrm{Adj}(R)\,R) = \pi((\lambda\,\mathrm{Adj}(R))\,R) = 0$, which finally proves that $\det(R) \in \mathrm{ann}_D(M)$.

2. Let $m = \pi(\lambda) \in M \setminus \{0\}$, where $\lambda \in D^{1\times p}$, and $d \in D$ satisfy $d\,m = 0$, i.e., $\pi(d\,\lambda) = 0$. Thus, there exists $\mu \in D^{1\times p}$ such that $d\,\lambda = \mu\,R$. Post-multiplying the latter equality by $\mathrm{Adj}(R)$, we obtain $d\,\lambda\,\mathrm{Adj}(R) = \mu\,R\,\mathrm{Adj}(R) = \mu\,\det(R)$. Setting $\nu := \lambda\,\mathrm{Adj}(R) \in D^{1\times p}$, we get $d\,\nu = \mu\,\det(R)$. In particular, $\det(R)$ divides $d\,\nu$. Now, if $\det(R)$ is irreducible, then either $\det(R)$ divides each entry of ν or d is a multiple of $\det(R)$. In the first case, we get $\nu = \alpha\,\det(R)$ for a certain $\alpha \in D^{1\times p}$, which yields $\lambda\,\mathrm{Adj}(R) = \alpha\,\det(R) = \alpha\,R\,\mathrm{Adj}(R)$. This implies that $\lambda = \alpha\,R$ since $\mathrm{Adj}(R)$ has full row rank, and proves that $m = \pi(\lambda) = 0$, which contradicts the fact that $m \neq 0$. □

We can now to state the main result.

Theorem 19. *Let $D = \mathbb{Q}(\rho_0, \Omega_0)[d_t, d_1, d_2, d_3]$, $R \in D^{4\times 4}$ be defined by (85) and $M = D^{1\times 4}/(D^{1\times 4}\,R)$ the D-module finitely presented by R and associated with (84). Then, M is an indecomposable D-module.*

Proof. If $M = M_1 \oplus M_2$ for two submodules M_1 and M_2 of M, then we get

$$D_t \otimes_D M = (D_t \otimes_D M_1) \oplus (D_t \otimes_D M_2)$$

(see, e.g., Rotman (2009)). Now, we proved above that $D_t \otimes_D M$ is an indecomposable D_t-module so that either $D_t \otimes_D M_1 = 0$ or $D_t \otimes_D M_2 = 0$. Without loss of generality, let us suppose $D_t \otimes_D M_1 = 0$. If $M_1 \neq 0$, then $\mathrm{ann}_D(M_1)$ must contain a non-zero polynomial $d \in \mathbb{Q}(\rho_0, \Omega_0)[d_t]$. Since M_1 is a D-submodule of M and $d_t^2\,\Delta + 4\,\Omega_0^2\,d_3^2$ is irreductible over $\mathbb{Q}(\rho_0, \Omega_0)$, by Lemma 6, we get $d = \beta\,\det(R) = \beta\,(\rho_0^2\,\left(d_t^2\,\Delta + 4\,\Omega_0^2\,d_3^2\right))$ for some $\beta \in D$, which contradicts the fact that $d \in \mathbb{Q}(\rho_0, \Omega_0)[d_t]$. Consequently, we have $M_1 = 0$, and thus $M = M_2$, which proves that the D-module M is indecomposable. □

8 Appendix

In this appendix, we first give the explicit computations for the wind tunnel model studied in Examples 6, 7, 9 and 10 of Sections 5 and 6, and then for standard linear PD systems encountered in hydrodynamics studied in Section 7.

The computations are obtained by means of the OreMorphisms package (Cluzeau and Quadrat (2009)). The OreMorphisms package is based on the OreModules package (Chyzak et al. (2007)). To handle linear algebra operations, we use the Maple package `linalg`.

```
> with(OreModules):
> with(OreMorphisms):
> with(linalg):
```

Since the symbol D is protected in Maple, in what follows, we shall use A instead of D as a name for (the data representing) an Ore algebra.

8.1 Wind tunnel model: decomposition

Let us consider the ring A of OD time-delay operators with coefficients in the field $\mathbb{Q}(a,k,\omega,\zeta)$, i.e.,

```
> A := DefineOreAlgebra(diff=[d,t],dual_shift=[delta,s],polynom=[t,s],
> comm=[a,k,omega,zeta]):
```

and the matrix $R \in A^{3\times 4}$ defined in Example 6, i.e.,

```
> R := evalm([[d+a,k*a*delta,0,0],[0,d,-1,0],[0,omega^2,d+2*zeta*omega,-omega^2]]);
```

$$R := \begin{bmatrix} d+a & k\,a\,\delta & 0 & 0 \\ 0 & d & -1 & 0 \\ 0 & \omega^2 & d+2\,\zeta\,\omega & -\omega^2 \end{bmatrix}$$

which finitely presents the A-module $M := A^{1\times 4}/(A^{1\times 3}\,R)$.

We can easily check that the following matrix $\Delta \in D^{4\times 3}$

```
> Delta := evalm([[0,-1,0],[0,0,0],[0,1,0],[1/omega^2,2*zeta/omega-a/omega^2,
> 1/omega^2]]);
```

$$\Delta := \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{\omega^2} & 2\,\frac{\zeta}{\omega} - \frac{a}{\omega^2} & \frac{1}{\omega^2} \end{bmatrix}$$

satisfies the algebraic Riccati equation $\Delta\,R\,\Delta = -\Delta$:

```
> simplify(evalm(Mult(Delta,R,Delta,A)+Delta));
```

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We note that the matrix Δ can be obtained by means of the `RiccatiConstCoeff` command of OreMorphisms (Cluzeau and Quadrat (2009)).

Hence, the matrix $\overline{P} := I_4 + \Lambda\,R$ defined by

```
> P_bar := simplify(evalm(1+Mult(Delta,R,A)));
```

$$P\_bar := \begin{bmatrix} 1 & -d & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & d & 0 & 0 \\ \frac{d+a}{\omega^2} & -\frac{-\omega^2-k\,a\,\delta-2\,d\,\zeta\,\omega+a\,d}{\omega^2} & \frac{d+a}{\omega^2} & 0 \end{bmatrix}$$

and the matrix $\overline{Q} := I_4 + R\,\Delta$ defined by

```
> Q_bar := simplify(evalm(1+Mult(R,Delta,A)));
```

$$Q\_bar := \begin{bmatrix} 1 & -d-a & 0 \\ 0 & 0 & 0 \\ -1 & d+a & 0 \end{bmatrix}$$

satisfy the identities $\overline{P}^2 = \overline{P}$, $\overline{Q}^2 = \overline{Q}$ and $R\,\overline{P} = \overline{Q}\,R$:

```
> simplify(evalm(Mult(P_bar,P_bar,A)-P_bar));
```

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

```
> simplify(evalm(Mult(Q_bar,Q_bar,A)-Q_bar));
```

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

```
> simplify(evalm(Mult(R,P_bar,A)-Mult(Q_bar,R,A)));
```

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since A is a commutative polynomial ring, we know that the matrix R is equivalent to a block diagonal matrix. Let us compute this block diagonal matrix. To do that, we first compute a basis of the free A-module $\ker_A(.\Delta)$:

```
> U2 := SyzygyModule(Delta,A);
```

$$U2 := \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

We get $\operatorname{im}_A(.\overline{P}) = \ker_A(.\Delta) = \operatorname{im}_A(.U_2)$, where U_2 has full row rank since:

```
> SyzygyModule(U2,A);
```

$$INJ\,(2)$$

Hence, the rows of U_2 form a basis of the free A-module $\ker_A(.\Delta)$ of rank 2. Now, let us compute a basis of the free A-module $\operatorname{im}_A(.\Delta) \cong \operatorname{coker}_A(.U_2)$ of rank 2. We first compute a minimal parametrization $Q_2 \in A^{4\times 2}$ of $\operatorname{coker}_A(.U_2)$

```
> Q2 := MinimalParametrization(U2,A);
```

$$Q2 := \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ -1 & 0 \\ 0 & 1 \end{bmatrix}$$

i.e., we have $\ker_A(.Q_2) = \operatorname{im}_A(.U_2)$ and Q_2 admits a left inverse $T_2 \in A^{2\times 4}$ defined by:

```
> T2 := LeftInverse(Q2,A);
```

$$T2 := \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

If $\kappa : A^{1\times 4} \longrightarrow \operatorname{coker}_A(.U_2)$ is the canonical projection, then the family $\{\kappa((T_2)_{1\bullet}), \kappa((T_2)_{2\bullet})\}$ formed by the residue classes of the rows $(T_2)_{1\bullet}$ and $(T_2)_{2\bullet}$ of T_2 defines a basis of $\operatorname{coker}_A(.U_2)$. Now, using the isomorphism $\phi : \mathrm{coker}_A(.U_2) \longrightarrow \mathrm{im}_A(.\Delta)$ defined by $\phi(\kappa(\mu)) = \mu\,\Delta$ for all $\mu \in A^{1\times 4}$, we obtain that $V_1 := T_2\,\Delta$, i.e.,

```
>  V1 := Mult(T2,Delta,A);
```

$$V1 := \begin{bmatrix} 0 & -1 & 0 \\ \frac{1}{\omega^2} & -\frac{-2\,\zeta\,\omega+a}{\omega^2} & \frac{1}{\omega^2} \end{bmatrix}$$

defines a basis of $\mathrm{im}_A(.\Delta)$, i.e., $\ker_A(.\overline{Q}) = \mathrm{im}_A(.\Delta) = \mathrm{im}_A(.V_1)$, where V_1 has full row rank:

```
>  SyzygyModule(V1,A);
```

$$INJ\,(2)$$

Now, we know that the matrix $U_1 := V_1\,R$ defined by

```
>  U1 := Mult(V1,R,A);
```

$$U1 := \begin{bmatrix} 0 & -d & 1 & 0 \\ \frac{d+a}{\omega^2} & -\frac{-\omega^2-k\,a\,\delta-2\,d\,\zeta\,\omega+a\,d}{\omega^2} & \frac{d+a}{\omega^2} & -1 \end{bmatrix}$$

forms a basis of $\ker_A(.\overline{P})$, i.e., $\ker_A(.\overline{P}) = \mathrm{im}_A(.(V_1\,R))$. Let us now compute a basis of the free A-module $\mathrm{coker}_A(.\Delta)$ of rank 1. Computing a minimal parametrization of $\mathrm{coker}_A(.\Delta)$, we obtain that $\Psi \in A^3$ defined by

```
>  psi := MinimalParametrization(Delta,A);
```

$$\Psi := \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

is such that $\ker_A(.\Psi) = \mathrm{im}_A(.\Delta)$ and Ψ admits a left inverse $\Phi \in A^{1\times 3}$ defined by:

```
>  Phi := LeftInverse(psi,A);
```

$$\Phi := \begin{bmatrix} 0 & 0 & -1 \end{bmatrix}$$

In particular, the residue class of Φ in the A-module $\mathrm{coker}_A(.\Delta)$ is a basis. Thus, the full row rank matrix $V_2 := \Phi\,\overline{Q}$ defined by

```
>  V2 := Mult(Phi,Q_bar,A);
```

$$V2 := \begin{bmatrix} 1 & -d-a & 0 \end{bmatrix}$$

defines a basis of $\mathrm{im}_A(.\overline{Q})$, i.e., $\mathrm{im}_A(.\overline{Q}) = \mathrm{im}_A(.V_2)$. Now, we know that Δ can be factorized by V_1

```
>  Y := Factorize(Delta,V1,A);
```

$$Y := \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ -1 & 0 \\ 0 & 1 \end{bmatrix}$$

i.e., we have $\Delta = Y\,V_1$. Similarly, the matrix $\overline{P}$ can be factorized by U_2

```
>  Z := Factorize(P_bar,U2,A);
```

$$Z := \begin{bmatrix} 1 & -d \\ 0 & 1 \\ 0 & d \\ \frac{d}{\omega^2}+\frac{a}{\omega^2} & 1+\frac{k\,a\,\delta}{\omega^2}+\frac{(2\,\zeta\,\omega-a)\,d}{\omega^2} \end{bmatrix}$$

i.e., we have $\overline{P} = Z\,U_2$. Let us define the matrices $X_1 = -R\,Y$, $X_2 = \Psi$, $W_1 = -Y$ and $W_2 = Z$, i.e.:

```
> X1 := Mult(evalm(-R),Y,A);
```

$$X1 := \begin{bmatrix} -d-a & 0 \\ -1 & 0 \\ d+2\,\zeta\,\omega & \omega^2 \end{bmatrix}$$

```
> X2 := evalm(psi);
```

$$X2 := \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

```
> W1 := evalm(-Y);
```

$$W1 := \begin{bmatrix} -1 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & -1 \end{bmatrix}$$

```
> W2 := evalm(Z);
```

$$W2 := \begin{bmatrix} 1 & -d \\ 0 & 1 \\ 0 & d \\ \frac{d}{\omega^2} + \frac{a}{\omega^2} & 1 + \frac{k\,a\,\delta}{\omega^2} + \frac{(2\,\zeta\,\omega - a)\,d}{\omega^2} \end{bmatrix}$$

If we form the matrix $U := (U_1^T \quad U_2^T)^T$, i.e.,

```
> U := stackmatrix(U1,U2);
```

$$U := \begin{bmatrix} 0 & -d & 1 & 0 \\ \frac{d+a}{\omega^2} & -\frac{-\omega^2 - k\,a\,\delta - 2\,d\,\zeta\,\omega + a\,d}{\omega^2} & \frac{d+a}{\omega^2} & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

then the matrix U is unimodular, i.e., $U \in \mathrm{GL}_4(A)$, and its inverse U^{-1} is defined by:

```
> U_inv := LeftInverse(U,A);
```

$$U\_inv := \begin{bmatrix} -1 & 0 & 1 & -d \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & d \\ 0 & -1 & \frac{d+a}{\omega^2} & -\frac{-\omega^2 - k\,a\,\delta - 2\,d\,\zeta\,\omega + a\,d}{\omega^2} \end{bmatrix}$$

If we note $W = (W_1 \quad W_2)$, i.e.,

```
> W := augment(W1,W2);
```

$$W := \begin{bmatrix} -1 & 0 & 1 & -d \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & d \\ 0 & -1 & \frac{d}{\omega^2} + \frac{a}{\omega^2} & 1 + \frac{k\,a\,\delta}{\omega^2} + \frac{(2\,\zeta\,\omega - a)d}{\omega^2} \end{bmatrix}$$

then we can check that $W = U^{-1}$:

```
> simplify(evalm(U_inv-W));
```

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Similarly, if we define the matrix $V := (V_1^T \quad V_2^T)^T$, i.e.,

```
> V := stackmatrix(V1,V2);
```

$$V := \begin{bmatrix} 0 & -1 & 0 \\ \frac{1}{\omega^2} & -\frac{-2\,\zeta\,\omega + a}{\omega^2} & \frac{1}{\omega^2} \\ 1 & -d-a & 0 \end{bmatrix}$$

then the matrix V is unimodular, i.e., $V \in \mathrm{GL}_3(A)$, and its inverse V^{-1} is defined by:

```
> V_inv := LeftInverse(V,A);
```

$$V\_inv := \begin{bmatrix} -d-a & 0 & 1 \\ -1 & 0 & 0 \\ d + 2\,\zeta\,\omega & \omega^2 & -1 \end{bmatrix}$$

If we note $X := (X_1 \quad X_2)$, i.e.,

```
> X := augment(X1,X2);
```

$$X := \begin{bmatrix} -d-a & 0 & 1 \\ -1 & 0 & 0 \\ d + 2\,\zeta\,\omega & \omega^2 & -1 \end{bmatrix}$$

then we can check that $X = V^{-1}$:

```
> simplify(evalm(V_inv-X));
```

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The matrix R is then equivalent to the block diagonal matrix $\overline{R} := V\,R\,W$, i.e.:

```
> R_bar := Mult(V,R,W,A);
```

$$R\_bar := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & d+a & -d^2 - a\,d + k\,a\,\delta \end{bmatrix}$$

We note that the second diagonal block of $\overline{R}$ is equal to $\Phi\,R\,Z$:

```
> Mult(Phi,R,Z,A);
```

$$\begin{bmatrix} d+a & -d^2 - a\,d + k\,a\,\delta \end{bmatrix}$$

Finally, if we form the two matrices defined in (71), i.e.,

```
> J := stackmatrix(augment(R,-X2),augment(U2,evalm([[0]$2])));
```

$$J := \begin{bmatrix} d+a & k\,a\,\delta & 0 & 0 & -1 \\ 0 & d & -1 & 0 & 0 \\ 0 & \omega^2 & d+2\,\zeta\,\omega & -\omega^2 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

and

```
> K := stackmatrix(augment(Mult(W1,V1,A),W2),augment(-V2,Mult(V2,R,W2,A)));
```

$$K := \begin{bmatrix} 0 & 1 & 0 & 1 & -d \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & d \\ -\frac{1}{\omega^2} & \frac{-2\,\zeta\,\omega+a}{\omega^2} & -\frac{1}{\omega^2} & \frac{d}{\omega^2}+\frac{a}{\omega^2} & 1+\frac{k\,a\,\delta}{\omega^2}+\frac{(2\,\zeta\,\omega-a)\,d}{\omega^2} \\ -1 & d+a & 0 & d+a & -d^2-d\,a+k\,a\,\delta \end{bmatrix}$$

then we can check that $J\,K = I_5$:

```
> Mult(J,K,A);
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Hence, it shows that we can take $\Lambda = X_2$ for Serre's reduction. Moreover, Λ admits the left inverse V_2:

```
> Mult(V2,X2,A);
```

$$\begin{bmatrix} 1 \end{bmatrix}$$

Now, let us consider another solution $\Delta_2 \in A^{4\times 3}$ of the Riccati equation $\Delta\,R\,\Delta = -\Delta$ defined by:

```
> Delta2 := evalm([[0,0,0],[0,0,0],[0,1,0],[0,(d+2*zeta*omega)/omega^2,1/omega^2]]);
```

$$\Delta 2 := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{d+2\,\zeta\,\omega}{\omega^2} & \frac{1}{\omega^2} \end{bmatrix}$$

We note that Δ_2 is a first order matrix contrary to Δ which is a zero order matrix, i.e., $\Delta \in \mathbb{Q}(a,k,\omega,\zeta)^{4\times 3}$. Let us redo the above computations with Δ_2 instead of Δ. Let us check again that $\Delta_2\,R\,\Delta_2 = -\Delta_2$:

```
> simplify(evalm(Mult(Delta2,R,Delta2,A)+Delta2));
```

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Now, let us define the matrix $\overline{P}_2 := I_4 + \Delta_2\,R$

```
> P_bar2 := simplify(evalm(1+Mult(Delta2,R,A)));
```

$$P\_bar2 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & \frac{\omega^2+d^2+2\,d\,\zeta\,\omega}{\omega^2} & 0 & 0 \end{bmatrix}$$

and the matrix $\overline{Q}_2 := I_4 + R\,\Delta_2$:

```
> Q_bar2 := simplify(evalm(1+Mult(R,Delta2,A)));
```

$$Q\_bar2 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Then, we can check again the identities $\overline{P}_2^2 = \overline{P}_2$, $\overline{Q}_2^2 = \overline{Q}_2$ and $R\,\overline{P}_2 = \overline{Q}_2\,R$:

```
> simplify(evalm(Mult(P_bar2,P_bar2,A)-P_bar2));
```

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

```
> simplify(evalm(Mult(Q_bar2,Q_bar2,A)-Q_bar2));
```

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

```
> simplify(evalm(Mult(R,P_bar2,A)-Mult(Q_bar2,R,A)));
```

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Let us now compute the decomposition of the matrix R. First, let us compute a basis of the free A-module $\ker_A(.\Delta_2)$. We first have

```
> U22 := SyzygyModule(Delta2,A);
```

$$U22 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

i.e., we have $\ker_A(.\Delta_2) = \mathrm{im}_A(.U_{22})$, where the matrix U_{22} has full row rank:

```
> SyzygyModule(U2,A);
```

$$INJ\,(2)$$

Hence, the rows of U_{22} define a basis of $\ker_A(.\Delta_2)$. Now, let us compute a basis of the free A-module $\mathrm{im}_A(.\Delta_2) \cong \mathrm{coker}_A(.U_{22})$ of rank 2. We first compute a minimal parametrization $Q_{22} \in A^{4\times 2}$ of $\mathrm{coker}_A(.U_{22})$

```
> Q22 := MinimalParametrization(U22,A);
```

$$Q22 := \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

i.e., we have $\ker_A(.Q_{22}) = \mathrm{im}_A(.U_{22})$ and Q_{22} admits a left inverse $T_{22} \in A^{2\times 4}$ defined by:

```
>  T22 := LeftInverse(Q22,A);
```

$$T22 := \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

If $\kappa_2 : A^{1\times 4} \longrightarrow \mathrm{coker}_A(.U_{22})$ is the canonical projection, then the family $\{\kappa_2((T_{22})_{1\bullet}), \kappa_2((T_{22})_{2\bullet})\}$ formed by the residue classes of the rows $(T_{22})_{1\bullet}$ and $(T_{22})_{2\bullet}$ of T_{22} defines a basis of $\mathrm{coker}_A(.U_{22})$. Now, using the isomorphism $\phi_2 : \mathrm{coker}_A(.U_{22}) \longrightarrow \mathrm{im}_A(.\Delta_2)$ defined by $\phi_2(\kappa_2(\mu)) = \mu\,\Delta_2$ for all $\mu \in A^{1\times 4}$, we get that $V_{12} := T_{22}\,\Delta_2$

```
>  V12 := Mult(T22,Delta2,A);
```

$$V12 := \begin{bmatrix} 0 & 1 & 0 \\ 0 & \frac{d+2\,\zeta\,\omega}{\omega^2} & \frac{1}{\omega^2} \end{bmatrix}$$

defines a basis of $\mathrm{im}_A(.\Delta_2)$, i.e., $\ker_A(.\overline{Q}_2) = \mathrm{im}_A(.\Delta_2) = \mathrm{im}_A(.V_{12})$, where V_{12} has full row rank:

```
>  SyzygyModule(V12,A);
```

$$INJ\,(2)$$

Now, we know that the matrix $U_{12} := V_{12}\,R$ defined by

```
>  U12 := Mult(V12,R,A);
```

$$U12 := \begin{bmatrix} 0 & d & -1 & 0 \\ 0 & \frac{\omega^2+d^2+2\,d\,\zeta\,\omega}{\omega^2} & 0 & -1 \end{bmatrix}$$

formed a basis of $\ker_A(.\overline{P}_2)$, i.e., $\ker_A(.\overline{P}_2) = \mathrm{im}_A(.(V_{12}\,R))$. Let us now compute a basis of the free A-module $\mathrm{coker}_A(.\Delta_2)$ of rank 1. Computing a minimal parametrization of $\mathrm{coker}_A(.\Delta_2)$, we obtain that $\Psi_2 \in A^3$ defined by

```
>  psi2 := MinimalParametrization(Delta2,A);
```

$$\Psi2 := \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

is such that $\ker_A(.\Psi_2) = \mathrm{im}_A(.\Delta_2)$ and Ψ_2 admits a left inverse $\Phi_2 \in A^{1\times 3}$ defined by:

```
>  Phi2 := LeftInverse(psi2,A);
```

$$\Phi2 := \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

In particular, the residue class of Φ_2 in the A-module $\mathrm{coker}_A(.\Delta_2)$ is a basis. Thus, the full row rank matrix $V_{22} := \Phi_2\,\overline{Q}_2$ defined by

```
>  V22 := Mult(Phi2,Q_bar2,A);
```

$$V22 := \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

defines a basis of $\mathrm{im}_A(.\overline{Q}_2)$, i.e., $\mathrm{im}_A(.\overline{Q}_2) = \mathrm{im}_A(.V_{22})$. Now, we know that Δ_2 can be factorized by V_{12}

```
> Y2 := Factorize(Delta2,V12,A);
```

$$Y2 := \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

i.e., we have $\Delta_2 = Y_2\, V_{12}$. Similarly, the matrix $\overline{P}_2$ can be factorized by U_{22}

```
> Z2 := Factorize(P_bar2,U22,A);
```

$$Z2 := \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & d \\ 0 & \frac{d^2}{\omega^2} + 2\,\frac{\zeta\, d}{\omega} + 1 \end{bmatrix}$$

i.e., we have $\overline{P}_2 = Z_2\, U_{22}$. Let us define the matrices $X_{12} = -R\, Y_2$, $X_{22} = \Psi_2$, $W_{12} = -Y_2$ and $W_{22} = Z_2$, i.e.:

```
> X12 := Mult(evalm(-R),Y2,A);
```

$$X12 := \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ -2\,\zeta\,\omega - d & \omega^2 \end{bmatrix}$$

```
> X22 := evalm(psi2);
```

$$X22 := \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

```
> W12 := evalm(-Y2);
```

$$W12 := \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}$$

```
> W22 := evalm(Z2);
```

$$W22 := \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & d \\ 0 & \frac{d^2}{\omega^2} + 2\,\frac{\zeta\, d}{\omega} + 1 \end{bmatrix}$$

If we form the matrix $U_2 := (U_{12}^T \quad U_{22}^T)^T$, i.e.,

```
> U2 := stackmatrix(U12,U22);
```

$$U2 := \begin{bmatrix} 0 & d & -1 & 0 \\ 0 & \frac{\omega^2 + d^2 + 2\,d\,\zeta\,\omega}{\omega^2} & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

then the matrix U_2 is unimodular, i.e., $U_2 \in \mathrm{GL}_4(A)$, and its inverse U_2^{-1} is defined by:

```
> U2_inv := LeftInverse(U2,A);
```

$$U2\_inv := \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & d \\ 0 & -1 & 0 & \frac{\omega^2+d^2+2\,d\,\zeta\,\omega}{\omega^2} \end{bmatrix}$$

If we note $W_2 = (W_{12} \quad W_{22})$, i.e.,

```
> W2 := augment(W12,W22);
```

$$W2 := \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & d \\ 0 & -1 & 0 & \frac{d^2}{\omega^2} + 2\,\frac{\zeta\,d}{\omega} + 1 \end{bmatrix}$$

then we can check that $W_2 = U_2^{-1}$:

```
> simplify(evalm(U2_inv-W2));
```

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Similarly, if we define the matrix $V_2 := (V_{12}^T \quad V_{22}^T)^T$, i.e.,

```
> V2 := stackmatrix(V12,V22);
```

$$V2 := \begin{bmatrix} 0 & 1 & 0 \\ 0 & \frac{d+2\,\zeta\,\omega}{\omega^2} & \frac{1}{\omega^2} \\ 1 & 0 & 0 \end{bmatrix}$$

then the matrix V_2 is unimodular, i.e., $V_2 \in \mathrm{GL}_3(A)$, and its inverse V_2^{-1} is defined by:

```
> V2_inv := LeftInverse(V2,A);
```

$$V2\_inv := \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ -2\,\zeta\,\omega - d & \omega^2 & 0 \end{bmatrix}$$

If we note $X_2 := (X_{12} \quad X_{22})$, i.e.,

```
> X2 := augment(X12,X22);
```

$$X2 := \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ -2\,\zeta\,\omega - d & \omega^2 & 0 \end{bmatrix}$$

then we can check that $X_2 = V_2^{-1}$:

```
> simplify(evalm(V2_inv-X2));
```

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The matrix R is then equivalent to the block diagonal matrix $\overline{R}_2 := V_2\, R\, W_2$, i.e.:

```
> R2_bar := Mult(V2,R,W2,A);
```

$$R2\_bar := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & d+a & k\,a\,\delta \end{bmatrix}$$

We note that the second diagonal block of $\overline{R}_2$ is equal to $\Phi_2\, R\, Z_2$:

```
> Mult(Phi2,R,Z2,A);
```

$$\begin{bmatrix} d+a & k\,a\,\delta \end{bmatrix}$$

Finally, if we form the two matrices defined in (71), i.e.,

```
> J2 := stackmatrix(augment(R,-X22),augment(U22,evalm([[0]$2])));
```

$$J2 := \begin{bmatrix} d+a & k\,a\,\delta & 0 & 0 & -1 \\ 0 & d & -1 & 0 & 0 \\ 0 & \omega^2 & d+2\,\zeta\,\omega & -\omega^2 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

and

```
> K2 := stackmatrix(augment(Mult(W12,V12,A),W22),augment(-V22,Mult(V22,R,W22,A)));
```

$$K2 := \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & d \\ 0 & -\frac{d+2\,\zeta\,\omega}{\omega^2} & -\frac{1}{\omega^2} & 0 & \frac{d^2}{\omega^2}+2\,\frac{\zeta\,d}{\omega}+1 \\ -1 & 0 & 0 & d+a & k\,a\,\delta \end{bmatrix}$$

then we can check that $J_2\, K_2 = I_5$:

```
> Mult(J2,K2,A);
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Hence, it shows that we can take $\Lambda = X_{22}$ for Serre's reduction. Moreover, Λ admits the left inverse V_{22}:

```
> Mult(V22,X22,A);
```

$$\begin{bmatrix} 1 \end{bmatrix}$$

8.2 Wind tunnel model: Serre's reduction

Let us consider the ring A of OD time-delay operators with coefficients in the field $\mathbb{Q}(a, k, \omega, \zeta)$, i.e.,

```
> A := DefineOreAlgebra(diff=[d,t],dual_shift=[delta,s],polynom=[t,s],
> comm=[a,k,omega,zeta]):
```

and the matrix $R \in A^{3\times 4}$ defined in Example 6, i.e.,

```
> R := evalm([[d+a,k*a*delta,0,0],[0,d,-1,0],[0,omega^2,d+2*zeta*omega,-omega^2]]);
```

$$R := \begin{bmatrix} d+a & k\,a\,\delta & 0 & 0 \\ 0 & d & -1 & 0 \\ 0 & \omega^2 & d+2\,\zeta\,\omega & -\omega^2 \end{bmatrix}$$

which finitely presents the A-module $M := A^{1\times 4}/(A^{1\times 3}\,R)$. Let us introduce the matrix R^T

```
> R_trans := transpose(R);
```

$$R\_trans := \begin{bmatrix} d+a & 0 & 0 \\ k\,a\,\delta & d & \omega^2 \\ 0 & -1 & d+2\,\zeta\,\omega \\ 0 & 0 & -\omega^2 \end{bmatrix}$$

and the A-module $N := A^{1\times 3}/(A^{1\times 4}\,R^T)$ finitely presented by R^T. We can check that N is a finite-dimensional $\mathbb{Q}(a, k, \omega, \zeta)$-vector space:

```
> KBasis(R_trans,A);
```

$$[\lambda_1]$$

The above result means that the residue class of the vector $(1 \quad 0 \quad 0)^T$ in N generates N as a $\mathbb{Q}(a, k, \omega, \zeta)$-vector space. Hence, let us consider the vector Λ defined by:

```
> Lambda := evalm([[1],[0],[0]]);
```

$$\Lambda := \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Now, we can check that the matrix $P := (R \quad -\Lambda) \in A^{3\times 5}$ defined by

```
> P := augment(R,-Lambda);
```

$$P := \begin{bmatrix} d+a & k\,a\,\delta & 0 & 0 & -1 \\ 0 & d & -1 & 0 & 0 \\ 0 & \omega^2 & d+2\,\zeta\,\omega & -\omega^2 & 0 \end{bmatrix}$$

admits a right inverse $S \in A^{5\times 3}$ defined by:

```
> S := RightInverse(P,A);
```

$$S := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & -\frac{d+2\,\zeta\,\omega}{\omega^2} & -\frac{1}{\omega^2} \\ -1 & 0 & 0 \end{bmatrix}$$

Thus, the A-module $E := A^{1\times 5}/(A^{1\times 3}\, P)$ is stably free, i.e., free by the Quillen-Suslin theorem (see 2 of Theorem 2). Let us compute a basis of E.

```
> Q := MinimalParametrization(P,A);
```

$$Q := \begin{bmatrix} 1 & 0 \\ 0 & \omega^2 \\ 0 & d\,\omega^2 \\ 0 & \omega^2 + d^2 + 2\,\zeta\,d\,\omega \\ d+a & k\,a\,\delta\,\omega^2 \end{bmatrix}$$

We first have $\ker_A(.Q) = \mathrm{im}_A(.P)$. Let us now check whether or not the matrix Q admits a left inverse:

```
> T := LeftInverse(Q,A);
```

$$T := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\omega^2} & 0 & 0 & 0 \end{bmatrix}$$

We have $T\,Q = I_2$, which proves that the residue classes of the two rows of T in E define a basis of E.

Let us write $Q := (Q_1^T \quad Q_2^T)^T$, where $Q_1 \in A^{4\times 2}$ is defined by

```
> Q1 := submatrix(Q,1..coldim(R),1..2);
```

$$Q1 := \begin{bmatrix} 1 & 0 \\ 0 & \omega^2 \\ 0 & d\,\omega^2 \\ 0 & \omega^2 + d^2 + 2\,\zeta\,d\,\omega \end{bmatrix}$$

and $Q_2 \in A^{1\times 2}$ by:

```
> Q2 := submatrix(Q,coldim(R)+1..coldim(R)+1,1..2);
```

$$Q2 := \begin{bmatrix} d+a & k\,a\,\delta\,\omega^2 \end{bmatrix}$$

By Theorem 11, we have $M \cong A^{1\times 2}/(A\,Q_2)$, i.e., M can be generated by 2 generators and 1 relation defined by Q_2, i.e., the wind tunnel model is equivalent to $\dot{z}(t) + a\,z(t) + k\,a\,\omega^2\,v(t-1) = 0$.

If we form the matrix $Z = (S \quad Q)$, i.e.,

```
> Z := augment(S,Q);
```

$$Z := \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \omega^2 \\ 0 & -1 & 0 & 0 & d\,\omega^2 \\ 0 & -\frac{d+2\,\zeta\,\omega}{\omega^2} & -\frac{1}{\omega^2} & 0 & \omega^2 + d^2 + 2\,\zeta\,d\,\omega \\ -1 & 0 & 0 & d+a & k\,a\,\delta\,\omega^2 \end{bmatrix}$$

then we can check again that $P\,Z = (I_3 \quad 0)$

```
> Mult(P,Z,A);
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

and that the matrix Z is unimodular, i.e., $Z \in \mathrm{GL}_5(A)$. In particular, its inverse Z^{-1} is defined by:

```
> Z_inv := LeftInverse(Z,A);
```

$$Z\_inv := \begin{bmatrix} d+a & k\,a\,\delta & 0 & 0 & -1 \\ 0 & d & -1 & 0 & 0 \\ 0 & \omega^2 & d+2\,\zeta\,\omega & -\omega^2 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\omega^2} & 0 & 0 & 0 \end{bmatrix}$$

We can check that the vector Λ admits a left inverse Γ defined by:

```
> Gamma := LeftInverse(Lambda,A);
```

$$\Gamma := \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

By Theorem 12 and Remark 9, the matrix R is equivalent to the matrix $\mathrm{diag}(I_2, Q_2)$. Let us compute two unimodular matrices $V \in \mathrm{GL}_3(A)$ and $W \in \mathrm{GL}_4(A)$ such that $V\,R\,W = \mathrm{diag}(I_2, Q_2)$. Since Λ admits a left inverse, we know that $\ker_A(.\Lambda)$ is a stably free A-module, i.e., free by the Quillen-Suslin theorem. Let us compute a basis of $\ker_A(.\Lambda)$. Let us first compute $\ker_A(.\Lambda)$.

```
> Theta := SyzygyModule(Lambda,A);
```

$$\Theta := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We get $\ker_A(.\Lambda) = \mathrm{im}_A(.\Theta)$. Let us now check whether or not the rows of Θ are A-linearly independent, i.e., if they define a basis of $\ker_A(.\Lambda)$. We have:

```
> SyzygyModule(Theta,A);
```

$$INJ\,(2)$$

Thus, the matrix Θ has full row rank, which shows that the rows of Θ define a basis of $\ker_A(.\Lambda)$.

Let us now note the matrix Θ by V_1, i.e.:

```
> V1 := evalm(Theta);
```

$$V1 := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now, if we form the matrix $U_1 := \Theta\,R$, i.e.,

```
> U1 := Mult(Theta,R,A);
```

$$U1 := \begin{bmatrix} 0 & d & -1 & 0 \\ 0 & \omega^2 & d+2\,\zeta\,\omega & -\omega^2 \end{bmatrix}$$

then the matrix U_1 defines a basis of $\ker_A(.Q_1)$, i.e., $\ker_A(.Q_1) = \mathrm{im}_A(.U_1)$ and U_1 has full row rank:

```
> SyzygyModule(U1,A);
```

$$INJ\,(2)$$

In particular, the matrix U_1 admits a right inverse $W_1 \in A^{4\times 2}$:

```
> W1 := RightInverse(U1,A);
```

$$W1 := \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -1 & 0 \\ -\frac{d+2\,\zeta\,\omega}{\omega^2} & -\frac{1}{\omega^2} \end{bmatrix}$$

Let us now compute the matrix U_2 which is such that $I_4 - W_1\, U_1 = Q_1\, U_2$:

```
> U2 := Involution(Factorize(Involution(evalm(1-Mult(W1,U1,A)),A),
> Involution(Q1,A),A),A);
```

$$U2 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\omega^2} & 0 & 0 \end{bmatrix}$$

Now, let us define the matrix $X_1 := R\, W_1$, i.e.,

```
> X1 := Mult(R,W1,A);
```

$$X1 := \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and the matrix $V_2 := \Gamma\,(I_3 - X_1\, V_1)$:

```
> V2 := Mult(Gamma,evalm(1-Mult(X1,V1,A)),A);
```

$$V2 := \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

Let us note $X_2 := \Lambda$

```
> X2 := evalm(Lambda);
```

$$X2 := \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

and $W_2 := Q_1$:

```
> W2 := evalm(Q1);
```

$$W2 := \begin{bmatrix} 1 & 0 \\ 0 & \omega^2 \\ 0 & d\,\omega^2 \\ 0 & \omega^2 + d^2 + 2\,\zeta\, d\,\omega \end{bmatrix}$$

Now, we can check again that we have $Q_2 = \Gamma\, R\, Q_1$:

```
> Q2 := Mult(Gamma,R,Q1,A);
```

$$Q2 := \begin{bmatrix} d + a & k\, a\, \delta\, \omega^2 \end{bmatrix}$$

If we note $U := (U_1^T \quad U_2^T)^T$, i.e.,

```
> U := stackmatrix(U1,U2);
```

$$U := \begin{bmatrix} 0 & d & -1 & 0 \\ 0 & \omega^2 & d + 2\,\zeta\,\omega & -\omega^2 \\ 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\omega^2} & 0 & 0 \end{bmatrix}$$

then we can check that U is unimodular, i.e., $U \in \mathrm{GL}_4(A)$. In particular, its inverse is defined by:

```
> U_inv := LeftInverse(U,A);
```

$$U\_inv := \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \omega^2 \\ -1 & 0 & 0 & d\,\omega^2 \\ -\frac{d+2\,\zeta\,\omega}{\omega^2} & -\frac{1}{\omega^2} & 0 & \omega^2+d^2+2\,\zeta\,d\,\omega \end{bmatrix}$$

If we note $W := (W_1 \quad W_2)$, i.e.:

```
>  W := augment(W1,W2);
```

$$W := \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \omega^2 \\ -1 & 0 & 0 & d\,\omega^2 \\ -\frac{d+2\,\zeta\,\omega}{\omega^2} & -\frac{1}{\omega^2} & 0 & \omega^2+d^2+2\,\zeta\,d\,\omega \end{bmatrix}$$

then we can check again that we have $W = U^{-1}$:

```
>  simplify(evalm(W-U_inv));
```

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

In particular, it proves that $W \in \mathrm{GL}_4(A)$. Now, let us define the matrix $V := (V_1^T \quad V_2^T)^T$.

```
>  V := stackmatrix(V1,V2);
```

$$V := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

The matrix V is unimodular, i.e., $V \in \mathrm{GL}_3(A)$, and its inverse is defined by:

```
>  V_inv := LeftInverse(V,A);
```

$$V\_inv := \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Let us note $X := (X_1 \quad X_2)$, i.e.:

```
>  X := augment(X1,X2);
```

$$X := \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

We can check again that we have $X = V^{-1}$.

```
>  simplify(evalm(X-V_inv));
```

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We then have $\overline{R} := V\,R\,W = \mathrm{diag}(I_2, Q_2)$:

```
> R_bar := Mult(V,R,W,A);
```

$$R\_bar := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & d+a & k\,a\,\delta\,\omega^2 \end{bmatrix}$$

Now, let us check again that the matrix $\Delta := -W_1\,V_1$, i.e.,

```
> Delta := Mult(-1,W1,V1,A);
```

$$\Delta := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{d+2\,\zeta\,\omega}{\omega^2} & \frac{1}{\omega^2} \end{bmatrix}$$

satisfies the algebraic Riccati equation $\Delta\,R\,\Delta = -\Delta$:

```
> simplify(evalm(Mult(Delta,R,Delta,A)+Delta));
```

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, if we define the matrix $\overline{P} := Q_1\,U_2$, i.e.,

```
> P_bar := Mult(Q1,U2,A);
```

$$P\_bar := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & \frac{\omega^2+d^2+2\,\zeta\,d\omega}{\omega^2} & 0 & 0 \end{bmatrix}$$

then we can check again that $\overline{P} = I_4 + \Delta\,R$

```
> simplify(evalm(P_bar-evalm(1+Mult(Delta,R,A))));
```

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and that $\overline{P}$ is an idempotent matrix, i.e., $\overline{P}^2 = \overline{P}$:

```
> simplify(evalm(Mult(P_bar,P_bar,A)-P_bar));
```

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Similarly, if we note $\overline{Q} := X_2\,V_2$, i.e.,

```
> Q_bar := Mult(X2,V2,A);
```

$$Q\_bar := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

then we can check again that we have $\overline{Q} = I_3 + R\,\Delta$

```
> simplify(evalm(Q_bar-evalm(1+Mult(R,Delta,A))));
```

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and that the matrix $\overline{Q}$ is an idempotent matrix:

```
> simplify(evalm(Mult(Q_bar,Q_bar,A)-Q_bar));
```

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Moreover, we have $R\,\overline{P} = \overline{Q}\,R$:

```
> simplify(evalm(Mult(R,P_bar,A)-Mult(Q_bar,R,A)));
```

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Finally, let us check again that the A-modules $\ker_A(.\overline{P})$, $\mathrm{im}_A(.\overline{P})$, $\ker_A(.\overline{Q})$ and $\mathrm{im}_A(.\overline{Q})$ are free. Let us first compute $\ker_A(.\overline{P})$.

```
> SyzygyModule(P_bar,A);
```

$$\begin{bmatrix} 0 & \omega^2 & d + 2\,\zeta\,\omega & -\omega^2 \\ 0 & d & -1 & 0 \end{bmatrix}$$

We then have $\ker_A(.\overline{P}) = \mathrm{im}_A(U_1)$, where the full row rank matrix U_1 is defined by:

```
> evalm(U1);
```

$$\begin{bmatrix} 0 & d & -1 & 0 \\ 0 & \omega^2 & d + 2\,\zeta\,\omega & -\omega^2 \end{bmatrix}$$

Let us now compute $\mathrm{im}_A(.\overline{P}) = \ker_A(.(I_4 - \overline{P}))$.

```
> SyzygyModule(evalm(1-P_bar),A);
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

We get $\mathrm{im}_A(.\overline{P}) = \mathrm{im}_A(.U_2)$, where the full row rank matrix U_2 is defined by:

```
> evalm(U2);
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\omega^2} & 0 & 0 \end{bmatrix}$$

Similarly, let us compute $\ker_A(.\overline{Q})$.

```
> SyzygyModule(Q_bar,A);
```

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We get $\ker_A(.\overline{Q}) = \mathrm{im}_A(.V_1)$, where the full row rank matrix V_1 is defined by:

```
> evalm(V1);
```

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Finally, let us compute $\mathrm{im}_A(.\overline{Q}) = \ker_A(.(I_3 - \overline{Q}))$.

```
> SyzygyModule(evalm(1-Q_bar),A);
```

$$\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

We have $\mathrm{im}_A(.\overline{Q}) = \mathrm{im}_A(.V_2)$, where V_2 is the full row rank matrix defined by:

```
> evalm(V2);
```

$$\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

Hence, we obtain that the A-modules $\ker_A(.\overline{P})$, $\mathrm{im}_A(.\overline{P})$, $\ker_A(.\overline{Q})$ and $\mathrm{im}_A(.\overline{Q})$ are free and can respectively be generated by means of the full row rank matrices U_1, U_2, V_1 and V_2.

8.3 Oseen equations

Let us consider the ring A of PD operators with coefficients in the field $\mathbb{Q}(\nu, b_1, b_2)$

```
> A := DefineOreAlgebra(diff=[dt,t],diff=[dx,x],diff=[dy,y],polynom=[t,x,y],
> comm=[nu,b1,b2]):
```

the Laplacian operator Δ defined by

```
> Delta := dx^2+dy^2;
```

$$\Delta := dx^2 + dy^2$$

and the matrix $R \in A^{3\times 3}$ defined by (76), i.e.,

```
> R := evalm([[dt+b1*dx+b2*dy-nu*Delta,0,dx],[0,dt+b1*dx+b2*dy-nu*Delta,dy],
> [dx,dy,0]]);
```

$$R := \begin{bmatrix} dt + b1\ dx + b2\ dy - \nu\left(dx^2 + dy^2\right) & 0 & dx \\ 0 & dt + b1\ dx + b2\ dy - \nu\left(dx^2 + dy^2\right) & dy \\ dx & dy & 0 \end{bmatrix}$$

which finitely presents the A-module $M = A^{1\times 3}/(A^{1\times 3}\,R)$. Let us compute a family of generators of the A-module $\mathrm{end}_A(M)$:

```
> E := MorphismsConstCoeff(R,R,A):
> F := E[1];
```

$$F := [\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & dy & 0 \\ 0 & -dx & 0 \\ 0 & 0 & -dx \end{bmatrix}, \begin{bmatrix} 0 & -dy & 0 \\ 0 & dx & 0 \\ 0 & 0 & dx \end{bmatrix}, \begin{bmatrix} 0 & 0 & dx \\ 0 & 0 & dy \\ 0 & 0 & -dt - b1\, dx - b2\, dy \end{bmatrix},$$
$$\begin{bmatrix} 0 & dy\,\nu\,dx & 0 \\ 0 & \nu\,dy^2 - dt - b1\,dx - b2\,dy & -dy \\ 0 & 0 & \nu\,dy^2 \end{bmatrix}, \begin{bmatrix} 0 & -\nu\,dy^3 + dy\,dt + dy^2\,b2 & dy^2 \\ 0 & -b2\,dx\,dy - dx\,dt + dy^2\,dx\,\nu & -dy\,dx \\ 0 & 0 & -b1\,dy^2 + dy^2\,dx\,\nu \end{bmatrix}]$$

Then, the A-module $\mathrm{end}_A(M)$ is generated by the A-endomorphisms f_i's defined by $f_i(\pi(\lambda)) = \pi(\lambda\, F_i)$, where $\pi : A^{1\times 3} \longrightarrow M$ is the canonical projection and the F_i's are the above matrices.

The relations between the generators $\{f_i\}_{i=1,\ldots,6}$ of $\mathrm{end}_A(M)$ are defined by:

```
> E[2];
```

$$E[2] := \begin{bmatrix} dx & 0 & -1 & 0 & 0 & 0 \\ \nu\,dy^2 - dt - b2\,dy & 0 & -b1 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\nu\,dx & 0 & -1 & 0 \\ 0 & 0 & \nu^2 dy^2 - b1\,dx\,\nu - dt\,\nu - b2\,dy\,\nu & -\nu\,dx & -b1 & -\nu \\ 0 & 0 & -dx\,\nu^2 dy^2 & -\nu\,dy^2 & -b2\,dy - dt & \nu\,dx \\ 0 & 0 & 0 & 0 & -b1 + \nu\,dx & -\nu \end{bmatrix}$$

We note that the second and third entries of F are the same up to a sign. Hence, we can remove the second entry of F to obtain the following family of generators:

```
> G := [F[1],F[3],F[4],F[5],F[6]];
```

$$G := [\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -dy & 0 \\ 0 & dx & 0 \\ 0 & 0 & dx \end{bmatrix}, \begin{bmatrix} 0 & 0 & dx \\ 0 & 0 & dy \\ 0 & 0 & -dt - b1\,dx - b2\,dy \end{bmatrix},$$
$$\begin{bmatrix} 0 & dy\,\nu\,dx & 0 \\ 0 & \nu\,dy^2 - dt - b1\,dx - b2\,dy & -dy \\ 0 & 0 & \nu\,dy^2 \end{bmatrix}, \begin{bmatrix} 0 & -\nu\,dy^3 + dy\,dt + dy^2\,b2 & dy^2 \\ 0 & -b2\,dx\,dy - dx\,dt + dy^2\,dx\,\nu & -dy\,dx \\ 0 & 0 & -b1\,dy^2 + dy^2\,dx\,\nu \end{bmatrix}]$$

Hence, if $f_i \in \mathrm{end}_A(M)$ is defined by $f_i(\pi(\lambda)) = \pi(\lambda\, G_i)$ for $i = 1, \ldots, 5$, then $\{f_i\}_{i=1,\ldots,5}$ is a family of generators of $\mathrm{end}_A(M)$. Let us compute the A-linear relations among the new generators.

```
> L := RelationsMatrix(R,R,G,A);
```

$$L := \begin{bmatrix} dx & -1 & 0 & 0 & 0 \\ \nu\,dy^2 - dt - b2\,dy & -b1 & -1 & -1 & 0 \\ 0 & -\nu\,dx & 0 & -1 & 0 \\ 0 & \nu^2 dy^2 - b1\,dx\,\nu - dt\,\nu - b2\,dy\,\nu & -\nu\,dx & -b1 & -\nu \\ 0 & -dx\,\nu^2 dy^2 & -\nu\,dy^2 & -b2\,dy - dt & \nu\,dx \\ 0 & 0 & 0 & -b1 + \nu\,dx & -\nu \end{bmatrix}$$

We obtain that $\mathrm{end}_A(M) \cong A^{1\times 5}/(A^{1\times 6}\, L)$. Now, let us prove that $\mathrm{end}_A(M)$ is a cyclic A-module. To do that, let us introduce the following vector $\Lambda \in A^{1\times 5}$

```
> Lambda := evalm([[1,0,0,0,0]]);
```

$$\Lambda := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and consider the matrix $P := (L^T \quad \Lambda^T)^T$ defined by:

```
> P := stackmatrix(L,Lambda);
```

$$P := \begin{bmatrix} dx & -1 & 0 & 0 & 0 \\ \nu\, dy^2 - dt - b2\, dy & -b1 & -1 & -1 & 0 \\ 0 & -\nu\, dx & 0 & -1 & 0 \\ 0 & \nu^2\, dy^2 - b1\, dx\, \nu - dt\, \nu - b2\, dy\, \nu & -\nu\, dx & -b1 & -\nu \\ 0 & -dx\, \nu^2\, dy^2 & -\nu\, dy^2 & -b2\, dy - dt & \nu\, dx \\ 0 & 0 & 0 & -b1 + \nu\, dx & -\nu \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Now, we can check that the matrix P admits a left inverse $X \in A^{5\times 7}$ defined by:

```
> X := LeftInverse(P,A);
```

$$X := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & dx \\ b1 - \nu\, dx & -1 & 1 & 0 & 0 & 0 & -dt - b1\, dx - b2\, dy + \nu\, dx^2 + \nu\, dy^2 \\ \nu\, dx & 0 & -1 & 0 & 0 & 0 & -\nu\, dx^2 \\ -dx\, (b1 - \nu\, dx) & 0 & \frac{b1 - \nu\, dx}{\nu} & 0 & 0 & -\nu^{-1} & dx^2\, (b1 - \nu\, dx) \end{bmatrix}$$

Let us note $X = (X_1^T \quad X_2^T)^T$, where $X_1 \in A^{5\times 6}$ and $X_2 \in A^5$, i.e.:

```
> X1 := submatrix(X,1..rowdim(X),1..6);
```

$$X1 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ b1 - \nu\, dx & -1 & 1 & 0 & 0 & 0 \\ \nu\, dx & 0 & -1 & 0 & 0 & 0 \\ -dx\, (b1 - \nu\, dx) & 0 & \frac{b1 - \nu\, dx}{\nu} & 0 & 0 & -\nu^{-1} \end{bmatrix}$$

```
> X2 := submatrix(X,1..rowdim(X),7..7);
```

$$X2 := \begin{bmatrix} 1 \\ dx \\ -dt - b1\, dx - b2\, dy + \nu\, dx^2 + \nu\, dy^2 \\ -\nu\, dx^2 \\ dx^2\, (b1 - \nu\, dx) \end{bmatrix}$$

Hence, we obtain that $\mathrm{end}_A(M)$ is a cyclic A-module defined by the generator $f_1 = \Lambda\, (f_1 \ \dots \ f_5)^T = \mathrm{id}_M$. All these results can directly be obtained by using the command ReducedModuleHom.

```
> Pp := ReducedModuleHom(R,R,F,A);
```

$$Pp := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Let us now compute the annihilator $\operatorname{ann}_A(\mathrm{id}_M) := \{a \in A \mid a\, \mathrm{id}_M = 0\}$ of id_M:

```
> Lp := RelationsMatrix(R,R,[Pp],A);
```

$$Lp := \left[\; 2\,\nu\, dx^2 dy^2 - dy^2 b1\; dx - dx^2 b2\; dy - dy^3 b2 + \nu\, dy^4 - dy^2 dt + \nu\, dx^4 - b1\; dx^3 - dx^2 dt \;\right]$$

```
> d := simplify(Lp[1,1]/Delta)*Delta
```

$$d := \left(-dt - b1\; dx - b2\; dy + \nu\, dx^2 + \nu\, dy^2\right)\left(dx^2 + dy^2\right)$$

We have $\operatorname{ann}_A(\mathrm{id}_M) = A\, d$, which shows that $\operatorname{end}_A(M) \cong A/(d)$. From that, following the arguments developed in Section 7.1, we can prove that M is an indecomposable A-module.

Finally, let us give two strict factorizations of the matrix R. First, let us consider the endomorphism $g_1 = \Delta\, \mathrm{id}_M$ defined by the matrices $P_1 = Q_1 = \Delta\, I_3$, i.e.:

```
> P1 := diag(Delta$3);
```

$$P1 := \begin{bmatrix} dx^2 + dy^2 & 0 & 0 \\ 0 & dx^2 + dy^2 & 0 \\ 0 & 0 & dx^2 + dy^2 \end{bmatrix}$$

The A-module $\operatorname{coim} g_1$ is then finitely presented by the following matrix S_1

```
> S1 := CoimMorphism(R,R,P1,A);
```

$$S1 := \begin{bmatrix} \nu\, dx & \nu\, dy & -1 \\ dx & dy & 0 \\ -dt - b1\; dx - b2\; dy + \nu\, dx^2 + \nu\, dy^2 & 0 & -dx \\ 0 & -dt - b1\; dx - b2\; dy + \nu\, dx^2 + \nu\, dy^2 & -dy \end{bmatrix}$$

i.e., we have $\operatorname{coim} g_1 = A^{1\times 3}/(A^{1\times 4}\, S_1)$. Then, we have the factorization $R = L_1\, S_1$, where the matrix $L_1 \in A^{3\times 4}$ is defined by:

```
> L1 := Factorize(R,S1,A);
```

$$L1 := \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

We can check that this factorization $R = L_1\, S_1$ is strict, i.e., $D^{1\times 3}\, R \subsetneq D^{1\times 4}\, S_1$:

```
> Factorize(S1,R,A);
```

$$[\,]$$

Now, if we consider the endomorphism $g_2 = (d_t + b_1\, d_x + b_2\, d_y - \nu\, \Delta)\, \mathrm{id}_M$ defined by the matrices $P_2 = Q_2 = (d_t + b_1\, d_x + b_2\, d_y - \nu\, \Delta)\, I_3$, i.e.

```
> P2 := diag(dt+b1*dx+b2*dy-nu*Delta$3);
```

$$P2 :=$$

$$\begin{bmatrix} dt + b1\; dx + b2\; dy - \nu\, \left(dx^2 + dy^2\right) & 0 & 0 \\ 0 & dt + b1\; dx + b2\; dy - \nu\, \left(dx^2 + dy^2\right) & 0 \\ 0 & 0 & dt + b1\; dx + b2\; dy - \nu\, \left(dx^2 + dy^2\right) \end{bmatrix}$$

then the A-module $\operatorname{coim} g_2$ is finitely presented by the following matrix S_2:

```
> S2 := CoimMorphism(R,R,P2,A);
```

$$S2 := \begin{bmatrix} -dy & dx & 0 \\ dy\,\nu\,dx - b1\;dy & \nu\,dy^2 - dt - b2\;dy & -dy \\ -dx & -dy & 0 \\ -dt - b1\;dx - b2\;dy + \nu\,dx^2 + \nu\,dy^2 & 0 & -dx \end{bmatrix}$$

i.e., we have $\operatorname{coim} g_2 = A^{1\times 3}/(A^{1\times 4}\,S_2)$. Then, we have the factorization $R = L_2\,S_2$, where the matrix $L_2 \in A^{3\times 4}$ is defined by:

```
>  L2 := Factorize(R,S2,A);
```

$$L2 := \begin{bmatrix} 0 & 0 & 0 & -1 \\ b1 - \nu\,dx & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

Finally, we can check that the factorization $R = L_2\,S_2$ is strict, i.e., $D^{1\times 3}\,R \subsetneq D^{1\times 4}\,S_2$:

```
>  Factorize(S2,R,A);
```

$$[\,]$$

8.4 Implicit scheme for the Oseen equations

Let us consider the ring A of PD operators with coefficients in the field $\mathbb{Q}(\nu, c)$

```
>  A := DefineOreAlgebra(diff=[dx,x],diff=[dy,y],polynom=[x,y],comm=[nu,c]):
```

the Laplacian operator Δ defined by

```
>  Delta := dx^2+dy^2;
```

$$\Delta := dx^2 + dy^2$$

and the matrix $R \in A^{3\times 3}$ defined by (80), i.e.,

```
>  R := evalm([[c-nu*(dx^2+dy^2),0,dx],[0,c-nu*(dx^2+dy^2),dy],[dx,dy,0]]);
```

$$R := \begin{bmatrix} c - \nu\left(dx^2 + dy^2\right) & 0 & dx \\ 0 & c - \nu\left(dx^2 + dy^2\right) & dy \\ dx & dy & 0 \end{bmatrix}$$

which finitely presents the A-module $M = A^{1\times 3}/(A^{1\times 3}\,R)$. Let us consider $f \in \operatorname{end}_A(M)$ defined by $f(\pi(\lambda)) = \pi(\lambda\,P)$, where $\pi : A^{1\times 3} \longrightarrow M$ is the canonical projection and the matrix $P = Q = \left(1 - \frac{\nu}{c}\,\Delta\right) I_3$, i.e.:

```
>  P := diag((1-nu*Delta/c)$3);
```

$$P := \begin{bmatrix} 1 - \frac{\nu\left(dx^2+dy^2\right)}{c} & 0 & 0 \\ 0 & 1 - \frac{\nu\left(dx^2+dy^2\right)}{c} & 0 \\ 0 & 0 & 1 - \frac{\nu\left(dx^2+dy^2\right)}{c} \end{bmatrix}$$

The A-module $\operatorname{coim} f$ is finitely presented by the following matrix

```
>  S := CoimMorphism(R,R,P,A);
```

$$S := \begin{bmatrix} -dy\,c & dx\,c & 0 \\ \nu\,dx\,dy\,c & \nu\,dy^2 c - c^2 & -dy\,c \\ -dx\,c & -dy\,c & 0 \\ \nu\,dx^2 c - c^2 + \nu\,dy^2 c & 0 & -dx\,c \end{bmatrix}$$

i.e., we have $\operatorname{coim} f = A^{1\times 3}/(A^{1\times 4}\,S)$. Then, we have the factorization $R = L\,S$, where the matrix $L \in A^{3\times 4}$ is defined by:

```
> L := Factorize(R,S,A);
```

$$L := \begin{bmatrix} 0 & 0 & 0 & -\frac{1}{c} \\ -\frac{\nu\,dx}{c} & -\frac{1}{c} & 0 & 0 \\ 0 & 0 & -\frac{1}{c} & 0 \end{bmatrix}$$

To compute a presentation of $\ker f = (A^{1\times 4}\,S)/(A^{1\times 3}\,R)$, let us first compute $\ker_A(.S)$:

```
> S2 := SyzygyModule(S,A);
```

$$S2 := \begin{bmatrix} -c + \nu\,dy^2 & -dx & 0 & dy \end{bmatrix}$$

We obtain that $\ker_A(.S) = \operatorname{im}_A(.S_2)$. Then, we have $\ker f \cong A^{1\times 4}/(A^{1\times 4}\,(L^T \quad S_2^T)^T)$, i.e., a presentation matrix of $\ker f$ is defined by:

```
> Lp := stackmatrix(L,S2);
```

$$Lp := \begin{bmatrix} 0 & 0 & 0 & -\frac{1}{c} \\ -\frac{\nu\,dx}{c} & -\frac{1}{c} & 0 & 0 \\ 0 & 0 & -\frac{1}{c} & 0 \\ -c + \nu\,dy^2 & -dx & 0 & dy \end{bmatrix}$$

We recall that the short exact sequence $0 \longrightarrow \ker f \xrightarrow{i} M \xrightarrow{\rho} \operatorname{coim} f \longrightarrow 0$ splits if and only if there exist matrices $U_1 \in A^{4\times 3}$, $U_2 \in A^4$ and $V \in A^{3\times 4}$ such that $U_1\,L + U_2\,S_2 + S\,V = I_4$ (Quadrat and Robertz (2007b)). Let us check if this last identity holds.

```
> M1 := KroneckerProduct(diag(1$4),Lp,A):
> M2 := KroneckerProduct(transpose(S),diag(1$4),A):
> M := stackmatrix(M1,M2):
> K := transpose(convert(convert(diag(1$4),vector),matrix)):
> J := Factorize(K,M,A);
```

$$J := [\; 0 \;\; 0 \;\; 0 \;\; -\tfrac{1}{c} \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; dx \;\; dy \;\; -c \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; \tfrac{\nu\,dy}{c} \;\; -\tfrac{\nu\,dy}{c^2} \;\; 0 \;\; 0 \;\; -\tfrac{1}{c^2} \;\; 0 \;\; -\tfrac{1}{c^2} \;\; 0 \;\; 0 \;\; -\tfrac{\nu^2\,dx\,dy}{c^2} \;\; -\tfrac{\nu\,dy}{c^2} \;\; 0 \;\; -\tfrac{\nu\,dx}{c^2} \;]$$

```
> u := convert(convert(submatrix(J,1..1,1..16),vector),list);
```

$$u := [0, 0, 0, -\tfrac{1}{c}, 0, 0, 0, 0, dx, dy, -c, 0, 0, 0, 0, \tfrac{\nu\,dy}{c}]$$

```
> v := convert(convert(submatrix(J,1..1,17..coldim(J)),vector),list);
```

$$v := [-\tfrac{\nu\,dy}{c^2}, 0, 0, -\tfrac{1}{c^2}, 0, -\tfrac{1}{c^2}, 0, 0, -\tfrac{\nu^2\,dx\,dy}{c^2}, -\tfrac{\nu\,dy}{c^2}, 0, -\tfrac{\nu\,dx}{c^2}]$$

Hence, if we define the following matrix $U \in A^{4\times 4}$ formed by the entries of the row vector u

```
> U := matrix(4,4,u);
```

$$U := \begin{bmatrix} 0 & 0 & 0 & -\frac{1}{c} \\ 0 & 0 & 0 & 0 \\ dx & dy & -c & 0 \\ 0 & 0 & 0 & \frac{\nu\, dy}{c} \end{bmatrix}$$

and the matrix $V \in A^{3\times 4}$ formed by the entries of the row vector v

```
> V := matrix(3,4,v);
```

$$V := \begin{bmatrix} -\frac{\nu\, dy}{c^2} & 0 & 0 & -\frac{1}{c^2} \\ 0 & -\frac{1}{c^2} & 0 & 0 \\ -\frac{\nu^2\, dx\, dy}{c^2} & -\frac{\nu\, dy}{c^2} & 0 & -\frac{\nu\, dx}{c^2} \end{bmatrix}$$

then we have the identity $U\,(L^T \quad S_2^T)^T + S\,V = I_4$, a fact which can be checked again:

```
> simplify(evalm(Mult(U,Lp,A)+Mult(S,V,A)));
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Thus, we have $M \cong \ker f \oplus \operatorname{coim} f$. All these results can be directly obtained as follows.

```
> C := ComplementConstCoeff(S,R,A);
```

$$C := [\begin{bmatrix} -\frac{c-\nu\, dx^2}{c} + 1 & \frac{\nu\, dx\, dy}{c} & -\frac{dx}{c} \\ \frac{\nu\, dx\, dy}{c} & -\frac{c-\nu\, dy^2}{c} + 1 & -\frac{dy}{c} \\ -\frac{\nu\, dx\,(c-\nu\, dx^2-\nu\, dy^2)}{c} & -\frac{\nu\, dy\,(c-\nu\, dx^2-\nu\, dy^2)}{c} & 1 - \frac{\nu\,(dx^2+dy^2)}{c} \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ dx & dy & -c \\ 0 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} -\frac{\nu\, dy}{c^2} & 0 & 0 & -\frac{1}{c^2} \\ 0 & -\frac{1}{c^2} & 0 & 0 \\ -\frac{\nu^2\, dx\, dy}{c^2} & -\frac{\nu\, dy}{c^2} & 0 & -\frac{\nu\, dx}{c^2} \end{bmatrix}]$$

```
> U1 := C[2];
```

$$U1 := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ dx & dy & -c \\ 0 & 0 & 0 \end{bmatrix}$$

```
> V := C[3];
```

$$V := \begin{bmatrix} -\frac{\nu\, dy}{c^2} & 0 & 0 & -\frac{1}{c^2} \\ 0 & -\frac{1}{c^2} & 0 & 0 \\ -\frac{\nu^2\, dx\, dy}{c^2} & -\frac{\nu\, dy}{c^2} & 0 & -\frac{\nu\, dx}{c^2} \end{bmatrix}$$

```
> U2 := Factorize(evalm(1-Mult(S,V,A)-Mult(U1,L,A)),S2,A);
```

$$U2 := \begin{bmatrix} -\frac{1}{c} \\ 0 \\ 0 \\ \frac{\nu\, dy}{c} \end{bmatrix}$$

```
> U := augment(U1,U2);
```

$$U := \begin{bmatrix} 0 & 0 & 0 & -\frac{1}{c} \\ 0 & 0 & 0 & 0 \\ dx & dy & -c & 0 \\ 0 & 0 & 0 & \frac{\nu\, dy}{c} \end{bmatrix}$$

Now, we can check that we have $P^2 = P + Z\, R$, where $Z \in A^{3\times 3}$ is defined by:

```
> Z := map(factor,Factorize(evalm(Mult(P,P,A)-P),R,A));
```

$$Z := \begin{bmatrix} -\frac{\nu\, dy^2}{c^2} & \frac{\nu\, dx\, dy}{c^2} & -\frac{dx\, \nu\left(c-\nu\, dx^2-\nu\, dy^2\right)}{c^2} \\ \frac{\nu\, dx\, dy}{c^2} & -\frac{\nu\, dx^2}{c^2} & -\frac{dy\, \nu\left(c-\nu\, dx^2-\nu\, dy^2\right)}{c^2} \\ -\frac{dx\, \nu\left(c-\nu\, dx^2-\nu\, dy^2\right)}{c^2} & -\frac{dy\, \nu\left(c-\nu\, dx^2-\nu\, dy^2\right)}{c^2} & \frac{\nu\left(c-\nu\, dx^2-\nu\, dy^2\right)^2}{c^2} \end{bmatrix}$$

Thus, the matrix P defines an idempotent A-endomorphism f of M.

We can check that the matrix $\Lambda \in A^{3\times 3}$ defined by

```
> Lambda := evalm(evalm([[-1,0,0],[0,-1,0],[nu*dx,nu*dy,nu*(nu*Delta-c)]])/c);
```

$$\Lambda := \begin{bmatrix} -\frac{1}{c} & 0 & 0 \\ 0 & -\frac{1}{c} & 0 \\ \frac{\nu\, dx}{c} & \frac{\nu\, dy}{c} & \frac{\nu\left(\nu\left(dx^2+dy^2\right)-c\right)}{c} \end{bmatrix}$$

satisfies the algebraic Riccati equation $\Lambda\, R\, \Lambda + (P - I_3)\, \Lambda + \Lambda\, Q + Z = 0$:

```
> simplify(evalm(Mult(Lambda,R,Lambda,A)+Mult(evalm(P-1),Lambda,A)
> +Mult(Lambda,P,A)+Z));
```

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, the idempotent endomorphism f of M can be defined by the idempotent matrix $\overline{P} := P + \Lambda\, R$

```
> P_bar := simplify(evalm(P+Mult(Lambda,R,A)));
```

$$P\_bar := \begin{bmatrix} 0 & 0 & -\frac{dx}{c} \\ 0 & 0 & -\frac{dy}{c} \\ 0 & 0 & 1 \end{bmatrix}$$

and the idempotent matrix $\overline{Q} := Q + R\, \Lambda$ defined by:

```
> Q_bar := simplify(evalm(P+Mult(R,Lambda,A)));
```

$$Q\_bar := \begin{bmatrix} \frac{\nu\, dx^2}{c} & \frac{\nu\, dx\, dy}{c} & -\frac{dx\, \nu\, (c-\nu\, dx^2-\nu\, dy^2)}{c} \\ \frac{\nu\, dx\, dy}{c} & \frac{\nu\, dy^2}{c} & -\frac{dy\, \nu\, (c-\nu\, dx^2-\nu\, dy^2)}{c} \\ -\frac{dx}{c} & -\frac{dy}{c} & \frac{c-\nu\, dx^2-\nu\, dy^2}{c} \end{bmatrix}$$

Let us check that we have $\overline{P}^2 = \overline{P}$:

```
> simplify(evalm(Mult(P_bar,P_bar,A)-P_bar));
```

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Let us also check that we have $\overline{Q}^2 = \overline{Q}$:

```
> simplify(evalm(Mult(Q_bar,Q_bar,A)-Q_bar));
```

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Moreover, let us check that we have $R\,\overline{P} = \overline{Q}\,R$:

```
> simplify(evalm(Mult(R,P_bar,A)-Mult(Q_bar,R,A)));
```

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, the matrix R is equivalent to a block diagonal matrix. Let us compute this block diagonal matrix.

Since $\overline{P}$ and $\overline{Q}$ are idempotent matrices of $A^{3\times 3}$, we know that $\ker_A(.\overline{P})$ and $\ker_A(.\overline{Q})$ are projective, i.e., free A-modules by the Quillen-Suslin theorem (see 2 of Theorem 2). Let us compute a basis of $\ker_A(.\overline{P})$ and of $\ker_A(.\overline{Q})$. To do that, let us first compute $\ker_A(.\overline{P})$ and of $\ker_A(.\overline{Q})$.

```
> X := SyzygyModule(P_bar,A);
```

$$X := \begin{bmatrix} c & 0 & dx \\ -dy & dx & 0 \\ 0 & c & dy \end{bmatrix}$$

We obtain $\ker_A(.\overline{P}) = \mathrm{im}_A(.X)$. Now, let us compute $\ker_A(.\overline{Q})$.

```
> Y := SyzygyModule(Q_bar,A);
```

$$Y := \begin{bmatrix} 1 & 0 & \nu\, dx \\ -dy & dx & 0 \\ 0 & 1 & \nu\, dy \end{bmatrix}$$

We obtain that $\ker_A(.\overline{Q}) = \mathrm{im}_A(.Y)$. Now, we can check that neither X nor Y has full row rank:

```
> SyzygyModule(X,A);
```

$$\begin{bmatrix} -dy & -c & dx \end{bmatrix}$$

```
> SyzygyModule(Y,A);
```

$$\begin{bmatrix} -dy & -1 & dx \end{bmatrix}$$

In particular, X (resp., Y) does not define a basis of $\ker_A(.\overline{P})$ (resp., $\ker_A(.\overline{Q})$). But, from the last but one matrix, we obtain that the second row of X is a A-linear combination of the first and third rows of X. Thus, if we consider the matrix $U_1 \in A^{2\times 3}$ formed by the first and third rows of X, i.e.,

```
>  U1 := submatrix(X,[1,3],1..3);
```

$$U1 := \begin{bmatrix} c & 0 & dx \\ 0 & c & dy \end{bmatrix}$$

then U_1 defines a basis of $\ker_A(.\overline{P})$, i.e., $\ker_A(.\overline{P}) = \operatorname{im}_A(.U_1)$. Similarly, we can check that the second row of Y is a A-linear combination of the first and third rows of Y. Thus, if we consider the matrix $V_1 \in A^{2\times 3}$ formed by the first and third rows of Y, i.e.,

```
>  V1 := submatrix(Y,[1,3],1..3);
```

$$V1 := \begin{bmatrix} 1 & 0 & \nu\, dx \\ 0 & 1 & \nu\, dy \end{bmatrix}$$

then V_1 defines a basis of $\ker_A(.\overline{Q})$, i.e., $\ker_A(.\overline{Q}) = \operatorname{im}_A(.V_1)$. The matrices U_1 and V_1 can directly be computed by the QuillenSuslin package (Fabiańska and Quadrat (2007)).

Now, let us compute a basis of $\operatorname{im}_A(.\overline{P}) = \ker_A(.(I_3 - \overline{P}))$.

```
>  U2 := SyzygyModule(evalm(1-P_bar),A);
```

$$U2 := \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

Thus, we have $\operatorname{im}_A(.\overline{P}) = \operatorname{im}_A(.U_2)$, where U_2 has full row rank, i.e., U_2 defines a basis of $\operatorname{im}_A(.\overline{P})$. Similarly, let us compute a basis of $\operatorname{im}_A(.\overline{Q}) = \ker_A(.(I_3 - \overline{Q}))$.

```
>  V2 := SyzygyModule(evalm(1-Q_bar),A);
```

$$V2 := \begin{bmatrix} dx & dy & -c + \nu\, dx^2 + \nu\, dy^2 \end{bmatrix}$$

Thus, we have $\operatorname{im}_A(.\overline{Q}) = \operatorname{im}_A(.V_2)$, where V_2 has full row rank, i.e., V_2 defines a basis of $\operatorname{im}_A(.\overline{Q})$. Now, if we define the matrix $U := (U_1^T \quad U_2^T)^T$, i.e.,

```
>  U := stackmatrix(U1,U2);
```

$$U := \begin{bmatrix} c & 0 & dx \\ 0 & c & dy \\ 0 & 0 & 1 \end{bmatrix}$$

and the matrix $V := (V_1^T \quad V_2^T)^T$ defined by

```
>  V := stackmatrix(V1,V2);
```

$$V := \begin{bmatrix} 1 & 0 & \nu\, dx \\ 0 & 1 & \nu\, dy \\ dx & dy & -c + \nu\, dx^2 + \nu\, dy^2 \end{bmatrix}$$

then we can check that these two matrices belong to $\mathrm{GL}_3(A)$:

```
>  U_inv := LeftInverse(U,A);
```

$$U\_inv := \begin{bmatrix} \frac{1}{c} & 0 & -\frac{dx}{c} \\ 0 & \frac{1}{c} & -\frac{dy}{c} \\ 0 & 0 & 1 \end{bmatrix}$$

```
>  V_inv := LeftInverse(V,A);
```

$$V\_inv := \begin{bmatrix} \frac{-\nu\, dx^2+c}{c} & -\frac{\nu\, dx\, dy}{c} & \frac{\nu\, dx}{c} \\ -\frac{\nu\, dx\, dy}{c} & \frac{c-\nu\, dy^2}{c} & \frac{\nu\, dy}{c} \\ \frac{dx}{c} & \frac{dy}{c} & -\frac{1}{c} \end{bmatrix}$$

Finally, the matrix R is equivalent to the block diagonal matrix $\overline{R} = V\, R\, U^{-1}$ defined by:

```
> R_bar := Mult(V,R,U_inv,A);
```

$$R\_bar := \begin{bmatrix} \frac{c-\nu\, dy^2}{c} & \frac{\nu\, dx\, dy}{c} & 0 \\ \frac{\nu\, dx\, dy}{c} & \frac{-\nu\, dx^2+c}{c} & 0 \\ 0 & 0 & dx^2+dy^2 \end{bmatrix}$$

Let us now study whether or not the first block diagonal matrix $T \in A^{2\times 2}$ of $\overline{R}$ is equivalent to a block diagonal matrix.

```
> T := submatrix(R_bar,1..2,1..2);
```

$$T := \begin{bmatrix} \frac{c-\nu\, dy^2}{c} & \frac{\nu\, dx\, dy}{c} \\ \frac{\nu\, dx\, dy}{c} & \frac{-\nu\, dx^2+c}{c} \end{bmatrix}$$

Let us note $O := A^{1\times 2}/(A^{1\times 2}\, T)$. We first compute a presentation of the A-module $\mathrm{end}_A(O)$.

```
> E := MorphismsConstCoeff(T,T,A):
```

We obtain that a family of generators of $\mathrm{end}_A(O)$ is defined by $\{g_i\}_{i=1,\ldots,4}$, where $g_i(\kappa(\nu)) = \kappa(\nu\, P_i)$, $\kappa : A^{1\times 2} \longrightarrow O$ is the canonical projection and the matrices P_i's are defined by:

```
> E[1]:
```

$$\left[\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & dy\, c \\ 0 & -dx\, c \end{bmatrix}, \begin{bmatrix} 0 & -\nu\, dy^2 \\ 0 & \nu\, dx\, dy \end{bmatrix}, \begin{bmatrix} 0 & \nu\, dx\, dy \\ 0 & -c+\nu\, dy^2 \end{bmatrix}\right]$$

The A-linear relations among the generators g_i's of the A-module $\mathrm{end}_A(O)$ are defined by:

```
> E[2];
```

$$\begin{bmatrix} dx\, c & 1 & 0 & 0 \\ -c+\nu\, dy^2 & 0 & 0 & -1 \\ 0 & \nu\, dy & c & 0 \\ 0 & \nu\, dx & 0 & -c \\ 0 & c & dy\, c & -dx\, c \end{bmatrix}$$

Hence, we get $\mathrm{end}_A(O) \cong A^{1\times 4}/(A^{1\times 5}\, E_2)$, i.e., E_2 is a presentation matrix of $\mathrm{end}_A(O)$. Let us check that $\mathrm{end}_A(O)$ is a cyclic A-module generated by $g_1 = \mathrm{id}_O$. If we define the following vector

```
> lambda := evalm([[1,0,0,0]]);
```

$$\lambda := \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$$

corresponding to a representative of $g_1 = \mathrm{id}_O$ in the A-module $\mathrm{end}_A(O) \cong A^{1\times 4}/(A^{1\times 5}\, E_2)$, then we can check that the matrix $(\lambda^T \quad E_2^T)^T$ admits the following left inverse

```
> LeftInverse(stackmatrix(lambda,E[2]),A);
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -dx\, c & 1 & 0 & 0 & 0 & 0 \\ \nu\, dx\, dy & -\frac{\nu\, dy}{c} & 0 & \frac{1}{c} & 0 & 0 \\ -c+\nu\, dy^2 & 0 & -1 & 0 & 0 & 0 \end{bmatrix}$$

which shows that g_1 generates the A-module $\mathrm{end}_A(O)$. This result can directly be obtained by using the command `ReducedModuleHom`:

```
>  ReducedModuleHom(T,T,E[1],A);
```

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Let us now compute the A-linear relations of g_1, i.e., its annihilator:

```
>  Rp := collect(RelationsMatrix(T,T,[E[1][1]],A),[nu,c]);
```

$$Rp := \left[\; c\,\nu\,\left(dx^2+dy^2\right)-c^2 \;\right]$$

We obtain that $\mathrm{end}_A(O) \cong A/(A\,(c\,(\nu\,\Delta - c)))$. From that, following the arguments developed in Section 7.2, we can easily check that $\mathrm{end}_A(O)$ does not admit non-trivial idempotents, which proves that O is an indecomposable A-module.

8.5 Rotating fluid

Let us consider the ring A of PD operators with coefficients in the field $\mathbb{Q}(\rho_0, \Omega_0)$, i.e.,

```
>  A := DefineOreAlgebra(diff=[dt,t],diff=[d1,x1],diff=[d2,x2],diff=[d3,x3],
>  polynom=[t,x1,x2,x3],comm=[rho0,Omega0]):
```

and the matrix $R \in A^{4\times 4}$ defined by (85), i.e.,

```
>  R := evalm([[rho0*dt,-2*rho0*Omega0,0,d1],[2*rho0*Omega0,rho0*dt,0,d2],
>  [0,0,rho0*dt,d3],[d1,d2,d3,0]]);
```

$$R := \begin{bmatrix} \rho 0\, dt & -2\,\rho 0\,\Omega 0 & 0 & d1 \\ 2\,\rho 0\,\Omega 0 & \rho 0\, dt & 0 & d2 \\ 0 & 0 & \rho 0\, dt & d3 \\ d1 & d2 & d3 & 0 \end{bmatrix}$$

which finitely presents the A-modue $M = A^{1\times 4}/(A^{1\times 4}\,R)$. Let us characterize the A-module $\mathrm{end}_A(M)$.

```
>  E := MorphismsConstCoeff(R,R,A):
```

We obtain that the A-module $\mathrm{end}_A(M)$ is generated by $f_i(\pi(\lambda)) = \pi(\lambda\, P_i)$, where $\pi : A^{1\times 4} \longrightarrow M$ is the canonical projection and the matrices P_i are defined by:

```
>  P := E[1];
```

$$P := [\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & d3 & -d2 & 0 \\ -d3 & 0 & d1 & 0 \\ d2 & -d1 & 0 & 0 \\ 0 & 0 & 2\,\rho0\,\Omega0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & d2 & d3 & 0 \\ 0 & -d1 & 0 & 0 \\ 0 & 0 & -d1 & 0 \\ 0 & 0 & 0 & -d1 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 2\,\rho0\,\Omega0 & 0 & -d1 \\ -2\,\rho0\,\Omega0 & 0 & 0 & -d2 \\ 0 & 0 & 0 & -d3 \\ 0 & 0 & 0 & \rho0\,dt \end{bmatrix}, \begin{bmatrix} 0 & -d2\,d1 & -d1\,d3 & 0 \\ 0 & d1^2 & 0 & 0 \\ 0 & 0 & d1^2 & 0 \\ -2\,\rho0\,\Omega0\,d2 & 2\,\Omega0\,d1\,\rho0 & 0 & -d2^2-d3^2 \end{bmatrix}]$$

The A-linear relations among the generators $f_1, \ldots, f_5$ of $\mathrm{end}_A(M)$ are defined by:

```
> L := E[2];
```

$$L := \begin{bmatrix} d1 & 1 & 0 & 0 & 0 \\ 2\,\rho0\,\Omega0\,d3 & 0 & \rho0\,dt & 0 & 0 \\ \rho0\,dt & 0 & 0 & -1 & 0 \\ 0 & -d1 & 0 & 0 & -1 \\ 0 & \rho0\,dt & 0 & d1 & 0 \\ 0 & 0 & 2\,\rho0\,\Omega0\,d3 & -d2^2-d3^2 & -\rho0\,dt \end{bmatrix}$$

We obtain $\mathrm{end}_A(M) \cong A^{1\times 5}/(A^{1\times 6}\,L)$. From the A-linear relations among the generators of $\mathrm{end}_A(M)$, we obtain $f_2 = -d_1\,f_1$, $f_4 = \rho_0\,d_t\,f_1$, $f_5 = -d_1\,f_2 = d_1^2\,f_1$. Hence, the A-module $\mathrm{end}_A(M)$ can only be generated by $f_1 = \mathrm{id}_M$ and f_3. Let us compute the A-linear relations among f_1 and f_3 to obtain a smaller presentation matrix for the A-module $\mathrm{end}_A(M)$.

```
> Lp := collect(RelationsMatrix(R,R,[E[1][1],E[1][3]],A),[rho0,dt]);
```

$$Lp := \begin{bmatrix} 2\,\rho0\,\Omega0\,d3 & \rho0\,dt \\ (d3^2+d2^2+d1^2)\,dt\,\rho0 & -2\,\rho0\,\Omega0\,d3 \end{bmatrix}$$

We get $\mathrm{end}_A(M) \cong A^{1\times 2}/(A^{1\times 2}\,L')$, where L' is the above matrix and the two generators of $\mathrm{end}_A(M)$ are f_1 and f_3. Note that the above fact can be obtained directly using the `ReducedModuleHom` command:

```
> GenFam:=ReducedModuleHom(R,R,E,A);
```

$$[\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & d3 & -d2 & 0 \\ -d3 & 0 & d1 & 0 \\ d2 & -d1 & 0 & 0 \\ 0 & 0 & 2\,\rho0\,\Omega0 & 0 \end{bmatrix}]$$

```
> collect(RelationsMatrix(R,R,GenFam,A),[rho0,dt]);
```

$$\begin{bmatrix} -2\,\Omega0\,d3\,\rho0 & -\rho0\,dt \\ (d1^2+d2^2+d3^2)\,dt\,\rho0 & -2\,\Omega0\,d3\,\rho0 \end{bmatrix}$$

Finally, if we note $B = \mathbb{Q}(\rho_0, \Omega_0, d_t)[d_1, d_2, d_3]$,

```
> B := DefineOreAlgebra(diff=[d1,x1],diff=[d2,x2],diff=[d3,x3],polynom=[x1,x2,x3],
> comm=[dt,rho0,Omega0]):
```

then let us prove that the $B \otimes_A \mathrm{end}_A(M) \cong B^{1\times 2}/(B^{1\times 2}\,L')$ is a cyclic B-module generated by $\mathrm{id}_{B\otimes_A M}$. If we consider the following vector which is the representative of $\mathrm{id}_{B\otimes_A M}$ in $B \otimes_A \mathrm{end}_A(M)$.

```
> Lambda := evalm([[1,0]]);
```

$$\Lambda := \begin{bmatrix} 1 & 0 \end{bmatrix}$$

and define the matrix $P = (\Lambda^T \quad L'^T)^T$, i.e.,

```
> P := stackmatrix(Lambda,Lp);
```

$$P := \begin{bmatrix} 1 & 0 \\ 2\,\rho0\,\Omega0\,d3 & \rho0\,dt \\ \left(d3^2 + d2^2 + d1^2\right) dt\,\rho0 & -2\,\rho0\,\Omega0\,d3 \end{bmatrix}$$

then we can check that P admits a left inverse $S \in B^{2\times 3}$ defined by

```
> S := LeftInverse(P,B);
```

$$S := \begin{bmatrix} 1 & 0 & 0 \\ -2\,\frac{\Omega0\,d3}{dt} & \frac{1}{\rho0\,dt} & 0 \end{bmatrix}$$

which proves that $B \otimes_A \mathrm{end}_A(M) \cong B^{1\times 2}/(B^{1\times 2}\,L')$ is a cyclic B-module generated by $\mathrm{id}_{B\otimes_A M}$. Let us now compute the annihilator $\mathrm{ann}_B(\mathrm{id}_{B\otimes_A M}) = \{b \in B \mid b\,\mathrm{id}_{B\otimes_A M} = 0\}$ of $\mathrm{id}_{B\otimes_A M}$. To do that, let us first factorize L' by P.

```
> F := Factorize(Lp,P,B);
```

$$F := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We get $L' = F\,P$. Now, let us compute $\ker_A(.P)$.

```
> P2 := collect(SyzygyModule(P,B),[rho0,dt]);
```

$$P2 := \begin{bmatrix} \left(\left(d3^2 + d2^2 + d1^2\right) dt^2 + 4\,\Omega0^2\,d3^2\right)\rho0 & -2\,\Omega0\,d3 & -dt \end{bmatrix}$$

We obtain $\ker_A(.P) = \mathrm{im}_A(.P_2)$. If we define the matrix $Q = (F^T \quad P_2^T)^T$, i.e.,

```
> Q := stackmatrix(F,P2);
```

$$Q := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \left(\left(d3^2 + d2^2 + d1^2\right) dt^2 + 4\,\Omega0^2\,d3^2\right)\rho0 & -2\,\Omega0\,d3 & -dt \end{bmatrix}$$

then we have $B \otimes_A \mathrm{end}_A(M) \cong B^{1\times 3}/(B^{1\times 3}\,Q) \cong B/(B\,b)$, where $b := Q_{31} = -\frac{1}{\rho_0}\,\det(R)$ since we have:

```
> collect(-det(R),[rho0,dt]);
```

$$\left(\left(d3^2 + d2^2 + d1^2\right) dt^2 + 4\,\Omega0^2\,d3^2\right)\rho0^2$$

This result can directly be obtained as follows:

```
> collect(RelationsMatrix(R,R,[E[1][1]],A),[rho0,dt]);
```

$$\begin{bmatrix} \left(\left(d3^2 + d2^2 + d1^2\right) dt^2 + 4\,\Omega0^2\,d3^2\right)\rho0 \end{bmatrix}$$

Finally, following the arguments developed in Section 7.3, we can prove that M is an indecomposable A-module.

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RESEARCH CENTRE
SACLAY – ÎLE-DE-FRANCE

Parc Orsay Université
4 rue Jacques Monod
91893 Orsay Cedex

Publisher
Inria
Domaine de Voluceau - Rocquencourt
BP 105 - 78153 Le Chesnay Cedex
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