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## ► To cite this version:

Marie Doumic, Miguel Escobedo. Time Asymptotics for a Critical Case in Fragmentation and Growth-Fragmentation Equations. 2015. hal-01080361v2

**HAL Id: hal-01080361**

**<https://inria.hal.science/hal-01080361v2>**

Preprint submitted on 19 Jan 2015 (v2), last revised 13 Oct 2015 (v3)

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# Time Asymptotics for a Critical Case in Fragmentation and Growth-Fragmentation Equations

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January 19, 2015

## Abstract

Fragmentation and growth-fragmentation equations is a family of problems with varied and wide applications. This paper is devoted to description of the long time time asymptotics of two critical cases of these equations, when the division rate is constant and the growth rate is linear or zero. The study of these cases may be reduced to the study of the following fragmentation equation:

$$\frac{\partial}{\partial t}u(t, x) + u(t, x) = \int_x^\infty k_0\left(\frac{x}{y}\right)u(t, y)dy.$$

Using the Mellin transform of the equation, we determine the long time behavior of the solutions. Our results show in particular the strong dependence of this asymptotic behavior with respect to the initial data.

**MSC Classification:** 35B40 Partial differential equations, Asymptotic behavior of solutions, 35Q92 PDEs in connection with biology and other natural sciences, 45K05 integro-partial differential equations, 92D25 Population dynamics [see also 92C37, 82D60]

**Keywords:** Structured populations; growth-fragmentation equations; cell division; self-similarity; long-time asymptotics; rate of convergence

**Short title:** A Critical Case in Fragmentation Models

## Introduction

Fragmentation and growth-fragmentation equations is a family of problems with varied and wide applications: phase transition, aerosols, polymerization processes, bacterial growth, chemostat etc. [5, 7, 9, 16, 17]. That explains the continuing interest they meet. This paper is devoted to the description of the long time asymptotic behavior of two critical cases of these equations that have been left open in the previous literature.

Under its general form, the linear growth-fragmentation equation may be written as follows.

$$\frac{\partial}{\partial t}u(t, x) + \frac{\partial}{\partial x}(\tau(x)u(t, x)) + B(x)u(t, x) = \int_x^\infty k(y, x)B(y)u(t, y)dy, \quad (1)$$

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where  $u$  represents the concentration of particles of size  $x$  at time  $t$ ,  $\tau$  their growth speed (equal to zero in the case of the pure fragmentation equation),  $B(x)$  the total fragmentation rate of particles of size  $x$ , and  $\frac{1}{2}k(y, x)$  the probability of a particle of size  $y$  to break into two particles of respective sizes  $x$  and  $y - x$ . For the sake of simplicity, we focus here on binary fragmentation, where each agent splits into two parts, but generalization to  $k$  fragments does not present any difficulty. In the case when  $\tau \equiv 0$ , Equation 1 is the pure linear fragmentation equation.

Existence, uniqueness and asymptotic behaviour have been studied and improved in many articles, let us refer to [15] for a most recent one, which also presents an exhaustive review of the literature. Let us indicate only the results most closely related to our work.

- In the case  $B(x) = x^\gamma$  and  $\tau(x) = x^\nu$ , existence and uniqueness of a steady nonnegative profile  $(\mathcal{U}, \lambda)$  with  $\lambda > 0$ ,  $\mathcal{U}(x > 0) > 0$ ,  $\mathcal{U} \in L^1(xdx)$  and trend of  $u(t, x)e^{-\lambda t}$  towards  $\mathcal{U}$  for an appropriate weighted norm [14] is established in [13], under the assumption  $1 + \gamma - \nu > 0$ . This profile  $\mathcal{U}$  and  $\lambda$  are solutions of the eigenvalue problem

$$\begin{cases} \frac{\partial}{\partial x}(\tau(x)\mathcal{U}(x)) + (B(x) + \lambda)\mathcal{U}(x) = \int_x^\infty k(y, x)B(y)\mathcal{U}(y)dy, & x \geq 0 \\ \tau\mathcal{U}(x=0) = 0, \quad \mathcal{U}(x) \geq 0, \quad \int_0^\infty \mathcal{U}(x)dx = 1. \end{cases} \quad (2)$$

Though this result may be generalized or refined in different directions [6, 4], the assumption linking  $B$  and  $\tau$  for  $x$  vanishing or tending to infinity remains of the same kind: if  $B(x) \sim x^\gamma$  and  $\tau(x) \sim x^\nu$ , it is necessary that  $1 + \gamma - \nu > 0$ . This may be understood as a balance between growth and division: enough growth is necessary in the neighbourhood of zero to counterbalance fragmentation, whereas enough division for large  $x$  is necessary to avoid mass loss to infinity. We refer to [6] for more details; in particular, counter-examples may be given where no steady profile exist if one of the assumptions is not fulfilled.

- The fragmentation equation, *i.e.* when  $\tau \equiv 0$ ,

$$\frac{\partial}{\partial t}u(t, x) + B(x)u(t, x) = \int_x^\infty k(y, x)B(y)u(t, y)dy, \quad (3)$$

was considered in [8] and [11] for a total fragmentation rate of the form  $B(x) = x^\gamma$  and for

$$\begin{aligned} k(x, y) &= \frac{1}{x}k_0\left(\frac{y}{x}\right); \quad k_0 \text{ is a non negative measure,} \\ \text{supp } k_0 &\subset [0, 1], \quad \int_0^1 zk_0(z)dz = 1, \quad \int_0^1 k_0(z)dz > 1 \end{aligned} \quad (4)$$

(i) When  $\gamma > 0$ , it was proved in [8] that for initial data  $u_0$  in  $L^1(xdx)$  the function  $xu(t, x)$  converges to a Dirac measure, and it does so in a self similar way. (This term is used here and in all the following in a slightly different way than in [11], see below in Section 3.1 for its precise meaning.) The self similar profiles are solutions of a particular case of equation (2), with  $\tau(x) = x$ , so  $\nu = 1$ , and  $\lambda = 1$ . The condition  $\gamma > 0$  may then be seen as  $\gamma + 1 - \nu > 0$  again.

(ii) For  $\gamma < 0$ , it was shown in [11] that the behaviour is not self similar and strongly depends on the initial data  $u_0$ . The precise convergence in the sense of measure of suitably re scalings of the solutions where proved for initial data of compact support or with exponential and algebraic decay at infinity.

- Fragmentation equations may be seen as deterministic linear rate equations that describe the mass distribution of the particles involved in a fragmentation phenomenon when such phenomenon is described by a “fragmentation” stochastic process (see [2]). The homogeneous self similar fragmentation process, whose associated deterministic rate equation for the mass distribution of particles is the fragmentation equation (3), (4) with  $\gamma = 0$ , has been studied in [1]. We present in Section 4.4 a brief description of some of the results obtained in [1] and their relation with ours, in particular Corollary 1.4.

In this article, we investigate the equation (1) when  $\gamma = 0$ ,  $\nu = 1$  and the function  $k(x, y)$  is still given by (4). This is one critical case where  $\gamma + 1 - \nu = 0$ . It is already known that under such conditions, there is no solution to the eigenvalue problem (2) in the case of homogeneous fragmentation ( $k_0 \equiv 2$ ) or if  $\tau(x) = cx$  and  $B(x) = B$  with two constants  $c, B > 0$  and  $c \neq B$  [6]. On the other hand, for the fragmentation equation, if  $\gamma = 0$  the arguments of [8] based on a suitable scaling of the variables break down. How can we expect to characterize the asymptotic behaviour of the population in that case? Our first remark is that under such conditions on  $\gamma$  and  $\nu$  the aggregation fragmentation equation and the fragmentation equation are related by a very simple change of dependent variable. Suppose that  $u$  satisfies:

$$\frac{\partial}{\partial t}u + u = \int_x^\infty \frac{1}{y}k_0\left(\frac{x}{y}\right)u(t, y)dy, \quad (5)$$

$$u(0, x) = u_0(x), \quad (6)$$

then the function

$$v(t, x) = e^{-ct}u(t, xe^{-ct}) \quad (7)$$

satisfies

$$\frac{\partial}{\partial t}v + \frac{\partial}{\partial x}(cxv) + v = \int_x^\infty \frac{1}{y}k_0\left(\frac{x}{y}\right)v(t, y)dy, \quad v(t = 0, x) = u_0(x). \quad (8)$$

From the behaviour of one of them it is then easy to deduce the behaviour of the other.

An important quantity for the solutions of the fragmentation equation (5) is the following:

$$M(t) = \int_0^\infty xu(t, x)dx \quad (9)$$

called sometimes the mass of the solution  $u$  at time  $t$ . After multiplication of the equation (5) by  $x$ , integration on  $(0, \infty)$  and applying Fubini's theorem, if all these operations are well defined, it follows that,

$$\frac{d}{dt}M(t) = 0. \quad (10)$$

All the solutions of (5) considered in this work satisfy that property (c.f. Theorem 2.1). Since on the other hand, in the pure fragmentation equation the particles may only fragment into smaller ones, it

is natural to expect  $xu(t, x)$  to converge to a Dirac mass at the origin as  $t \rightarrow \infty$ . This property is proved in Theorem 3.1 below under suitable conditions on the initial data.

The main goal of this work is to determine how does this convergence take place. Our first observation (in Theorem 3.2 below) is that equation (5) has no solution of the form  $w(t, x) = f(t)\Phi(xg(t))$  such that  $\Phi \in L^1(xdx) \cap L^\alpha(xdx)$  for some  $\alpha > 1$ . (Although it has a one parameter family of self similar solutions that do not satisfy these conditions.) Our main results actually show that the long time asymptotic behaviour of the solutions of (5), (6) strongly depend on their initial data. It also appears that this dependence is determined by the measure  $k_0$ . This is seen by exhibiting a large set of initial data, for which the solutions to equation (5) are given by means of the Mellin transform and where such a dependence is seen very explicitly.

## 1 Assumptions and Main Results

### 1.1 Representation formula by the Mellin transform

The solutions of equation (5) may be explicitly computed for a large class of initial data by means of the Mellin transform. Given a function  $f$  defined on  $(0, \infty)$ , its Mellin transform is defined as follows:

$$\mathcal{M}_f(s) = \int_0^\infty x^{s-1} f(x) dx \quad (11)$$

whenever this integral converges. In the following we denote, for the sake of simplicity

$$U(t, s) := \mathcal{M}_{u(t, \cdot)}(s).$$

It is easy to check that if we multiply all the terms of equation (5) by  $x^{s-1}$  and integrate on  $(0, \infty)$ , assuming that all the integrals converge and Fubini's theorem may be applied, we obtain:

$$\frac{\partial}{\partial t} U(t, s) + U(t, s) = K(s) U(t, s) \quad (12)$$

where, by (4),

$$K(s) = \mathcal{M}_{k_0}(s) = \int_0^1 k_0(z) z^{s-1} dz \quad (13)$$

is continuous on  $\Re(s) \geq 1$ , analytic in  $\Re(s) > 1$  with  $K(2) = 1$  and  $K(1) > 1$  due to (4). We have then, formally at least:

$$U(t, s) = U_0(s) e^{(K(s)-1)t} \quad (14)$$

$$U_0(s) = \mathcal{M}_{u_0}(s). \quad (15)$$

It only remains, in principle, to invert to Mellin transform to recover  $u(t, x)$ . As it is well known, in order to have an explicit formula for the inverse Mellin transform, some hypothesis on  $U_0 = \mathcal{M}_{u_0}$  and  $K(s)$  are needed. If such conditions are fulfilled then,

$$u(t, x) = \frac{1}{2\pi i} \int_{\nu-i\infty}^{\nu+i\infty} U_0(s) e^{(K(s)-1)t} x^{-s} ds \quad (16)$$

for some  $\nu \in \mathbb{R}$  suitably choosed. Theorem 2.1 in Section 2 below provides a rigorous setting where (16) holds true. Our results on the long time behaviour of the solutions to (5) are based on this explicit expression of these solutions.

## 1.2 Asymptotic formula

Since we are mainly interested in the asymptotic behaviour of the solution as  $t \rightarrow \infty$  it is enough to be able to extract such information from the inverse Mellin transform in (16). In order to understand the long time behaviour of the function  $u$ , it is readily seen on equation (16) that a key function is

$$\phi(s, t, x) = -s \log(x) + tK(s). \quad (17)$$

We are then led to consider different regions of the real half line  $x > 0$ , determined by the different behaviour of the  $x$  variable with respect to  $t$ . It will be divided in several subdomains that are determined by the relative values of two parameters, that we shall denote as  $p_0$ ,  $q_0$ , with respect to a third, that we call  $s_+$ . The two first,  $p_0$  and  $q_0$ , only depend on the initial data  $u_0$ . The third parameter  $s_+ = s_+(t, x)$  depends on the kernel  $k_0$  as well as on  $t$  and  $x$ . In order to define these three parameters we need to precise the initial data  $u_0$  and the kernel  $k_0$  that we shall consider.

The initial data  $u_0$  will be supposed to be a function, satisfying first the following conditions

$$u_0 \geq 0, \quad \int_0^\infty u_0(x)(1+x)dx < \infty. \quad (18)$$

By condition (18), the Mellin transform of the initial data  $\mathcal{M}(0, \alpha)$  is analytic on the strip  $\Re(\alpha) \in (1, 2)$  at least. Let us denote:

$$I(u_0) = \left\{ p \in \mathbb{R}; \quad \int_0^\infty u_0(x)x^{p-1}dx < \infty \right\}. \quad (19)$$

This is necessarily an interval of  $\mathbb{R}$  and, by hypothesis (18),  $[1, 2] \subset I(u_0)$ . We then define:

$$p_0 = \inf\{p; p \in I(u_0)\} \quad (20)$$

$$q_0 = \sup\{q; q \in I(u_0)\}. \quad (21)$$

It follows from Fatou's Lemma that  $p_0 \notin I(u_0)$  and  $q_0 \notin I(u_0)$ . By (18) we have  $p_0 \leq 1$  and  $q_0 \geq 2$ . Next, we assume:

$$\forall \nu \in (p_0, q_0) : \int_0^\infty |u_0'(x)|x^\nu dx < \infty, \quad \lim_{x \rightarrow 0} x^\nu u_0(x) = \lim_{x \rightarrow \infty} x^\nu u_0(x) = 0, \quad (22)$$

$$\forall \nu \in (p_0, q_0) : \int_0^\infty |u_0''(x)|x^{1+\nu} dx < \infty, \quad \lim_{x \rightarrow 0} x^{1+\nu} u_0'(x) = \lim_{x \rightarrow \infty} x^{1+\nu} u_0'(x) = 0. \quad (23)$$

These hypothesis will be used in order to have the explicit expression (16) for the solution  $u$  of (5), (6). Some of our results on the asymptotic behaviour need furthermore the following conditions on

the initial data.

$$\exists a_0 > 0, \exists r > q_0; \quad |u_0(x) - a_0 x^{-q_0}| \leq C x^{-r}, \quad \forall x > 1, \quad (24)$$

$$|u'_0(x) + a_0 q_0 x^{-q_0-1}| \leq C x^{-r-1} \quad \forall x > 1 \quad (25)$$

$$\int_1^\infty x^{\nu+1} |u''_0(x) - a_0 q_0(q_0 + 1) x^{-q_0-2}| dx < \infty, \quad \forall \nu \in (q_0, r). \quad (26)$$

$$\exists b_0 > 0, \exists \rho < p_0; \quad |u_0(x) - b_0 x^{-p_0}| \leq C x^{-\rho}, \quad \forall x \in (0, 1), \quad (27)$$

$$|u'_0(x) + b_0 p_0 x^{-p_0-1}| \leq C x^{-\rho-1} \quad \forall x \in (0, 1) \quad (28)$$

$$\int_0^1 x^{\nu+1} |u''_0(x) - b_0 p_0(p_0 + 1) x^{-p_0-2}| dx < \infty, \quad \forall \nu \in (\rho, p_0). \quad (29)$$

The kernel  $k(x, y)$  is defined by (4), from where we already saw that  $K(s)$  is analytic in  $(1, \infty)$ , continuous and strictly decreasing on  $[1, \infty)$ . We complete this assumption by considering, as for  $u_0$ , the integral  $I(k_0)$  defined by (19) and define

$$p_1 = \inf\{p; p \in I(k_0)\}. \quad (30)$$

By Fatou's Lemma,  $p_1 \notin I(k_0)$ . The relative position of  $p_0$  and  $p_1$  is discussed after Theorem 1.2. Under these conditions on  $k_0$ , it may be checked that the function  $K$  is strictly convex on  $(p_1, +\infty)$  and that for any  $t > 0$ ,  $x > 0$  there exists a unique  $s = s_+(t, x)$  such that

$$s_+(t, x) = (K')^{-1} \left( \frac{\log x}{t} \right). \quad (31)$$

(c.f. Lemma 6 in the Appendix).

We may now come to the main results of this work, that is the description of the long time behaviour of the solutions  $u$  of (5)(6) given by (16).

The first region where the behaviour of  $u$  may be easily described is where  $x > 1$ . That is because for  $t > 0$  and  $x > 1$ , the function  $\phi(s, t, x)$  is decreasing with respect to the real part of  $s$ . We have then in that region

**Theorem 1.1** *Suppose that  $u_0$  satisfies (18), (22)-(26). Then, there exists  $T$  large enough such that*

$$u(t, x) = a_0 x^{-q_0} e^{(K(q_0)-1)t} \left( 1 + \mathcal{O} \left( e^{(K(q_0+\delta)-K(q_0))T} \right) \right) \quad \forall t > T, \forall x > 1. \quad (32)$$

As it is shown in Lemma 6, when  $x \in [0, 1]$  the behaviour of the function  $\phi$ , and as a consequence that of the solution  $u(t, x)$ , is not so simple. Let us first describe the behaviour of the solutions of (5)(6) in the region where  $x \in [0, 1]$  and the balance between  $x$  and  $t$  leads to a result similar to Theorem 1.1.

**Theorem 1.2** *Suppose that  $u_0$  satisfies (18), (22), (23),  $k_0$  satisfies (4), (30),  $s_+$  be defined in (31).*

*Suppose moreover that  $u_0$  satisfies (27)-(29). Then,*

$$(i) \quad u(t, x) = b_0 x^{-p_0} e^{(K(p_0)-1)t} \left( 1 + \mathcal{O} \left( x^{-\nu+p_0} e^{(K(\nu)-K(p_0))t} \right) \right) \text{ as } t \rightarrow \infty, \text{ and } s_+(t, x) < p_0$$

*for any  $\nu \in (\rho, p_0)$  with  $\rho$  given in (27).*

Suppose moreover that  $u_0$  satisfies (24)–(26). Then,

$$(ii) \quad u(t, x) = a_0 x^{-q_0} e^{(K(q_0)-1)t} \left( 1 + \mathcal{O} \left( x^{-\nu'+q_0} e^{(K(\nu')-K(q_0))t} \right) \right) \quad \text{as } t \rightarrow \infty, \text{ and } q_0 < s_+(t, x).$$

for any  $\nu' \in (q_0, r)$  with  $r$  given in (24).

Notice that if  $p_0 \leq p_1$ , the point (i) of Theorem 1.2 never happens.

Given the results stated in Theorems 1.1 and 1.2, the only region which remains to study is the region  $(t, x)$  defined by  $p_0 < s_+(t, x) < q_0$ . To do so, it is necessary to distinguish a very particular type of singular discrete measures  $k_0$  whose support  $\Sigma$  satisfies the condition that is defined below.

**Condition H.** We say that a subset  $\Sigma$  of  $(0, 1)$  satisfies condition H if:

$$\begin{aligned} \exists L \in \mathbb{N}^* \cup \{+\infty\}, \quad \exists \theta \in (0, 1), \quad \exists (p_\ell)_{\ell \in \mathbb{N}, \ell \leq L}, \quad 0 < p_\ell < p_{\ell+1} \quad \forall \ell \in \mathbb{N}, \ell \leq L-1, \\ \Sigma = \{\sigma_\ell \in (0, 1); \sigma_\ell = \theta^{p_\ell}\}, \quad (p_\ell)_{0 \leq \ell \leq L} \text{ are setwise coprime.} \end{aligned}$$

Note that the assumption of a coprime and ordered sequence of integers is not a restriction (we may always order a given sequence of integers, and up to a change  $\theta$  to  $\theta^{gcd(p_\ell)}$ , we can choose a coprime sequence). See Propositions 1 to 3 in the Appendix for more properties of Condition H, which lead to define a number  $v_*$  and a set  $Q$  by

$$v_* = \frac{2\pi}{\log \theta}, \quad Q = v_* \mathbb{Z}. \quad (33)$$

Before we state the last main theorem let us remind that by the Lebesgue decomposition of a non negative bounded measure, the measure  $k_0$  may be decomposed as follows:

$$k_0(x) = g(x)dx + d\mu + d\nu$$

where  $g \in L^1(\mathbb{R}^+)$ ,  $d\mu$  is a singular continuous measure and  $d\nu$  is a singular discrete measure (cf. for example [12]).

**Theorem 1.3** Suppose that  $u_0$  satisfies (18), (22), (23). Then,

(a) If the measure  $k_0$  has non zero absolutely continuous part or if it is a discrete singular measure whose support does not satisfy the condition H and for which 1 is not a limit point of  $(\sigma_l)$ , then, as  $t \rightarrow +\infty$ ,  $\varepsilon \rightarrow 0$ ,  $t\varepsilon^2 \rightarrow \infty$ , and for all  $\delta > 0$  such that  $p_0 + \delta < s_+(t, x) < q_0 - \delta$ :

$$u(t, x) = x^{-s_+(t, x)} e^{(K(s_+(t, x))-1)t} \left( \frac{U_0(s_+(t, x))}{\sqrt{2\pi t K''(s_+)}} \Theta_1(t, \varepsilon) + \mathcal{O} \left( e^{-\gamma(\varepsilon)t} \right) \right) \quad (34)$$

$$\Theta_1(t, \varepsilon) = 1 + \omega(\varepsilon) \left( 1 + \mathcal{O} \left( \frac{e^{-\frac{\varepsilon^2}{2} t K''(q_0)}}{\sqrt{\varepsilon^2 t K''(q_0)}} \right) \right) \quad (35)$$

where  $\gamma(\varepsilon) \sim C\varepsilon^2$  is given by Lemma 3, and  $\omega(\varepsilon)$  defined in (68) satisfies

$$|\omega(\varepsilon)| \leq C(p_0 + \delta, q_0 - \delta) \left( \int_{R(\varepsilon)}^\infty \left| z^{q_0-\delta-1} u_0(z) \right| dz + \int_0^{\frac{1}{R(\varepsilon)}} \left| z^{p_0+\delta-1} u_0(z) \right| dz + \varepsilon \log R(\varepsilon) \right), \quad (36)$$

where the constant  $C(p_0 + \delta, q_0 + \delta)$  depends on the integrability properties of  $u_0$  (see (59)),  $R$  is any function of  $\varepsilon$  such that  $R(\varepsilon) \rightarrow \infty$ ,  $\varepsilon \log R(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ .

(b) If the measure  $k_0$  is a discrete singular measure whose support satisfies the condition  $H$  then, as  $t \rightarrow \infty$ ,  $\varepsilon \rightarrow 0$ ,  $t\varepsilon^2 \rightarrow \infty$  and  $p_0 < s_+(t, x) < q_0$ :

$$u(t, x) = x^{-s_+(t, x)} e^{(K(s_+(t, x)) - 1)t} \left( \frac{\sum_{k \in \mathbb{Z}} U_0(s_k) e^{-\frac{2i\pi k}{\log \theta} \log x}}{\sqrt{2\pi t K''(s_+)}} + \Theta_2(t, \varepsilon) + \mathcal{O}\left(e^{-\gamma(\varepsilon)t}\right) \right), \quad (37)$$

where  $s_k = s_+ + ikv_*$ ,  $v_*$  defined by (33),  $\theta$  defined in condition  $H$ ,  $\gamma(\varepsilon) \sim C\varepsilon^2$  is given by Lemma 4 and

$$\Theta_2(t, \varepsilon) = \left( 1 + \mathcal{O}\left( \frac{e^{-\frac{\varepsilon^2}{2} t K''(q_0)}}{\sqrt{\varepsilon^2 t K''(q_0)}} \right) \right) S(\varepsilon) \quad (38)$$

$$|S(\varepsilon)| \leq C(p_0 + \delta, q_0 - \delta) \left( (\varepsilon \log R(\varepsilon)) L(\varepsilon) + L(\varepsilon) \left( \int_{R(\varepsilon)}^{\infty} |z^{q_0 - \delta - 1} u_0(z)| dz + \int_0^{\frac{1}{R(\varepsilon)}} |z^{p_0 + \delta - 1} u_0(z)| dz \right) \right) \quad (39)$$

where  $L, R$  are any functions of  $\varepsilon$  such that  $L(\varepsilon) \rightarrow \infty$ ,  $R(\varepsilon) \rightarrow \infty$ ,

$L(\varepsilon) \left( \int_{R(\varepsilon)}^{\infty} z^{q_0 - \delta - 1} u_0(z) dz + \int_0^{\frac{1}{R(\varepsilon)}} z^{p_0 + \delta - 1} u_0(z) dz \right) \rightarrow 0$  and  $\varepsilon L(\varepsilon) \log R(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ .

**Remark 1.1 (The leading terms of the solution.)** The two functions  $a_0 x^{-q_0} e^{(K(q_0) - 1)t}$  and  $b_0 x^{-p_0} e^{(K(p_0) - 1)t}$  that appear in the long time behaviour of the solution  $u$  of (5), (6) in Theorem 1.1, and in Theorem 1.2 are self-similar solutions of the equation (cf. Remark 3.2).

Since  $q_0 > 0$  and  $K(q_0) < 1$ , it follows from Theorem 1.1 and point (ii) of Theorem 1.2 that, as  $t \rightarrow \infty$ , the leading terms of the solution  $u(t, x)$  decays exponentially fast uniformly in the domain  $x > 1$  and  $q_0 < s_+(t, x)$ . Since  $xu(t, x)$  converges to a Dirac mass at  $x = 0$ , the long time asymptotic behaviour of the leading term of  $u(t, x)$  for  $x \in (0, 1)$  is more involved. In particular in point (i) of Theorem 1.2 and formulas (34) and (37) of Theorem 1.3 some balance exists between the power law of  $x$  and the time exponential term. This is described in some detail in Section 5. Let us just say here that the scaling law of the Dirac mass formation in  $t, x$  variables is of exponential type in all the cases.

**Remark 1.2 (Rates of convergence.)** It follows from Theorem 1.1 and Theorem 1.2 that, after suitable re scaling by the leading term, the solution  $u$  converges to zero, exponentially fast in time, uniformly for  $x$  in the three domains  $x > 1$ ,  $s_+(t, x) < p_0$  and  $q_0 < s_+(t, x)$ . When  $p_0 < s_+(t, x) < q_0$  the rate of convergence to one of the solution after that re scaling may be much lower, depending on the measure  $k_0$  and the initial data  $u_0$ . This may be seen from the way in which  $\gamma(\varepsilon)$  is obtained in Lemma 3 and in the expressions (36) for  $\omega(\varepsilon)$  and (39) for  $S(\varepsilon)$ . It is actually easy to see that the rates in that case can only be at best algebraic. This follows from the following. Since  $\varepsilon^2 t$  must tend to  $\infty$ ,  $\varepsilon$  must tend to zero more slowly than  $t^{-1/2}$ . Since the term  $\varepsilon \log R(\varepsilon)$  cannot decay to zero faster than  $\varepsilon$  it then follows that  $\omega(\varepsilon)$  and  $S(\varepsilon)$  decay at most like a power law in time, as  $t \rightarrow \infty$ . This constant is uniquely determined by the measure  $k_0$  (through its Mellin transform  $K(s)$ ). On the other

hand, the best possible choice of the functions  $R(\varepsilon)$  and  $L(\varepsilon)$ , depends on the initial data  $u_0$  and is not known. One possibility is to choose  $R$  by defining  $R(\varepsilon)$  such that

$$\varepsilon = \frac{1}{\log R(\varepsilon)} \left( \int_{R(\varepsilon)}^{\infty} z^{q_0-\delta-1} u_0(z) dz + \int_0^{\frac{1}{R(\varepsilon)}} z^{p_0+\delta-1} u_0(z) dz \right).$$

Such a value of  $R$  is uniquely defined since the function in the right hand side is strictly decreasing with respect to  $R > 1$ . It is then clear that  $\varepsilon \log R(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ , and  $|\omega(\varepsilon)| \leq 2C(p_0 + \delta, q_0 - \delta)\varepsilon \log R(\varepsilon)$ .

**Remark 1.3** The hypothesis on the measure  $k_0$  stated in Theorem 1.3 are rather restrictive. The cases where  $k_0$  has an absolutely continuous part or is singular discrete measure without 1 as a limit point are covered, but not the case when the measure  $k_0$  has no absolutely continuous part but has a singular continuous one or where 1 is a limit point.

**Remark 1.4** Condition  $H$  may be seen as a generalization of the "mitotic" fragmentation kernel, where  $k_0(x) = 2\delta_{x=\frac{1}{2}}$ . Similar assumptions have been found in other related studies, see [3], Appendix D. Equation (37) can be interpreted in terms of Fourier series in  $y = \log x$ , and exhibits a limit which has a  $\log \theta$ -periodic part in  $y$ . The Poisson summation formula applied to the function  $u_0(e^y)e^{s+y}$ , for  $s_+$  fixed, leads at first order to

$$u(t, x) \sim \log \theta \frac{e^{(K(s_+)-1)t}}{\sqrt{2\pi t K''(s_+)}} \sum_{n \in \mathbb{Z}} u_0(\theta^n x).$$

From the pointwise results, we obtain the following weak convergence results for suitably rescaled densities.

**Corollary 1.4** Let us define

$$r(t, y) := tye^{2ty}u(t, e^{ty}), \quad \tilde{r}(t, z) := r(t, y_0 + \frac{\sigma z}{\sqrt{t}})\frac{\sigma}{\sqrt{t}}, \quad (40)$$

with  $y_0 := K'(2)$  and  $\sigma^2 := K''(2)$ . Under the assumptions of Theorems 1.1, 1.2 and 1.3 a),  $r(t, \cdot) \rightharpoonup \delta_{K'(2)}U_0(2)$  and  $\tilde{r}(t, \cdot) \rightharpoonup U_0(2)G$ , with  $G(z) = \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}$ , in the weak sense of measures: for any bounded continuous function  $\phi$  on  $\mathbb{R}$ , we have

$$\int_{-\infty}^{+\infty} \phi(y)r(t, y)dy \rightarrow U_0(2)\phi(K'(2)), \quad \int_{-\infty}^{+\infty} \phi(z)\tilde{r}(t, z)dz \rightarrow U_0(2) \int_{-\infty}^{+\infty} \phi(z)\frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}dz,$$

with  $U_0(2) = \int_0^{\infty} xu_0(x)dx$  the initial mass.

This Corollary may be related with some of the results in [1]. See Section 4.4 for some details.

**Remark 1.5** The case  $\tau(x) = x^{1+\gamma}$  and  $B(x) = x^\gamma$  may be studied following similar lines. The two equations (growth fragmentation and pure fragmentation) are not related anymore as they were before, when  $\gamma = 0$ . If we take Mellin transform in the pure fragmentation equation we obtain:

$$\frac{\partial}{\partial t}U(t, s) = (1 - K(s))U(t, s + \gamma).$$

A similar calculation may be done for the growth-fragmentation equation. Although these are not ordinary differential equations anymore, since they are not local with respect to the  $s$  variable, they may still be explicitly solved using a Wiener Hopf type argument.

It is straightforward to deduce from Theorem 1.1, Theorem 1.2 and Theorem 1.3 the asymptotic behaviour of the solution  $v$  of the growth fragmentation equation. But this has to be done in terms of  $e^{ct}v(t, xe^{ct})$ . As it is shown in detail in Section 5.2, if the method is unchanged, the shape of the interesting domain is modified.

The plan of the remaining of this article is as follows. In Section 2 we explicitly solve equation (5) under suitable conditions on the initial data  $u_0$  and the kernel  $k_0$ , using the Mellin transform. In Section 3 we prove that the solutions  $u$  of the equation (5) obtained in Section 2 are such that  $xu(t, x)$  converges to a Dirac mass at the origin as  $t \rightarrow \infty$ . We also prove that equation (5) has no self similar solutions of the form  $u(t, x) = f(t)\Phi(xg(t))$  for  $\Phi \in L((1+x)dx)$ . In Section 4 we prove Theorem 1.1, Theorem 1.2, Theorem 1.3 and Corollary 1.4. Then, we relate our results with some of those obtained in [1]. In Section 5, we describe in some detail the regions of the  $(x, t)$  plane where the solutions of fragmentation equation (5) and the growth fragmentation equation (8) are concentrated as  $t$  increases. We also present some numerical simulations where such regions may be observed.

## 2 Explicit solution to the fragmentation equation (5).

In this section we rigorously perform the arguments presented in the introduction leading to the explicit formula (16).

Of course, it is possible to obtain existence and uniqueness of suitable types of solutions to the Cauchy problem for the fragmentation equation (5) with kernel  $k(x, y)$  given by (4) under much weaker assumptions than we are assuming in Theorem 2.1. It is easily seen for example that if  $u_0$  satisfies (18), there exists a unique mild solution  $u \in C([0, \infty); L^1((1+x)dx)) \cap C^1((0, \infty); L^1((1+x)dx))$ . More general situations are considered in [10]. Conditions (22) and (23) are imposed in order to have the representation formula (16).

The main result of this Section is the following.

**Theorem 2.1** *Suppose that  $u_0$  satisfies the condition (18), (22) and (23) Suppose that the kernel  $k$  is of the form given by (4). Then, there exists a unique function  $u$  in  $C([0, \infty); L^1((1+x)dx)) \cap C^1((0, \infty); L^1((1+x)dx))$  that satisfies equation (5) and condition (6) for all  $t > 0$  and almost every  $x > 0$ . This solution is given by (16) for all  $\nu \in (p_0, q_0)$  and satisfies property (10).*

**Proof.** Suppose that a function  $u(t, x) \in C([0, \infty); L^1((1+x)dx))$  satisfies (5). Then, for all  $t > 0$  the Mellin transform of  $u$  is well defined for  $\Re(s) \in [1, 2]$  and is analytic in  $\Re(s) \in (1, 2)$ . If we multiply both sides of (5) by  $x^{s-1}$  and integrate over  $(0, \infty)$  we obtain:

$$\begin{aligned} \frac{d}{dt} \int_0^\infty u(t, x) x^{s-1} dx + \int_0^\infty u(t, x) x^{s-1} dx &= \int_0^\infty x^{s-1} \int_x^\infty \frac{1}{y} k_0\left(\frac{x}{y}\right) u(t, y) dy \\ &= \int_0^\infty u(t, y) \int_0^y x^{s-1} k_0\left(\frac{x}{y}\right) dx \frac{dy}{y} = \int_0^\infty u(t, y) y^{s-1} dy \int_0^1 k_0(z) z^{s-1} dz. \end{aligned}$$

In the second step we have applied Fubini's Theorem, since by hypothesis  $k_0(z)z^{s-1}u(t,y)y^{s-1} \in L^1((0,1) \times (0,\infty))$ . This is nothing but equation (12) from where we deduce (14). Since for all  $t > 0$ ,  $u(t)$  is analytic on the strip  $\Re(s) \in (1,2)$  we will immediately deduce (16), for all  $\nu \in (1,2)$  as soon as we show that

$$\int_{\nu-i\infty}^{\nu+i\infty} \left| U_0(s) e^{(K(s)-1)t} x^{-s} \right| ds < \infty, \quad (41)$$

To this end, we notice first that, by definition of  $K(s)$ ,  $K(s) \leq K(1)$ ,  $\forall s \in [1,2]$ . Therefore:

$$\left| U_0(s) e^{(K(s)-1)t} x^{-s} \right| \leq \left| U_0(s) x^{-\nu} \right| e^{(K(1)-1)t}$$

Let us check that, under the assumptions made on  $u_0$  we actually have

$$\int_{-\infty}^{\infty} |U_0(\nu + iv)| dv < \infty. \quad (42)$$

for all  $\nu \in (p_0, q_0)$ . To this end we write:

$$\begin{aligned} U_0(\nu + iv) &= \int_0^\infty u_0(x) x^{\nu-1} x^{iv} dx = -\frac{1}{1+iv} \int_0^\infty (u_0(x) x^{\nu-1})' x^{iv+1} dx \\ &= -\frac{\nu-1}{1+iv} \int_0^\infty u_0(x) x^{\nu-1} x^{iv} dx - \frac{1}{1+iv} \int_0^\infty (u_0)'(x) x^\nu x^{iv} dx \\ &= -\frac{\nu-1}{1+iv} U_0(\nu + iv) - \frac{1}{1+iv} \int_0^\infty (u_0)'(x) x^\nu x^{iv} dx, \end{aligned}$$

where we have used (22). Therefore,

$$U_0(\nu + iv) = -\frac{1}{\nu + iv} \int_0^\infty (u_0)'(x) x^{\nu+iv} dx.$$

By the same argument, using now (23):

$$\lim_{x \rightarrow 0} (u_0)'(x) x^{\nu+1} = \lim_{x \rightarrow \infty} (u_0)'(x) x^{\nu+1} = 0,$$

we obtain:

$$U_0(\nu + iv) = \frac{1}{(\nu + iv)(1 + \nu + iv)} \int_0^\infty (u_0)''(x) x^{1+\nu+iv} dx.$$

We deduce

$$|U_0(\nu + iv)| \leq \frac{1}{\sqrt{(\nu^2 + v^2)((1+\nu)^2 + v^2)}} \int_0^\infty |(u_0)''(x)| x^{1+\nu} dx \quad (43)$$

and by (23), (42) follows.

On the other hand, it is easy to check that if  $u_0$  satisfies (18)-(23), then the function  $u$  defined by (16) satisfies  $u \in C([0, \infty); L^1((1+x)dx)) \cap C^1((0, \infty); L^1((1+x)dx))$  and solves (5), (6) for all  $t > 0$  and almost every  $x > 0$ .  $\square$

### 3 General convergence results

#### 3.1 Convergence to a Dirac Mass.

**Theorem 3.1** *Suppose that  $u \in C([0, \infty); L^1((1+x)dx)) \cap C^1((0, \infty); L^1((1+x)dx))$  is a solution of (5) with  $k_0$  satisfying (4). Then:*

$$\forall \varphi \in C_0(\mathbb{R}^+), \quad \lim_{t \rightarrow \infty} \int_0^\infty xu(t, x)\varphi(x)dx = M\varphi(0),$$

where

$$M = \int_0^\infty xu_0(x)dx.$$

**Proof.** Multiply equation (5) by  $x$  and integrate between  $z$  and  $\infty$  where  $z \geq 0$ :

$$\begin{aligned} \frac{d}{dt} \int_z^\infty xu(t, x)dx + \int_z^\infty xu(t, x)dx &= \int_z^\infty x \int_x^\infty \frac{1}{y} k_0\left(\frac{x}{y}\right) u(t, y) dy dx \\ &= \int_z^\infty u(t, y) \frac{1}{y} \int_z^y x k_0\left(\frac{x}{y}\right) dx dy = \int_z^\infty u(t, y) \frac{1}{y} \int_{z/y}^1 yr k_0(r) y dr dy \\ &= \int_z^\infty yu(t, y) \int_{z/y}^1 r k_0(r) dr dy. \end{aligned}$$

We first notice that if  $z = 0$ ,

$$\frac{d}{dt} \int_0^\infty xu(t, x)dx = 0$$

and the initial value  $M$  of  $\int_0^\infty xu(t, x)dx$  is preserved for all time  $t > 0$ .

Denoting now  $F(x) = \int_0^x yk_0(y)dy$ , we may see  $yk_0(y)$  as a probability distribution on  $(0, 1)$ , and  $F$  as its cumulative distribution function. The previous formula may be written as

$$\frac{d}{dt} \int_z^\infty xu(t, x)dx = - \int_z^\infty F\left(\frac{z}{x}\right) xu(t, x)dx \leq 0.$$

The function  $\int_z^\infty xu(t, x)dx$  is thus a nonnegative nonincreasing function. Hence, as  $t \rightarrow \infty$ , it tends to a limit  $l(z) \geq 0$ , and this implies that its time derivative tends to 0 as  $t \rightarrow \infty$ :

$$\int_z^\infty F\left(\frac{z}{x}\right) xu(t, x)dx \rightarrow 0, \quad \text{as } t \rightarrow \infty.$$

As soon as  $k_0$  is not equal to  $\delta_1$  (excluded by Assumption (4)) there is a neighbourhood  $(1 - \varepsilon, 1)$  of  $y = 1$  around which  $F(y)$  is bounded from below by a strictly positive constant. This implies that for all  $z > 0$ , the integral  $\int_z^{\frac{z}{1-\varepsilon}} xu(t, x)dx \rightarrow 0$ , which is sufficient to ensure that  $xu(t, x) \rightharpoonup M\delta$ , as  $t \rightarrow +\infty$ , in  $\mathcal{D}'(\mathbb{R}^+)$ .  $\square$

**Remark 3.1** *By Theorem 3.1, the solution  $u$  obtained in Theorem 2.1 is such that  $xu(t, x)$  converges to  $M\delta$  in  $\mathcal{D}'(\mathbb{R}^+)$  as  $t \rightarrow +\infty$ , where  $M = \int_0^\infty xu_0(x)dx$ .*

Theorem 3.1 does not give any information about how the convergence takes place. When the fragmentation rate is  $B(x) = x^\gamma$ , with  $\gamma > 0$ , it was proved in [8] that the convergence to a Dirac mass is self similar. First it was proved that for any  $M > 0$  there is a unique non negative integrable solution  $g \in L^1(xdx)$  to:

$$\begin{cases} g(z) + \frac{\partial}{\partial z}(zg(z)) + B(z)g(z) = \gamma \int_z^\infty k(\rho, z)B(\rho)g(\rho)d\rho \\ \int_0^\infty xg(x)dx = M. \end{cases} \quad (44)$$

The first equation in (44) is a particular case of Equation (2), with  $\tau(x) = x$ , so  $\nu = 1$ , and  $\lambda = 1$ .

Moreover, the solution  $u$  of the fragmentation equation with initial data satisfying  $\int_0^\infty xu_0(x)dx = M$ , is such that

$$\lim_{t \rightarrow \infty} \int_0^\infty \left| u(t, y) - t^{-\frac{2}{\gamma}} g\left(t^{\frac{1}{\gamma}} y\right) \right| y dy = 0.$$

The function  $t^{-\frac{2}{\gamma}} g\left(t^{\frac{1}{\gamma}} y\right)$  is a solution of the fragmentation equation, called self similar due to its invariance by the natural scaling associated to the equation. We show in the next section that there is no such type of solutions to equation (5).

When  $\gamma < 0$  it is well known that the mass  $M(t) = \int xu(t, x)dx$  of solutions of the fragmentation equation (5) is not constant anymore but, on the contrary, is a decreasing function of time. The asymptotic behaviour of the solutions of (5) for  $\gamma < 0$  has been described in [11], and it was proved that it is not self similar. It was also shown that such asymptotic behaviour strongly depends on the initial data  $u_0$ . For example, if  $u_0$  has compact support and the measure  $k_0$  satisfies:

$$\int_0^{1-z} xk_0(x)dx \sim z^{-\beta}, \text{ as } z \rightarrow 0$$

for some  $\beta \in [0, 1)$ , then the measure  $\mu(t, x)$ , solution of the fragmentation equation (5) satisfies, for some positive constant  $C_{\alpha, \beta}$  only depending on  $\alpha$  and  $\beta$ :

$$\frac{1}{M(t)} \frac{1}{C_{\alpha, \beta} t^{\frac{\beta}{(1-\beta)|\alpha|}}} \mu\left(t, \frac{dx}{C_{\alpha, \beta} t^{\frac{\beta}{(1-\beta)|\alpha|}}}\right) \rightharpoonup \mu_\infty(dx), \text{ as } t \rightarrow \infty$$

in the weak sense of measures, where

$$M(t) \sim \exp\left(-\frac{1-\beta}{|\alpha|} C_{\alpha, \beta}^{|\alpha|} |\alpha| t^{\frac{\beta}{(1-\beta)|\alpha|}}\right), \text{ as } t \rightarrow \infty.$$

### 3.2 No Self similar solutions.

The goal of this section is to prove the following result.

**Theorem 3.2** *Suppose that the kernel  $k$  is of the form given by (4). Then, equation (5) has no solution of the form*

$$\omega(t, x) = f(t)\Phi(xg(t)). \quad (45)$$

where  $f$  and  $g$  are continuously derivable functions from  $\mathbb{R}^+$  to itself and  $\Phi$  in  $L^1((1+x)dx)$ .

**Proof.** We denote  $\Phi(s) = \int_0^\infty x^{s-1}\Phi(x)dx$ , that by hypothesis is well defined for  $s \in [1, 2]$ . The Mellin transform of  $u(t)$  takes now the form:

$$\mathcal{M}_u(t, s) = (g(t))^{-s} f(t)\Phi(s),$$

and the equation (5) gives

$$\frac{\partial \mathcal{M}_u}{\partial t}(t, s) = \frac{f'(t)}{f(t)} \mathcal{M}_u(t, s) - s \frac{g'(t)}{g(t)} \mathcal{M}_u(t, s) = (-1 + K(s)) \mathcal{M}_u(t, s)$$

Therefore:

$$(-1 + K(s)) = \frac{f'(t)}{f(t)} - s \frac{g'(t)}{g(t)}.$$

and

$$K'(s) = -\frac{g'(t)}{g(t)}.$$

We deduce that  $K'(s)$  and  $\frac{g'(t)}{g(t)}$  must be constants. Therefore,  $g(t) = Ae^{-at}$ , for some constants  $A$  and  $a$ , and  $K(s) = a(s-1) + 2$ . Since  $K(s) \geq 0$  for every  $s \geq 1$ , we must have  $a \geq 0$ . Since on the other hand,  $K(s)$  is decreasing, we deduce  $a \leq 0$ , and therefore  $a = 0$ . But that is impossible because  $K(1) > 1$  and  $K(2) = 1$ .  $\square$

**Remark 3.2** *All the functions of the form*

$$u_s(t, x) := x^{-s} e^{(K(s)-1)t} = e^{(K(s)-1)t - s \log x}$$

are pointwise solutions of equation (5) for all  $t > 0$  and  $x > 0$  since:

$$\begin{aligned} \frac{\partial u_s}{\partial t} &= (K(s) - 1)x^{-s} e^{(K(s)-1)t} \\ u_s(t, x) - \int_x^\infty \frac{dy}{y} k_0\left(\frac{x}{y}\right) u_s(t, y) &= x^{-s} e^{(K(s)-1)t} - e^{(K(s)-1)t} \int_x^\infty \frac{dy}{y} k_0\left(\frac{x}{y}\right) y^{-s} \\ &= x^{-s} e^{(K(s)-1)t} - e^{(K(s)-1)t} x^{-s} \int_0^1 k_0(z) z^s dz = x^{-s} e^{(K(s)-1)t} (1 - K(s)). \end{aligned}$$

Such solutions are invariant by the change of variables:

$$u_\lambda(t, x) = u\left(t + \frac{s \log(\lambda)}{K(s)}, \lambda x\right)$$

since:

$$\begin{aligned} u_{s,\lambda}(t, x) &= u_s\left(t + \frac{s \log(\lambda)}{K(s)}, \lambda x\right) = e^{-t} e^{K(s)\left(t + \frac{s \log(\lambda)}{K(s)}\right) - s(\log \lambda x + \log \lambda)} \\ &= e^{-t} e^{K(s)t - s \log x} = u_s(t, x). \end{aligned}$$

These solutions are then self similar. Notice although that they do not belong to  $L^1(0, \infty)$  nor  $L^1(x dx)$ .

## 4 Proof of the main results

To study the long time behaviour of the solutions  $u$  of (5) we use their explicit representation (16), that holds under the assumptions (18), (22), (23) on the initial data  $u_0$  as shown in Theorem 2.1. In order to simplify as much as possible the presentation we define the new function:

$$w(t, x) = e^t u(t, x) \quad (46)$$

that satisfies:

$$\frac{\partial}{\partial t} w = \int_x^\infty \frac{1}{y} k_0\left(\frac{x}{y}\right) w(t, y) dy, \quad w(0, x) = u_0(x), \quad (47)$$

and, since  $M_w(0, s) = U_0(s)$ , it is given by:

$$w(t, x) = \frac{1}{2\pi i} \int_{\nu-i\infty}^{\nu+i\infty} x^{-s} U_0(s) e^{K(s)t} ds \quad (48)$$

for any  $\nu \in (1, 2)$ .

### 4.1 behaviour for $x > 1$ .

The long time behaviour of  $u$  for  $x > 1$  is given in Theorem 1.1. Its proof is based on the following lemma.

**Lemma 1** *Suppose that  $u_0$  satisfies (18), (22)-(25). Then the function  $U_0(s)$ , defined by*

$$U_0(s) = \int_0^\infty x^{s-1} u_0(x) dx \quad (49)$$

*that is by hypothesis analytic on the strip  $\Re(s) \in (p_0, q_0)$  may be extended to a meromorphic function, still denoted  $U_0(s)$ , on  $\Re(s) \in (\rho, r)$  for  $r > 0$  given in (24) and  $\rho > 0$  given in (27). Moreover,*

$$\int_{\nu-i\infty}^{\nu+i\infty} |U_0(s)| ds < +\infty \quad (50)$$

*for all  $\nu \in (\rho, p_0) \cup (p_0, q_0) \cup (q_0, r)$ .*

**Proof.** By (18) we already have that  $U_0(s)$  is analytic on  $\Re(s) \in (p_0, q_0)$ . For such an  $s$  we may write:

$$U_0(s) = \int_0^1 x^{s-1} u_0(x) dx + \int_1^\infty x^{s-1} u_0(x) dx = I_1(s) + I_2(s).$$

The integral  $I_1(s)$  is analytic on the half plane  $\Re(s) > p_0$ , and then, for the part  $(p_0, r)$ , it only remains to consider  $I_2$ . This term may be written as follows

$$I_2(s) = \int_1^\infty x^{s-1} a_0 x^{-q_0} dx + \int_1^\infty x^{s-1} (u_0(x) - a_0 x^{-q_0}) dx.$$

If  $s \in \mathbb{R}^+$  and  $s < q_0 - 1$ , we have:

$$\int_1^\infty x^{s-1}(a_0 x^{-q_0}) dx = \frac{a_0}{q_0 - s},$$

and that function has a meromorphic extension to all the complex plane, with a single pole at  $s = q_0$ .

On the other hand, by hypothesis (24),

$$\int_1^\infty |x^{s-1}(u_0(x) - a_0 x^{-q_0})| dx < C \int_1^\infty x^{-r+\Re(s)-1} dx$$

The integral in the right-hand side converges for  $q_0 \leq \Re(s) < r$ . We deduce that

$$\int_1^\infty x^{s-1}(u_0(x) - a_0 x^{-q_0}) dx$$

is analytic in the strip  $\Re(s) \in [q_0, r)$ .

If we define now :

$$H(s) = \int_0^1 x^{s-1} u_0(x) - \frac{a_0}{s - q_0} + \int_1^\infty x^{s-1}(u_0(x) - a_0 x^{-q_0}) dx.$$

$H$  is meromorphic in the strip  $\Re(s) \in (p_0, q_0)$ . By definition of  $U_0(s)$  we have, for all  $\Re(s) \in (p_0, q_0)$ :

$$U_0(s) = \int_0^1 x^{s-1} u_0(x) - \frac{a_0}{s - q_0} + \int_1^\infty x^{s-1}(u_0(x) - a_0 x^{-q_0}) dx.$$

And in the strip  $\Re(s) \in (q_0, r)$  we have:

$$\frac{a_0}{s - q_0} = \int_0^1 x^{s-1} a_0 x^{-q_0} dx$$

then, if  $\Re(s) \in (q_0, r)$ :

$$H(s) = \int_0^\infty x^{s-1}(u_0(x) - a_0 x^{-q_0}) dx := G(s)$$

We have then:

$$U_0(s) = H(s), \quad \forall s; \Re(s) \in (a, q_0)$$

$$H(s) = G(s), \quad \forall s; \Re(s) \in (q_0, r)$$

and  $G(s)$  is the analytic extension of  $U_0(s)$  to  $\Re(s) \in (q_0, r)$ . We still denote  $U_0(s)$  the meromorphic function that is obtained in that way on  $\Re(s) \in (p_0, r)$ .

We prove now (50) for  $\nu \in (p_0, r)$ . By (43) we already know that it holds for  $\nu \in (p_0, q_0)$ . For all  $s = u + iv$  in the strip  $(q_0, r)$ , we have:

$$\begin{aligned} U_0(s) &= \int_0^\infty x^{iv} x^{u-1} (u_0(x) - a_0 x^{-q_0}) dx \\ &= \frac{1}{iv + 1} x^{iv+1} x^{u-1} (u_0(x) - a_0 x^{-q_0}) \Big|_{x=0}^{x=\infty} - \\ &\quad - \frac{1}{iv + 1} \int_0^\infty x^{iv+1} \{ (u-1)x^{u-2} (u_0(x) - a_0 x^{-q_0}) + x^{u-1} ((u_0)'(x) + a_0 q_0 x^{-q_0-1}) \} dx. \end{aligned}$$

Since  $u > q_0$ , and using (22), we have:

$$|x^{iv+1}x^{u-1}(u_0(x) - a_0x^{-q_0})| \leq x^u(|u_0(x)| + |a_0x^{-q_0}|) \rightarrow 0 \text{ as } x \rightarrow 0.$$

On the other hand, using that  $u < r$  and (24):

$$|x^{iv+1}x^{u-1}(u_0(x) - a_0x^{-q_0})| \leq Cx^{u-r} \rightarrow 0 \text{ as } x \rightarrow +\infty.$$

Therefore,

$$\begin{aligned} U_0(s) &= \\ &= -\frac{1}{iv+1} \int_0^\infty x^{iv+1} \{ (u-1)x^{u-2} (u_0(x) - a_0x^{-q_0}) + x^{u-1} ((u_0)'(x) + a_0q_0x^{-q_0-1}) \} dx \\ &= -\frac{1}{iv+1} \int_0^\infty x^{iv} \{ (u-1)x^{u-1} (u_0(x) - a_0x^{-q_0}) + x^u ((u_0)'(x) + a_0q_0x^{-q_0-1}) \} dx \\ &= J_1(s) + J_2(s). \end{aligned}$$

Since

$$\begin{aligned} J_1(s) &= -\frac{u-1}{iv+1} \int_0^\infty x^{iv}x^{u-1} (u_0(x) - a_0x^{-q_0}) dx \\ &= -\frac{u-1}{iv+1} \int_0^\infty x^{s-1} (u_0(x) - a_0x^{-q_0}) dx = -\frac{u-1}{iv+1} U_0(s), \end{aligned}$$

we have:

$$\begin{aligned} -\frac{(iv+1)}{u-1} J_1(s) &= J_1(s) + J_2(s) \\ \frac{(u+iv)}{u-1} J_1(s) &= -J_2(s) \\ &= \frac{1}{iv+1} \int_0^\infty x^{iv} \{ x^u ((u_0)'(x) + a_0q_0x^{-q_0-1}) \} dx \\ J_1(s) &= \frac{u-1}{(iv+1)(u+iv)} \int_0^\infty x^{iv} \{ x^u ((u_0)'(x) + a_0q_0x^{-q_0-1}) \} dx \\ |J_1(s)| &\leq \frac{|u-1|}{|(iv+1)(u+iv)|} \int_0^\infty |x^u ((u_0)'(x) + a_0q_0x^{-q_0-1})| dx. \end{aligned}$$

Therefore:

$$U_0(s) = \frac{-1}{(u+iv)} \int_0^\infty x^{iv} \{ x^u ((u_0)'(x) + a_0q_0x^{-q_0-1}) \} dx \quad (51)$$

By (22), (24) and (25), and since  $u > q_0$ , the integral in the right-hand side of (51) is absolutely convergent for all  $s$  such that  $\Re(s) \in (q_0, r)$ .

We may repeat the integration by parts with the integral in the right-hand side of (51):

$$J_3(s) = \int_0^\infty x^{iv} \{ x^u ((u_0)'(x) + a_0q_0x^{-q_0-1}) \} dx \quad (52)$$

with  $s = u + iv$  and  $u \in (q_0, r)$ . As before,

$$\begin{aligned} J_3(s) &= \frac{1}{iv+1} x^{iv+1} x^u \left( (u_0)'(x) + a_0 q_0 x^{-q_0-1} \right) \Big|_{x=0}^{x=\infty} - \\ &\quad - \frac{1}{iv+1} \int_0^\infty x^{iv+1} \{ u x^{u-1} \left( (u_0)'(x) + a_0 q_0 x^{-q_0-1} \right) + \\ &\quad + x^u \left( (u_0)''(x) - a_0 q_0 (q_0 + 1) x^{-q_0-2} \right) \} dx. \end{aligned}$$

By (23) and the fact that  $u > q_0$  we have:

$$|x^{iv+1} x^u \left( (u_0)'(x) + a_0 q_0 x^{-q_0-1} \right)| \leq x^{u+1} |(u_0)'(x)| + |a_0| q_0 x^{u-q_0} \rightarrow 0, \text{ as } x \rightarrow 0.$$

Using now (25)

$$|x^{iv+1} x^u \left( (u_0)'(x) + a_0 q_0 x^{-q_0-1} \right)| \leq C x^{u+1-r-1} \rightarrow 0, \text{ as } x \rightarrow \infty.$$

Then,

$$\begin{aligned} J_3(s) &= \frac{-1}{iv+1} \int_0^\infty x^{iv+1} \{ u x^{u-1} \left( (u_0)'(x) + a_0 q_0 x^{-q_0-1} \right) + \\ &\quad + x^u \left( (u_0)''(x) - a_0 q_0 (q_0 + 1) x^{-q_0-2} \right) \} dx \\ &= \frac{-1}{iv+1} J_3(s) - \frac{1}{iv+1} \int_0^\infty x^{s+1} \left( (u_0)''(x) - a_0 q_0 (q_0 + 1) x^{-q_0-2} \right) dx \\ J_3(s) &= -\frac{1}{iv+2} \int_0^\infty x^{s+1} \left( (u_0)''(x) - a_0 q_0 (q_0 + 1) x^{-q_0-2} \right) dx \\ &= -\frac{1}{iv+2} \left( \int_0^1 (\dots) dx + \int_1^\infty (\dots) dx \right). \end{aligned}$$

The first integral in the right-hand side is absolutely convergent by (23) and because  $\Re(s) - q_0 - 1 > -1$ . The second integral converges absolutely by the hypothesis (26).

We have then, for  $\Re(s) \in (q_0, r)$ :

$$U_0(s) = \frac{1}{(u+iv)(iv+2)} \int_0^\infty x^{s+1} \left( (u_0)''(x) - a_0 q_0 (q_0 + 1) x^{-q_0-2} \right) dx$$

and (50) follows for  $\Re(s) \in (q_0, r)$ .

A similar argument, using (27), (28) and (29), shows (50) also for  $\Re(s) \in (\rho, p_0)$ .  $\square$

We may prove now Theorem 1.1.

**Proof.** [Proof of Theorem 1.1.] Let us consider the function  $w(t, x)$  defined in (46) and given by (48) for all  $\nu \in (p_0, q_0)$ . By Lemma 1 we may deform the integration contour in (48), cross the straight line  $\Re(s) = q_0$  to obtain:

$$w(t, x) = \text{Res}(U_0(s); s = q_0) x^{-q_0} e^{2K(q_0)t} + \frac{1}{2\pi i} \int_{\nu'-i\infty}^{\nu'+i\infty} x^{-s} U_0(s) e^{K(s)t} ds \quad (53)$$

with  $\nu' \in (q_0, r)$ . We notice now, for  $s = \nu' + iv$  :

$$\begin{aligned} K(s) &= \int_0^1 x^{\nu'} \cos(v \log(x)) k_0(x) dx + i \int_0^1 x^{\nu'} \sin(v \log(x)) k_0(x) dx \\ e^{K(s)t} &= e^{\int_0^1 x^{\nu'} \cos(v \log(x)) k_0(x) dx} e^{it \int_0^1 x^{\nu'} \sin(v \log(x)) k_0(x) dx} \end{aligned}$$

$$\begin{aligned} \left| \int_{\nu'-i\infty}^{\nu'+i\infty} x^{-s} U_0(s) e^{K(s)t} d\alpha \right| &\leq e^{\int_0^1 x^{\nu'} k_0(x) dx} \left| \int_{\nu'-i\infty}^{\nu'+i\infty} x^{-s} U_0(s) e^{it \int_0^1 x^{\nu'} \sin(v \log(x)) k_0(x) dx} d\alpha \right| \\ &\leq x^{-\nu'} e^{tK(\nu')} \left| \int_{-\infty}^{+\infty} x^{iv} U_0(\nu' + iv) e^{it \int_0^1 x^{\nu'} \sin(v \log(x)) k_0(x) dx} dv \right| \\ &\leq x^{-\nu'} e^{tK(\nu')} \int_{-\infty}^{+\infty} |U_0(\nu' + iv)| dv \end{aligned} \quad (54)$$

Combining (53) and (54) we obtain, for  $x > 1$  and  $t \rightarrow \infty$ :

$$w(t, x) = \text{Res}(U_0(s); s = q_0) x^{-q_0} e^{K(q_0)t} \left( 1 + \mathcal{O} \left( x^{-\nu' + q_0} e^{(K(\nu') - K(q_0))t} \right) \right) \quad (55)$$

□

## 4.2 $x$ small and $t$ large

In order to understand the long time behaviour of the function  $u$ , or  $w$ , let us recall the definition (17) of the key function  $\phi$  :

$$\phi(s, t, x) = -s \log(x) + tK(s). \quad (56)$$

Let us consider for a while the function  $\phi$  only for real values of  $s$ . When  $t > 0$  and  $x > 1$ , this function is decreasing with respect to the real part of  $s$ . That is the key property that, in Subsection 4.1, makes the rest term to be negligible with respect to the residue at  $s = q_0$ , once we have that  $U_0(s)$  is integrable for  $s \in (q_0, r)$ . If we maintain  $t$  fixed and take  $x < 1$ , the function  $\phi$  is now increasing with respect to  $s$ . We could then easily obtain a result similar to Theorem 1.1 for  $t$  fixed and  $x \rightarrow 0$ . But that is not a region particularly interesting. These monotonicity properties of  $\phi$  do not hold exactly when  $t$  is large and  $x$  small although that is exactly the interesting domain to consider for the long time behaviour of  $u$ . Lemma 6 shows the behaviour of  $\phi(t, \cdot, x)$  : it is decreasing for  $s \in (p_1, s_+(t, x))$  and increasing for  $s \in (s_+(t, x), +\infty)$  where  $s_+(t, x)$  is defined by (88).

Since  $s_+$  depends on  $t$  and  $x$ , is an increasing function of  $t > 0$  and an increasing function of  $x \in (0, 1)$  we have several domains of different long time asymptotic behaviour, that correspond to the different points (i) and (ii) of Theorem 1.2 and Theorem 1.3.

**Proof.** [Proof of Theorem 1.2]. We only consider in detail the case (i) since case (ii) is completely similar. Suppose then that we are in the region where  $t \rightarrow \infty$  and  $x \rightarrow 0$  in such a way that

$$s_+(t, x) = (K')^{-1} \left( \frac{\log(x)}{t} \right) > q_0.$$

By (48):

$$w(t, x) = \frac{1}{2\pi i} \int_{\nu-i\infty}^{\nu+i\infty} U_0(s) e^{\phi(s, t, x)} ds \quad (57)$$

for any  $\nu \in (p_0, q_0)$ .

Using Lemma 1 we deform the integration contour in (57) towards the right and cross the pole  $q_0$ , i.e. that we have:

$$\begin{aligned} w(t, x) &= e^{\phi(q_0, t, x)} \left( \text{Res}(U_0(s); s = q_0) + \mathcal{R} \right) \\ \mathcal{R} &= \frac{1}{2\pi i} \int_{\nu'-i\infty}^{\nu'+i\infty} U_0(s) e^{-\phi(q_0, t, x) + \phi(s, t, x)} ds \end{aligned}$$

with  $\nu' = q_0 + \delta$  for some  $\delta > 0$  such that  $q_0 + \delta < s_+(t, x)$ . Then,

$$|\mathcal{R}| \leq \int_{\nu'-i\infty}^{\nu'+i\infty} |U_0(s)| e^{\Re(-\phi(q_0, t, x) + \phi(s, t, x))} ds,$$

where

$$\begin{aligned} \Re(-\phi(q_0, t, x) + \phi(s, t, x)) &= \Re(-(s - q_0) \log x + t(K(s) - K(q_0))) \\ &= -\delta \log x + t \Re \left( \int_0^1 z^{s-1} k_0(z) dz - \int_0^1 z^{q_0-1} k_0(z) dz \right) \\ &= -\delta \log x + t \left( \int_0^1 z^{q_0+\delta-1} \cos(v \log z) k_0(z) dz - \int_0^1 z^{q_0-1} k_0(z) dz \right) \\ &\leq -\delta \log x + t \left( \int_0^1 z^{q_0+\delta-1} k_0(z) dz - \int_0^1 z^{q_0-1} k_0(z) dz \right) \\ &= -\phi(q_0, t, x) + \phi(q_0 + \delta, t, x). \end{aligned}$$

This ends the proof of the point (ii) of Theorem 1.2. The proof of point (i) is similar. If we want to be more specific:

$$\begin{aligned} \Re(-\phi(q_0, t, x) + \phi(s, t, x)) &\leq \delta \frac{\partial \phi(q_0 + \delta, t, x)}{\partial s}, \text{ by convexity of } \phi(\cdot, t, x) \\ &= \delta (-\log x + t K'(q_0 + \delta)) \\ &= \delta t (-K'(s_+(t, x)) + K'(q_0 + \delta)) \end{aligned}$$

If  $s_+(t, x) \geq q_0 + 2\delta$  then

$$-K'(s_+(t, x)) + K'(q_0 + \delta) \leq -K'(q_0 + 2\delta) + K'(q_0 + \delta) = -\delta K''(\eta)$$

for some  $\eta \in (q_0 + \delta, q_0 + 2\delta)$ . Since  $K''' < 0$ ,  $K''(\eta) \geq K''(q_0 + 2\delta)$  and therefore:

$$\Re(-\phi(q_0, t, x) + \phi(\alpha, t, x)) \leq -\delta^2 t K''(q_0 + 2\delta).$$

We deduce

$$\begin{aligned} \left| e^{-\phi(q_0, t, x)} \mathcal{R}(t, x) \right| &\leq C e^{-\delta^2 t K''(q_0 + 2\delta)} \\ \text{for all } t > 0, x > 0; s_+(t, x) &> q_0 + 2\delta > q_0. \end{aligned}$$

□

### 4.3 Proof of Theorem 1.3.

In order to estimate the function  $w(t, x)$  defined in (48) when  $s_+(t, x) \in (p_0, q_0)$  we first take  $\nu = s_+(t, x)$  in that formula. Then, we will use several estimates on  $U_0(s)$  and  $\Re e(\phi(s, t, x))$  along the integration curve  $\Re e(s) = s_+(t, x) \in (p_0 + \delta, q_0 - \delta)$ .

The estimate on  $U_0(s)$  follows from (43):

$$\begin{aligned} |U_0(s_+ + iv)| &\leq \frac{1}{\sqrt{(s_+^2 + v^2)((1 + s_+)^2 + v^2)}} \int_0^\infty |(u_0)''(y)| y^{1+s_+} dy \\ &\leq \frac{1}{\sqrt{(p_0^2 + v^2)((1 + p_0)^2 + v^2)}} \left( \int_0^1 |(u_0)''(y)| y^{1+p_0+\delta} dy + \int_1^\infty |(u_0)''(y)| y^{1+q_0-\delta} dy \right). \end{aligned} \quad (58)$$

We then have

$$\begin{aligned} \int_{\Re e(s)=s_+, |\Im m(s)| \geq \varepsilon} |U_0(s)| dv &\leq \int_{\Re e(s)=s_+} \frac{dv}{\sqrt{(p_0^2 + v^2)((1 + p_0)^2 + v^2)}} \times \\ &\times \left( \int_0^1 |(u_0)''(y)| y^{1+p_0+\delta} dy + \int_1^\infty |(u_0)''(y)| y^{1+q_0-\delta} dy \right) = C_0(p_0 + \delta, q_0 - \delta) \end{aligned} \quad (59)$$

In order to estimate  $\Re e(\phi)$  we wish to use the method of stationary phase. To this end we must consider different cases, depending on the measure  $k_0$ .

**Lemma 2** *For all  $\varepsilon \in (0, 1)$ , for all  $t > 0$ ,  $x > 0$  and all  $s \in \mathbb{C}$  such that  $s = s_+(t, x) + iv$ ,  $v \in \mathbb{R}$  and  $|s - s_+(t, x)| \leq \varepsilon$ :*

$$\Re e(K(s)) = K(s_+(t, x)) - \frac{v^2}{2} K''(s_+(t, x)) + \mathcal{O}(\varepsilon^3). \quad (60)$$

*If  $k_0$  is a discrete measure whose support  $\Sigma$  satisfies the condition H, then (60) holds for all  $\varepsilon > 0$ ,  $t > 0$ ,  $x > 0$  and for all  $s \in s_+ + i\mathbb{R}$  such that  $|s - (s_+(t, x) + iv)| \leq \varepsilon$ , for some  $v \in Q$ .*

**Proof.** By Taylor's formula

$$\begin{aligned} K(s) &= K(s_+(t, x)) + (s - s_+(t, x))K'(s_+(t, x)) + \frac{(s - s_+(t, x))^2}{2} K''(s_+(t, x)) \\ &\quad + \frac{(s - s_+(t, x))^3}{6} K'''(\zeta(t, x)) \end{aligned}$$

for some  $\zeta(t, x) \in \mathbb{C}$  between  $s$  and  $s_+(t, x)$ . Since  $|s - s_+(t, x)| \leq \varepsilon$ ,  $|\zeta(t, x) - s_+(t, x)| \leq \varepsilon$ . Since  $s_+ \in [p_0, q_0]$ ,  $|\zeta(t, x)| \leq q_0 + \varepsilon$ . We have then

$$\begin{aligned} |K'''(\zeta)| &= \left| \int_0^1 (\log z)^3 k_0(z) e^{(\zeta-1) \log z} dz \right| \\ &\leq \int_0^1 |\log z|^3 |k_0(z)| e^{\Re e(\zeta-1) \log z} dz \\ &\leq \int_0^1 |\log z|^3 |k_0(z)| e^{-(p_0+1) \log z} dz \end{aligned}$$

Using that  $K'(s_+) \in \mathbb{R}$ , (60) follows.

Suppose now that  $k_0$  satisfies condition H. We have:

$$\begin{aligned} k_0(x) &= \sum_{k \in \mathbb{N}} a_k \delta(x - \sigma_k), \quad \sum_{k \in \mathbb{N}} a_k \sigma_k = 1 \\ K(s) &= \sum_{\ell \in \mathbb{N}} a_\ell \sigma_\ell^{s-1}, \quad K'(s) = \sum_{\ell \in \mathbb{N}} a_\ell (\log \sigma) \sigma_\ell^{s-1} \\ \phi(s, t, x) &= -s \log x + t \sum_{\ell \in \mathbb{Z}} a_{k\ell} \sigma_\ell^{s-1}. \end{aligned}$$

Therefore, for all  $v \in Q$

$$\begin{aligned} K(s_+ + iv) &= \sum_{\ell \in \mathbb{N}} a_\ell \sigma_\ell^{s_+ - 1} e^{iv \log \sigma_\ell} \\ &= \sum_{\ell \in \mathbb{N}} a_\ell \sigma_\ell^{s_+ - 1} e^{2i\pi k(\ell)} = \sum_{\ell \in \mathbb{N}} a_\ell \sigma_\ell^{s_+ - 1} = K(s_+) \end{aligned} \tag{61}$$

And for the same reason

$$\begin{aligned} K'(s_+ + iv) &= K'(s_+), \quad \forall v \in Q, \\ K''(s_+ + iv) &= K''(s_+), \quad \forall v \in Q, \end{aligned}$$

from where (60) follows for all  $s \in \mathbb{C}$  such that  $|s - (s_+(t, x) + iv)| \leq \varepsilon$  for some  $v \in Q$  by Taylor's formula around the point  $s_+(t, x) + iv$ .  $\square$

We prove now estimates on  $\Re(\Phi)$  for  $s$  far from the points  $s_k = s_+ + kv_*$  with  $v_*$  defined by Equation 33.

**Lemma 3** *Suppose that the measure  $k_0$  has an absolutely continuous part or is a singular discrete measure whose support  $\Sigma$  does not satisfies condition H. Then, for all  $\varepsilon > 0$  there exists  $\gamma(\varepsilon) > 0$  such that for all  $t > 0$  and  $x > 0$ :*

$$\sup_{|v| \geq \varepsilon} \Re \left( \phi(s_+ + iv, t, x) \right) \leq \Re \left( \phi(s_+, t, x) \right) - \gamma(\varepsilon)t$$

Moreover, when  $\varepsilon \rightarrow 0$ ,  $\gamma(\varepsilon) \sim C\varepsilon^2$  for a constant  $C > 0$ .

**Proof.** For all  $v \in \mathbb{R}$ :

$$\begin{aligned} \Re \left( \phi(s_+ + iv, t, x) - \phi(s_+, t, x) \right) &= \Re \left( -iv \log(x) + t(K(s_+ + iv) - K(s_+)) \right) \\ &= t \Re(K(s_+ + iv) - K(s_+)) \\ &= t \Re \left( \int_0^1 x^{s_+ - 1} k_0(x) (x^{iv} - 1) dx \right) \\ &= \int_0^1 x^{s_+ - 1} k_0(x) \left( \cos(v \log(x)) - 1 \right) dx \\ &= tG(v) \leq 0. \end{aligned}$$

The function  $G(v)$  is a continuous nonpositive function of  $v$ . If  $k_0$  has an absolutely continuous part  $g$  then:

$$\forall v \in \mathbb{R} \setminus \{0\}, \quad \int_0^1 x^{s+1} \cos(v \log x) g(x) dx < \int_0^1 x^{s+1} g(x) dx,$$

and then, for all  $v \in \mathbb{R} \setminus \{0\}$ :

$$\begin{aligned} \int_0^1 x^{s+1} \cos(v \log x) k_0(x) dx &= \int_0^1 x^{s+1} \cos(v \log x) (g(x) + d\mu(x)) dx \\ &< \int_0^1 x^{s+1} (g(x) dx + d\mu(x)) = \int_0^1 x^{s+1} k_0(x) dx \end{aligned}$$

By continuity of the function  $G(v)$  it follows that for all  $R > 0$  and  $\varepsilon > 0$  there exists  $\delta > 0$  such that

$$\sup_{\varepsilon \leq |v| \leq R} G(v) \leq -\delta$$

Lemma (3) will follow if we prove that the limit of  $G$  at infinity is also strictly negative. To this end let us write  $G$  under the form

$$\begin{aligned} G(v) &= \int_0^1 x^{s+1} k_0(x) \left( \cos(v \log(x)) - 1 \right) dx = \int_0^\infty e^{-ys+1} k_0(e^{-y}) \left( \cos(vy) - 1 \right) dy \\ &= \int_0^\infty e^{-ys+1} \left( \cos(vy) - 1 \right) \left( g(e^{-y}) dy + d\mu(e^{-y}) \right) \end{aligned}$$

By Riemann Lebesgue theorem,

$$\lim_{v \rightarrow \pm\infty} \int_0^\infty e^{-ys+1} \cos(vy) g(e^{-y}) dy = 0$$

and then

$$\begin{aligned} \lim_{v \rightarrow \pm\infty} G(v) &= \lim_{v \rightarrow \pm\infty} \int_0^\infty e^{-ys+1} \left( \cos(vy) - 1 \right) \left( g(e^{-y}) dy + d\mu(e^{-y}) \right) \\ &= - \int_0^\infty e^{-ys+1} g(e^{-y}) dy + \lim_{v \rightarrow \pm\infty} \int_0^\infty e^{-ys+1} \left( \cos(vy) - 1 \right) d\mu(e^{-y}) \\ &\leq - \int_0^\infty e^{-ys+1} g(e^{-y}) dy = - \int_0^1 x^{s+1} g(x) dx < 0. \end{aligned}$$

Suppose now, that the measure  $k_0$  is a singular discrete measure whose support  $\Sigma$  is given by the countable set of points  $\sigma$ :

$$\begin{aligned} k_0(x) &= \sum_{\ell \in \mathbb{N}} a_\ell \delta(x - \sigma_\ell), \quad \sum_{\ell \in \mathbb{N}} a_\ell \sigma_\ell = 1 \\ K(s) &= \sum_{\ell \in \mathbb{N}} a_\ell \sigma_\ell^{s-1}, \quad K'(s) = \sum_{\ell \in \mathbb{N}} a_\ell (\log \sigma) \sigma_\ell^{s-1} \\ \Phi(s, t, x) &= -s \log x + t \sum_{\ell \in \mathbb{N}} a_\ell \sigma_\ell^{s-1}. \end{aligned}$$

By definition of  $p_1$ :

$$\sum_{\ell \in \mathbb{N}} a_\ell \sigma_\ell^s < +\infty, \quad \forall s; \quad \Re s > p_1.$$

Suppose also that the points  $\sigma_\ell$  do not satisfy the condition H. Then, let us see first that for any  $v \neq 0$ ,

$$\int_0^1 x^{s+1} k_0(x) \cos(v \log(x)) < \int_0^1 x^{s+1} k_0(x) \cos(v \log(x)).$$

Suppose on the contrary that this is not true. Then, by the definition of  $k_0$ :

$$\sum_{\ell \in \mathbb{N}} a_\ell \sigma_\ell^{s+1} \cos(v \log \sigma_\ell) = \sum_{\ell \in \mathbb{N}} a_\ell \sigma_\ell^{s+1}$$

Since  $a_\ell \geq 0$ ,  $\sigma_\ell \geq 0$  and  $\sigma_\ell$  we than have

$$\cos(v \log \sigma_\ell) = 1, \quad \forall \ell \in \mathbb{N},$$

or, in other terms,

$$\forall \ell \in \mathbb{N}, \exists k(\ell) \in \mathbb{N}; \quad v \log \sigma_\ell = 2k(\ell)\pi.$$

But, by Proposition 1 this contradicts the assumption that  $\Sigma$  does not satisfy condition H. Suppose now once again that there exists a sequence  $v_n \rightarrow \infty$  such that

$$\int_0^1 x^{s+1} k_0(x) \cos(v_n \log(x)) = \sum_{\ell \in \mathbb{N}} a_\ell \sigma_\ell^{s+1} \cos(v_n \log \sigma_\ell) \rightarrow \sum_{\ell \in \mathbb{N}} a_\ell \sigma_\ell^{s+1}$$

If  $\min_\ell \sigma_\ell = \sigma_{\ell_0} > 0$ , for all  $n$  there exists  $w_n \in [0, 2\pi/\log \sigma_{\ell_0}]$  such that  $\cos(w_n \log \sigma_\ell) = \cos(v_n \log \sigma_\ell)$  for all  $\ell$ . Then, there exists a subsequence  $(w_n)_{n \in \mathbb{N}}$  and  $w_* \in [0, 2\pi/\log \sigma_{\ell_0}]$  such that  $w_n \rightarrow w_*$  as  $n \rightarrow \infty$ . By continuity this would imply

$$\sum_{\ell \in \mathbb{N}} a_\ell \sigma_\ell^{s+1} \cos(w_* \log \sigma_\ell) = \sum_{\ell \in \mathbb{N}} a_\ell \sigma_\ell^{s+1}$$

Since  $a_\ell \geq 0$  and  $\sigma_\ell \geq 0$  this implies that:

$$\cos(w_* \log \sigma_\ell) = 1, \quad \forall \ell \in \mathbb{N}$$

which is impossible since the set of points  $\sigma_\ell$  does not satisfies condition H. Therefore

$$\sup_{|v| \geq \varepsilon} \int_0^1 x^{s_+-1} \cos(v \log x) d\mu_c < \int_0^1 x^{s_+-1} d\mu_c$$

and this concludes the proof of Lemma 3. As concerns the asymptotic behaviour when  $\varepsilon \rightarrow 0$ , it is given by the Taylor's formula of Lemma 2, since  $0 < K''(q_0) < K''(s_+) < K''(p_0)$ , so that  $\gamma \sim C\varepsilon^2$  with  $K''(q_0)/2 < C < K''(p_0)/2$ .  $\square$

**Remark 4.1** *We do not know whether Lemma 3 holds when the absolutely continuous part of the measure  $k_0$  is zero but its singular continuous part is not zero. For our purpose it would be enough to know if, given a singular continuous probability measure  $d\mu$  with support contained in  $[0, 1]$ :*

$$\overline{\lim}_{v \rightarrow \infty} \int_0^1 e^{ivx} d\mu(x) < 1.$$

**Remark 4.2** *As it is seen in the proof of Lemma 3, condition H arises very naturally when looking for the existence of at least one point  $v \in \mathbb{R}$  such that for all  $\ell \in \mathbb{N}$  there exists  $k(\ell) \in \mathbb{Z}$  such that  $v \log \sigma_\ell = 2\pi k(\ell)$ . For more details see Proposition 1 in the Appendix.*

We consider now the case where the measure  $k_0$  is discrete and its support satisfies the condition H. Let us first observe the following.

**Lemma 4** *Suppose that  $k_0$  is a singular discrete measure whose support  $\Sigma$  satisfies the condition H. Then,*

$$\begin{aligned} \forall \varepsilon \in (0, |\log \theta|^{-1}), \exists \gamma(\varepsilon, \theta) > 0, \quad \gamma \sim_{\varepsilon \rightarrow 0} C\varepsilon^2, \quad C > 0; \\ \Re \left( \phi(s_+ + iv, t, x) \right) \leq \Re \left( \phi(s_+, t, x) \right) - \gamma(\varepsilon, \theta)t, \end{aligned} \tag{62}$$

$$\forall v \in \mathbb{R} \setminus \{0\}, \text{ such that, } d(v, Q) \geq \varepsilon. \tag{63}$$

**Proof.** Since  $k_0$  is a discrete measure whose support satisfies condition H,

$$\begin{aligned} \Re \left( \phi(s_+ + iv, t, x) \right) &= -s_+ \log x + t \Re \left( \sum_{\ell \in \mathbb{N}} a_\ell \sigma_\ell^{s_+-1} \right) \\ &= -s_+ \log x + t \sum_{\ell \in \mathbb{N}} a_\ell e^{(s_+-1) \log \sigma_\ell} \cos(v \log \sigma_\ell). \end{aligned}$$

We now apply Proposition (3) with  $b_\ell = a_\ell e^{(s_+-1) \log \sigma_\ell}$ . Then, for all  $\varepsilon > 0$  there exists  $\gamma = \gamma(\varepsilon, \theta) > 0$  such that

$$\sum_{\ell \in \mathbb{N}} a_\ell e^{(s_+-1) \log \sigma_\ell} \cos(v \log \sigma_\ell) \leq \sum_{\ell \in \mathbb{N}} a_\ell e^{(s_+-1) \log \sigma_\ell} - \gamma$$

and therefore

$$\Re \left( \phi(s_+ + iv, t, x) \right) \leq \Re \left( \phi(s_+, t, x) \right) - \gamma t.$$

The asymptotic behaviour of  $\gamma(\varepsilon, \theta)$  when  $\varepsilon \rightarrow 0$  comes from Taylor's formula given in Lemma 2.  $\square$

We prove now Theorem 1.3.

**Proof.** [Theorem 1.3.] We start proving point (a).

Let us split the integral defining  $w(t, x)$  as follows:

$$\begin{aligned} w(t, x) &= \frac{1}{2\pi i} \int_{s_+(t, x) - i\infty}^{s_+(t, x) + i\infty} U_0(s) e^{\phi(s, t, x)} ds \\ &= \frac{1}{2\pi i} \int_{s_+(t, x) - i\varepsilon}^{s_+(t, x) + i\varepsilon} U_0(s) e^{\phi(s, t, x)} ds + \frac{1}{2\pi i} \int_{\Re(s)=s_+, |\Im(s)| \geq \varepsilon} U_0(s) e^{\phi(s, t, x)} ds \end{aligned} \quad (64)$$

We estimate the two integrals in the right hand side of (64) using Lemma 2 and Lemma 3. First of all we consider the integral near  $s_+$ , and write:

$$\begin{aligned} \frac{1}{2\pi i} \int_{s_+(t, x) - i\varepsilon}^{s_+(t, x) + i\varepsilon} U_0(s) e^{\phi(s, t, x)} ds &= \frac{U_0(s_+)}{2\pi i} \int_{s_+(t, x) - i\varepsilon}^{s_+(t, x) + i\varepsilon} e^{\phi(s, t, x)} ds + \\ &\quad + \frac{1}{2\pi i} \int_{s_+(t, x) - i\varepsilon}^{s_+(t, x) + i\varepsilon} (U_0(s) - U_0(s_+)) e^{\phi(s, t, x)} ds \\ &= I_1 + I_2. \end{aligned}$$

By Lemma 2, for all  $\varepsilon \in (0, 1)$ ,  $x > 0$ ,  $t > 0$  and  $s = s_+ + iv$  such that  $|s - s_+(t, x)| \leq \varepsilon$ ,

$$\Re(\phi(s, t, x)) = \Re(\phi(s_+(t, x), t, x)) + t \left( \frac{(s - s_+(t, x))^2}{2} K''(s_+(t, x)) + \mathcal{O}(\varepsilon^3) \right). \quad (65)$$

and

$$\int_{s_+(t, x) - i\varepsilon}^{s_+(t, x) + i\varepsilon} e^{\phi(s, t, x)} ds = e^{\phi(s_+(t, x), t, x)} \int_{s_+(t, x) - i\varepsilon}^{s_+(t, x) + i\varepsilon} e^{t \left( \frac{(s - s_+(t, x))^2}{2} K''(s_+(t, x)) + \mathcal{O}(\varepsilon^3) \right)} ds.$$

Since  $s = s_+(t, x) + iv$ , then  $(s - s_+(t, x))^2 = -v^2$  and

$$\begin{aligned} \int_{s_+(t, x) - i\varepsilon}^{s_+(t, x) + i\varepsilon} e^{\phi(s, t, x)} ds &= i e^{\phi(s_+(t, x), t, x)} \int_{-\varepsilon}^{\varepsilon} e^{-t \left( \frac{v^2}{2} K''(s_+(t, x)) + \mathcal{O}(\varepsilon^3) \right)} dv \\ &= i \frac{e^{\phi(s_+(t, x), t, x)}}{\sqrt{t K''(s_+)}} \int_{-\varepsilon \sqrt{t K''(s_+) + \mathcal{O}(\varepsilon)}}^{\varepsilon \sqrt{t K''(s_+) + \mathcal{O}(\varepsilon)}} e^{-\frac{y^2}{2}} dy \end{aligned}$$

Since  $s_+ \in (p_0, q_0)$ ,  $K''(s_+) \in (K''(q_0), K''(p_0))$  and  $\varepsilon \sqrt{t K''(s_+) + \mathcal{O}(\varepsilon)} \rightarrow \infty$  as  $t \rightarrow \infty$  and  $\varepsilon \rightarrow 0$ , if we choose them such that  $\varepsilon \sqrt{t} \rightarrow \infty$

$$\begin{aligned} \int_{-\varepsilon \sqrt{t K''(s_+) + \mathcal{O}(\varepsilon)}}^{\varepsilon \sqrt{t K''(s_+) + \mathcal{O}(\varepsilon)}} e^{-\frac{y^2}{2}} dy &= \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} dy - 2 \int_{\varepsilon \sqrt{t K''(s_+) + \mathcal{O}(\varepsilon)}}^{\infty} e^{-\frac{y^2}{2}} dy \\ &= \sqrt{2\pi} - 2e^{-\frac{t(\varepsilon^2 K''(s_+) + \mathcal{O}(\varepsilon^3))}{2}} \times \mathcal{O} \left( \frac{1}{\varepsilon \sqrt{t K''(s_+) + \mathcal{O}(\varepsilon)}} \right), \end{aligned}$$

as  $t \rightarrow \infty$ ,  $\varepsilon \rightarrow 0$ ,  $\varepsilon \sqrt{t} \rightarrow \infty$  and  $s_+(t, x) \in (p_0, q_0)$ .

$$\frac{1}{2\pi i} \int_{s_+(t, x) - i\varepsilon}^{s_+(t, x) + i\varepsilon} e^{\phi(s, t, x)} ds = \frac{1}{\sqrt{2\pi}} \frac{e^{\phi(s_+(t, x), t, x)}}{\sqrt{t K''(s_+)}} \left( 1 + \mathcal{O} \left( \frac{e^{-\frac{t(\varepsilon^2 K''(s_+) + \mathcal{O}(\varepsilon^3))}{2}}}{\varepsilon \sqrt{t K''(s_+) + \mathcal{O}(\varepsilon)}} \right) \right)$$

Since  $K'''(s) \leq 0$  and  $s_+(t, x) \in (p_0, q_0)$ ,  $K''(s_+) \geq K''(q_0)$  and then,

$$I_1 = \frac{u_0(s_+(t, x))}{\sqrt{2\pi}} \frac{e^{\phi(s_+(t, x), t, x)}}{\sqrt{tK''(s_+)}} \left( 1 + \mathcal{O} \left( \frac{e^{-\frac{\varepsilon^2 t K''(q_0)}{2}}}{\varepsilon \sqrt{tK''(q_0)}} \right) \right) \quad (66)$$

We consider now  $I_2$ .

$$\begin{aligned} & \left| \int_{s_+(t, x) - i\varepsilon}^{s_+(t, x) + i\varepsilon} (U_0(s) - U_0(s_+)) e^{\phi(s, t, x)} ds \right| \leq \\ & \leq \sup_{|v| \leq \varepsilon} |U_0(s_+ + iv) - U_0(s_+)| \int_{s_+(t, x) - i\varepsilon}^{s_+(t, x) + i\varepsilon} |e^{\phi(s, t, x)}| ds \end{aligned}$$

As before in Lemma 2, if  $s = s_+(t, x) + iv$  with  $v \in (-\varepsilon, \varepsilon)$ :

$$|e^{\phi(s, t, x)}| = e^{\phi(s_+(t, x), t, x)} e^{-t \left( \frac{v^2}{2} K''(s_+(t, x)) + \mathcal{O}(\varepsilon^3) \right)}$$

and we may then estimate  $I_2$  as

$$\begin{aligned} |I_2| & \leq \omega(\varepsilon) \frac{e^{\phi(s_+(t, x), t, x)}}{\sqrt{2\pi t K''(s_+)}} \left( 1 + \mathcal{O} \left( \frac{e^{-\frac{t(\varepsilon^2 K''(s_+) + \mathcal{O}(\varepsilon^3))}{2}}}{\varepsilon \sqrt{tK''(s_+) + \mathcal{O}(\varepsilon)}} \right) \right) \\ & = \omega(\varepsilon) \frac{e^{\phi(s_+(t, x), t, x)}}{\sqrt{2\pi t K''(s_+)}} \left( 1 + \mathcal{O} \left( \frac{e^{-\frac{\varepsilon^2 t K''(q_0)}{2}}}{\varepsilon \sqrt{tK''(q_0)}} \right) \right). \end{aligned} \quad (67)$$

where  $\omega(\varepsilon)$  is defined by

$$\omega(\varepsilon) := \sup_{|v| \leq \varepsilon} |U_0(s_+ + iv) - U_0(s_+)|. \quad (68)$$

Moreover, for  $R > 1$ :

$$\begin{aligned} |U_0(s_+ + iv) - U_0(s_+)| & = \left| \int_0^\infty z^{s_+ - 1} u_0(z) (z^{iv} - 1) dz \right| \\ & \leq \int_{\frac{1}{R}}^R |z^{s_+ - 1} u_0(z) (z^{iv} - 1)| dz + \int_R^\infty |z^{s_+ - 1} u_0(z) (z^{iv} - 1)| dz + \\ & \quad + \int_0^{\frac{1}{R}} |z^{s_+ - 1} u_0(z) (z^{iv} - 1)| dz \\ & \leq \int_{\frac{1}{R}}^R |z^{s_+ - 1} u_0(z) (z^{iv} - 1)| dz + 2 \int_R^\infty |z^{q_0 - \delta - 1} u_0(z)| dz + \\ & \quad + 2 \int_0^{\frac{1}{R}} |z^{p_0 + \delta - 1} u_0(z)| dz \end{aligned}$$

Since  $z^{q_0 - \delta - 1} u_0, z^{p_0 + \delta - 1} u_0 \in L^1(\mathbb{R}^+)$ , we have  $\int_R^\infty |z^{q_0 - \delta - 1} u_0(z)| dz \rightarrow 0$  and  $\int_0^{\frac{1}{R}} |z^{p_0 + \delta - 1} u_0(z)| dz \rightarrow 0$  as  $R \rightarrow \infty$ .

On the other hand, for all  $R > 1 > \text{fixed}$ ,

$$\int_{\frac{1}{R}}^R |z^{s_+ - 1} u_0(z)(z^{iv} - 1)| dz = \int_{\frac{1}{R}}^R |z^{s_+ - 1} u_0(z)(e^{iv \log z} - 1)| dz. \quad (69)$$

We use now that for  $s \in \mathbb{R}$ :

$$\begin{aligned} |e^{is} - 1| &= \left| \int_0^s e^{ir} dr \right| = \left| s \int_0^1 e^{is\rho} d\rho \right| \leq |s| \\ |e^{iv \log z} - 1| &\leq |v \log z| \leq (\varepsilon \log R), \quad \forall z \in \left( \frac{1}{R}, R \right), \end{aligned}$$

and then

$$\begin{aligned} \int_{\frac{1}{R}}^R |z^{s_+ - 1} u_0(z)(z^{iv} - 1)| dz &\leq (\varepsilon \log R) \int_0^\infty |z^{s_+ - 1} u_0(z)| dz \\ &\leq (\varepsilon \log R) \int_0^\infty |(z^{p_0 + \delta - 1} + z^{q_0 - \delta - 1}) u_0(z)| dz. \end{aligned}$$

We deduce

$$|\omega(\varepsilon)| \leq C(p_0 + \delta, q_0 - \delta) \left( \int_R^\infty |z^{q_0 - \delta - 1} u_0(z)| dz + \int_0^{\frac{1}{R}} |z^{p_0 + \delta - 1} u_0(z)| dz + \varepsilon \log R \right)$$

and (36) follows choosing  $R = R(\varepsilon)$  such that  $R(\varepsilon) \rightarrow \infty$  and  $\varepsilon \log R(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ .

We now estimate the second term of the right hand side in (64). By Lemma (3) and (59):

$$\begin{aligned} \left| \frac{1}{2\pi i} \int_{\Re(s)=s_+, |\Im(s)| \geq \varepsilon} U_0(s) e^{\phi(s,t,x)} ds \right| &\leq \frac{e^{\phi(s_+(t,x),t,x) - \gamma(\varepsilon)t}}{2\pi} \int_{\Re(s)=s_+, |\Im(s)| \geq \varepsilon} |U_0(s)| ds \\ &\leq \frac{e^{\phi(s_+(t,x),t,x) - \gamma(\varepsilon)t}}{2\pi} C(p_0 + \delta, q_0 - \delta) \end{aligned}$$

This ends the proof of (a).

We prove now point (b). To this end we split now the integral in (57) as follows:

$$w(t, x) = \frac{1}{2\pi i} \sum_{k \in \mathbb{Z}} \int_{s_k(t,x) - i\varepsilon}^{s_k(t,x) + i\varepsilon} U_0(s) e^{\phi(s,t,x)} ds + \frac{1}{2\pi i} \int_{\Gamma_\varepsilon} U_0(s) e^{\phi(s,t,x)} ds \quad (70)$$

$$\Gamma_\varepsilon = \{s_+ + iv; |v - v_k| \geq \varepsilon \quad \forall k \in \mathbb{Z}\}, \quad v_k = kv_* \quad (71)$$

where  $\varepsilon > 0$  is small enough to have that the intervals  $[s_k(t, x) - i\varepsilon, s_k(t, x) + i\varepsilon]$  are disjoint. We first estimate the integral over  $\Gamma_\varepsilon$  using Lemma 4:

$$\begin{aligned} \left| \int_{\Gamma_\varepsilon} U_0(s) e^{\phi(s,t,x)} ds \right| &\leq \int_{\Gamma_\varepsilon} |U_0(s)| e^{\Re \phi(s,t,x)} ds \\ &\leq e^{\phi(s_+(t,x),t,x) - \gamma t} \int_{s_+ - i\infty}^{s_+ + i\infty} |U_0(s)| ds, \end{aligned}$$

and we conclude as for point (a) above.

For  $v \in [v_k - \varepsilon, v_k + \varepsilon]$  for some  $k \in \mathbb{Z}$ , we write  $s_k = s_+ + iv_k$  and

$$\begin{aligned} \frac{1}{2\pi i} \int_{s_k(t,x)-i\varepsilon}^{s_k(t,x)+i\varepsilon} U_0(s) e^{\phi(s,t,x)} ds &= \frac{U_0(s_k)}{2\pi i} \int_{s_k(t,x)-i\varepsilon}^{s_k(t,x)+i\varepsilon} e^{\phi(s,t,x)} ds + \\ &+ \frac{1}{2\pi i} \int_{s_k(t,x)-i\varepsilon}^{s_k(t,x)+i\varepsilon} (U_0(s) - U_0(s_k)) e^{\phi(s,t,x)} ds \\ &= I_1(k) + I_2(k). \end{aligned}$$

Using Lemma (2) we may repeat the analysis of the terms  $I_1$  and  $I_2$  performed in the proof of point (a) above to estimate  $I_1(k)$  and  $I_2(k)$  as follows:

$$\begin{aligned} I_1(k) &= \frac{U_0(s_k)}{\sqrt{2\pi}} \frac{e^{\phi(s_k(t,x),t,x)}}{\sqrt{tK''(s_k)}} \left( 1 + \mathcal{O} \left( \frac{e^{-\frac{t(\varepsilon^2 K''(q_0))}{2}}}{\varepsilon \sqrt{tK''(q_0)}} \right) \right), \\ |I_2(k)| &\leq \omega_k(\varepsilon) \frac{e^{\phi(s_k(t,x),t,x)}}{\sqrt{2\pi t K''(s_k)}} \left( 1 + \mathcal{O} \left( \frac{e^{-\frac{t(\varepsilon^2 K''(q_0))}{2}}}{\varepsilon \sqrt{tK''(q_0)}} \right) \right), \end{aligned} \quad (72)$$

$$\omega_k(\varepsilon) = \sup_{|v| \leq \varepsilon} |U_0(s_k + iv) - U_0(s_k)|. \quad (73)$$

To estimate the sum of  $I_1(k)$ , as we have seen in (61)

$$\phi(s_k, t, x) = -s_k \log x + tK(s_k) = \phi(s_+, t, x) - \frac{2i\pi k}{\log \theta} \log x.$$

We deduce:

$$e^{\phi(s_k(t,x),t,x)} = e^{\phi(s_+(t,x),t,x)} e^{-\frac{2i\pi k}{\log \theta} \log x}$$

and therefore,

$$\sum_{k \in \mathbb{Z}} I_1(k) = \left( \frac{1}{\sqrt{2\pi}} + \mathcal{O} \left( \frac{e^{-\frac{t(\varepsilon^2 K''(p_0))}{2}}}{\varepsilon \sqrt{tK''(p_0)}} \right) \right) \frac{e^{\phi(s_+(t,x),t,x)}}{\sqrt{tK''(s_+)}} \sum_{k \in \mathbb{Z}} U_0(s_k) e^{-\frac{2i\pi k}{\log \theta} \log x} \quad (74)$$

$$\sum_{k \in \mathbb{Z}} |I_2(k)| \leq \left( 1 + \mathcal{O} \left( \frac{e^{-\frac{\varepsilon^2 t K''(p_0)}{2}}}{\varepsilon \sqrt{tK''(p_0)}} \right) \right) \frac{e^{\phi(s_+(t,x),t,x)}}{\sqrt{2\pi t K''(s_+)}} \sum_{k \in \mathbb{Z}} |\omega_k(\varepsilon)| \quad (75)$$

We define  $y = \frac{\log x}{\log \theta}$ , (or  $x = \theta^y$ ), and first consider the series:  $\sum_{k \in \mathbb{Z}} U_0(s_k) e^{-2i\pi k y}$ , where

$$\begin{aligned} U_0(s_k) &= \int_0^\infty u_0(x) x^{s_k-1} dx = \int_0^\infty u_0(x) e^{(s_+-1+\frac{2i\pi k}{\log \theta}) \log x} dx \\ &= \int_{\mathbb{R}} u_0(\theta^y) \theta^{(s_+-1)y} e^{2i\pi k y} \theta^y \log \theta dy = \int_{\mathbb{R}} u_0(\theta^y) \theta^{s_+y} e^{2i\pi k y} \log \theta dy. \end{aligned}$$

Since the function  $g(y) = u_0(\theta^y)\theta^{s+y}$  is real valued,  $U_0(s_{-k}) = \overline{U_0(s_k)}$ , and then  $U_0(s_{-k})e^{2i\pi ky} = \overline{U_0(s_k)e^{-2i\pi ky}}$ . We first deduce, using (58), that the series is absolutely convergent. We may then rearrange the series to obtain:

$$\begin{aligned} \sum_{k \in \mathbb{Z}} U_0(s_k) e^{-2i\pi ky} &= U_0(s_0) + \sum_{k \in \mathbb{N}^*} \left( U_0(s_k) e^{-2i\pi ky} + \overline{U_0(s_k) e^{-2i\pi ky}} \right) \\ &= U_0(s_+) + 2 \sum_{k \in \mathbb{N}^*} \Re \left( U_0(s_k) e^{-2i\pi ky} \right), \end{aligned}$$

where we see that the series is real valued.

Let us now turn to the sum of  $I_2(k)$ . We have

$$\begin{aligned} \sum_{k \in \mathbb{Z}} \sup_{|v| \leq \varepsilon} |U_0(s_k + iv) - U_0(s_k)| &\leq \sum_{k=-L}^L \sup_{|v| \leq \varepsilon} |U_0(s_k + iv) - U_0(s_k)| + \\ &\quad + \sum_{k \in \mathbb{Z}, |k| > L} \sup_{|v| \leq \varepsilon} (|U_0(s_k + iv)| + |U_0(s_k)|) \\ &\leq \sum_{k=-L}^L \sup_{|v| \leq \varepsilon} |U_0(s_k + iv) - U_0(s_k)| + \\ &\quad + 2C_1(p_0 + \delta, q_0 - \delta) \sum_{k \in \mathbb{Z}, |k| > L} \frac{1}{|v_k|^2} \end{aligned}$$

Since by definition  $|v_k| = \left| \frac{2\pi k}{\log \theta} \right|$ ,

$$\lim_{L \rightarrow \infty} 2C_1(p_0 + \delta, q_0 - \delta) \sum_{k \in \mathbb{Z}, |k| > L} \frac{1}{|v_k|^2} = 0.$$

Moreover, for all  $L > 0$  fixed, all  $k \in \mathbb{Z}$ ,  $|k| \leq L$  and  $R > 1$ :

$$\begin{aligned} U_0(s_k + iv) - U_0(s_k) &= \int_0^\infty z^{s_+ + iv_k - 1} u_0(z) (z^{iv} - 1) dz \\ |U_0(s_k + iv) - U_0(s_k)| &\leq \int_{\frac{1}{R}}^R |z^{s_+ - 1} z^{iv_k} u_0(z) (z^{iv} - 1)| dz + \\ &\quad + 2 \int_R^\infty |z^{q_0 - \delta - 1} u_0(z)| dz + 2 \int_0^{\frac{1}{R}} |z^{p_0 + \delta - 1} u_0(z)| dz \end{aligned}$$

Arguing as above,

$$\lim_{R \rightarrow \infty} \left( \int_R^\infty |z^{q_0 - \delta - 1} u_0(z)| dz + \int_0^{\frac{1}{R}} |z^{p_0 + \delta - 1} u_0(z)| dz \right) = 0.$$

For all  $R > 1$  fixed,

$$\int_{\frac{1}{R}}^R |z^{s_+ - 1} z^{iv_k} u_0(z) (z^{iv} - 1)| dz \leq (\varepsilon \log R) \int_0^\infty |(z^{p_0 + \delta - 1} + z^{q_0 - \delta - 1}) u_0(z)| dz. \quad (76)$$

We finally have, for all  $L > 0$ ,  $R \rightarrow \infty$  and  $\varepsilon \rightarrow 0$  such that  $t\varepsilon^2 \rightarrow \infty$ :

$$\begin{aligned} \sum_{k \in \mathbb{Z}} |I_2(k)| &\leq \left( 1 + \mathcal{O} \left( \frac{e^{-\frac{\varepsilon^2 t K''(q_0)}{2}}}{\varepsilon \sqrt{t K''(q_0)}} \right) \right) \frac{e^{\phi(s_+(t,x), t, x)}}{\sqrt{2\pi t K''(s_+)}} S(L, R, \varepsilon), \\ S(L, R, \varepsilon) &\leq 2C_1(p_0, q_0) \sum_{k \in \mathbb{Z}, |k| > L} \frac{1}{|v_k|^2} + 2(\varepsilon \log R) L \int_0^\infty \left| (z^{p_0+\delta-1} + z^{q_0-\delta-1}) u_0(z) \right| dz \\ &\quad + 4L \left( \int_R^\infty \left| z^{q_0-\delta-1} u_0(z) \right| dz + \int_0^{\frac{1}{R}} \left| z^{p_0+\delta-1} u_0(z) \right| dz \right) \end{aligned}$$

By Lemma 4 and (59) we may estimate now the second integral in the right hand side of (70) as follows:

$$\begin{aligned} \left| \frac{1}{2\pi i} \int_{\Gamma_\varepsilon} U_0(s) e^{\phi(s, t, x)} ds \right| &\leq \frac{e^{\phi(s_+(t,x), t, x) - \gamma(\varepsilon)t}}{2\pi} \int_{\Re(s)=s_+, |\Im(s)| \geq \varepsilon} |U_0(s)| ds \\ &\leq \frac{e^{\phi(s_+(t,x), t, x) - \gamma(\varepsilon)t}}{2\pi} C(p_0 + \delta, q_0 - \delta). \end{aligned}$$

This proves point (b).  $\square$

#### 4.4 Proof of Corollary 1.4

In this Section we first prove Corollary 1.4. Then we comment on the links of Corollary 1.4 with Theorem 1 in [1].

**Proof.** [Corollary 1.4.] We only write the proof of the convergence for  $r(t, y)$  since the convergence of  $\tilde{r}(t, y)$  follows from the same arguments. For any continuous and bounded test function  $\phi$  on  $\mathbb{R}$ :

$$\int_{-\infty}^{+\infty} \phi(y) r(t, y) dy = \int_{-\infty}^{K'(p_0)} + \int_{K'(p_0)}^{K'(p_0+\delta)} + \int_{K'(p_0+\delta)}^{K'(q_0-\delta)} + \int_{K'(q_0-\delta)}^{K'(q_0)} + \int_{K'(q_0)}^{+\infty} \phi(y) r(t, y) dy.$$

For the first and the last integrals, are estimated using Theorem 1.2 to prove that they vanish. Consider for example the first. Using property (i) of Theorem 1.2 we deduce that there exists  $T > 0$  such that, for all  $t \geq T$ :

$$\begin{aligned} \int_{-\infty}^{K'(p_0)} \phi(y) r(t, y) dy &= \int_0^{e^{K'(p_0)t}} x u(t, x) \phi \left( \frac{\log(x)}{t} \right) dx \\ &\leq 2 \|\phi\|_\infty |b_0| e^{K(p_0)t-t} \int_0^{e^{K'(p_0)t}} x^{1-p_0} dx = \frac{2 \|\phi\|_\infty |b_0|}{2-p_0} e^{H(p_0)t}. \end{aligned} \tag{77}$$

where

$$H(\nu) = K(\nu) + (2 - \nu)K'(\nu) - 1.$$

By hypothesis (4),  $H(2) = 0$ . Moreover, since  $H'(\nu) = (2 - \nu)K''(\nu)$  and  $K''(\nu) \geq 0$  we deduce that  $H$  has a maximum at  $\nu = 2$  and therefore  $H(\nu) < 0$  for all  $\nu \neq 2$ . In particular, since  $p_0 < 1$ :  $H(p_0) < 0$ . The same argument gives:

$$\int_{K'(q_0)}^{\infty} \phi(y)r(t, y)dy \leq \frac{2\|\phi\|_{\infty}|a_0|}{q_0 - 2}e^{H(q_0)t}. \quad (78)$$

For the third integral, we use Theorem 1.3-a), which gives, for  $t \rightarrow +\infty$ ,  $\varepsilon \rightarrow 0$ ,  $t\varepsilon^2 \rightarrow \infty$ :

$$\int_{K'(p_0+\delta)}^{K'(q_0-\delta)} \phi(y)r(t, y)dy = t\Theta_1(t, \varepsilon) \int_{K'(p_0+\delta)}^{K'(q_0-\delta)} \phi(y)e^{\Psi(y)t} \frac{U_0(K'^{-1}(y))dy}{\sqrt{2\pi t K''(K'^{-1}(y))}} \quad (79)$$

with  $\Theta_1(t, \varepsilon)$  defined in (35) and

$$\Psi(y) = 2y - yK'^{-1}(y) + K(K'^{-1}(y)) - 1.$$

It is easy to check that  $\Psi$  has a unique maximum at  $y_0 = K'(2)$  with  $\Psi(y_0) = 0$  and  $\Psi''(y) = -\frac{1}{K''(K'^{-1}(y))}$ . We may then apply Laplace's method in the right hand side of 79 to obtain:

$$\lim_{t \rightarrow \infty} \int_{K'(p_0+\delta)}^{K'(q_0-\delta)} \phi(y)r(t, y)dy = U_0(2)\phi(K'(2)) \quad (80)$$

with  $U_0(2) = \int xu_0(x)dx$  the initial mass.

To bound the second integral (the fourth is similar), we go back to the expression in  $x$ :

$$\begin{aligned} \int_{K'(p_0)}^{K'(p_0+\delta)} \phi(y)r(t, y)dy &= \int_{e^{K'(p_0)t}}^{e^{K'(p_0+\delta)t}} xu(t, x)\phi\left(\frac{\log(x)}{t}\right)dx \\ &\leq \|\phi\|_{L^\infty(\mathbb{R})} \int_0^{e^{K'(p_0+\delta)t}} xu(t, x)dx. \end{aligned}$$

We now use Formula (16) to write for  $\nu \in (p_0, q_0)$

$$|u(t, x)| = \left| \frac{1}{2\pi i} \int_{\nu-i\infty}^{\nu+i\infty} U_0(s) e^{\phi(s, t, x)-t} ds \right| \leq \frac{1}{2\pi} e^{\Re(\phi(\nu, t, x))-t} \int_{\nu-i\infty}^{\nu+i\infty} |U_0(s)| ds.$$

We have by definition  $\phi(\nu, t, x) = -\nu \log(x) + tK(\nu)$ . We thus have for  $\nu \in (p_0, q_0)$

$$\int_0^{e^{K'(p_0+\delta)t}} xu(t, x)dx \leq C e^{tK(\nu)-t} \int_0^{e^{K'(p_0+\delta)t}} x e^{-\nu \log(x)} dx = C e^{tK(\nu)-t} e^{(2-\nu)K'(p_0+\delta)t}.$$

Consider now the function

$$F(\nu) = K(\nu) + (2 - \nu)K'(p_0 + \delta) - 1.$$

By hypothesis (4),  $F(2) = 0$ . Moreover:

$$F'(\nu) = K'(\nu) - K'(p_0 + \delta) = \int_0^1 k_0(x) \log x \left( x^{\nu-1} - x^{p_0+\delta-1} \right) dx$$

and then, if  $\delta > 0$  is such that  $p_0 + \delta - 1 < 1$ :

$$F'(2) = \int_0^1 k_0(x) \log x \left( x - x^{p_0+\delta-1} \right) dx > 0.$$

There exists therefore  $\nu_1 \in (p_0, 2)$  such that  $F(\nu_1) < 0$  and

$$\int_0^{e^{K'(p_0+\delta)t}} xu(t, x) dx \leq C e^{F(\nu_1)t} \rightarrow 0, \text{ as } t \rightarrow \infty. \quad (81)$$

In order to bound the fourth integral we write:

$$\begin{aligned} \int_{K'(q_0-\delta)}^{K'(q_0)} \phi(y) r(t, y) dy &= \int_{e^{K'(q_0-\delta)t}}^{e^{K'(q_0)t}} xu(t, x) \phi\left(\frac{\log(x)}{t}\right) dx \\ &\leq \|\phi\|_{L^\infty(\mathbb{R})} \int_{e^{K'(q_0-\delta)t}}^{e^{K'(q_0)t}} xu(t, x) dx. \end{aligned}$$

Using again Formula 16 we deduce, for  $\nu \in (2, q_0)$ :

$$\begin{aligned} \int_{e^{K'(q_0-\delta)t}}^{e^{K'(q_0)t}} xu(t, x) dx &\leq C e^{tK(\nu)-t} \int_{e^{K'(q_0-\delta)t}}^{e^{K'(q_0)t}} x e^{-\nu \log(x)} dx \\ &= \frac{C}{\nu-2} e^{tK(\nu)-t} \left( e^{(2-\nu)K'(q_0-\delta)t} - e^{(2-\nu)K'(q_0)t} \right) \\ &\leq \frac{C}{\nu-2} e^{tK(\nu)-t} e^{(2-\nu)K'(q_0-\delta)t}. \end{aligned}$$

If we consider now the function

$$G(\nu) = K(\nu) + (2-\nu)K'(q_0-\delta) - 1.$$

By hypothesis (4),  $G(2) = 0$  and since  $K'$  is increasing,  $G'(\nu) = K'(\nu) - K'(q_0-\delta)$  is positive for  $\nu > q_0 - \delta$  and negative for  $\nu < q_0 - \delta$ , so if  $q_0 - \delta > 2$  we have  $G(\nu) < 0$  for any  $\nu \in (2, q_0 - \delta)$ . Taking simply  $\nu = q_0 - \delta$  we have

$$\int_{e^{K'(q_0-\delta)t}}^{e^{K'(q_0)t}} xu(t, x) dx \leq \frac{C}{q_0 - \delta - 2} e^{tK(q_0-\delta)-t} e^{(2-q_0+\delta)K'(q_0-\delta)t} = \frac{C e^{G(q_0-\delta)t}}{q_0 - \delta - 2}, \quad (82)$$

and the right-hand side vanishes exponentially when  $t \rightarrow \infty$ .

**Remark 4.3** Notice that (77), (78), (81) and (82) give rates of convergence to zero as  $t \rightarrow \infty$ . A careful application of Laplace's method to the third integral would also give a rate of convergence in the limit (80) under suitable assumptions on the function

$$h_\phi(y) = \phi(y) \frac{U_0(K'^{-1}(y)) dy}{\sqrt{2\pi K''(K'^{-1}(y))}}$$

in a neighborhood of  $y_0$ . For example, if for some  $\alpha > 0$ :

$$h_\phi(y) = h_\phi(y_0) + \mathcal{O}((y - y_0)^\alpha) \text{ as } y \rightarrow y_0$$

then

$$\int_{K'(p_0+\delta)}^{K'(q_0-\delta)} \phi(y) r(t, y) dy = U_0(2) \phi(K'(2)) + \mathcal{O}(t^{-\frac{\alpha+1}{2}} + t^{-1}) \text{ as } t \rightarrow \infty.$$

As we said in the introduction the fragmentation equation with breaking probability  $k(x, y)$  of the form (4) and total fragmentation rate homogeneous of degree  $\gamma = 0$  has not been previously considered in the literature. However the corresponding fragmentation process  $X = (X(t), t \geq 0)$  has been studied in [1]. Let us briefly recall some of the results obtained therein.

The goal is to describe a system of particles that split independently of each other to give smaller particles and each obtained particle splits in turn, independent of the past and of other particles etc. To this end a stochastic process,  $X = (X(t), t \geq 0)$ , is introduced that takes values in the state space denoted as  $\mathcal{S}^\downarrow(y)$ , the set of all sequences  $Y = (y_i)_{i \in \mathbb{N}^*}$  such that the following holds:

$$y_1 \geq \dots \geq y_i \geq y_{i+1} \geq \dots \geq 0 \quad \text{and} \quad y = \sum_{i=1}^{\infty} y_i \leq 1,$$

Under suitable assumptions on the splitting measure (giving the rate at which a particle with mass one splits) it is then proved in Theorem 1 (i) of [1] that the random measure  $\rho_t(dy)$  defined by

$$\rho_t(dy) = \sum_{i=1}^{\infty} X_i(t) \delta_{\frac{1}{t} \log X_i(t)}(dy)$$

converges to  $\delta_{-\mu}$  in probability for some  $\mu < \infty$ . Moreover, if  $\tilde{\rho}_t$  is the random measure defined as the image of  $\rho_t$  by the map  $x \rightarrow \sqrt{t}(x + \mu)/\sigma$ , it is proved in Theorem 1 (ii) that  $\tilde{\rho}_t$  converges in probability to the standard normal distribution  $\mathcal{N}(0, 1)$ .

It is not easy to compare with some precision the stochastic results obtained in [1] on the microscopic level and our deterministic results for the density of particles. But, although the two approaches consider different objects and use different methods, it is possible nevertheless to establish some relations between the asymptotic behavior in law of the stochastic probabilities  $\rho_t(dy)$  and  $\tilde{\rho}_t(dy)$  and the long time behavior in the weak sense of the density function  $u$ .

The laws of the random measures  $\rho_t(dy)$  and  $\tilde{\rho}_t(dy)$  considered in [1] correspond to the following rescalings of our function  $u$ :

$$r(t, y) dy = t y e^{2ty} u(t, e^{ty}) dy$$

and

$$\tilde{r}(t, z) dz = r\left(t, y_0 + \frac{\sigma z}{\sqrt{t}}\right) \frac{\sigma dz}{\sqrt{t}}$$

respectively. The results of Corollary 1.4 above correspond then to the convergence in law of the random measures  $\rho_t(dy)$  and  $\tilde{\rho}_t(dy)$  towards  $\delta_{-\mu}$  and  $\mathcal{N}(0, 1)$  respectively. The convergences in law for  $\rho_t(dy)$  and  $\tilde{\rho}_t(dy)$  follow from their convergence in probability proved in Theorem 1 of [1].

## 5 Further remarks on the asymptotic results

### 5.1 Study of the different regions of asymptotic behaviour.

It is proved in Theorem 3.1 that if  $u$  is a solution of the fragmentation equation (5) with suitable integrability properties,  $xu(t, x)$  converges to a Dirac mass at the origin as  $t \rightarrow +\infty$ . The formation of that Dirac mass may be followed using the descriptions of the long time behaviour of  $u$  in Theorem 1.1 and Theorem 1.2. As already said, Theorem 1.1 implies a uniform time exponential decay for  $x > 1$ .

In this section we give some indications about the domain where, following Theorem 1.2, the function  $xu(t, x)$  concentrates towards a Dirac measure, as  $t$  increases.

All the cases of Theorem 1.2 may be described, roughly speaking, as giving a formula for  $xu(t, x)$  of the form

$$xu(t, x) \approx A(t, \sigma)x^{-\sigma+1}e^{(K(\sigma)-1)t}, \quad (83)$$

with  $A = a_0$  and  $\sigma = q_0$  in Theorem 1.1 and Theorem 1.2 (ii),  $A = b_0$  and  $\sigma = p_0$  in Theorem 1.2 (i), and  $\sigma = s_+(t, x)$ ,  $A(t, s_+) = \frac{U_0(s_+)}{\sqrt{2\pi t K''(s_+)}}$  in Theorem 1.2 (iii).

Since  $A$  is either a constant or a function bounded by a power law of  $t$ , the long-time behaviour is essentially given by the (exponential) behaviour of the term  $x^{-\sigma+1}e^{(K(\sigma)-1)t}$ .

In order to understand the behaviour of the function  $xu(t, x)$  as  $t$  increases we may consider the curves  $\mathcal{C}_r$  in the plane  $(x, t)$  defined by  $s_+(t, x) = r$  for any constant  $r \in \mathbb{R}$ . The curve  $\mathcal{C}_r$  also correspond to the set of points  $(x, t)$  where  $x = e^{K'(r)t}$ . Since moreover, the function  $K'$  is negative and increasing, it is then easy to understand the behaviour of the function  $x^{-\sigma+1}e^{(K(\sigma)-1)t}$  along such curves.

Let us begin by the case of Theorem 1.3 when  $p_0 < s_+ < q_0$ : we may rewrite (83) as

$$xu(t, x) \approx A(t, s_+)e^{tF(s_+(t, x))}$$

where we define  $F$  by

$$F(s) = K(s) - 1 - (s - 1)K'(s). \quad (84)$$

Let us now turn to the cases  $s_+ > q_0$  and  $s_+ < p_0$ . Equation (83) becomes

$$xu(t, x) \approx A(p)e^{tG(p, s_+)},$$

with  $p = p_0$  or  $p = q_0$  and  $G$  defined by

$$G(p, s) = K(p) - 1 - (p - 1)K'(s). \quad (85)$$

The behaviour of the signs of the functions  $G$  and  $F$ , which determine the exponential growth or decay of  $xu(t, x)$  along the lines  $x = e^{K'(s_+)t}$ , is given in the following lemma. Its proof is given in the Appendix (Lemma 7).

**Lemma 5** *The function  $F(p)$  defined by (84) has two zeros  $\bar{p} \in (p_1, 1)$  and  $\bar{q} \in (2, \infty)$ . It is negative on  $(p_1, \bar{p}) \cap (\bar{q}, \infty)$  and positive on  $(\bar{p}, \bar{q})$ .*

If  $p_0 < \bar{p}$ , the function  $G(p_0, s)$  defined by (85) is negative for  $s \in (p_1, p_0)$ . If  $\bar{p} < p_0$ , the function  $G(p_0, \cdot)$  has a unique zero  $\bar{s}(p_0) \in (p_1, p_0)$ , so that  $G(p, s)$  is negative for  $s < \bar{s}(p_0)$  and positive for  $\bar{s}(p_0) < s < p_0$ .

Similarly: If  $q_0 > \bar{q}$ , the function  $G(q_0, s)$  is negative for  $s \in (q_0, \infty)$ . If  $\bar{q} > q_0$ , the function  $G(q_0, \cdot)$  has a unique zero  $\bar{s}(q_0) \in (q_0, \infty)$ , so that  $G(q, s)$  is negative for  $s > \bar{s}(q_0)$  and positive for  $q_0 < s < \bar{s}(q_0)$ .

The sign of  $F$  implies the exponential growth or decay of  $xu(t, x)$  on the domain where Theorem 1.3 applies.

Let us now use this lemma to follow the behaviour of  $xu(t, x)$  along the curves  $\mathcal{C}_r$ , which correspond to  $x = e^{tK'(r=s_+)}$ .

- $p_0 < \bar{p} < 1$  (resp.  $2 < \bar{q} < q_0$ ):  $G(p_0, s_+)$  is negative for  $s_+ \in (p_1, p_0)$  (resp.  $G(q_0, s_+)$  is negative for  $s_+ \in (q_0, \infty)$ ), we have an exponential decay on  $s_+ < p_0$  (resp.  $q_0 < s_+$ ). For  $s$  on the interval  $(p_0, 2)$  (resp.  $(1, q_0)$ ), the behaviour is given by the function  $F$ : exponential decay for  $s_+ \in (p_0, \bar{p})$  (resp.  $s_+ \in (\bar{q}, q_0)$ ) and exponential growth for  $s_+ \in (\bar{p}, 2)$  (resp.  $s_+ \in (1, \bar{q})$ ).

All together, this gives a limit for the zone of exponential growth given on the left by  $s_+ = \bar{p}$  (resp. on the right by  $s_+ = \bar{q}$ ).

- $\bar{p} < p_0 < 1$  (resp.  $2 < q_0 < \bar{q}$ ):  $G(p_0, \cdot)$  has a unique zero  $\bar{s}(p_0) \in (p_1, p_0)$  (resp.  $G(q_0, \cdot)$  has a unique zero  $\bar{s}(q_0)$  on  $(q_0, \infty)$ ), and we have an exponential decay for  $s_+ < \bar{s}(p_0)$  (resp.  $\bar{s}(q_0) < s_+$ ) and an exponential increase for  $\bar{s}(p_0) < s_+ < p_0$  (resp.  $q_0 < s_+ < \bar{s}(q_0)$ ). For  $s_+ \in (p_0, 2)$  (resp.  $s_+ \in (1, q_0)$ ), the behaviour is given by the function  $F$ : since  $\bar{p} < p_0$  (resp.  $q_0 < \bar{q}$ ), there is an exponential growth for  $s_+ \in (p_0, 2)$  (resp.  $s_+ \in (1, q_0)$ ).

All together, this gives a limit for the zone of exponential growth given on the left by  $s_+ = \bar{s}(p_0)$  (resp. on the right by  $s_+ = \bar{s}(q_0)$ ).

Of course, any combination of these two cases, for the respective positions of  $\bar{p}$  and  $p_0$  for "  $s_+$  small " and of  $\bar{q}$  and  $q_0$  for "  $s_+$  large " is possible, which finally leads to four possible situations.

We also notice that for the case of very regular initial data, *i.e.*  $p_0 < \bar{p}$  and  $\bar{q} < q_0$ , the zone of exponential growth or decay depends only on the fragmentation kernel properties, which define the values of  $\bar{p}$  and  $\bar{q}$ , and endly the dependence on the initial condition only appears as a correction term (presence or not of  $\frac{1}{\sqrt{t}}$ ).

### Example 1: homogeneous kernel.

For the homogeneous kernel,  $k_0 = 2$  so that Equation (13) gives  $K(s) = \frac{2}{s}$ , so that  $p_1 = 0$  and we have

$$s_+(t, x) = \sqrt{\frac{2t}{-\log(x)}}, \quad K'(s_+) = -\frac{2}{s_+^2}.$$

We calculate easily that

$$F(s) = \frac{2}{s} - 1 + 2\frac{s-1}{s^2} = \frac{-s^2 + 4s - 2}{s^2} = -\frac{(s - (2 + \sqrt{2}))(s - (2 - \sqrt{2}))}{s^2},$$

so that  $\bar{p} = 2 - \sqrt{2}$ ,  $\bar{q} = 2 + \sqrt{2}$ , and

$$G(p, s) = \frac{2}{p} - 1 + 2\frac{p-1}{s^2} = \frac{1}{s^2} \left( s^2 \left( \frac{2}{p} - 1 \right) + 2(p-1) \right),$$

so that  $\bar{s}(p_0) = \sqrt{\frac{2p_0(1-p_0)}{2-p_0}}$ ,  $\bar{s}(q_0) = \sqrt{\frac{2q_0(q_0-1)}{q_0-2}}$ . Notice that  $\bar{s}(p_0) \rightarrow 0 = p_1$  when  $p_0 \rightarrow 1$ , and  $\bar{s}(q_0) \rightarrow \infty$  when  $q_0 \rightarrow 2$ : the “less regular” the initial data, the largest the domain of exponential growth.

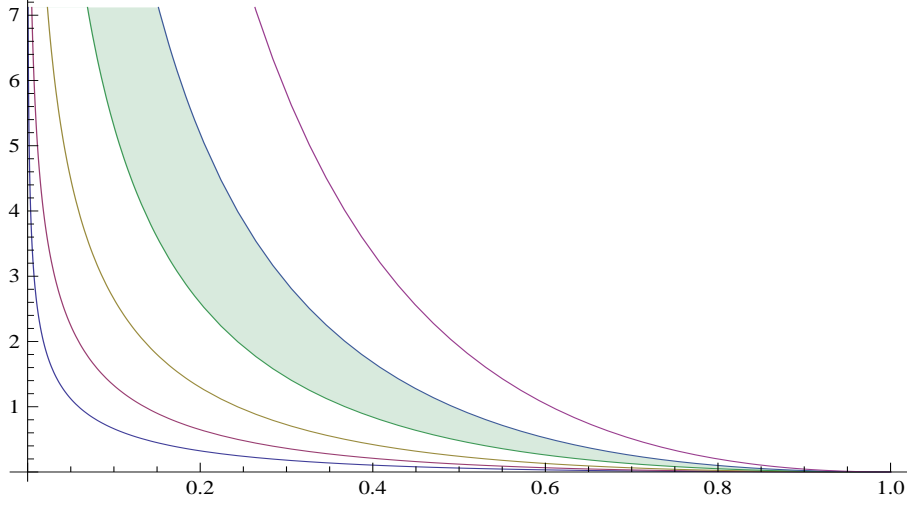


Figure 1: Different curves of the form  $s_+ = \gamma$  for different values of  $\gamma > 0$ , so that  $2t = -\gamma^2 \log x$ . The function  $xu(t, x)$  tends to zero exponentially fast uniformly out of the green region contained between the light blue and the green curves. It tends to infinity inside that region. As  $t \rightarrow \infty$ , the function  $xu(t, x)$  concentrates in the interval  $x \in \left( e^{-\frac{2t}{\gamma_\ell^2}}, e^{-\frac{2t}{\gamma_r^2}} \right)$ .

The values of  $\gamma_r$  in the light blue curve and  $\gamma_\ell$  in the green curve in Figure 1 depend on the relative values of  $q_0$  and  $\bar{q} = 2 + \sqrt{2}$  and of  $p_0$  and  $\bar{p} = 2 - \sqrt{2}$  respectively, as detailed above for the general case.

## Example 2: Mitosis kernel.

For the mitosis kernel,  $k_0 = 2\delta_{z=\frac{1}{2}}$ : Condition H is satisfied, and  $K(s) = 2^{2-s}$ ,  $p_1 = -\infty$  and we have  $s_+(t, x)$  defined by

$$2^{2-s_+(t,x)} = e^{(2-s_+)\log(2)} = -\frac{\log(x)}{t \log(2)}, \quad s_+(t, x) = 2 - \frac{\log\left(-\frac{\log(x)}{t \log(2)}\right)}{\log(2)}.$$

We calculate easily that

$$F(s) = 2^{2-s} - 1 + (s-1) \log(2) 2^{2-s},$$

and  $\bar{p} < 1$  and  $\bar{q} > 2$  are defined by  $F(\bar{p}/\bar{q}) = 0$ , see Figure 2.

$$G(p, s) = 2^{2-p} - 1 + (p-1) \log(2) 2^{2-s},$$

so that  $\bar{s}(p_0) = 2 - \frac{1}{\log(2)} \log\left(\frac{1-2^{2-p_0}}{(p_0-1) \log(2)}\right)$ ,  $\bar{s}(q_0) = 2 - \frac{1}{\log(2)} \log\left(\frac{1-2^{2-q_0}}{(q_0-1) \log(2)}\right)$ .

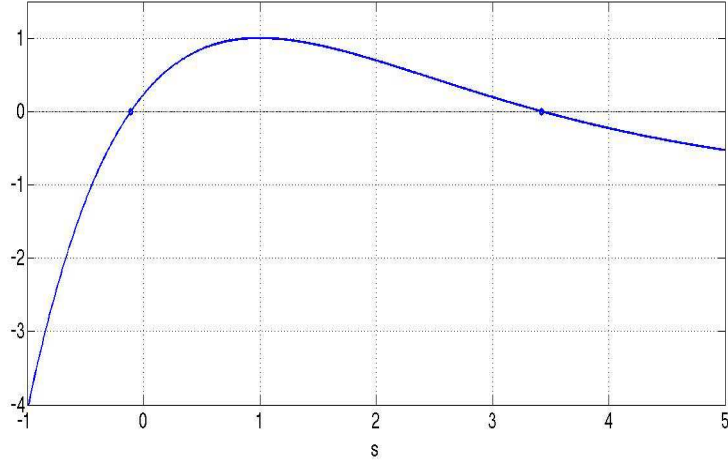


Figure 2: Function  $F(s)$  for  $k_0 = 2\delta_{y=\frac{1}{2}}$ . We see the two zeros  $\bar{p} < 0$  and  $\bar{q} > 3$ .

## 5.2 The asymptotic behaviour for the growth-fragmentation equation.

The asymptotic behaviour for the growth-fragmentation equation (8) may also be deduced from Theorems 1.1 and 1.2. Using Formula 7, we know that the solution  $v$  satisfies  $v(t, x) = e^{-ct}u(t, xe^{-ct})$  : we can apply directly the results of Theorems 1.1 and 1.2. It remains to analyse in which part of the plane  $(x, t)$  there is an exponential growth or decay.

First, for  $x = \alpha e^{ct}$  with  $\alpha > 1$ , Theorem 1.1 implies, for a constant  $C > 0$ ,

$$xv(t, x) = \alpha u(t, \alpha) \sim C\alpha^{1-q_0}e^{(K(q_0)-1)t},$$

so that there is an exponential decrease for the domain  $x > e^{ct}$ .

For the domain  $x < e^{ct}$ , the lines where we can follow the mass concentration are now given by  $s_+(t, xe^{-ct})$  constant, *i.e.*  $x = e^{(c+K'(s_+))t}$ . Notice that contrarily to the fragmentation equation, these lines either go to infinity or to zero, with a limit case for  $s_+ = K'^{-1}(-c)$ .

To avoid too long and repetitive considerations, we focus on the case of very smooth data, where  $p_0 < s_+ < q_0$  is the main domain of interest in Theorem 1.2. On that domain, for a constant  $C > 0$ ,

$$xv(t, x) = e^{K'(s_+)t}u(t, e^{K'(s_+)t}) \sim \frac{C}{\sqrt{t}}e^{(K'(s)(1-s)+K(s)-1)t},$$

where we recognize  $e^{F(s)t}$  with  $F$  defined by (84). We can thus apply directly the study done for the fragmentation equation: the domain of exponential growth is for  $\bar{p} < s_+(t, xe^{-ct}) < \bar{q}$ , with  $\bar{p}, \bar{q}$  defined in Lemma 5. This corresponds to curves  $x = e^{(c+K'(s_+))t}$  with  $K'(s_+) \in (K'(\bar{p}), K'(\bar{q}))$ . What is new here is to investigate whether these curves go to zero or to infinity in large times. We also notice that the curve of maximal exponential growth is given for  $F(s) = F(1) = K(1) - 1$ . We have the following cases.

- $c > -K'(\bar{p})$  : the zone of exponential growth goes to infinity.
- $-K'(\bar{q}) < c < -K'(\bar{p})$  : the domain of exponential growth covers a wide range of lines  $x$ , going to infinity or to zero. In particular, the line of maximal growth, given by  $x = e^{(c+K'(1))t}$ , may either go to zero if  $c < -K'(1)$  or to infinity if  $c > -K'(1)$ .

- $c < -K'(\bar{q})$  : the domain of exponential growth goes to zero.

### 5.3 Numerical illustration

To visualize the long-time behaviour of Equation (5) as described in the previous results, we choose to simulate it in the log-variable  $y = \log(x)$ . The equation becomes, for  $n(t, y) = u(t, x)$ ,

$$\frac{\partial}{\partial t} n(t, y) + n(t, y) = \int_0^\infty \kappa_0(z) n(t, y + z) dz, \quad (86)$$

where  $\kappa_0(z) = k_0(e^{-z})$ . In the case of the mitosis kernel  $k_0 = 2\delta_{x=\frac{1}{2}}$ , we have

$$\frac{\partial}{\partial t} n(t, y) + n(t, y) = 4n(t, y + \log 2), \quad n(0, y) = n^{in}(y). \quad (87)$$

We choose a gaussian for the initial data  $n^{in} = e^{-\frac{(y-y_0)^2}{2}}$ , with  $y = -5$ , and following the solution in time we draw the limits of the zone where  $n(t, y) \geq 0.1 \max_x n(t, y)$ . This is given in Figure 3. A linear fit of the form  $t = a_1 y + a_2$  gives excellent results. This corresponds to curves  $x = e^{\frac{t-a_2}{a_1}}$ , which are of the predicted shape.

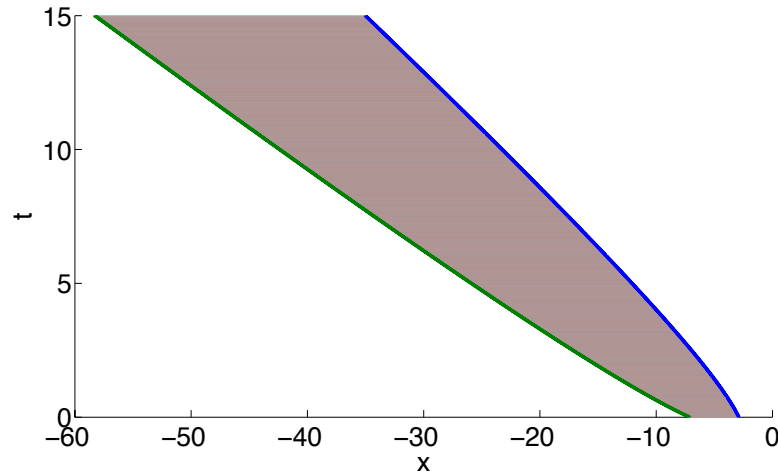


Figure 3: The solution of the growth-fragmentation equation in a log-scale. The two curves in blue and green represent limits for the major part of the support of  $u(t, x)$  : inside these two curves, the solution  $u(t, x)$  is larger than 10% of its maximal value in  $x$  at time  $t$ .

## Appendix

We give in this appendix the statements and proofs of some important auxiliary results.

**Lemma 6** Let  $k_0$  be defined by (4),  $K$  its Mellin transform defined by (13),  $p_1 \in [-\infty, 1)$  defined by (30), and  $\phi(s, t, x)$  defined by (17). Then for  $x < 1$ ,  $\phi(\cdot, t, x)$  is convex, it decreases for  $s \in (p_1, s_+(t, x))$  and increases for  $s \in (s_+(t, x), +\infty)$  where  $s_+(t, x)$  is defined by

$$s_+(t, x) = (K')^{-1} \left( \frac{\log(x)}{t} \right). \quad (88)$$

**Proof.** We consider the derivative of  $\phi$  with respect to  $s$

$$\frac{\partial \phi}{\partial s}(s, t, x) = -\log(x) + tK'(s) = -\log(x) + t \int_0^1 (\log(z)) z^{s-1} k_0(z) dz.$$

The second derivative of  $\phi$  with respect to  $s$  is

$$\frac{\partial^2 \phi}{\partial s^2}(s, t, x) = tK''(s) = t \int_0^1 (\log(z))^2 z^{s-1} k_0(z) dz > 0,$$

so that  $\phi(\cdot, t, x)$  is convex. By definition,

$$K'(\cdot) : (p_1, \infty) \longrightarrow (-\infty, 0), \quad K'(s) = \int_0^1 (\log(z)) z^{s-1} k_0(z) dz$$

is an increasing bijective function from  $(p_1 \leq 1, \infty)$  to  $(-\infty, 0)$ . Then, for each  $t > 0$ ,  $x > 0$ :

$$\frac{\partial \phi}{\partial s}(\cdot, t, x) : (p_1, \infty) \longrightarrow (-\infty, -\log x > 0)$$

is also increasing and bijective as a function of  $s$ , so that for all  $x \in (0, 1)$ , the function  $\frac{\partial \phi}{\partial s}(\cdot, t, x)$  has a unique zero on  $(p_1, \infty)$  given by (88). Since  $K'$  is an increasing function of  $s$ ,  $s_+(\cdot, x)$  is an increasing function of  $t$  and  $x \in (0, 1)$ .

**Lemma 7 (Lemma 5)** The function  $F(p)$  defined by (84) has two zeros  $\bar{p} \in (p_1, 1)$  and  $\bar{q} \in (2, \infty)$ . It is negative on  $(p_1, \bar{p}) \cap (\bar{q}, \infty)$  and positive on  $(\bar{p}, \bar{q})$ .

If  $p_0 < \bar{p}$ , the function  $G(p_0, s)$  defined by (85) is negative for  $s \in (p_1, p_0)$ . If  $\bar{p} < p_0$ , the function  $G(p_0, \cdot)$  has a unique zero  $\bar{s}(p_0) \in (p_1, p_0)$ , so that  $G(p, s)$  is negative for  $s < \bar{s}(p_0)$  and positive for  $\bar{s}(p_0) < s < p_0$ .

Similarly: If  $q_0 > \bar{q}$ , the function  $G(q_0, s)$  is negative for  $s \in (q_0, \infty)$ . If  $\bar{q} > q_0$ , the function  $G(q_0, \cdot)$  has a unique zero  $\bar{s}(q_0) \in (q_0, \infty)$ , so that  $G(q, s)$  is negative for  $s > \bar{s}(q_0)$  and positive for  $q_0 < s < \bar{s}(q_0)$ .

**Proof.** We first notice that  $F(p_1) = -\infty$  (this may be viewed by going back to the definition of  $K$ ,  $p_1$  and  $K'$ ),  $F(1) = K(1) - 1 > 0$ ,  $F(2) = -K'(2) > 0$  and  $F(+\infty) = -1$ , so that by continuity  $F$  has at least two zeros  $\bar{p} \in (p_1, 1)$  and  $\bar{q} \in (2, \infty)$ . Since  $F'(s) = (1-s)K'(s)$ ,  $F$  is increasing on  $(p_1, 1)$  and decreasing on  $(1, \infty)$ :  $\bar{p}$  and  $\bar{q}$  are its only zeros. This implies that  $F$  is negative on  $(p_1, \bar{p}) \cup (\bar{q}, \infty)$  and positive on  $(\bar{p}, \bar{q})$ .

We notice that  $G(s, s) = F(s)$ . Considering  $G$  as a function of  $s$  we have

$$\frac{\partial}{\partial s} G(p, s) = (1 - p)K''(s),$$

so that  $\frac{\partial}{\partial s} G(p, s) > 0$  for  $p < 1$ ,  $\frac{\partial}{\partial s} G(p, s) < 0$  for  $p > 1$ :  $G(p_0, \cdot)$  is an increasing function of  $s$  for  $p_0 \in (p_1, 1)$  and  $G(q_0, \cdot)$  is a decreasing function of  $s$  for  $q_0 \in (2, \infty)$ . Since  $G(p, p) = F(p)$  is positive on  $(\bar{p}, \bar{q})$  and negative on  $(p_1, \bar{p}) \cup (\bar{q}, \infty)$ , and  $G(p_0 < 1, p_1) = -\infty$ , and  $G(q_0 > 2, \infty) = K(q_0) - 1 < 0$ , we have two cases for each part  $(p_1, 1)$  and  $(2, \infty)$  according to the respective position of  $\bar{p}$  and  $p_0, \bar{q}$  and  $q_0$ , as stated in the lemma.

### Weak solutions of the fragmentation equation (5).

**Definition 1** A function  $u \in L^\infty([0, \infty); L^1((1+x)dx))$  is said a weak solution of the fragmentation equation (5) if for all  $\varphi \in C_c([0, \infty))$  and a.e.  $t > 0$ :

$$\int_0^\infty u(t, y) \varphi(y) dy = \int_0^\infty u_0(y) \varphi(y) dy + \int_0^t \int_0^\infty u(s, y) \left( \int_0^1 \varphi(zy) k_0(z) - \varphi(y) \right) dz dy ds \quad (89)$$

Notice that by definition, if  $u \in L^\infty([0, \infty); L^1((1+x)dx))$  then  $u \in C([0, \infty); L^1((1+x)dx))$ .

If  $u_0 \in L^1((1+x)dx)$  we may obtain a weak solution as follows. Define:

$$T(u)(t, x) = u_0(x) + \int_0^t \int_x^\infty \frac{1}{y} k_0\left(\frac{x}{y}\right) u(t, y) dy ds$$

We claim that  $T$  is a contraction from  $L^\infty([0, \tau]; L^1((1+x)dx))$  into itself for any  $0 < \tau < 1$ .

$$\begin{aligned} \|T(u)(t)\|_{L^1} &\leq \|u_0\|_{L^1} + \int_0^t \int_0^\infty \int_x^\infty \frac{1}{y} k_0\left(\frac{x}{y}\right) |u(t, y)| dy dx ds \\ &= \|u_0\|_{L^1} + \int_0^t \int_0^\infty \int_0^y \frac{1}{y} k_0\left(\frac{x}{y}\right) u(t, y) dx dy ds \\ &= \|u_0\|_{L^1} + \int_0^t \int_0^\infty \int_0^1 k_0(z) u(t, y) dz dy ds \\ &\leq \|u_0\|_{L^1} + \tau \sup_{t \in (0, \tau)} \|u(t)\|_{L^1} \end{aligned}$$

$$\begin{aligned} \|T(u)(t)\|_{L^1(xdx)} &\leq \|u_0\|_{L^1(xdx)} + \int_0^t \int_0^\infty x \int_x^\infty \frac{1}{y} k_0\left(\frac{x}{y}\right) u(t, y) dy dx ds \\ &= \|u_0\|_{L^1(xdx)} + \int_0^t \int_0^\infty \int_0^y x \frac{1}{y} k_0\left(\frac{x}{y}\right) u(t, y) dx dy ds \\ &= \|u_0\|_{L^1(xdx)} + \int_0^t \int_0^\infty \int_0^1 z k_0(z) y u(t, y) dz dy ds \\ &\leq \|u_0\|_{L^1(xdx)} + \tau \sup_{t \in (0, \tau)} \|u(t)\|_{L^1(xdx)} \end{aligned}$$

If  $u$  and  $v$  are in  $L^1((1+x)dx)$  and  $w = u - v$ :

$$\begin{aligned}
\|T(u)(t) - T(v)(t)\|_{L^1} &\leq \int_0^t \int_0^\infty \int_x^\infty \frac{1}{y} k_0\left(\frac{x}{y}\right) |w(t, y)| dy dx ds \\
&= \int_0^t \int_0^\infty \int_0^y \frac{1}{y} k_0\left(\frac{x}{y}\right) |w(t, y)| dx dy ds \\
&= \int_0^t \int_0^\infty \int_0^1 k_0(z) |w(t, y)| dz dy ds \\
&\leq \tau \sup_{t \in (0, \tau)} \|w(t)\|_{L^1}
\end{aligned}$$

Therefore, there is a fixed point  $u \in L^\infty([0, \tau]; L^1((1+x)dx))$ . Since the time of existence is independent of the initial data, this procedure may be iterated to obtain a solution  $u \in L^\infty([0, \infty); L^1((1+x)dx))$  of the equation

$$u(t, x) = u_0 + \int_0^t \int_x^\infty \frac{1}{y} k_0\left(\frac{x}{y}\right) u(t, y) dy ds, \quad \text{a.e. } t > 0$$

in  $L^1((1+x)dx)$ . We immediately deduce  $u \in C([0, \infty); L^1((1+x)dx)) \cap W_{loc}^{1, \infty}((0, s); L^1((1+x)dx))$ . Then, the function

$$v(t, x) = e^{-t} u(t, x)$$

is such that, for a. e.  $t > 0$

$$v(t, x) = v_0 + \int_0^t \int_x^\infty \frac{1}{y} k_0\left(\frac{x}{y}\right) v(t, y) dy ds \quad (90)$$

in  $L^1((1+x)dx)$  and satisfies, for a.e.  $t > 0$ :

$$\begin{aligned}
v_t + v &= -e^{-t} u + e^{-t} u_t + v = e^{-t} \int_x^\infty \frac{1}{y} k_0\left(\frac{x}{y}\right) u(t, y) dy \\
&\iff v_t + v = \int_x^\infty \frac{1}{y} k_0\left(\frac{x}{y}\right) v(t, y) dy
\end{aligned}$$

in  $L^1((1+x)dx)$ . In particular, after multiplication of both sides of the equation by a test function  $\varphi \in L^\infty[0, \infty)$  we obtain for a.e.  $t > 0$ :

$$\int_0^\infty v(t, y) \varphi(y) dy = \int_0^\infty u_0(y) \varphi(y) dy + \int_0^t \int_0^\infty v(s, y) \left( \int_0^1 \varphi(zy) k_0(z) dz - \varphi(y) \right) dy ds.$$

We deduce that  $v$  is a weak solution. If we choose  $\varphi(x) = x$  we obtain for a.e.  $t > 0$ :

$$\begin{aligned}
\int_0^\infty y v(t, y) dy &= \int_0^\infty y u_0(y) dy + \int_0^t \int_0^\infty v(s, y) \left( \int_0^1 zy k_0(z) dz - y \right) dy ds \\
&= \int_0^\infty y u_0(y) dy
\end{aligned}$$

So the mass is constant for all  $t > 0$ .

Suppose finally that we have two functions  $u_i \in L^1$ ,  $i = 1, 2$ , satisfying (90). Then, if  $w = u_1 - u_2$ :

$$w(t, x) = \int_0^t \int_x^\infty \frac{1}{y} k_0\left(\frac{x}{y}\right) w(t, y) dy$$

and,

$$\begin{aligned} \int_0^\infty |w(t, x)| dx &\leq \int_0^t \int_0^\infty \int_x^\infty \frac{1}{y} k_0\left(\frac{x}{y}\right) |w(t, y)| dy dx \\ &= \int_0^t \int_0^\infty \frac{1}{y} \int_0^y k_0\left(\frac{x}{y}\right) |w(t, y)| dx dy \\ &= \int_0^t \int_0^\infty \int_0^1 k_0(z) |w(t, y)| dz dy \\ &= \int_0^t \int_0^\infty |w(t, y)| dy \int_0^1 k_0(z) dz \end{aligned}$$

And we deduce  $\int_0^\infty |w(t, x)| dx = 0$  and  $u_1(t) = u_2(t)$  in  $L^1$  for a. e.  $t > 0$ .

### On the condition H.

We present here some useful remarks on the condition H.

**Proposition 1** *Given a sequence  $(\sigma_\ell)_{\ell \in \mathbb{N}}$  in  $(0, 1)$ , condition H is a necessary and sufficient condition in order to have at least one real number  $v \neq 0$  with the following property:*

$$\forall \ell \in \mathbb{N}, \exists k(\ell) \in \mathbb{N}; |v \log \sigma_\ell| = 2k(\ell)\pi. \quad (91)$$

**Proof.** Notice that  $v = 0$  satisfies property (91). If condition H is satisfied, the number

$$v_* = \frac{2\pi}{\log \theta} < 0 \quad (92)$$

has property (91) since, for all  $\ell$ ,  $\sigma_\ell = \theta^{p_\ell}$  and then,

$$v_* \log \sigma_\ell = v_* p_\ell \log \theta = 2p_\ell \pi.$$

Suppose on the other hand that such a  $v \in \mathbb{R} \setminus \{0\}$  exists. Then, for all  $\ell \neq 0$ :

$$-|v| = \frac{2\pi k(\ell)}{\log \sigma_\ell} = \frac{2\pi k(0)}{\log \sigma_0}$$

and therefore

$$\sigma_\ell = \sigma_0^{\frac{k(\ell)}{k(0)}} = \left( \sigma_0^{\frac{1}{k(0)}} \right)^{k(\ell)},$$

and condition H follows with  $\theta = \sigma_0^{\frac{\gcd(k(\ell))}{k(0)}}$  and  $p_\ell = k(\ell)$ .

**Proposition 2** *Given a sequence  $(\sigma_\ell)_{\ell \in \mathbb{N}}$  of real numbers in  $(0, 1)$  satisfying condition H and  $v_*$ ,  $Q$  defined by (92), the set of real numbers  $v$  that satisfy property (91) is given by the set  $Q = v_* \mathbb{Z}$ .*

**Proof.** Suppose that  $v \neq v_*$  is a real number satisfying (91) and fix  $\ell \in \mathbb{N}$ . Then,

$$v_* \log \sigma_\ell = 2p_\ell \pi, \quad v \log \sigma_\ell = 2k(\ell)\pi \quad (93)$$

and we must then have:

$$v = \frac{k(\ell)}{p_\ell} v_*.$$

Suppose therefore that

$$v = \frac{p}{q} v_*$$

where  $p/q$  is irreducible. We deduce from (93) that for any  $\ell \in \mathbb{N}$ :

$$v \log \sigma_\ell = \frac{p}{q} v_* \log \sigma_\ell = 2 \frac{p}{q} p_\ell \pi = 2k(\ell)\pi.$$

and  $p_\ell$  must be a multiple of  $q$  for any  $\ell$ . Since, by assumption, 1 is the only common divisor of all  $p_\ell$ , we deduce that  $q = 1$  and  $v \in \mathbb{Z}v_* = Q$ .

If, on the other hand  $v \in \mathbb{Z}v_*$ , property (91) is immediate.

**Proposition 3** *Suppose that  $b_\ell > 0$  for all  $\ell \in \mathbb{N}$  and  $B = \sum_{\ell \in \mathbb{N}} b_\ell < \infty$ . Then, for all  $\varepsilon > 0$  there exists  $\delta > 0$  such that:*

$$\sup_{v, d(v, Q) \geq \varepsilon} \Re \left( \sum_{\ell \in \mathbb{N}} b_\ell e^{iv \log \sigma_\ell} \right) \leq B(1 - \delta).$$

**Proof.** Without lack of generality, we assume  $B = 1$ . By definition of  $Q$ , if  $v \notin Q$ , there exists  $\ell_0 \in \mathbb{N}$  such that

$$\frac{v \log \sigma_{\ell_0}}{2\pi} \notin \mathbb{Z}.$$

Then,

$$\begin{aligned} \Re \left( \sum_{\ell \in \mathbb{N}} b_\ell e^{iv \log \sigma_\ell} \right) &= b_{\ell_0} \cos(v \log \sigma_{\ell_0}) + \sum_{\ell \in \mathbb{N}, \ell \neq \ell_0} b_\ell \cos(v \log \sigma_\ell) \\ &\leq 1 + b_{\ell_0} (\cos(v \log \sigma_{\ell_0}) - 1) \\ &= 1 - b_{\ell_0} (1 - \cos(v \log \sigma_{\ell_0})) \end{aligned}$$

Since the function  $v \rightarrow \Re \left( \sum_{\ell \in \mathbb{N}} b_\ell e^{iv \log \sigma_\ell} \right)$  is continuous, for all  $R > 0$ :

$$\sup_{d(v, Q) \geq \varepsilon, |v| \leq R} \Re \left( \sum_{\ell \in \mathbb{N}} b_\ell e^{iv \log \sigma_\ell} \right) < 1.$$

Suppose now that for some sequence  $v_n \rightarrow \infty$ ,

$$\lim_{n \rightarrow \infty} \sum_{\ell \in \mathbb{N}} b_\ell e^{iv_n \log \sigma_\ell} = 1.$$

Since  $\sigma_\ell = \theta^{p_\ell}$ ,

$$\lim_{n \rightarrow \infty} \sum_{\ell \in \mathbb{N}} b_\ell e^{iv_n p_\ell \log \theta} = 1$$

Since  $\theta \in \mathbb{R}$  and  $v_n \in \mathbb{R}$  for all  $n$ ,  $(e^{iv_n \log \theta})_{n \in \mathbb{N}}$  is bounded in  $\mathbb{C}$ . Then, there is a subsequence  $(v_k)_{k \in \mathbb{N}}$  and  $A \in \mathbb{C}$  such that  $|A| = 1$  and

$$e^{iv_n \log \theta} \rightarrow A,$$

and then, for all  $\ell \in \mathbb{N}$ :

$$e^{iv_n p_\ell \log \theta} \rightarrow A^{p_\ell}.$$

On the other hand, since  $v_n \in \mathbb{R}$ ,  $p_\ell \in \mathbb{N}$  and  $\theta \in \mathbb{R}$ :

$$\left| b_\ell e^{iv_n p_\ell \log \theta} \right| \leq b_\ell \in \ell^1(\mathbb{N})$$

and then, by the Lebesgue convergence theorem:

$$\sum_{\ell \in \mathbb{N}} b_\ell A^{p_\ell} = 1. \quad (94)$$

For some  $\sigma \in [0, 2\pi)$ ,  $A = e^{i\sigma}$  and since  $\sum_{\ell \in \mathbb{N}} b_\ell e^{i\sigma p_\ell} = 1$  the calculation above implies that  $\sigma \in Q$ , which contradicts the fact that  $d(v_n, Q) \geq \varepsilon$ .

We deduce that

$$\sup_{d(v, Q) \geq \varepsilon} \Re \left( \sum_{\ell \in \mathbb{N}} b_\ell e^{iv \log \sigma_\ell} \right) < 1.$$

**Acknowledgments.** We thank Bénédicte Haas for fruitful discussions. The research of M. Doumic was supported by the ERC Starting Grant SKIPPER<sup>AD</sup> (number 306321). The research of M. Escobedo is supported by DGES Grant 2011-29306-C02-00 and Basque Government Grant IT641-13.

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