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Size-Constrained Tree Decompositions [★]

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Abstract. *Tree-Decompositions* are the corner-stone of many dynamic programming algorithms for solving graph problems. Since the complexity of such algorithms generally depends exponentially on the *width* (size of the *bags*) of the decomposition, much work has been devoted to compute tree-decompositions with small width. However, practical algorithms computing tree-decompositions only exist for graphs with *treewidth* less than 4. In such graphs, the time-complexity of dynamic programming algorithms based on tree-decompositions is dominated by the *size* (number of bags) of the tree-decompositions. It is then interesting to minimize the size of the tree-decompositions. In this report, we consider the problem of computing a tree-decomposition of a graph with width at most k and minimum size. More precisely, we focus on the following problem: given a fixed $k \geq 1$, what is the complexity of computing a tree-decomposition of width at most k with minimum size in the class of graphs with treewidth at most k ? We prove that the problem is NP-complete for any fixed $k \geq 4$ and polynomial for $k \leq 2$; for $k = 3$, we show that it is polynomial in the class of trees and 2-connected outerplanar graphs.

1 Introduction

A *tree-decomposition* of a graph [13] is a way to represent G by a family of subsets of its vertex-set organized in a tree-like manner and satisfying some connectivity property. The *treewidth* of G measures the proximity of G to a tree. More formally, a tree decomposition of $G = (V, E)$ is a pair (T, \mathcal{X}) where $\mathcal{X} = \{X_t | t \in V(T)\}$ is a family of subsets, called *bags*, of V , and T is a tree, such that:

- $\bigcup_{t \in V(T)} X_t = V$;
- for any edge $uv \in E$, there is a bag X_t (for some node $t \in V(T)$) containing both u and v ;
- for any vertex $v \in V$, the set $\{t \in V(T) | v \in X_t\}$ induces a subtree of T .

The *width* of a tree-decomposition (T, \mathcal{X}) is $\max_{t \in V(T)} |X_t| - 1$ and its *size* is order $|V(T)|$ of T . The *treewidth* of G , denoted by $tw(G)$, is the minimum width over all possible tree-decompositions of G .

If T is constrained to be a path, (T, \mathcal{X}) is called a *path-decomposition* of G . The *pathwidth* of G , denoted by $pw(G)$, is the minimum width over all possible path-decompositions of G .

Tree-Decompositions are the corner-stone of many dynamic programming algorithms for solving graph problems. As an example, the famous Courcelle’s Theorem states that any problem expressible in MSOL can be solved in linear-time in the class of bounded treewidth graphs [6]. Another framework based on graph decompositions is the *bi-dimensionality theory* that allowed the design of sub-exponential-time algorithms for many problems in the class of graphs excluding some fixed graph as a minor (e.g., [7]). Given a tree-decomposition with width w and size n , the time-complexity of most of such dynamic programming algorithms can be expressed as $O(2^w n)$ (or $O(2^{w \log w} n)$ in the case of *global* problems). Therefore, the problem of computing tree-decompositions with small width has drawn much attention in the last decades. It has been extensively studied and investigated from different angles: parametrized complexity, exact or approximation algorithms.

The above mentioned algorithms have mainly a theoretical interest because, on the one hand, their time-complexity exponential depends on the treewidth of graphs and, on the other hand, as far as we know, no practical algorithm exists that computes a “good” tree-decomposition for graphs with treewidth at least 5. However, in case of small (≤ 4)

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treewidth graphs, efficient (i.e., practical) algorithms exist to compute tree-decompositions with optimal width. Moreover, in such case, the time-complexity of above-mentioned dynamic programming algorithms becomes dominated by the size of the tree-decompositions and, therefore, it becomes interesting to minimize it.

In report, we study the problem of computing tree-decompositions with minimum size. Obviously, if the width is not constrained, then the problem is trivial since there always exists a tree-decomposition of a graph with one bag (the full vertex-set). Hence, given a graph G and an integer $k \geq tw(G)$, we consider the problem of minimizing the size of a tree-decomposition of G with width at most k .

Related Work. The problem of computing “good” tree-decompositions has been extensively studied. Computing optimal tree-decomposition - i.e., with width $tw(G)$ - is NP-complete in the class of general graphs G [1]. For any fixed $k \geq 1$, Bodlaender designed an algorithm that computes, in time $O(k^{k^3} n)$, a tree-decomposition of width k of any n -node graph with treewidth at most k [3]. Very recently, a single-exponential (in k) algorithm has been proposed that computes a tree-decomposition with width at most $5k$ in the class of graphs with treewidth at most k [4]. As far as we know, the only practical algorithms for computing optimal tree-decompositions hold for graphs with treewidth at most 1 (trivial since $tw(G) = 1$ if and only if G is a tree), 2 (graphs excluding K_4 as a minor) [16], 3 [2, 11, 12] and 4 [14].

We are not aware of any work dealing with the computation of tree-decompositions with minimum size. In [8], Dereniowski *et al.* consider the problem of size-constrained path-decompositions. Given any positive integer k and any graph G with pathwidth at most k . Let $l_k(G)$ denote the smallest size (length) of a path-decomposition of G with width at most k . For any fixed $k \geq 4$, computing l_k is NP-complete in the class of general graphs and it is NP-complete, for any fixed $k \geq 5$, in the class of connected graphs [8]. Moreover, computing l_k can be solved in polynomial-time in the class of graphs with pathwidth at most k for any $k \leq 3$. Finally, the “dual” problem is also hard: for any fixed $s \geq 2$, it is NP-complete in general graphs to compute the minimum width of a tree-decomposition with size s [8]⁶.

Our results. Let k be any positive integer and G be any graph. If $tw(G) > k$, let us set $s_k(G) = \infty$. Otherwise, let $s_k(G)$ denote the minimum size of a tree-decomposition of G with width at most k . See a simple example in Fig. 1. We first prove in Section 2 that, for any (fixed) $k \geq 4$, the problem of computing s_k is NP-hard in the class of graphs with treewidth at most k . Moreover, the computation of s_k for $k \geq 5$ is NP-hard in the class of connected graphs with treewidth at most k . Furthermore, the computation of s_4 is NP-complete in the class of planar graphs with treewidth 3. In Section 3, we present a general approach for computing s_k for any $k \geq 1$. In the rest of the report, we prove that computing s_2 can be solved in polynomial-time. Finally, we prove that s_3 can be computed in polynomial time in the class of trees and 2-connected outerplanar graphs.

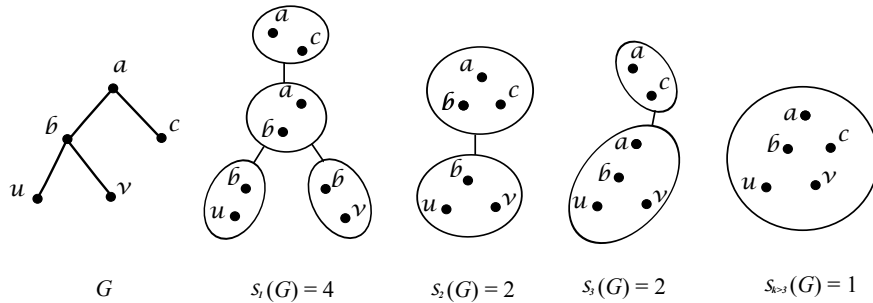


Fig. 1. Given a tree G with five vertices, for any $k \geq 1$, a minimum size tree decomposition of width at most k is shown. So we see that $s_1(G) = 4$, $s_2(G) = s_3(G) = 2$, $s_{k>3}(G) = 1$.

2 NP-hardness in the class of bounded treewidth graphs

In this section, we prove that:

⁶ This result was proved in [8] in terms of path-decomposition but it is straightforward to extend it to tree-decomposition.

Theorem 1. *For any fixed integer $k \geq 4$ (resp., $k \geq 5$), the problem of computing s_k is NP-complete in the class of graphs (resp., of connected graphs) with treewidth at most k .*

Note that the corresponding decision problem is clearly in NP. Hence, we only need to prove it is NP-hard.

Our proof mainly follows the one of [8] for size-constrained path-decompositions. Hence, we recall here the two steps of the proof in [8]. First, it is proved that, if computing l_k is NP-hard for any $k \geq 1$ in general graphs, then the computation of l_{k+1} is NP-hard in the class of connected graphs. Second, it is shown that computing l_4 is NP-hard in general graphs with pathwidth 4. In particular, this implies that computing l_5 is NP-hard in the class of connected graphs with pathwidth 5. The second step consists of a reduction from the 3-PARTITION problem [9] to the one of computing l_4 . Precisely, for any instance \mathcal{I} of 3-PARTITION, a graph $G_{\mathcal{I}}$ is built such that \mathcal{I} is a YES instance if and only if $l_4(G_{\mathcal{I}})$ equals a defined value $\ell_{\mathcal{I}}$.

Our contribution consists first in showing that the first step of [8] directly extends to the case of tree-decompositions. That is, it directly implies that, if computing s_k is NP-hard for some $k \geq 4$ in general graphs, then so is the computation of s_{k+1} in the class of connected graphs. Our main contribution of this section is to show that, for the graphs $G_{\mathcal{I}}$ built in the reduction proposed in [8], any tree-decomposition of $G_{\mathcal{I}}$ with width at most 4 and minimum size is a path decomposition. Hence, in this class of graphs, $l_4 = s_4$ and, for any instance \mathcal{I} of 3-PARTITION, \mathcal{I} is a YES instance if and only if $s_4(G_{\mathcal{I}})$ equals a defined value $\ell_{\mathcal{I}}$. We describe the details in what follows.

Lemma 1. *If the problem of computing s_k for an integer $k \geq 1$ is NP-complete in general graphs, then the computation of s_{k+1} is NP-complete in the class of connected graphs.*

Proof. Let G be any graph. We construct an auxiliary connected graph G' from G by adding a vertex a adjacent to all vertices in $V(G)$. Given two integers $k, s \geq 1$, in the following, we prove that there is a tree decomposition of G with width at most k and size at most s if and only if there is a tree decomposition of G' with width at most $k + 1$ and size at most s .

First, assume that (T, \mathcal{X}) is a tree decomposition of G with width at most k and size at most s . Add a in each bag of \mathcal{X} . Then we obtain a tree decomposition of G' with width at most $k + 1$ and size at most s .

Now let (T', \mathcal{X}') be a tree decomposition of G' with width at most $k + 1$ and size at most s . We are going to find a tree decomposition of G with width at most k and size at most s . Let \mathcal{X}_a be the set of all bags in \mathcal{X}' containing a . Let T_a be the subtree of T' induced by the bags in \mathcal{X}_a . Every vertex $v \in V(G)$ is contained in a bag in \mathcal{X}_a because $va \in E(G')$. For any edge $uv \in E(G)$, there is a bag $X \supseteq \{a, u, v\}$ in \mathcal{X}' since $\{a, u, v\}$ induces a clique in G' . So $X \in \mathcal{X}_a$. Delete a in each bag of \mathcal{X}_a and denote by \mathcal{X}^- the obtained set of bags. So (T_a, \mathcal{X}^-) is a tree decomposition of G with width at most k and size at most s .

Before doing the reduction from the 3-PARTITION problem to the problem of computing s_4 , let us first recall its definition.

Definition 1. [3-PARTITION]

Instance: A multiset S of $3m$ positive integers $S = (w_1, \dots, w_{3m})$ and an integer b .

Question: Is there a partition of the set $\{1, \dots, 3m\}$ into m sets S_1, \dots, S_m such that $\sum_{i \in S_j} w_i = b$ for each $j = 1, \dots, m$?

This problem is NP-complete even if $|S_j| = 3$ for all $j = 1, \dots, m$ [9].

Given an instance of 3-PARTITION, in the following, we construct a disconnected graph $G(S, b)$ as in [8]. First, for each $i \in \{1, \dots, 3m\}$, we construct a connected graph H_i as follows. Take w_i copies of K_3 , denoted by $K_3^{i,q}$, $q = 1, \dots, w_i$, and $w_i - 1$ copies of K_4 , denoted by $K_4^{i,q}$, $q = 1, \dots, w_i - 1$ (the copies are mutually disjoint). Then for each $q = 1, \dots, w_i - 1$, we identify two different vertices of $K_4^{i,q}$ with a vertex of $K_3^{i,q}$ and with a vertex of $K_3^{i,q+1}$, respectively. This is done in such a way that each vertex of each $K_3^{i,q}$ is identified with at most one vertex from other cliques. Informally the cliques form a 'chain' in which the cliques of size 3 and 4 alternate. See Figure 2(a) for an example of H_i where $w_i = 3$.

Second, we construct a graph $H_{m,b}$ as follows. Take $m + 1$ copies of K_5 , denoted by K_5^1, \dots, K_5^{m+1} , and m copies of the path graph P_b of length b (P_b has b edges and $b + 1$ vertices), denoted by P_b^1, \dots, P_b^m . (Again, the copies are taken to be mutually disjoint.) Now, for each $j = 1, \dots, m$, identify one of the endpoints of P_b^j with a vertex of

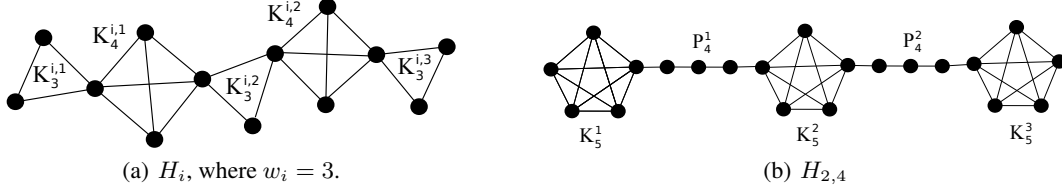


Fig. 2. Examples of gadgets in graph $G(S, b)$.

K_5^j , and identify the other endpoint with a vertex of K_5^{j+1} . Moreover, do this in a way that ensures that, for each j , no vertex of K_5^j is identified with the endpoints of two different paths. See Figure 2(b) for an example of $H_{2,4}$.

Let $G(S, b)$ be the graph obtained by taking the disjoint union of the graphs H_1, \dots, H_{3m} and the graph $H_{m,b}$. In the following, we prove that there is a tree decomposition of $G(S, b)$ of width 4 and size at most $s = 1 - 2m + 2 \sum_{i=1}^{3m} w_i$ if and only if there is a partition of the set $\{1, \dots, 3m\}$ into m sets S_1, \dots, S_m such that $\sum_{i \in S_j} w_i = b$ for each $j = 1, \dots, m$ in the instance of 3-PARTITION.

In Lemma 2.2 of [8], a path decomposition of $G(S, b)$ of width 4 and length $1 - 2m + 2 \sum_{i=1}^{3m} w_i$ is constructed if there is a partition of the set $\{1, \dots, 3m\}$ into m sets S_1, \dots, S_m such that $\sum_{i \in S_j} w_i = b$ for each $j = 1, \dots, m$ in the instance of 3-PARTITION. Obviously, this path decomposition is also a tree decomposition of $G(S, b)$ of width 4 and size s . So we have the following lemma.

Lemma 2. *Given a multiset S of $3m$ positive integers $S = (w_1, \dots, w_{3m})$ and an integer b , if there is a partition of the set $\{1, \dots, 3m\}$ into m sets S_1, \dots, S_m such that $\sum_{i \in S_j} w_i = b$ for each $j = 1, \dots, m$, then $G(S, b)$ has a tree decomposition of width at most 4 and size at most $s = 1 - 2m + 2 \sum_{i=1}^{3m} w_i$.*

Now we prove the other direction.

Lemma 3. *If $G(S, b)$ has a tree decomposition (T, \mathcal{X}) of width at most 4 and size at most $s = 1 - 2m + 2 \sum_{i=1}^{3m} w_i$, then there is a partition of the set $\{1, \dots, 3m\}$ into m sets S_1, \dots, S_m such that $\sum_{i \in S_j} w_i = b$ for each $j = 1, \dots, m$.*

Proof. Lemma 2.6 in [8] proved that if $G(S, b)$ has a path decomposition (T, \mathcal{X}) of width at most 4 and length at most $1 - 2m + 2 \sum_{i=1}^{3m} w_i$, then there is a partition of the set $\{1, \dots, 3m\}$ into m sets S_1, \dots, S_m such that $\sum_{i \in S_j} w_i = b$ for each $j = 1, \dots, m$. So it is enough to prove that any tree decomposition (T, \mathcal{X}) of $G(S, b)$ of width at most 4 and size at most $s = 1 - 2m + 2 \sum_{i=1}^{3m} w_i$ is a path decomposition of $G(S, b)$.

As proved in Lemma 2.3 of [8], each bag in (T, \mathcal{X}) contains exactly one of the cliques $K_3^{i,q}, K_4^{i,q}, K_5^j$. Indeed, each of these cliques has size at least 3. Moreover, any two of them share at most one vertex, and no two cliques of size 3 ($K_3^{i,q}$) share a vertex. So each bag of (T, \mathcal{X}) contains at most one of the cliques $K_3^{i,q}, K_4^{i,q}, K_5^j$. However, for any clique, there is a bag in (T, \mathcal{X}) containing its vertices. Since s equals the number of the cliques $K_3^{i,q}, K_4^{i,q}, K_5^j$, each bag of (T, \mathcal{X}) contains exactly one of them.

Moreover let us prove that any edge in $K_4^{i,q}, K_5^j, P_b^j$ (i.e. both the two endpoints of the edge) is contained in exactly one bag. Since each bag in (T, \mathcal{X}) contains exactly one of the cliques $K_3^{i,q}, K_4^{i,q}, K_5^j$, the two endpoints of any edge in the paths P_b^1, \dots, P_b^m are contained in a bag containing some $K_3^{i,q}$. (The bags containing a $K_4^{i,q}$ (resp., K_5^j) cannot add another two vertices (resp., one vertex) since (T, \mathcal{X}) is a tree decomposition of width at most 4.) Every bag containing some $K_3^{i,q}$ contains at most one edge in the paths P_b^1, \dots, P_b^m , because the bag can add at most another two vertices and any $K_3^{i,q}$ and P_b^j are disjoint. There are mb edges in the paths P_b^1, \dots, P_b^m and there are mb bags containing some $K_3^{i,q}$, so every bag containing a $K_3^{i,q}$ contains exactly one edge in the paths P_b^1, \dots, P_b^m . So any edge in the paths P_b^1, \dots, P_b^m is contained in exactly one bag. Also each bag containing some $K_3^{i,q}$ contains 5 vertices, so it does not contain any edge (i.e. both its two endpoints) in $K_4^{i,q}$ or K_5^j . Therefore, any edge on $K_4^{i,q}, K_5^j$ is contained in exactly one bag.

Now we prove that there are only two leaves in T and so T is a path. If a bag containing some $K_3^{i,q}$ and an edge uv on some path P_b^j is a leaf bag in T , then its neighbor bag also contains u, v because both u and v are incident to

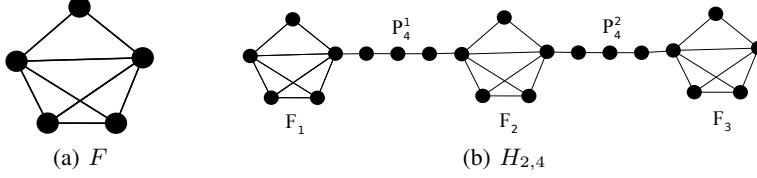


Fig. 3. Example of the new gadget in $G(S, b)$.

other edges in $G(S, b)$. This is a contradiction with any edge (its two endpoints) on P_b^j are contained only in one bag. So any bag containing some $K_3^{i,q}$ is not a leaf bag in T . Similarly, we can prove that any bag containing any $K_4^{i,q}$ or K_5^j for $1 < j < m + 1$ is not a leaf bag in T . Thus there are only two bags containing K_5^1 and K_5^{m+1} are leaves in T .

Then we get the following corollary.

Corollary 1. *It is NP-complete to compute s_4 in the class of graphs of treewidth at most 4.*

Theorem 1 follows from Lemma 1 and Corollary 1.

We furthermore modify the reduction to prove theorem 2.

Theorem 2. *It is NP-complete to compute s_4 in the class of planar graphs of treewidth at most 3.*

Proof. As in the previous reduction, we build a graph $G(S, b)$ for an instance of 3-PARTITION; we keep the subgraphs H_i as they are and modify the graph $H_{m,b}$ as follows. We replace the $m + 1$ copies of K_5 by $m + 1$ copies of the graph F that consists of a K_4 and a K_3 sharing an edge as depicted in Figure 3(a). We denote the copies by F_1, F_2, \dots, F_{m+1} . The new graph $G(S, b)$ we obtain is planar and has treewidth 3.

Lemma 2 is still true and for 3 to be correct we need to prove that if $G(S, b)$ has a tree decomposition (T, \mathcal{X}) of width at most 4 and size at most $s = 1 - 2m + 2 \sum_{i=1}^{3m} w_i$, then there is a bag of (T, \mathcal{X}) containing F_i , for each F_i , $i \in 1, \dots, m + 1$. Let us denote by K_3^i and K_4^i the two cliques sharing exactly one edge that form F_i . Each of these cliques, should appear in one bag. Note that among all the cliques of $G(S, b)$, the only cliques that can coexist in a bag are of the form K_3^i and K_4^i since the sum of the number of vertices of any other two cliques is more than 5. Let us suppose that there exists $j \in 1, \dots, m + 1$ such that no bag of (T, \mathcal{X}) contains F_j , i.e. K_3^j and K_4^j are not in the same bag. In this case the number of bags of (T, \mathcal{X}) is at least the number of the cliques $K_3^{i,q}, K_4^{i,q}, K_4^{i'}$ ($i' \neq j$), plus the two bags containing K_3^j and K_4^j . This gives a size of at least $2 - 2m + 2 \sum_{i=1}^{3m} w_i$ which is not possible.

3 Notations and preliminaries

In this section, we present the definitions and notations used throughout the report and some well-known facts about tree-decompositions.

3.1 Notations

Given a graph $G = (V, E)$, for any $S \subseteq V$, For an integer $c \geq 0$, a graph $G = (V, E)$ is c -connected if $|V| > c$ and no subset $V' \subseteq V$ with $|V'| < c$ is a separator in G . A 2-connected component of G is a maximal 2-connected subgraph.

Let (T, \mathcal{X}) be any tree-decomposition of G . Abusing the notations, we will identify a node $t \in V(T)$ and its corresponding bag $X_t \in \mathcal{X}$. This means that, e.g., instead of saying $t \in V(T)$ is adjacent to $t' \in V(T)$ in T , we can also say that $X_t \in \mathcal{X}$ is adjacent to $X_{t'} \in \mathcal{X}$ in T . A bag $B \in \mathcal{X}$ is called a *leaf-bag* if B has degree one in T . Let $k \geq 1$ and G be a graph with $tw(G) \leq k$. A subset $B \subseteq V(G)$ is a k -potential-leaf if there is a tree-decomposition (T, \mathcal{X}) with width at most k and size $s_k(G)$ such that B is a leaf bag of (T, \mathcal{X}) . A subgraph $H \subseteq V$ is a k -potential-leaf of G if $V(H)$ is a k -potential-leaf of G . Note that a k -potential-leaf has size at most $k + 1$. Given a class of graphs \mathcal{C} and integer $k \in \mathbb{N}^*$, a set of graphs \mathcal{P} is called a *complete set of k -potential-leaves* of \mathcal{C} , if for any graph $G \in \mathcal{C}$, there exists a graph $H \in \mathcal{P}$ such that H is a k -potential-leaf of G .

A tree-decomposition is *reduced* if no bag is contained in another one. It is straightforward that, in any leaf-bag B of reduced tree-decomposition, there is $v \in V$ such that v appears only in B and so $N[v] \subseteq B$. Note that it implies that any reduced tree-decomposition has at most $n - 1$ bags.

In the following we define two *transformation rules*, that take a tree-decomposition (T, \mathcal{X}) of a graph G , and computes another one without increasing the width nor the size.

Leaf. Let $X \in \mathcal{X}$ and $N_T(X) = \{X_1, \dots, X_d\}$. Assume that, for any $1 < i \leq d$, $X_i \cap X \subseteq X_1$. Let $(T^*, \mathcal{X}^*) = \text{Leaf}(X, X_1, (T, \mathcal{X}))$ denote the tree-decomposition of G obtained by replacing each edge $X_i X \in E(T)$ by an edge $X_i X_1$ for any $1 < i \leq d$. Note that X becomes a leaf-bag after the operation. See in Fig. 4.

Reduce. Let $XX' \in E(T)$ with $X \subseteq X'$. Let $(T^*, \mathcal{X}^*) = \text{Reduce}(X, X', (T, \mathcal{X}))$ denote the tree-decomposition of G obtained by deleting the bag X from the tree-decomposition $\text{Leaf}(X, X', (T, \mathcal{X}))$. Note that the size of the tree decomposition is decreased by one after the operation.

From any tree-decomposition of G with width k and size s , it is easy to obtain a reduced tree-decomposition of G with width at most k and size at most $s - 1$ by applying the Reduce operation while it is possible (i.e., while a bag is contained in another one). In particular, any minimum size tree decomposition is reduced.

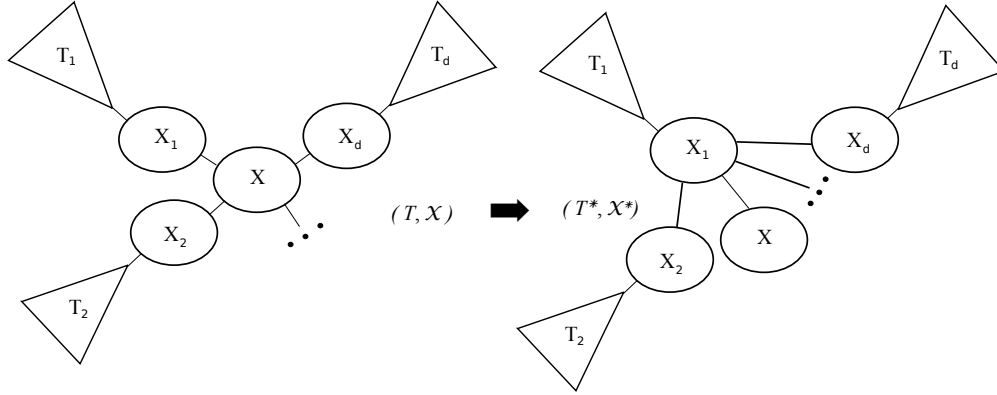


Fig. 4. In a tree decomposition (T, \mathcal{X}) , $N_T(X) = \{X_1, \dots, X_d\}$ and for any $1 < i \leq d$, $X_i \cap X \subseteq X_1$. For $1 \leq i \leq d$, $T_i \cup X_i$ induces the subtree containing X_i in $T \setminus \{XX_i\}$. Replace each edge $X_i X \in E(T)$ by an edge $X_i X_1$ for any $1 < i \leq d$. This gives a tree decomposition $(T^*, \mathcal{X}^*) = \text{Leaf}(X, X_1, (T, \mathcal{X}))$. X is a leaf-bag in (T^*, \mathcal{X}^*) .

We conclude this section by a general lemma on tree-decompositions. This lemma is known as folklore, we recall it for completeness.

Lemma 4. Let (T, \mathcal{X}) be a tree decomposition of a graph G . Let $X \in \mathcal{X}$ and $v, w \in X$. If there exists a connected component in $G \setminus X$ containing a neighbor of v and a neighbor of w , then there is a neighbor bag of X in (T, \mathcal{X}) containing v and w .

Proof. First, let us note that, for any connected subgraph H of G , the set of bags of T that contain a vertex of H induces a subtree of T (the proof is easy by induction on $|V(H)|$).

Let C be a connected component in $G \setminus X$ containing a neighbor of v and a neighbor of w . By above remark, let T'_C be the subtree of T induced by the bags that contain some vertex of C . Moreover, because no vertices of C are contained in the bag X , then T'_C is a subtree of $T \setminus X$. Let T_C be the connected component of $T \setminus X$ that contains T'_C . Let $Y \in V(T_C)$ be the bag of T_C which is a neighbor of X in T . Let $x \in N(v) \cap C$ be a neighbor of v in C . Then there exists a bag $Z \in \mathcal{X}$ in T_C containing both x and v . So both X and Z contain vertex v . Then the bag Y , which is on the path between X and Z in T , also contains v . Similarly, we can prove that $w \in Y$.

Corollary 2. Let (T, \mathcal{X}) be a tree decomposition of a 2-connected graph G . Let $X \in \mathcal{X}$ and $|X| \leq 2$. Then there is a neighbor bag Y of X in (T, \mathcal{X}) such that $X \subseteq Y$.

Proof. Since G is 2-connected, $|V(G)| \geq 3$. So there exist at least another bag except X in \mathcal{X} .

If $|X| = 1$, let $X = \{v\}$. Then there is a neighbor bag Y of X containing v , since G is 2-connected and v is adjacent to some vertices in G . So $X \subseteq Y$.

Otherwise $|X| = 2$ and let $X = \{v, w\}$. Let G_1 be any connected component in $G \setminus X$. If v is not adjacent to any vertex in G_1 , then $\{w\}$ separates $V(G_1)$ from $\{v\}$. It contradicts with the assumption that G is 2-connected. So any connected component in $G \setminus X$ containing a neighbor of v and a neighbor of w . From Lemma 4, there is a neighbor bag Y of X containing v, w , i.e. $X \subseteq Y$.

3.2 General approach

In what follows, we propose polynomial-time algorithms to compute minimum-size tree-decompositions of graphs with small treewidth. Our algorithms mainly use the notion of potential-leaf.

Let $k \geq 1$ and $G = (V, E)$ be a graph with $tw(G) \leq k$. The key idea of our algorithms is to identify a finite complete set of potential-leaves. Then, our algorithms are recursive: given a graph G and a k -potential-leaf H from the complete set, we compute a minimum-size tree-decomposition of G by adding H to a minimum-size tree-decomposition of a smaller graph.

The next lemmas formalize the above paragraph.

Lemma 5. *Let $k \geq 1$ and $G = (V, E)$ be a graph with $tw(G) \leq k$. Let $B \subseteq V$ be a k -potential-leaf of G . Let $S \subset B$ be the set of vertices of B that have a neighbor in $V \setminus B$. Then $s_k(G) = s_k(G_S \setminus (B \setminus S)) + 1$.*

Proof. Let us first prove $s_k(G) \leq s_k(G_S \setminus (B \setminus S)) + 1$. Suppose that (T_S, \mathcal{X}_S) is a minimum size tree decomposition of width at most k of the graph $G_S \setminus (B \setminus S)$. Then there exists a bag $X \in \mathcal{X}_S$ containing S because S induces a clique in the graph $G_S \setminus (B \setminus S)$. So add the bag B and make it adjacent to X in the tree decomposition (T_S, \mathcal{X}_S) . Then we obtain a tree decomposition of width at most k for graph G of size $s_k(G_S \setminus (B \setminus S)) + 1$.

Now we prove that $s_k(G) \geq s_k(G_S \setminus (B \setminus S)) + 1$. Let (T, \mathcal{X}) be a minimum size tree decomposition of G of width at most k such that B is a leaf bag in it. Note that, if $B = V$ then $G_S \setminus (B \setminus S)$ is the empty graph. Let us assume that $B \subset V$. Then (T, \mathcal{X}) is also a tree decomposition of G_S . Let B be adjacent to the bag Y in (T, \mathcal{X}) . Then $S \subset Y$ since each vertex in S is contained in another bag in (T, \mathcal{X}) . Let (T', \mathcal{X}') be the tree decomposition obtained by deleting the vertices in $B \setminus S$ in all the bags of (T, \mathcal{X}) . Then B is changed to $B' = S \in \mathcal{X}'$ and let Y be changed to $Y' \in \mathcal{X}'$. So $B' \subseteq Y'$. Then the tree decomposition $Reduce(B', Y', (T', \mathcal{X}'))$ is a tree decomposition of $G_S \setminus (B \setminus S)$ of size $s_k(G) - 1$. So $s_k(G) - 1 \geq s_k(G_S \setminus (B \setminus S))$.

This lemma implies the following corollary:

Corollary 3. *Let $k \in \mathbb{N}^*$ and \mathcal{C} be the class of graphs with treewidth at most k . If there is a $g(n)$ -time algorithm \mathcal{A}_k that, for any n -vertex-graph $G \in \mathcal{C}$, computes a k -potential-leaf of G . Then s_k can be computed in $O(g(n) \cdot n)$ time in the class of n -vertex graphs in \mathcal{C} . Moreover, a minimum size tree decomposition of width at most k can be constructed in the same time.*

Proof. Let $G \in \mathcal{C}$ be a n -vertex-graph. Let us apply Algorithm \mathcal{A}_k to find a subgraph H of G in $g(n)$ time, which is a k -potential-leaf of G . Let $S \subset V(H)$ be the set of vertices having a neighbor in $G \setminus H$ and $G' = G_S \setminus (V(H) \setminus S)$. Then, by Lemma 5, $s_k(G) = s_k(G') + 1$. Finally, $|V(G')| \leq n - 1$ and G' has treewidth at most k . We then proceed recursively. So the total time complexity is $O(g(n) \cdot n)$. Moreover, for any minimum size $(s_k(G'))$ tree decomposition (T', \mathcal{X}') of G' of width k , there is a bag X containing S since S induces a clique in G' . Add a new bag $N = V(H)$ adjacent to X in (T', \mathcal{X}') . The obtained tree decomposition is a minimum size $(s_k(G) = s_k(G') + 1)$ tree decomposition of G of width at most k .

4 Graphs with treewidth at most 2

In this section, we describe algorithm \mathcal{A}_2 computes a 2-potential-leaf of a given graph. In particular, all graphs considered in this section have treewidth at most 2, i.e. partial 2-trees. Please see a complete set of 2-potential-leaf of graphs of treewidth at most 2 in Fig. 5. We are going to prove that any of the subgraphs in Fig. 5 is a potential-leaf and then that each non-empty graph of treewidth at most 2 contains one of them as a 2-potential-leaf.

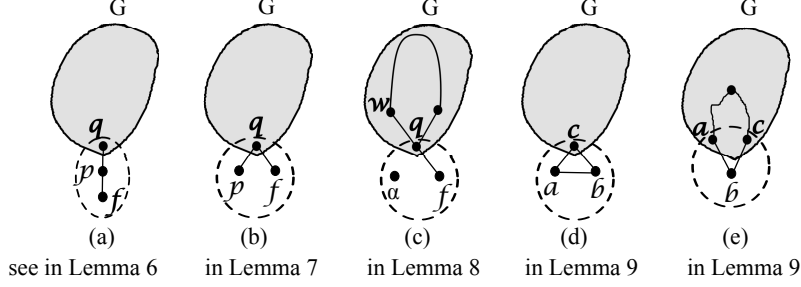


Fig. 5. Complete set of 2-potential-leaves of graphs of treewidth at most 2.

Lemma 6. Let G be a graph with treewidth at most 2 and $p \in V(G)$ such that $N(p) = \{f, q\}$ and f has degree one (see in Fig. 5(a)). Then $\{f, p, q\}$ is a 2-potential-leaf of G .

Proof. Let (T, \mathcal{X}) be any tree-decomposition of G with width at most 2 and size at most $s \geq 1$. We show how to modify it to obtain a tree-decomposition with width at most 2 and size at most s and in which $\{f, p, q\}$ is a leaf bag.

Since $fp \in E(G)$, there is a bag B in (T, \mathcal{X}) containing both f and p . We may assume that B is the single bag containing f (otherwise, delete f from any other bag). Similarly, since $pq \in E(G)$, let X be a bag in (T, \mathcal{X}) containing both p and q .

First, let us assume that $X = B = \{f, p, q\}$. In that case, we may assume that X is the single bag containing p (otherwise, delete p from any other bag). If X is a leaf bag, then the lemma is proved. Otherwise, let X_1, \dots, X_d be the neighbors of X in T . Since f and p appear only in X , then $X \cap X_i \subseteq \{q\}$ for any $1 \leq i \leq d$. If there is $1 \leq i \leq d$ such that $q \in X_i$, let us assume w.l.o.g., that $q \in X_1$. By definition of the operation *Leaf*, the tree-decomposition $\text{Leaf}(X, X_1, (T, \mathcal{X}))$ has width at most 2, same size as (T, \mathcal{X}) , and X is a leaf.

Second, consider the case when $X \neq B$. There are two cases to be considered. Either $B = \{f, p\}$ or $B = \{f, p, x\}$ with $x \neq q$. In the latter case, note that there is another bag B' , neighbor of B , that contains x unless x is an isolated vertex of G . In the former case or if x appears only in B (in which case, x is an isolated vertex), let B' be any neighbor of B . Let (T', \mathcal{X}') be obtained by deleting f, p in all bags of (T, \mathcal{X}) . Then, contract the edge BB' in T' , i.e., remove B and make any neighbor of B adjacent to B' . Note that, in the resulting tree-decomposition of $G \setminus \{f, p\}$, there is a bag X' containing q and with $|X'| \leq 2$ (the bag that results from X). Finally, add a bag $\{f, p, q\}$ adjacent to X' and, if node x was only in B , then add x to X' . The result is the desired tree-decomposition.

Lemma 7. Let G be a graph with treewidth at most 2 and $q \in V(G)$ such that q has at least two one-degree neighbors f and p (see in Fig. 5(b)). Then $\{f, p, q\}$ is a 2-potential-leaf of G .

Proof. Let (T, \mathcal{X}) be any tree-decomposition of G with width at most 2 and size at most $s \geq 1$. We show how to modify it to obtain a tree-decomposition with width at most 2 and size at most s and in which $\{f, p, q\}$ is a leaf bag.

Since $fq \in E(G)$, there is a bag B in (T, \mathcal{X}) containing both f and q . We may assume that B is the single bag containing f (otherwise, delete f from any other bag). Similarly, since $pq \in E(G)$, let X be a bag in (T, \mathcal{X}) containing both p and q . Again, we may assume that X is the single bag containing p (otherwise, delete p from any other bag).

First, let us assume that $X = B = \{f, p, q\}$. If X is a leaf bag, then the lemma is proved. Otherwise, let X_1, \dots, X_d be the neighbors of X in T . Since f and p appear only in X , then $X \cap X_i \subseteq \{q\}$ for any $1 \leq i \leq d$. If there is $1 \leq i \leq d$ such that $q \in X_i$, let us assume w.l.o.g., that $q \in X_1$. By definition of the operation *Leaf*, the tree-decomposition $\text{Leaf}(X, X_1, (T, \mathcal{X}))$ has width at most 2, same size as (T, \mathcal{X}) , and X is a leaf.

Second, let us assume that $X = \{f, q\}$ or $B = \{p, q\}$. In the former case, remove p from any bag and add p to X . In the latter case, remove f from any bag and add f to B . In both cases, we get a bag $\{f, p, q\}$ as in the first case.

Otherwise, let $B = \{f, q, x\}$, $x \neq p$, and $X = \{p, q, y\}$, $y \neq f$.

- If B and X are adjacent in T , then add a new bag $N = \{q, x, y\}$; remove B and X and make each of their neighbors adjacent to the new bag N and, finally, add a leaf-bag $\{f, p, q\}$ adjacent to N . See in Fig. 6(a). The obtained tree-decomposition has the desired properties.

- Otherwise, if there is a neighbor B' of B with $q, x \in B'$, then remove B , make all neighbors of B adjacent to B' and finally add a leaf-bag $\{f, p, q\}$ adjacent to X . The obtained tree-decomposition has the desired properties.
- Otherwise, let B' be the neighbor of B on the path between B and X . In this case, $q \in B'$ and $x \notin B'$. Moreover, q does not belong to any neighbor of B that contains x and the other way around: x does not belong to any neighbor of B that contains q . For any neighbor Y of B with $q \in Y$ (and hence $x \notin Y$), replace the edge $YB \in E(T)$ with the edge YB' . Finally, replace the edge $BB' \in E(T)$ by the edge BX . See in Fig. 6(b). In the resulting tree-decomposition of G , B and X are adjacent and we are back to the first item.

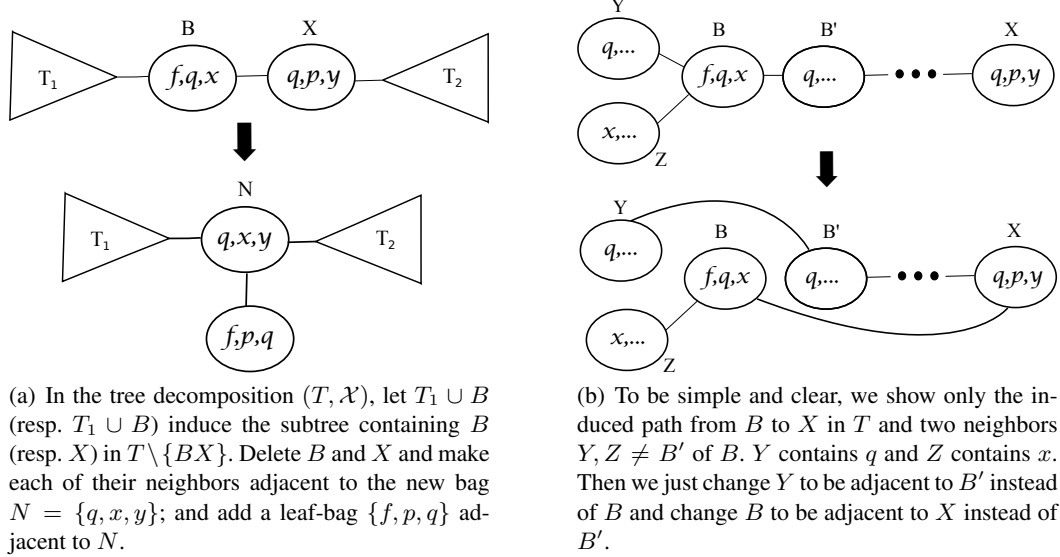


Fig. 6. Explanation of proof of Lemma 7.

Lemma 8. Let G be a graph with treewidth at most 2 and $q \in V(G)$ such that q has one neighbor f with degree 1 and for any vertex $w \in N(q) \setminus \{f\}$, $\{w, q\}$ belongs to a 2-connected component of G .

If G has an isolated vertex α , then $\{q, f, \alpha\}$ is a 2-potential-leaf; otherwise $\{q, f\}$ is a 2-potential-leaf (see in Fig. 5(c)).

Proof. Let (T, \mathcal{X}) be any tree-decomposition of G with width at most 2 and size at most $s \geq 1$. We show how to modify it to obtain a tree-decomposition with width at most 2 and size at most s and in which $\{f, q, \alpha\}$ is a leaf bag if G has an isolated vertex α ; and otherwise $\{f, q\}$ is a leaf bag.

Since $fq \in E(G)$, there is a bag B in (T, \mathcal{X}) containing both f and q . We may assume that B is the single bag containing f (otherwise, delete f from any other bag).

1. If $B = \{f, q\}$, then the intersection of B and any of its neighbor in T is empty or $\{q\}$. If there is a neighbor of B containing q , then let X be such a neighbor; otherwise let X be any neighbor of B . By definition of the operation *Leaf*, the tree-decomposition $\text{Leaf}(B, X, (T, \mathcal{X}))$ has width at most 2, same size as (T, \mathcal{X}) , and B is a leaf. If there are no isolated vertices, we are done. Otherwise, if there is an isolated vertex α in G , then delete α in all bags of the tree-decomposition $\text{Leaf}(B, X, (T, \mathcal{X}))$ and add α to bag B , i.e. make $B = \{f, p, \alpha\}$. The result is the desired tree decomposition.
2. Otherwise let $B = \{f, q, x\}$.
 - (a) If x is a neighbor of q , then x and q are in a 2-connected component of G . So there exists a connected component in $G \setminus B$ containing a vertex adjacent to x and a vertex adjacent to q . From Lemma 4, there is

a neighbor X of B in (T, \mathcal{X}) containing both x and q . Then by definition of the operation *Leaf*, the tree-decomposition $\text{Leaf}(B, X, (T, \mathcal{X}))$ has width at most 2, same size as (T, \mathcal{X}) , and B is a leaf. Then delete x in B , i.e. $B = \{f, q\}$. Finally, if α is an isolated vertex of G , remove it to any other bag and add it to B . The result is the desired tree decomposition.

- (b) Otherwise x is not adjacent to q . If there is a neighbor X of B in (T, \mathcal{X}) containing both x and q , then (T, \mathcal{X}) is modified as in case 2a. Otherwise, any neighbor of B in (T, \mathcal{X}) contains at most one of q and x . If there is a neighbor of B in T containing q , then let Y be such a neighbor of B ; otherwise let Y be any neighbor of B . Delete the edges between B and all its neighbors not containing x except Y in (T, \mathcal{X}) and make them adjacent to Y . If there is no neighbor of B containing x , then x is an isolated vertex and we get a tree decomposition of the same size and width as (T, \mathcal{X}) , in which there is a leaf bag $B = \{f, q, x\}$. It is a required tree decomposition. Otherwise let Z be a neighbor of B in (T, \mathcal{X}) containing x , then delete the edges between B and all its neighbors containing x except Z in (T, \mathcal{X}) and make them adjacent to Z . Now B has only two neighbors Y and Z and $B \cap Y \subseteq \{q\}$, $B \cap Z = \{x\}$ and $Y \cap Z = \emptyset$. Delete the edge between B and Z and make Z adjacent to Y . Delete x in B , i.e. make $B = \{f, q\}$. See the transformations in Fig. 7. Then we get a tree decomposition of the same size and width as (T, \mathcal{X}) , in which $B = \{f, q\}$ is a leaf bag. Again, if α is an isolated vertex of G , remove it to any other bag and add it to B . The result is the desired tree decomposition.

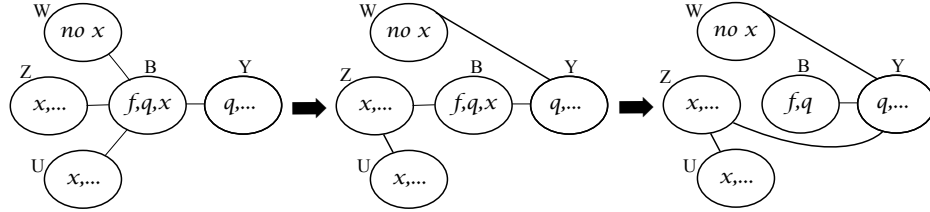


Fig. 7. To be simple and clear, we show only the subtree induced by B, Y and another three neighbors Z, W, U of B . Y contains q ; Z, U both contain x and W does not contain x . First we make the bag not containing x , e.g. W adjacent to Y instead of B ; and make the bag containing x except Z , e.g. U adjacent to Z instead of B . Second, make Z adjacent to Y instead of B and delete x in B . Then $B = \{f, q\}$ is a leaf-bag.

Lemma 9. Let G be a graph of treewidth at most 2. Let $b \in V(G)$ with $N(b) = \{a, c\}$. If $N(a) = \{b, c\}$ (see in Fig. 5(d)) or if there is a path, with at least one internal vertex, between a and c in $G \setminus \{b\}$ (see in Fig. 5(e)), then $\{a, b, c\}$ is a 2-potential-leaf of G .

Proof. Let $G = (V, E)$ be a graph of treewidth at most 2. Let $b \in V$ with exactly 2 neighbors $a, c \in V$ satisfy the hypotheses of the lemma. If $V = \{a, b, c\}$, the result holds trivially, so let us assume that $|V| \geq 4$.

Let (T, \mathcal{X}) be a reduced tree decomposition of width at most 2 of G . From (T, \mathcal{X}) , we will compute a tree decomposition (T^*, \mathcal{X}^*) of G without increasing the width or the size and such that $\{a, b, c\}$ is a leaf-bag of (T^*, \mathcal{X}^*) .

Let X be any bag of (T, \mathcal{X}) containing $\{a, b\}$ and Y be any bag containing $\{b, c\}$. The bags X, Y exist because $ab, bc \in E$. If $X = \{a, b\}$, then there exists a connected component in $G \setminus X$ containing a neighbor of a and a neighbor of b . By Lemma 4, there is a neighbor of X in (T, \mathcal{X}) that contains both a and b , contradicting the fact that (T, \mathcal{X}) is reduced. So $|X| = 3$ and, similarly, $|Y| = 3$.

- Let us first assume that $X = Y = \{a, b, c\}$. In particular, it is the case when $N(a) = \{b, c\}$ since $\{a, b, c\}$ induces a clique. We may assume that b only belongs to bag X (otherwise, remove b from any other bag). If $N(a) = \{b, c\}$, then we can also assume that a only belongs to X . Let Z be any neighbor of B containing c if exists; otherwise let Z be any neighbor of B (Z exists since $|V| \geq 4$). Otherwise, there exists a path P between a and c in $G \setminus \{b\}$ with at least one internal vertex. In this latter case, there exists a connected component in $G \setminus X$ containing a neighbor of a and a neighbor of c . So by Lemma 4, there is a neighbor bag Z of X in (T, \mathcal{X}) containing both a and c . In both cases, $\text{Leaf}(X, Z, (T, \mathcal{X}))$ is the desired tree-decomposition.

- Otherwise, $X = \{a, b, x\}$ and $Y = \{b, c, y\}$ with $x \neq c$ and $y \neq a$; and there exists a path P between a and c in $G \setminus \{b\}$ with at least one internal vertex. Let Q be the path between X and Y in (T, \mathcal{X}) . We may assume that b only belongs to the bags in Q , because otherwise b can be removed from any other bag.
 - If X is adjacent to Y , then by properties of tree-decomposition, $X \cap Y$ separates a and c . Since $\{b\}$ does not separate a and c , $X \cap Y = \{b, x\}$, i.e. $x = y$. In this case, (T^*, \mathcal{X}^*) is obtained by making $X = \{a, c, x\}$ and removing Y from (T, \mathcal{X}) , then making all neighbors of Y adjacent to X and finally, adding a bag $\{a, b, c\}$ adjacent to X .
 - Otherwise, let X' be the bag in the path Q containing a , which is closest to Y . Similarly, let Y' be the bag in the path Q containing c , which is closest to X . Finally, let Q' be the path from X' to Y' in T and note that b belongs to each bag in Q' and a and c do not belong to any internal bag in Q' . Then we may assume that b only belongs to the bags in Q' , because otherwise b can be removed from any other bag. If X' and Y' are adjacent in T , the proof is similar to the one in previous item. Otherwise, let Z be the neighbor of X' in Q' . By properties of tree-decompositions, $X' \cap Z$ separates a and c . Since $\{b\}$ does not separate a and c , let $X' \cap Z = \{b, x'\}$. Since $Z \neq \{b, x'\}$ because (T, \mathcal{X}) is reduced, then $Z = \{b, x', z\}$ for some $z \in V$. Replace b with a in all the bags. By doing this (T, \mathcal{X}) is changed to a tree decomposition (T^c, \mathcal{X}^c) of the graph G/ab obtained by contracting the edge ab in G . In (T^c, \mathcal{X}^c) , the bag X' has become $X^c = \{a, x'\}$ and Z is changed to be $Z^c = \{a, x', z\}$. So X^c can be reduced in (T^c, \mathcal{X}^c) . Moreover Y is changed to $Y^c = \{a, c, y\}$. To conclude, let us add the bag $\{a, b, c\}$ adjacent to Y^c in the tree decomposition $\text{Reduce}(X^c, Z^c, (T^c, \mathcal{X}^c))$. See in Fig. 8. The result is the desired tree-decomposition (T^*, \mathcal{X}^*) of G .

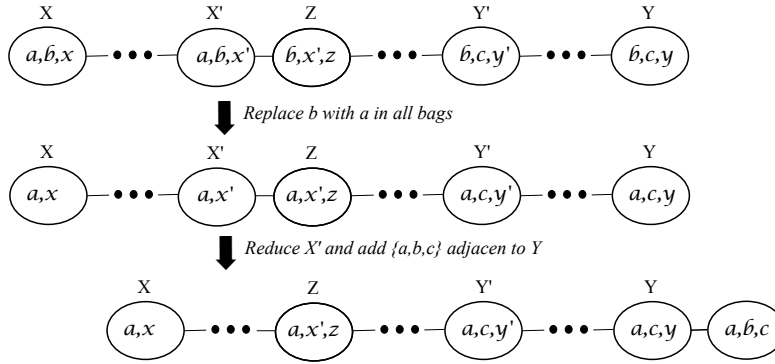


Fig. 8. To be simple and clear, we show only the path from X to Y . After the two transformations, $\{a, b, c\}$ is a leaf-bag.

Before going further, let us introduce some notations. A *bridge* in a graph $G = (V, E)$ is any subgraph induced by two adjacent vertices u and v of G (i.e., $uv \in E$) such that the number of connected components strictly increases when deleting the edge uv , but not the two vertices u, v in G , i.e., $G' = (V, E \setminus \{uv\})$ has strictly more connected components than G . A vertex $v \in V$ is a *cut vertex* if $\{v\}$ is a separator in G . A maximal connected subgraph without a cut vertex is called a *block*. Thus, every block of a graph $G = (V, E)$ is either a 2-connected component of G or a bridge or an isolated vertex. Conversely, every such subgraph is a block. Different blocks of G intersect in at most one vertex, which is a cut vertex of G . Hence, every edge of G lies in a unique block, and G is the union of its blocks.

Let $G = (V, E)$ be a connected graph and let $r \in V$. A spanning tree T of G is a BFS-tree of G if for any $v \in V(G)$, the distance from r to v in G is the same as the one in T . Let $\mathcal{B} = \{C : C \text{ is a block of } G\}$. The *block graph* of G is the graph $B(G)$ whose vertices are the blocks of G and two block-vertices of $B(G)$ are adjacent if the corresponding blocks intersect, that is, $B(G) = (\mathcal{B}, \{C_1 C_2 : C_1, C_2 \in \mathcal{B} \text{ and } C_1 \cap C_2 \neq \emptyset\})$. Note that $B(G)$ is connected. Finally, a *block-tree* of G is any BFS-tree F (with any arbitrary root) of $B(G)$. See an example in Fig. 9.

There is a linear (in the number of edges) algorithm for computing all blocks in a given graph [10]. Also a BFS-tree can be found in linear (in the number of vertices plus the number of edges) time. So given a graph $G = (V, E)$, we can compute a block tree F of G in $O(|V| + |E|)$ time.

Now we are ready to prove next theorem by using the Lemmas 6-9.

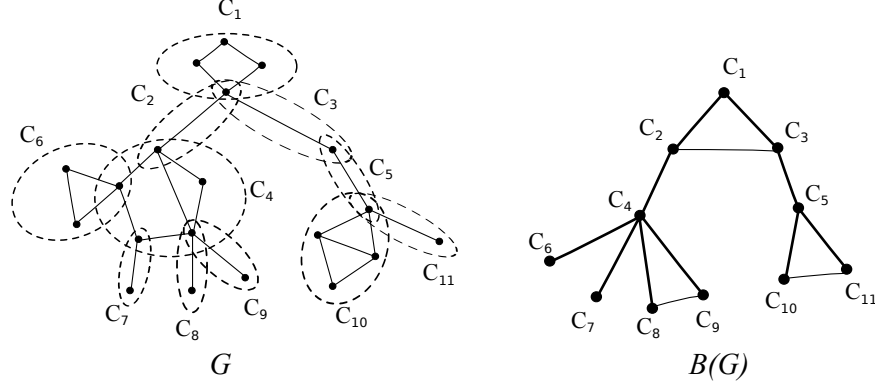


Fig. 9. Graph G is connected. For $i = 1, \dots, 11$, each C_i is a block of G . $B(G)$ is the block graph of G . The BFS tree of $B(G)$ with bold edges is a block tree of G with root C_1 .

Theorem 3. *There is an algorithm that, for any n -vertex- m -edge-graph G with treewidth at most 2, computes a 2-potential-leaf of G in time $O(n + m)$.*

Proof. If $n \leq 3$, then $V(G)$ is a 2-potential-leaf of G . Let us assume that $n \geq 4$. First, let us compute the set of isolated vertices in G , which can be done in $O(n)$ time. If G has only isolated vertices, then any three vertices induce a 2-potential-leaf of G . Otherwise, there is at least one edge in G .

Let G_1 be any connected component of G containing at least one edge. If $|V(G_1)| = 2$, then from Lemma 9, either G has an isolated vertex α and $\{\alpha, u, v\}$ is a 2-potential-leaf or $\{u, v\}$ is a 2-potential leaf.

Otherwise, $|V(G_1)| \geq 3$. Compute a block tree F of G_1 rooted in an arbitrary block R . This can be done in time $O(n + m)$. Note that any node in F corresponds to either a 2-connected component of G or a bridge $uv \in E(G)$. Let C be a leaf block in F , which is furthest from R and $|V(C)|$ is maximum. There are several cases to be considered.

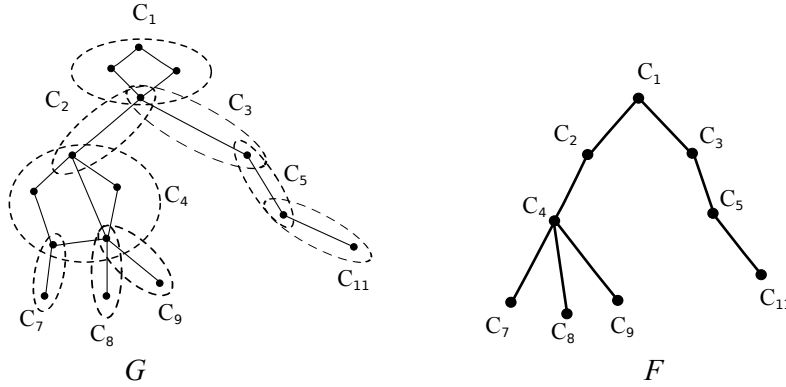


Fig. 10. This graph G is an induced subgraph of the graph in Fig. 9. Its block tree F , with root C_1 , has two blocks less than the one in Fig. 9 (the blocks C_6 and C_{10}). All leaf blocks, C_7, C_8, C_9, C_{11} , in F contains two vertices of G .

- let us first assume that C is a bridge in G , i.e. C consists of one edge $fp \in E(G)$ and p is a cut vertex. Then f has degree one in G because C is a leaf block in F . Let P be the parent block of C in F . Then any child block A of P in F consists of one edge because C has the maximum number of vertices among all the children of P ; and A is a leaf block in F because C is a furthest leaf from the root block R .

If P has another child block except C in F containing the cut vertex p , then this child block also consists of one edge $f'p \in E(G)$, where f' has degree one in G because this child is also a leaf block in F . (For example, in Fig. 10, take C as C_8 , which intersects C_9 with a cut vertex.) From Lemma 7, $\{f, p, f'\}$ is a 2-potential-leaf.

Otherwise P has only one child block C in F containing the cut vertex p . Then any vertex in $N_G(p) \setminus \{f\}$ belongs to P . If P is also a bridge in G , i.e., P consists of one edge $pq \in E(G)$, then p has degree 2 in G . (For example, in Fig. 10, take C as C_{11} , whose parent C_5 is also a bridge in G .) From Lemma 6, $\{f, p, q\}$ is a 2-potential-leaf of G . Otherwise, P is a 2-connected component of G and $p \in V(G)$ satisfies the hypothesis of Lemma 8. (For example, in Fig. 10, take C as C_7 , whose parent C_4 is a 2-connected component of G .) Hence, either G has an isolated vertex α and $\{\alpha, f, p\}$ is a 2-potential-leaf or $\{f, p\}$ is a 2-potential-leaf.

- Finally, let us assume that C is a 2-connected component of G . It is known that any graph with at least two vertices of treewidth k contains at least two vertices of degrees at most k [5]. There is no degree one vertex in C because C is 2-connected. So there exists two vertices with degree 2 in C . Since C is a leaf in F , there is only one cut vertex of G in C . So there exists a vertex b in C which has degree two in G . If $|V(C)| \geq 4$, then there exists a path between two neighbors a, c of b in $G \setminus \{b\}$ containing at least one internal vertex. (For example, in Fig. 9, take C as C_{10} .) From Lemma 9, $\{a, b, c\}$ is a 2-potential-leaf. Otherwise C is a triangle $\{a, b, c\}$ with at least two vertices with degree 2 in G . Again from Lemma 9, $\{a, b, c\}$ is a 2-potential-leaf.

So the total time complexity is $O(n + m)$.

Corollary 4. s_2 can be computed in polynomial-time in general graphs. Moreover, a minimum size tree decomposition can be constructed in polynomial-time in the class of partial 2-trees.

Proof. Let G be any graph. It can be checked in polynomial-time whether $tw(G) \leq 2$ (e.g. see [17]). If $tw(G) > 2$, then $s_2 = \infty$. Otherwise $tw(G) \leq 2$, then the result follows from Theorem 3 and Corollary 3.

5 Minimum-size tree-decompositions of width at most 3

In this section, we study the computation of s_3 in the class of trees and 2-connected outerplanar graphs.

5.1 computation of s_3 in trees

In this subsection, given a tree G , we show how to find a 3-potential-leaf in G . We characterize a complete set of 3-potential-leaves of trees in Fig. 11. We first prove that each of the subgraphs in Fig. 11 is a 3-potential-leaf and then that any tree with at least four vertices contains one of them.

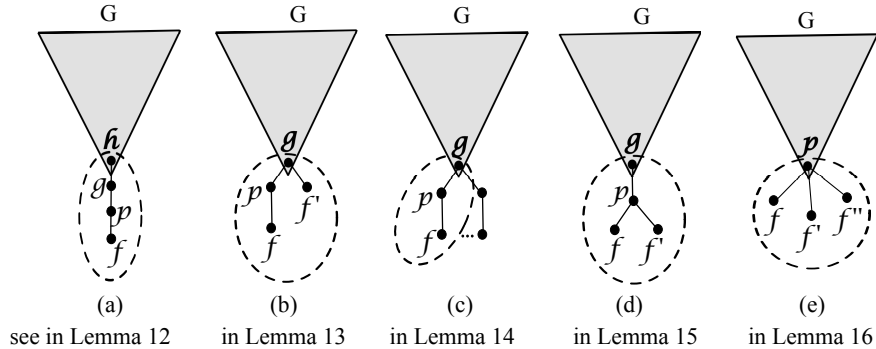


Fig. 11. Complete set of 3-potential-leaves of trees.

Lemma 10. Let (T, \mathcal{X}) be a tree decomposition of a tree G . Let $X \in \mathcal{X}$ and $N_T(X) = \{X_1, \dots, X_d\}$ for $d \geq 1$. Suppose that for any $1 \leq i \leq d$, $X_i \cap X \subseteq \{x\}$. Then there is a tree decomposition (T', \mathcal{X}') of G of the same width and size as (T, \mathcal{X}) such that X is a leaf bag.

Proof. If there is a bag X_i for $1 \leq i \leq d$ containing x , then let B be X_i . Otherwise let B be any neighbor of X . By definition of the operation *Leaf*, the tree-decomposition $\text{Leaf}(X, B, (T, \mathcal{X}))$ is a desired tree decomposition.

Lemma 11. Let G be a tree rooted at $r \in V(G)$. Let f be a leaf in G . Let p be the parent of f and let g be the parent of p in G . Let p have degree 2 in G . Let (T, \mathcal{X}) be a tree decomposition of G of width at most 3 and size at most $s \geq 1$. If there is no bag in (T, \mathcal{X}) containing all of f, p, g , then there is a tree decomposition (T', \mathcal{X}') of G of width at most 3 and size at most s such that $\{f, p, g\} \in \mathcal{X}'$ is a leaf bag.

Proof. Since $fp \in E(G)$, there is a bag B in (T, \mathcal{X}) containing both f and p . We may assume that B is the single bag containing f (otherwise, delete f from any other bag). Similarly, since $pg \in E(G)$, let X be a bag in (T, \mathcal{X}) containing both p and g . Let P be the path in T from B to X . Then p is contained in all bags on P and we may assume that p is not contained in any other bags (otherwise, delete p from any other bag). Let B' be the neighbor of B on P . Then $B \cap B' \supseteq \{p\}$. Note that it is possible that $B' = X$.

If $B = \{f, p\}$, then make all other neighbors of B adjacent to B' and delete B . Add a bag $\{f, p, g\}$ adjacent to X . The result is a desired tree decomposition (T', \mathcal{X}') .

Otherwise, B contains at least one vertex not in $\{f, p\}$. If $B \cap B' = \{p\}$, then $\{p\}$ separates g from any vertex in $B \setminus \{p\}$. So $B \setminus \{p\} = \{f\}$, i.e., $B = \{f, p\}$. It contradicts with the assumption.

So $|B \cap B'| \geq 2$ and let $\{p, x\} \subseteq B \cap B'$. Then create a bag $Z = (B \setminus \{f, p\}) \cup (B' \setminus \{p, x\})$. (Note that $x \in Z$ since $x \in B$.) So $|Z| \leq 4$. Make Z adjacent to all neighbors of B and all neighbors of B' ; and delete the two bags B and B' ; and delete f, p from all bags. Finally add another new bag $N = \{f, p, g\}$ adjacent to some bag containing g . The obtained tree decomposition has width at most 3, same size as (T, \mathcal{X}) , and a bag $N = \{f, p, g\}$.

Lemma 12. Let G be a tree rooted at $r \in V(G)$ and $|V(G)| \geq 4$. Let f be a leaf in G . Let p be the parent of f and let g be the parent of p in G . Suppose that both p and g have degree 2. Let h be the parent of g (see in Fig. 11(a)), then $H = G[\{f, p, g, h\}]$ is a 3-potential-leaf of G .

Proof. Let (T, \mathcal{X}) be any reduced tree decomposition of width at most 3 and size at most $s \geq 1$ of G . We show how to modify it to obtain a tree-decomposition with width at most 3 and size at most s and in which $\{f, p, g, h\}$ is a leaf bag.

From Lemma 11, we can assume that there is a bag B in (T, \mathcal{X}) containing all f, p, g . We may assume that B is the single bag containing f, p (otherwise, delete f, p from any other bag). Since $gh \in E(G)$, let Y be a bag in (T, \mathcal{X}) containing both h and g .

1. If $B = Y = \{f, p, g, h\}$, then the intersection of B and any of its neighbor in T is contained in $\{h\}$. A desired tree decomposition can be obtained from Lemma 10.
2. If $B = \{f, p, g\}$, then the intersection of B and any of its neighbors in T is contained in $\{g\}$. From Lemma 10, there is a tree-decomposition (T', \mathcal{X}') of the same width and size as the ones of (T, \mathcal{X}) such that $B = \{f, p, g\}$ is a leaf. Then delete B in the tree-decomposition $\text{Leaf}(B, B', (T, \mathcal{X}))$ and add a new bag $N = \{f, p, g, h\}$ adjacent to Y . The obtained tree decomposition has the desired properties.
3. Otherwise, $B = \{f, p, g, x\}$ where $x \neq h$. Then the intersection of B and any of its neighbor in T is contained in $\{g, x\}$. Let P be the path in T from B to Y . Then g is contained in all bags on P . Let B' be the neighbor of B on P . Note that it is possible that $B' = Y$. If $B \cap B' = \{g\}$, then $\{g\}$ separates h from x . So $x \in \{f, p\}$ i.e. $B = \{f, p, g\}$, a contradiction with the assumption. So we have $B \cap B' = \{g, x\}$. By definition of the operation *Leaf*, the tree-decomposition $\text{Leaf}(B, B', (T, \mathcal{X}))$ has width at most 3, same size as (T, \mathcal{X}) , and $B = \{f, p, g, x\}$ is a leaf. Then delete B in the tree-decomposition $\text{Leaf}(B, B', (T, \mathcal{X}))$ and add a new bag $N = \{f, p, g, h\}$ adjacent to Y . The obtained tree decomposition has the desired properties since $\{g, x\} \subseteq B'$ and $\{g, h\} \subseteq Y$.

Lemma 13. Let G be a tree rooted at $r \in V(G)$ and $|V(G)| \geq 4$. Let f a leaf in G . Let p be the parent of f and let g be the parent of p in G . If p has degree 2 and g has a child f' , which is a leaf in G (see in Fig. 11(b)), then $H = G[\{f, p, g, f'\}]$ is a 3-potential-leaf of G .

Proof. Let (T, \mathcal{X}) be any reduced tree decomposition of width at most 3 and size at most $s \geq 1$ of G . We show how to modify it to obtain a tree-decomposition with width at most 3 and size at most s and in which $\{f, p, g, f'\}$ is a leaf bag.

From Lemma 11, we can assume that there is a bag B in (T, \mathcal{X}) containing all f, p, g . We may assume that B is the single bag containing f, p (otherwise, delete f, p from any other bag). Since $gf' \in E(G)$, let Y be a bag in (T, \mathcal{X}) containing both f and g . We may assume that Y is the single bag containing f' (otherwise, delete f' from any other bag).

- If $B = Y = \{f, p, g, f'\}$, then the intersection of B and any of its neighbor in T is contained in $\{g\}$. A desired tree decomposition can be obtained from Lemma 10.
- If $B = \{f, p, g\}$, then delete f' in Y and add f' in B , then we are in the previous case.
- Otherwise, $B = \{f, p, g, x\}$ where $x \neq f'$. Then the intersection of B and any of its neighbor in T is contained in $\{g, x\}$. Let P be the path in T from B to Y . Then g is contained in all bags on P . Let B' be the neighbor of B on P . If $B \cap B' = \{g, x\}$, then by definition of the operation *Leaf*, the tree-decomposition $\text{Leaf}(B, B', (T, \mathcal{X}))$ has width at most 3, same size as (T, \mathcal{X}) , and $B = \{f, p, g, x\}$ is a leaf. Then in the tree-decomposition $\text{Leaf}(B, B', (T, \mathcal{X}))$, delete f' in Y and remove x from B and add f' to B , i.e. make $B = \{f, p, g, f'\}$. The obtained tree decomposition has the desired properties since $\{g, x\} \subseteq B'$.
Otherwise $B \cap B' = \{g\}$. Delete f' from the bag Y and add x in Y ; delete x from B and add f' in B , i.e., make $B = \{f, p, g, f'\}$; finally make all neighbors of B except B' adjacent to Y since now $\{g, x\} \subseteq Y$. The result is the desired tree decomposition.

Lemma 14. Let G be a tree rooted at $r \in V(G)$ and $|V(G)| \geq 3$. Let f be one of the furthest leaves from r . Let p be the parent of f and let g be the parent of p in G . If g has degree at least 3 and any child of g has degree 2 in G (see in Fig. 11(c)), then $H = G[\{f, p, g\}]$ is a 3-potential-leaf of G .

Proof. Let (T, \mathcal{X}) be any reduced tree decomposition of width at most 3 and size at most $s \geq 1$ of G . We show how to modify it to obtain a tree-decomposition with width at most 3 and size at most s and in which $\{f, p, g, f'\}$ is a leaf bag.

From Lemma 11, we can assume that there is a bag B in (T, \mathcal{X}) containing all f, p, g . We may assume that B is the single bag containing f, p (otherwise, delete f, p from any other bag).

1. If $B = \{f, p, g\}$, then the intersection of B and any of its neighbor in T is contained in $\{g\}$. A desired tree decomposition can be obtained from Lemma 10.
2. Otherwise, $B = \{f, p, g, x\}$. Then the intersection of B and any of its neighbor in T is contained in $\{g, x\}$.
 - (a) If there is a neighbor B' of B such that $B \cap B' = \{g, x\}$, then by definition of the operation *Leaf*, the tree-decomposition $\text{Leaf}(B, B', (T, \mathcal{X}))$ has width at most 3, same size as (T, \mathcal{X}) , and $B = \{f, p, g, x\}$ is a leaf. Then delete x in B in the tree-decomposition $\text{Leaf}(B, B', (T, \mathcal{X}))$ since $\{g, x\} \subseteq B'$. The obtained tree decomposition has the desired properties.
 - (b) Otherwise any neighbor of B contains at most one of g and x . If x is not adjacent to g , then there is a connected component in $G \setminus B$ containing a neighbor of g and a neighbor of x . From Lemma 4, there exists a neighbor bag of B in (T, \mathcal{X}) containing g and x . It is a contradiction. So we have x is adjacent to g in this case.
 - i. x is a child of g . Then x has exactly one child y , which is a leaf in G since f is one of the furthest leaves from r . Since $yx \in E(G)$, there is a bag Y in (T, \mathcal{X}) containing both y and x . We may assume that Y is the single bag containing y (otherwise, delete y from any other bag). Since $\{g, x\} \subset B$ and any neighbor of B contains at most one of g and x , any bag except B contains at most one of g and x . Then $g \notin Y$ because $x \in Y$. So y, x, g are not contained in one bag. From Lemma 11, we can modify (T, \mathcal{X}) to obtain a tree-decomposition (T', \mathcal{X}') of width at most 3 and size at most s having a leaf bag $X = \{y, x, g\}$. Note that x (resp. y) plays the same role as p (resp. f) in G , i.e., g, p, f and g, x, y are symmetric in G . Hence, the result is a desired tree decomposition.
 - ii. x is the parent of g . Let p' be another child of g and let f' be the child of p' , which is a leaf in G . Let B' be the bag in (T, \mathcal{X}) containing both f' and p' . We may assume that B' is the single bag containing f' (otherwise, delete f' from any other bag). Let X' be a bag containing both p' and g . Then we have

$X' \neq B$ (because $p' \notin B$). Since $g \in X'$ any bag except B contains at most one of g and x , we have $x \notin X'$. In the following, we modify (T, \mathcal{X}) to obtain a tree-decomposition (T', \mathcal{X}') with width at most 3 and size at most s having a bag $\{f', p', g\}$. Then we are in case 1, since g, p, f and g, p', f' are symmetric in G .

If $B' = X' = \{f', p', g\}$ then, we are done. If $B' = X' = \{f', p', g, x'\}$. Then x' is not x , which is the parent of g , since $x \notin X'$. So we can do as in case 2a or case 2(b)i.

Otherwise, $B' \neq X'$. From Lemma 11, we can modify (T, \mathcal{X}) to obtain a tree-decomposition with width at most 3 and size at most s having a leaf bag $\{f', p', g\}$.

Lemma 15. *Let G be a tree rooted at $r \in V(G)$ and $|V(G)| \geq 4$. Let f a leaf in G . Let p be the parent of f . If p has exactly two children f, f' in G and let g be the parent of p in G (see in Fig. 11(d)), then $H = G[\{f, f', p, g\}]$ is a 3-potential-leaf of G .*

Proof. Let (T, \mathcal{X}) be any reduced tree decomposition of width at most 3 and size at most $s \geq 1$ of G . We show how to modify it to obtain a tree-decomposition with width at most 3 and size at most s and in which $\{f, f', p, g\}$ is a leaf bag.

Since $fp \in E(G)$, there is a bag B in (T, \mathcal{X}) containing both f and p . We may assume that B is the single bag containing f (otherwise, delete f from any other bag). Similarly, let B' be the single bag in (T, \mathcal{X}) containing both f' and p . Let X be a bag containing both p and g .

1. If $B = B' = X = \{f, f', p, g\}$, then we can assume that B is the single bag containing p (otherwise, delete p from any other bag). So the intersection of B and any of its neighbor in T is contained in $\{g\}$. Then a desired tree decomposition can be obtained from Lemma 10.
2. If $B = B' = \{f, f', p\}$, then the intersection of B and any of its neighbor in T is contained in $\{p\}$. Let Y be a neighbor of B in T containing p . By definition of the operation *Leaf*, the tree-decomposition $Leaf(B, Y, (T, \mathcal{X}))$ has width at most 3, same size as (T, \mathcal{X}) , and $B = \{f, f', p\}$ is a leaf. Then delete B and add a new bag $N = \{f, f', p, g\}$ adjacent to X . The result is a desired tree decomposition.
3. If $B = B' = \{f, f', p, x\}$ and $x \neq g$, then the intersection of B and any of its neighbor in T is contained in $\{p, x\}$. Since $x \notin \{f, f', g\}$, p is not adjacent to x . There is a connected component in $G \setminus B$ containing a neighbor of p and a neighbor of x . From Lemma 4, there exists a neighbor bag of B in (T, \mathcal{X}) containing p and x . Let Y be such a neighbor of B in T . By definition of the operation *Leaf*, the tree-decomposition $Leaf(B, Y, (T, \mathcal{X}))$ has width at most 3, same size as (T, \mathcal{X}) , and $B = \{f, f', p, x\}$ is a leaf. Then delete x from B and we get a tree decomposition having a bag $\{f, f', p\}$. So we are in case 2.
4. If $B \neq B'$ and $|B| \leq 3$, then delete f' in B and add f' in B' . Then we are in case 2 or 3. It is proved similarly if $B \neq B'$ and $|B'| \leq 3$.
5. Otherwise $B \neq B'$ and $|B| = |B'| = 4$. Let $B = \{f, p, x, y\}$ and $B' = \{f', p, x', y'\}$. Let P be the path in T from B to B' . Then p is contained in all bags on P . Let Y be the neighbor of B on P . If $B \cap Y = \{p\}$, then $\{p\}$ separates x from x' . But p is not a separator between any two vertices in $V(G) \setminus \{f, f'\}$. It is a contradiction. So w.l.o.g. we can assume that $B \cap Y \supseteq \{p, x\}$. Deleting f, f', p in all bags of (T, \mathcal{X}) . Add a new bag $Z = \{x, y\} \cup Y \setminus \{p, x\}$ adjacent to all neighbors of the two bags B, Y and delete B and Y . Finally add another new bag $N = \{f, f', p, g\}$ adjacent to a bag containing g . The obtained tree decomposition has the desired properties.

Lemma 16. *Let G be a tree rooted at $r \in V(G)$ and $|V(G)| \geq 4$. Let all children of p be leaves in G and p have at least three children f, f', f'' (see in Fig. 11(e)). Then $H = G[\{p, f, f', f''\}]$ is a 3-potential-leaf of G .*

Proof. Let (T, \mathcal{X}) be any reduced tree decomposition of width at most 3 and size at most $s \geq 1$ of G . We show how to modify it to obtain a tree-decomposition with width at most 3 and size at most s and in which $\{p, f, f', f''\}$ is a leaf bag.

Since $fp \in E(G)$, there is a bag B in (T, \mathcal{X}) containing both f and p . We may assume that B is the single bag containing f (otherwise, delete f from any other bag). Similarly, let B' (resp. B'') be the single bag in (T, \mathcal{X}) containing both f' (resp. f'') and p .

1. If $B = B' = B'' = \{f, f', f'', p\}$, then the intersection of B and any of its neighbor in T is contained in $\{p\}$. A desired tree decomposition can be obtained from Lemma 10.

2. If $B = B' = \{f, f', p\}$, then delete f'' in B'' and add f'' in B . Then we are in case 1. It can be proved similarly if $B = B'' = \{f, f'', p\}$ or $B' = B'' = \{f', f'', p\}$.
3. If $B = B' = \{f, f', p, x\}$ and $x \neq f''$, then the intersection of B and any of its neighbor in T is contained in $\{p, x\}$.

If x is a child of p , then x is also a leaf in G and x play the same role as f'' . Then we are in case 1. So in the following we assume that x is not a child of p .

If x is not the parent of p , then p is not adjacent to x . So there is a connected component in $G \setminus B$ containing a neighbor of p and a neighbor of x . From Lemma 4, there exists a neighbor bag of B in (T, \mathcal{X}) containing p and x . Let Y be such a neighbor of B in T . By definition of the operation *Leaf*, the tree-decomposition $Leaf(B, Y, (T, \mathcal{X}))$ has width at most 3, same size as (T, \mathcal{X}) , and $B = \{f, f', p, x\}$ is a leaf. Then delete x from B and we get a tree decomposition having a bag $\{f, f', p\}$. So we are in case 2.

Otherwise x is the parent of p . Let P be the path in T from B to B'' . Then p is contained in all bags on P . Let Y be the neighbor of B on P . If $B \cap Y = \{p, x\}$, then by definition of the operation *Leaf*, the tree-decomposition $Leaf(B, Y, (T, \mathcal{X}))$ has width at most 3, same size as (T, \mathcal{X}) , and $B = \{f, f', p, x\}$ is a leaf. Then deleting x from B we are in case 2. Otherwise, $B \cap Y = \{p\}$. So $\{p\}$ separates x from all vertices in $B'' \setminus \{p\}$. Then all vertices in $B'' \setminus \{p\}$ are children of p and so they are leaves in G . So we can assume that any vertex in $B'' \setminus \{p\}$ are contained only in B'' (otherwise we can delete it in any other bags). Then delete f, f' from B and add vertices of $B'' \setminus \{f'', p\}$ in B ; and make $B'' = \{f, f', f'', p\}$. Then we are in case 1.

The cases $B = B'' = \{f, f'', p, x\}$ and $x \neq f'$ or $B' = B'' = \{f', f'', p, x\}$ and $x \neq f$ can be proved similarly.

4. Otherwise, none two of f, f', f'' are contained in a same bag.

If $|B| \leq 3$, then delete f' in B' and add f' in B . Then we are in case 2 or 3. It is proved similarly if $|B'| \leq 3$ or $|B''| \leq 3$.

Otherwise $|B| = |B'| = |B''| = 4$. In the following, we are going to modify (T, \mathcal{X}) to obtain a tree-decomposition with width at most 3 and size at most s having a bag X containing at least two of f, f', f'' or $f \in X$ and $|X| \leq 3$. Then we are in the above cases. Note that all children of p play the same role (they are all leaves) in G . So it is enough to get that X contains at least two children of p or that X contains one child of p and $|X| \leq 3$.

Let T_p be the subtree in T induced by all the bags containing p . If $|V(T_p)| \leq 2$, there exists one bag containing at least two children of p since p has at least three children. Then it is done. So we assume that $|V(T_p)| \geq 3$. There is a bag $R \in V(T_p)$ containing both p and g . Root T_p at R and let $L \in V(T_p)$ be one of the furthest leaf bag in T_p from R . If there is no child of p in L , then we can delete p in L and consider $T_p \setminus \{L\}$. So we can assume there is a vertex $l \in L$, which is a child of p in G . Let Y be the neighbor of L in T_p . If the intersection of $L \cap Y = \{p\}$, then p separate any vertex in $L \setminus \{p\}$ and any vertex in $Y \setminus \{p\}$. So at least one of L, Y , denoted as X , contains only p and children of p . Then either X contains at least two children of p or X contains only one children and $|X| = 2$. So (T, \mathcal{X}) and X satisfy the desired properties.

Otherwise, $|L \cap Y| \geq 2$. If Y has no other child except L in T_p , then $Y \neq R$ since $|V(T_p)| \geq 3$. Let $X = \{p, l\}$ if Y contains no child of p ; and $X = \{p, l, l'\}$ if Y contains one child l' of p . Add a new bag $Z = Y \cup L \setminus X$. Since $|Y \cap L| \geq 2$, $|Y \cup L| \leq 6$. Then $|Z| \leq 4$, since $X \subseteq Y \cup L$ and $|X| \geq 2$. Make Z adjacent to all neighbors of Y, L in T and delete Y, L . Finally make X adjacent to R . The obtained tree decomposition and X have the desired properties.

Otherwise, Y has at least another child L' in T_p . Then L' is also a furthest leaf from R in T_p , since L is a furthest leaf from R . For the same reason as L , there is a vertex $l' \in L'$, which is a child of p in G . Let $L = \{l, p, x, y\}$ and $L' = \{l', p, x', y'\}$. So the intersection of L (resp. L') and any of its neighbors except Y in T is contained in $\{x, y\}$ (resp. $\{x', y'\}$). Create a new bag $N = \{x, y, x', y'\}$ adjacent to all neighbor of L, L' and delete L, L' . Finally add another bag $X = \{p, l, l'\}$ adjacent to Y . The obtained tree decomposition and X have the desired properties.

From Lemmas 12- 16 and Corollary 3, we obtain the following result.

Corollary 5. s_3 and a minimum size tree decomposition of width at most 3 can be computed in polynomial-time in the class of trees.

Proof. From Corollary 3, it is enough to prove we can find a 3-potential-leaf in any tree in polynomial time.

Let G be any tree. If $|V(G)| \leq 4$, then $V(G)$ is a 3-potential-leaf. Let us assume that $|V(G)| \geq 5$. Root G at any vertex r . Let f be one of the furthest leaves from r in G . Let p, g, h be the first three vertices on the path from f to r in G (if exist), i.e. p is f 's parent; g is p 's parent; and h is g 's parent in G .

- If g, p both have only one child in G , then $\{f, p, g, h\}$ is a 3-potential-leaf of G from Lemma 12;
- If p has only one child and g has a child f' , which is a leaf in G , then $\{f, p, g, f'\}$ is a 3-potential-leaf of G from Lemma 13;
- If p has only one child and any child of g has exactly one child, then $\{f, p, g\}$ is a 3-potential-leaf of G from Lemma 14;
- If p has only one child and there exist a child p' of g , which has exactly two children f_1, f_2 , then $\{f_1, f_2, p', g\}$ is a 3-potential-leaf of G from Lemma 15;
- If p has only one child and there exist a child p' of g , which has at least three children f_1, f_2, f_3 , then $\{f_1, f_2, f_3, p'\}$ is a 3-potential-leaf of G from Lemma 16;
- If p has exactly two children f, f' , then $\{f, f', p, g\}$ is a 3-potential-leaf of G from Lemma 15;
- Otherwise p has at least three children f, f', f'' , then $\{f, f', f'', p\}$ is a 3-potential-leaf of G from Lemma 16.

In fact, the algorithm for trees can be extended to forests by consider their connected component, i.e., trees. The only difference is in Lemma 14 the 3-potential-leaf becomes $\{f, p, g, \alpha\}$ if there is an isolated vertex α in the given forest.

5.2 Computation of s_3 in 2-connected outerplanar graphs

In this subsection, given a 2-connected outerplanar graph G , we show how to find a 3-potential-leaf in G . See a complete set of 3-potential-leaves of 2-connected outerplanar graphs in Fig. 12. We first prove that each subgraph in the Fig. 12 is a 3-potential-leaf and then we show that any 2-connected outerplanar graphs contains one of them.

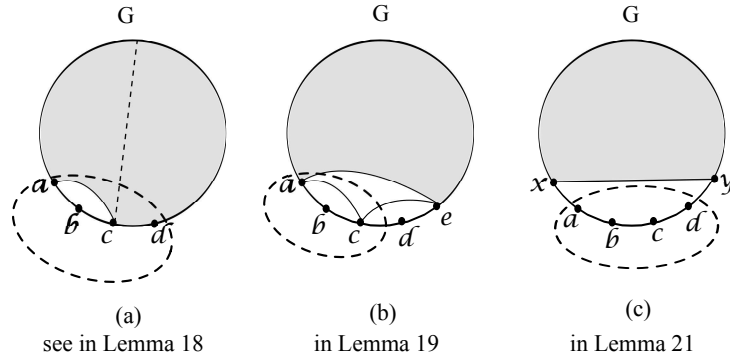


Fig. 12. Complete set of 3-potential-leaves of 2-connected outerplanar graphs.

The following fact is well known for 2-connected outerplanar graphs.

Lemma 17. [15] *A 2-connected outerplanar graph has the unique Hamilton cycle.*

In the rest of this subsection, let G be a 2-connected outerplanar graph and C be the Hamilton cycle in G .

Definition 2. Any edge in $E(G) \setminus E(C)$ is called a *chord* in G .

The vertices $v_1, \dots, v_j \in V(G)$, for $2 \leq j \leq |V(G)|$, are *consecutive* in C (we also say that they are consecutive in G) if $v_i v_{i+1} \in E(C)$ for $1 \leq i \leq j-1$; and $v_1, \dots, v_j \in V(G)$ are also called the consecutive vertices from v_1 to v_j .

Lemma 18. *Let $a, b, c, d \in V(G)$ be consecutive in C . If $\{a, b, c\}$ induces a clique and c has degree 3 in G (see in Fig. 12(a)), then $H = G[\{a, b, c, d\}]$ is a 3-potential-leaf of G .*

Proof. Let (T, \mathcal{X}) be any tree decomposition of width at most 3 and size at most $s \geq 1$ of G . We show how to modify it to obtain a tree-decomposition with width at most 3 and size at most s and in which $\{a, b, c, d\}$ is a leaf bag.

Since $\{a, b, c\}$ induces a clique in G , there is a bag B containing all a, b, c . Let X be a bag in (T, \mathcal{X}) containing both c and d (it exists since $cd \in E(G)$). Note that b is not incident to any chords, i.e. has degree 2. (Because if $by \in E(G)$ is a chord in G , then deleting all chords except ac, by in G and contracting the edges in C except ab, bc we get a K_4 -minor in G . It is a contradiction with the fact that G is outerplanar.)

Replace b, c with a in all bags of (T, \mathcal{X}) . Then (T, \mathcal{X}) becomes a tree decomposition (T', \mathcal{X}') of the graph G' obtained by contracting the edges ab and bc . The bag X becomes X' , which contains both a and d ; and B becomes $B' = \{a\}$ if $B = \{a, b, c\}$ or $B' = \{a, x\}$ if $B = \{a, b, c, x\}$. From Corollary 2, in both case there exists a neighbor Y of B' such that $B' \subseteq Y$. So B' can be reduced in (T', \mathcal{X}') . The tree decomposition $\text{Reduce}(B', Y, (T', \mathcal{X}'))$ has one bag less than (T, \mathcal{X}) . Finally, add a new bag $N = \{a, b, c, d\}$ adjacent to X' , which contained both a and d , in the tree decomposition $\text{Reduce}(B', Y, (T', \mathcal{X}'))$. The result is a desired tree decomposition, because b, c are not adjacent to any vertices in $V(G) \setminus N$.

Lemma 19. *Let $a, b, c, d, e \in V(G)$ be consecutive in C . If $\{a, b, c\}$ and $\{c, d, e\}$ induce two cliques respectively in G and $ae \in E(G)$ (see in Fig. 12(b)), then $H = G[\{a, b, c\}]$ is a 3-potential-leaf of G .*

Proof. Let (T, \mathcal{X}) be any tree decomposition of width at most 3 and size at most $s \geq 1$ of G . We show how to modify it to obtain a tree-decomposition with width at most 3 and size at most s and in which $\{a, b, c\}$ is a leaf bag.

Since $\{a, b, c\}$ (resp. $\{c, d, e\}$) induces a clique in G , there is a bag X (resp. Y) containing all a, b, c (resp. c, d, e). Note that b, c, d are not adjacent to any vertices in $V(G) \setminus \{a, b, c, d, e\}$.

Delete b, c, d in all bags of (T, \mathcal{X}) . Then (T, \mathcal{X}) becomes a tree decomposition (T', \mathcal{X}') of the graph $G' = G \setminus \{b, c, d\}$. The bag X becomes $X' = \{a\}$ if $X = \{a, b, c\}$ or $X' = \{a, x\}$ if $X = \{a, b, c, x\}$. From Corollary 2, in both case there exists a neighbor A of X' such that $X' \subseteq A$. So X' can be reduced in (T', \mathcal{X}') . Similarly, the bag Y becomes Y' , which can also be reduced in (T', \mathcal{X}') . After reducing the two bags X', Y' in (T', \mathcal{X}') , let the obtained tree decomposition be (T'', \mathcal{X}'') . Finally, add two new bags $N_1 = \{a, b, c\}$ and $N_2 = \{a, c, d, e\}$; make N_1 adjacent to N_2 and make N_2 adjacent to a bag Z containing both a and e in the tree decomposition (T'', \mathcal{X}'') . (Z exists because $ae \in E(G')$.) The result is a desired tree decomposition.

Lemma 20. *Let C_l be a cycle of $l \geq 4$ vertices. Let (T, \mathcal{X}) be a tree decomposition of C_l of width at most 3. Then there exist either a bag containing all vertices of $V(C_l)$ (only if $l = 4$) or two bags $X, Y \in \mathcal{X}$ such that X (resp. Y) contains at least three consecutive vertices x_1, x_2, x_3 (resp. y_1, y_2, y_3) and the two edge sets $\{x_1x_2, x_2x_3\} \cap \{y_1y_2, y_2y_3\} = \emptyset$.*

Proof. The treewidth of any cycle is bigger than 1, so there exists a bag in any tree decomposition of a cycle (with at least 4 vertices) containing two vertices not consecutive, equivalently they are not adjacent in the cycle. We prove the lemma by induction on l in the following.

First let us prove that it is true for $l = 4$. Let a, b, c, d be the four consecutive vertices in C_4 . Let (T, \mathcal{X}) be a tree decomposition of width at most 3. Then there exists a bag containing a, c or b, d . W.l.o.g assume a, c are contained in one bag. So (T, \mathcal{X}) is also a tree decomposition of H , obtained from C_4 by adding the edge ac . The set $\{a, b, c\}$ induces a clique in H . So there is a bag X containing a, b, c . For the same reason, there is a bag Y containing c, d, a . If $X = Y$ then there is a bag containing all a, b, c, d of $V(C_4)$. Otherwise there are two bags X, Y such that $X \supseteq \{a, b, c\}$ and $Y \supseteq \{c, d, a\}$. We see that $\{ab, bc\} \cap \{cd, da\} = \emptyset$. So the lemma is true for $l = 4$.

Now suppose it is true for $l \leq n - 1$ and we prove it for $l = n \geq 5$. Note that since (T, \mathcal{X}) has width 3 and $l \geq 5$, there is no bag containing all vertices of $V(C_l)$. So in the following we prove that there always exist two bags X, Y with the desired properties. Let v_1, \dots, v_n be the n consecutive vertices in C_n . Let (T, \mathcal{X}) be a tree-decomposition of width at most 3 of C_n . Then there exists a bag containing two non-adjacent vertices v_i, v_j for $1 \leq i < j \leq n$. So (T, \mathcal{X}) is also a tree decomposition of the graph H , obtained from C_n by adding the edge $v_i v_j$. The graph H is also the union of two subcycles C^1 induced by $\{v_i, \dots, v_j\}$ and C^2 induced by $\{v_j, \dots, v_n, \dots, v_i\}$. Then $\max\{|C^1|, |C^2|\} \leq n - 1$. Let (T^1, \mathcal{X}^1) (resp. (T^2, \mathcal{X}^2)) be the tree decomposition of C^1 (resp. C^2) obtained by deleting all vertices not in C^1 (resp. C^2) in the bags of (T, \mathcal{X}) .

If $|V(C^1)| = 3$ then there is a bag in (T^1, \mathcal{X}^1) containing $V(C^1) = \{v_i, v_{i+1}, v_j = v_{i+2}\}$. So $v_i v_j \notin \{v_i v_{i+1}, v_{i+1} v_j\}$.

If $|V(C^1)| \geq 4$ then, by induction, there exist either a bag in (T^1, \mathcal{X}^1) containing all vertices of $V(C^1) = \{v_i, v_{i+1}, v_{i+2}, v_j = v_{i+3}\}$ or two bags A, B in (T^1, \mathcal{X}^1) containing three consecutive vertices a_1, a_2, a_3 and b_1, b_2, b_3

respectively in C^1 ; moreover, $\{a_1a_2, a_2a_3\} \cap \{b_1b_2, b_2b_3\} = \emptyset$. So we have either $v_iv_j \notin \{a_1a_2, a_2a_3\}$ or $v_iv_j \notin \{b_1b_2, b_2b_3\}$.

In both cases ($|V(C^1)| = 3$ and $|V(C^1)| \geq 4$), there is at least one bag X in (T^1, X^1) containing three consecutive vertices in C^1 , denoted as x_1, x_2, x_3 , such that $v_iv_j \notin \{x_1x_2, x_2x_3\}$. So x_1, x_2, x_3 are also consecutive in C . Similarly, there is at least one bag Y in (T^2, X^2) containing three consecutive vertices in C^2 , denoted as y_1, y_2, y_3 , such that $v_iv_j \notin \{y_1y_2, y_2y_3\}$. So y_1, y_2, y_3 are also consecutive in C . Finally, we have $\{x_1x_2, x_2x_3\} \cap \{y_1y_2, y_2y_3\} = \emptyset$ because $E(C^1) \cap E(C^2) = \{v_iv_j\}$ and $v_iv_j \notin \{x_1x_2, x_2x_3\}$.

Lemma 21. *Let xy be a chord in G . Let C' be the set of all the consecutive vertices from x to y in C and $|C'| \geq 4$. If each vertex in $C' \setminus \{x, y\}$ has degree 2 in G , then for any consecutive vertices $a, b, c, d \in C'$ (see in Fig. 12(c)), $H = G[\{a, b, c, d\}]$ is a 3-potential-leaf of G .*

Proof. Let (T, \mathcal{X}) be any tree decomposition of width at most 3 and size at most $s \geq 1$ of G . We show how to modify (T, \mathcal{X}) to obtain a tree-decomposition of G , which has width at most 3, size at most s and a leaf bag $\{a, b, c, d\}$.

Note that the vertices of C' induce a cycle in G . Without confusion, we denote this cycle C' . Let (T', X') be the tree decomposition of C' obtained by deleting all vertices not in C' in the bags of (T, X) . From Lemma 20, there is either a bag containing all vertices in C' (only if $|C'| = 4$); or two bags X, Y containing three consecutive vertices in C' respectively and the two corresponding edge sets do not intersect.

In the former case, $V(C') = \{a, b, c, d\}$ and so (T, \mathcal{X}) is also a tree decomposition of $G \cup \{ac\}$, from Lemma 18, $\{a, b, c, d\}$ is a 3-potential-leaf of G .

In the latter case, let $X \supseteq \{u, v, w\}$ and $Y \supseteq \{u', v', w'\}$, where u, v, w (resp. u', v', w') are consecutive in C' . Since $\{uv, vw\} \cap \{u'v', v'w'\} = \emptyset$, we have either $xy \notin \{uv, vw\}$ or $xy \notin \{u'v', v'w'\}$. W.l.o.g. assume that $xy \notin \{uv, vw\}$. Then u, v, w are also consecutive in C and at least one of u, w has degree 2 in G . W.l.o.g. suppose w has degree 2 in G , i.e. $w \notin \{x, y\}$ (since x, y have degree at least 3 in G). Let $z \in C'$ be the other neighbor (except v) of w in C' . (z exists because $w \notin \{x, y\}$.)

(T, \mathcal{X}) is also a tree decomposition of $G \cup \{uw\}$, which is still an outerplanar graph by assumptions. Note that w has degree 3 in the graph $G \cup \{uw\}$. So from Lemma 18, we can modify (T, \mathcal{X}) to obtain a tree-decomposition (T', \mathcal{X}') of $G \cup \{uw\}$, which has width at most 3, size at most s and a leaf bag L containing four consecutive vertices $\{u, v, w, z\}$. Note that (T', \mathcal{X}') is also a tree decomposition of G . So we get a tree decomposition where a leaf bag contains 4 consecutive vertices of C' . It remains to show how to modify it to obtain a tree decomposition with a leaf bag $\{a, b, c, d\}$.

Let B be the neighbor of L in T . Then $u, z \in B$ since each of u, z is adjacent to some vertices in $G \setminus L$. We can assume that L is the single bag containing v, w in (T', \mathcal{X}') , because otherwise we can delete them in any other bags. Thus, deleting the bag L in (T', \mathcal{X}') , we get a tree decomposition (T_1, \mathcal{X}_1) of the graph G_1 , which is the graph obtained by deleting v, w and adding an edge uz in G . So (T_1, \mathcal{X}_1) has width at most 3 and size at most $s - 1$. Note that the graph G_1 is isomorphic to the graph $G_2 \equiv G \cup \{ad\} \setminus \{b, c\}$ since $z \in C'$. So from the tree decomposition (T_1, \mathcal{X}_1) of G_1 we can obtain a tree decomposition (T_2, \mathcal{X}_2) of G_2 with the same width and size. Note that since $ad \in E(G_2)$, there is a bag Y containing both a and d . Finally, add a new bag $N = \{a, b, c, d\}$ adjacent to Y in (T_2, \mathcal{X}_2) . The result is a desired tree decomposition.

Lemma 22. *There is an algorithm that, for any 2-connected outerplanar graph G , computes a 3-potential-leaf of G in polynomial time.*

Proof. Let G be a 2-connected outerplanar graph and C be the unique Hamilton cycle of G . If $|V(G)| \leq 4$, then $V(G)$ is a 3-potential-leaf of G . Otherwise, $|V(G)| \geq 5$ and consider the outerplanar embedding of G .

- If there exists an inner face f with at most one chord of G and f has at least four vertices, then from Lemma 21, the set of any four consecutive vertices in f , which are also consecutive in C , is a 3-potential-leaf in G .
- If there is an inner face $f = \{a, b, c\}$ with only one chord ac of G and c has degree 3, then let d be the other neighbor of c except b, a . From Lemma 18, the set of four consecutive vertices a, b, c, d , is a 3-potential-leaf in G .
- Otherwise, let \mathcal{F} be the set of all inner faces with only one chord of G . Then any face $f \in \mathcal{F}$ has three vertices and both the two endpoints of the chord in f have degree at least 4, i.e., they are incident to some other chords except this one. We can prove by induction on $|V(G)|$ that:

Claim. There exist two faces $f_1, f_2 \in \mathcal{F}$ such that (1) $f_1 = \{a, b, c\}$; (2) $f_2 = \{c, d, e\}$; (3) a, b, c, d, e are consecutive in G ; (4) there is a face f_0 containing both ac and ce and at most one chord, which is not in any face of \mathcal{F} . See in Fig. 13.

It is true when $|V(G)| = 5$. Assume that it is true for $|V(G)| \leq n - 1$. Now we prove it is true for $|V(G)| = n$. Note that $\mathcal{F} \neq \emptyset$ if there is at least one chords in G , which is valid in this case. Let $f \in \mathcal{F}$ have three consecutive vertices x, y, z and let $xz \in E(G)$ be the single chord in f . Then the graph $G \setminus y$ is a 2-connected outerplanar graph with $n - 1$ vertices. From the assumption, we have the desired faces f'_0, f'_1, f'_2 in $G \setminus y$. If xz is not an edge in any face of f'_1, f'_2 , then these faces are also the desired faces in G . Otherwise, let xz be an edge of f'_1 or $f'_2 = \{x, z, t\}$. Then z has degree 3 in G , i.e. it is not incident to any other chords except xz , since $xt \in E(G)$. So we are in second case above, which contradicts with the assumption.

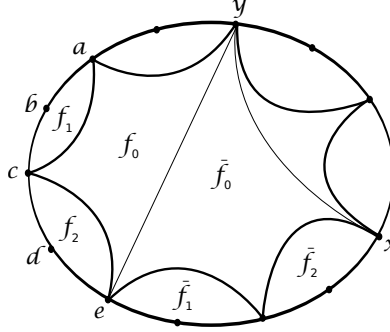


Fig. 13. \mathcal{F} is the set of all inner faces with only one chord of G , such as $f_1, f_2, \bar{f}_1, \bar{f}_2$. The faces f_0, f_1, f_2 satisfy the properties in Claim 5.2. But $\bar{f}_0, \bar{f}_1, \bar{f}_2$ does not satisfy the properties since \bar{f}_0 contains two edges ey, xy which are not in any face of \mathcal{F} .

In the following, let f_0, f_1, f_2 be the faces as in Claim 5.2. If $ae \in E(G)$, then from Lemma 19, $\{a, b, c\}$ is a 3-potential-leaf of G .

Otherwise, we can prove that any tree decomposition of G of width at most 3 can be modified to a tree decomposition of $G \cup \{ae\}$ with the same width and size in the following. So $\{a, b, c\}$ is a 3-potential-leaf of G .

Let (T, \mathcal{X}) be a tree decomposition of width at most 3 and size at most $s \geq 1$ of G . Let (T_0, \mathcal{X}_0) be the tree decomposition obtained by deleting all vertices not in f_0 . Then (T_0, \mathcal{X}_0) is a tree decomposition of f_0 (without confusion f_0 is used to denote the face and the cycle induced by vertices in f_0 as well). From Lemma 20, there is a bag containing three consecutive vertices u, v, w in f_0 and uv, vw are edges of some faces in \mathcal{F} . (Note that u, v, w are not consecutive in C .) So (T, \mathcal{X}) is also a tree decomposition of $G \cup uw$. The graph $G \cup uw$ and the graph $G \cup ae$ are isomorphic. So from (T, \mathcal{X}) we can obtain a tree decomposition (T', \mathcal{X}') of $G \cup ae$ with the same width and size. Then (T', \mathcal{X}') is the desired tree decomposition.

From Lemmas 22 and Corollary 3, we obtain the following result.

Corollary 6. s_3 can be computed and a minimum size tree decomposition of width at most 3 can be constructed in polynomial-time in the class of 2-connected outerplanar graphs.

6 Conclusion

In this report, we gave preliminary results on the complexity of minimizing the size of tree-decompositions with given width. Table 1 summarizes our results as well as the remaining open questions.

We currently investigate the case of s_3 in the class of connected graphs with treewidth 2 or 3 and we conjecture it is polynomially solvable. But it is more tricky than computing s_3 in trees and 2-connected outerplanar graphs. It seems that a global view of the graph needs to be considered to decide whether a subgraph is a 3-potential-leaf of the graph.

| | s_1 | s_2 | s_3 | s_4 | $s_k, k = \max\{tw + 1, 5\}$ |
|---|---------------|-------|-------|---------|------------------------------|
| Graphs of treewidth at most $tw = 1$ | P (trivial) | P | P | ? | ? |
| Graphs of treewidth at most $tw = 2$ | - | P | ? | ? | ? |
| Graphs of treewidth at most $tw = 3$ | - | - | ? | NP-hard | ? |
| Graphs of treewidth at most $tw \geq 4$ | - | - | - | NP-hard | NP-hard |

Table 1. Summary of the complexity results.

See an example in Fig. 14(a). In this example, G is a connected outerplanar graph. $\{r, a, b, c\}$ is not a 3-potential-leaf of G , but it is a 3-potential-leaf of $G \setminus \{yw\}$. Let $G' \equiv G \setminus \{a, b, c\}$. Then G' is 2-connected outerplanar. From the algorithm of computing s_3 in 2-connected outerplanar graphs in subsection 5.2, we can compute that $s_3(G') = 5$. So if $\{r, a, b, c\}$ is a potential-leaf of G , then $s_3(G) = 6$. But there exists a tree decomposition of G of width 3 and size 5, where the bags are $\{a, r, z, y\}, \{r, y, x, w\}, \{b, r, w, v\}, \{r, v, u, e\}, \{c, r, d, e\}$. So $\{r, a, b, c\}$ is not a 3-potential-leaf of G . However, in the graph $G'' \equiv G \setminus \{yw\}$, we can prove that $s_3(G'') = 5$ and there is a minimum size tree decomposition containing $\{r, a, b, c\}$ as a leaf bag, i.e. $\{r, a, b, c\}$ is 3-potential-leaf of G'' . So the existence of the edge yw , not incident to any vertex in $\{r, a, b, c\}$, changes the behavior of $\{r, a, b, c\}$.

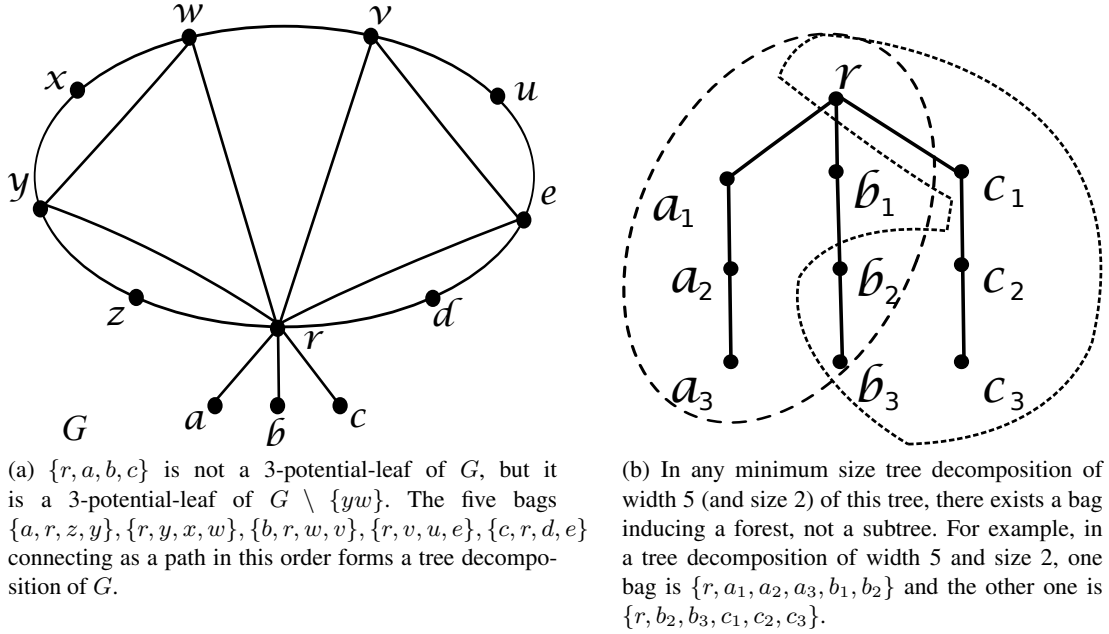


Fig. 14.

The problem of computing s_k , for $k \geq 4$, seems more intricate already in the case of trees. Indeed, our polynomial-time algorithms to compute s_k , $k \leq 3$, in trees mainly rely on the fact that, for any tree T , there exists a minimum-size tree-decomposition of T with width at most 3, where each bag induces a connected subtree. This is unfortunately not true anymore in the case of tree-decomposition with width 5. As an example, consider the tree G (with 10 nodes) obtained from a star with three 3 leaves by subdividing twice each edge. See in Fig. 14(b). $s_5(G) = 2$ and any minimum size tree decomposition has a bag X such that $G[X]$ is disconnected.

References

1. ARNBORG, S., CORNEIL, D. G., AND PROSKUROWSKI, A. Complexity of finding embeddings in a k-tree. *SIAM J. Algebraic*

- Discrete Methods* 8, 2 (Apr. 1987), 277–284.
2. ARNBORG, S., AND PROSKUROWSKI, A. Characterization and recognition of partial 3-trees. *SIAM J. Algebraic Discrete Methods* 7, 2 (Apr. 1986), 305–314.
 3. BODLAENDER, H. L. A linear-time algorithm for finding tree-decompositions of small treewidth. *SIAM J. Comput.* 25, 6 (1996), 1305–1317.
 4. BODLAENDER, H. L., DRANGE, P. G., DREGI, M. S., FOMIN, F. V., LOKSHTANOV, D., AND PILIPCZUK, M. A $o(c^k n)$ 5-approximation algorithm for treewidth. *CoRR abs/1304.6321* (2013).
 5. BODLAENDER, H. L., AND KOSTER, A. M. C. A. Treewidth computations ii. lower bounds. *Inf. Comput.* 209, 7 (2011), 1103–1119.
 6. COURCELLE, B. The monadic second-order logic of graphs. i. recognizable sets of finite graphs. *Information and Computation* 85, 1 (1990), 12 – 75.
 7. DEMAINE, E. D., AND HAJIAGHAYI, M. The bidimensionality theory and its algorithmic applications. *Comput. J.* 51, 3 (2008), 292–302.
 8. DERENIOWSKI, D., KUBIAK, W., AND ZWOLS, Y. Minimum length path decompositions. *CoRR abs/1302.2788* (2013).
 9. GAREY, M. R., AND JOHNSON, D. S. *Computers and Intractability: A Guide to the Theory of NP-Completeness*. W. H. Freeman & Co., New York, NY, USA, 1979.
 10. HOPCROFT, J., AND TARJAN, R. Algorithm 447: efficient algorithms for graph manipulation. *Commun. ACM* 16, 6 (June 1973), 372–378.
 11. KAJITANI, Y., ISHIZUKA, A., AND UENO, S. Characterization of partial 3-trees in terms of three structures. *Graphs and Combinatorics* 2, 1 (1986), 233–246.
 12. MATOUSEK, J., AND THOMAS, R. Algorithms finding tree-decompositions of graphs. *Journal of Algorithms* 12, 1 (1991), 1 – 22.
 13. ROBERTSON, N., AND SEYMOUR, P. D. Graph minors. II. algorithmic aspects of tree-width. *J. Algorithms* 7, 3 (1986), 309–322.
 14. SANDERS, D. P. On linear recognition of tree-width at most four. *SIAM J. Discret. Math.* 9, 1 (1996), 101–117.
 15. SYSTO, M. M. Characterizations of outerplanar graphs. *Discrete Mathematics* 26, 1 (1979), 47 – 53.
 16. WALD, J. A., AND COLBOURN, C. J. Steiner trees, partial 2-trees, and minimum ifi networks. *Networks* 13, 2 (1983), 159–167.
 17. WALD, J. A., AND COLBOURN, C. J. Steiner trees, partial 2-trees, and minimum ifi networks. *Networks* 13, 2 (1983), 159–167.