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# Size-Constrained Tree Decompositions <sup>★</sup>

Bi Li<sup>1,2,3</sup>, Fatima Zahra Moataz<sup>2,1</sup>, Nicolas Nisse<sup>1,2</sup>, and Karol Suchan<sup>4,5</sup>

<sup>1</sup> Inria, France

<sup>2</sup> Univ. Nice Sophia Antipolis, CNRS, I3S, UMR 7271, Sophia Antipolis, France

<sup>3</sup> Institute of Applied Mathematics, Chinese Academy of Sciences, Beijing, China

<sup>4</sup> FIC, Universidad Adolfo Ibáñez, Santiago, Chile

<sup>5</sup> WMS, AGH, University of Science and Technology, Krakow, Poland

**Abstract.** *Tree-Decompositions* are the corner-stone of many dynamic programming algorithms for solving graph problems. Since the complexity of such algorithms generally depends exponentially on the *width* (size of the *bags*) of the decomposition, much work has been devoted to compute tree-decompositions with small width. However, practical algorithms computing tree-decompositions only exist for graphs with *treewidth* less than 4. In such graphs, the time-complexity of dynamic programming algorithms based on tree-decompositions is dominated by the *size* (number of bags) of the tree-decompositions. It is then interesting to minimize the size of the tree-decompositions. In this report, we consider the problem of computing a tree-decomposition of a graph with width at most  $k$  and minimum size. More precisely, we focus on the following problem: given a fixed  $k \geq 1$ , what is the complexity of computing a tree-decomposition of width at most  $k$  with minimum size in the class of graphs with treewidth at most  $k$ ? We prove that the problem is NP-complete for any fixed  $k \geq 4$  and polynomial for  $k \leq 2$ ; for  $k = 3$ , we show that it is polynomial in the class of trees and 2-connected outerplanar graphs.

## 1 Introduction

A *tree-decomposition* of a graph [13] is a way to represent  $G$  by a family of subsets of its vertex-set organized in a tree-like manner and satisfying some connectivity property. The *treewidth* of  $G$  measures the proximity of  $G$  to a tree. More formally, a tree decomposition of  $G = (V, E)$  is a pair  $(T, \mathcal{X})$  where  $\mathcal{X} = \{X_t | t \in V(T)\}$  is a family of subsets, called *bags*, of  $V$ , and  $T$  is a tree, such that:

- $\bigcup_{t \in V(T)} X_t = V$ ;
- for any edge  $uv \in E$ , there is a bag  $X_t$  (for some node  $t \in V(T)$ ) containing both  $u$  and  $v$ ;
- for any vertex  $v \in V$ , the set  $\{t \in V(T) | v \in X_t\}$  induces a subtree of  $T$ .

The *width* of a tree-decomposition  $(T, \mathcal{X})$  is  $\max_{t \in V(T)} |X_t| - 1$  and its *size* is order  $|V(T)|$  of  $T$ . The *treewidth* of  $G$ , denoted by  $tw(G)$ , is the minimum width over all possible tree-decompositions of  $G$ .

If  $T$  is constrained to be a path,  $(T, \mathcal{X})$  is called a *path-decomposition* of  $G$ . The *pathwidth* of  $G$ , denoted by  $pw(G)$ , is the minimum width over all possible path-decompositions of  $G$ .

Tree-Decompositions are the corner-stone of many dynamic programming algorithms for solving graph problems. As an example, the famous Courcelle’s Theorem states that any problem expressible in MSOL can be solved in linear-time in the class of bounded treewidth graphs [6]. Another framework based on graph decompositions is the *bi-dimensionality theory* that allowed the design of sub-exponential-time algorithms for many problems in the class of graphs excluding some fixed graph as a minor (e.g., [7]). Given a tree-decomposition with width  $w$  and size  $n$ , the time-complexity of most of such dynamic programming algorithms can be expressed as  $O(2^w n)$  (or  $O(2^{w \log w} n)$  in the case of *global* problems). Therefore, the problem of computing tree-decompositions with small width has drawn much attention in the last decades. It has been extensively studied and investigated from different angles: parametrized complexity, exact or approximation algorithms.

The above mentioned algorithms have mainly a theoretical interest because, on the one hand, their time-complexity exponential depends on the treewidth of graphs and, on the other hand, as far as we know, no practical algorithm exists that computes a “good” tree-decomposition for graphs with treewidth at least 5. However, in case of small ( $\leq 4$ )

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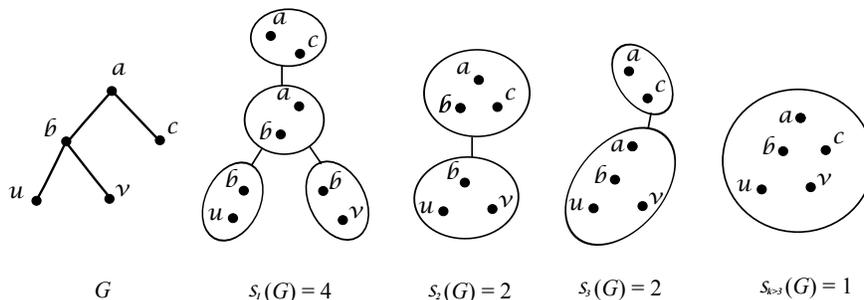
treewidth graphs, efficient (i.e., practical) algorithms exist to compute tree-decompositions with optimal width. Moreover, in such case, the time-complexity of above-mentioned dynamic programming algorithms becomes dominated by the size of the tree-decompositions and, therefore, it becomes interesting to minimize it.

In report, we study the problem of computing tree-decompositions with minimum size. Obviously, if the width is not constrained, then the problem is trivial since there always exists a tree-decomposition of a graph with one bag (the full vertex-set). Hence, given a graph  $G$  and an integer  $k \geq tw(G)$ , we consider the problem of minimizing the size of a tree-decomposition of  $G$  with width at most  $k$ .

**Related Work.** The problem of computing “good” tree-decompositions has been extensively studied. Computing optimal tree-decomposition - i.e., with width  $tw(G)$  - is NP-complete in the class of general graphs  $G$  [1]. For any fixed  $k \geq 1$ , Bodlaender designed an algorithm that computes, in time  $O(k^{k^3} n)$ , a tree-decomposition of width  $k$  of any  $n$ -node graph with treewidth at most  $k$  [3]. Very recently, a single-exponential (in  $k$ ) algorithm has been proposed that computes a tree-decomposition with width at most  $5k$  in the class of graphs with treewidth at most  $k$  [4]. As far as we know, the only practical algorithms for computing optimal tree-decompositions hold for graphs with treewidth at most 1 (trivial since  $tw(G) = 1$  if and only if  $G$  is a tree), 2 (graphs excluding  $K_4$  as a minor) [16], 3 [2, 11, 12] and 4 [14].

We are not aware of any work dealing with the computation of tree-decompositions with minimum size. In [8], Dereniowski *et al.* consider the problem of size-constrained path-decompositions. Given any positive integer  $k$  and any graph  $G$  with pathwidth at most  $k$ . Let  $l_k(G)$  denote the smallest size (length) of a path-decomposition of  $G$  with width at most  $k$ . For any fixed  $k \geq 4$ , computing  $l_k$  is NP-complete in the class of general graphs and it is NP-complete, for any fixed  $k \geq 5$ , in the class of connected graphs [8]. Moreover, computing  $l_k$  can be solved in polynomial-time in the class of graphs with pathwidth at most  $k$  for any  $k \leq 3$ . Finally, the “dual” problem is also hard: for any fixed  $s \geq 2$ , it is NP-complete in general graphs to compute the minimum width of a tree-decomposition with size  $s$  [8]<sup>6</sup>.

**Our results.** Let  $k$  be any positive integer and  $G$  be any graph. If  $tw(G) > k$ , let us set  $s_k(G) = \infty$ . Otherwise, let  $s_k(G)$  denote the minimum size of a tree-decomposition of  $G$  with width at most  $k$ . See a simple example in Fig. 1. We first prove in Section 2 that, for any (fixed)  $k \geq 4$ , the problem of computing  $s_k$  is NP-hard in the class of graphs with treewidth at most  $k$ . Moreover, the computation of  $s_k$  for  $k \geq 5$  is NP-hard in the class of connected graphs with treewidth at most  $k$ . Furthermore, the computation of  $s_4$  is NP-complete in the class of planar graphs with treewidth 3. In Section 3, we present a general approach for computing  $s_k$  for any  $k \geq 1$ . In the rest of the report, we prove that computing  $s_2$  can be solved in polynomial-time. Finally, we prove that  $s_3$  can be computed in polynomial time in the class of trees and 2-connected outerplanar graphs.



**Fig. 1.** Given a tree  $G$  with five vertices, for any  $k \geq 1$ , a minimum size tree decomposition of width at most  $k$  is shown. So we see that  $s_1(G) = 4$ ,  $s_2(G) = s_3(G) = 2$ ,  $s_{k>3}(G) = 1$ .

## 2 NP-hardness in the class of bounded treewidth graphs

In this section, we prove that:

<sup>6</sup> This result was proved in [8] in terms of path-decomposition but it is straightforward to extend it to tree-decomposition.

**Theorem 1.** *For any fixed integer  $k \geq 4$  (resp.,  $k \geq 5$ ), the problem of computing  $s_k$  is NP-complete in the class of graphs (resp., of connected graphs) with treewidth at most  $k$ .*

Note that the corresponding decision problem is clearly in NP. Hence, we only need to prove it is NP-hard.

Our proof mainly follows the one of [8] for size-constrained path-decompositions. Hence, we recall here the two steps of the proof in [8]. First, it is proved that, if computing  $l_k$  is NP-hard for any  $k \geq 1$  in general graphs, then the computation of  $l_{k+1}$  is NP-hard in the class of connected graphs. Second, it is shown that computing  $l_4$  is NP-hard in general graphs with pathwidth 4. In particular, this implies that computing  $l_5$  is NP-hard in the class of connected graphs with pathwidth 5. The second step consists of a reduction from the 3-PARTITION problem [9] to the one of computing  $l_4$ . Precisely, for any instance  $\mathcal{I}$  of 3-PARTITION, a graph  $G_{\mathcal{I}}$  is built such that  $\mathcal{I}$  is a YES instance if and only if  $l_4(G_{\mathcal{I}})$  equals a defined value  $\ell_{\mathcal{I}}$ .

Our contribution consists first in showing that the first step of [8] directly extends to the case of tree-decompositions. That is, it directly implies that, if computing  $s_k$  is NP-hard for some  $k \geq 4$  in general graphs, then so is the computation of  $s_{k+1}$  in the class of connected graphs. Our main contribution of this section is to show that, for the graphs  $G_{\mathcal{I}}$  built in the reduction proposed in [8], any tree-decomposition of  $G_{\mathcal{I}}$  with width at most 4 and minimum size is a path decomposition. Hence, in this class of graphs,  $l_4 = s_4$  and, for any instance  $\mathcal{I}$  of 3-PARTITION,  $\mathcal{I}$  is a YES instance if and only if  $s_4(G_{\mathcal{I}})$  equals a defined value  $\ell_{\mathcal{I}}$ . We describe the details in what follows.

**Lemma 1.** *If the problem of computing  $s_k$  for an integer  $k \geq 1$  is NP-complete in general graphs, then the computation of  $s_{k+1}$  is NP-complete in the class of connected graphs.*

*Proof.* Let  $G$  be any graph. We construct an auxiliary connected graph  $G'$  from  $G$  by adding a vertex  $a$  adjacent to all vertices in  $V(G)$ . Given two integers  $k, s \geq 1$ , in the following, we prove that there is a tree decomposition of  $G$  with width at most  $k$  and size at most  $s$  if and only if there is a tree decomposition of  $G'$  with width at most  $k + 1$  and size at most  $s$ .

First, assume that  $(T, \mathcal{X})$  is a tree decomposition of  $G$  with width at most  $k$  and size at most  $s$ . Add  $a$  in each bag of  $\mathcal{X}$ . Then we obtain a tree decomposition of  $G'$  with width at most  $k + 1$  and size at most  $s$ .

Now let  $(T', \mathcal{X}')$  be a tree decomposition of  $G'$  with width at most  $k + 1$  and size at most  $s$ . We are going to find a tree decomposition of  $G$  with width at most  $k$  and size at most  $s$ . Let  $\mathcal{X}'_a$  be the set of all bags in  $\mathcal{X}'$  containing  $a$ . Let  $T_a$  be the subtree of  $T'$  induced by the bags in  $\mathcal{X}'_a$ . Every vertex  $v \in V(G)$  is contained in a bag in  $\mathcal{X}'_a$  because  $va \in E(G')$ . For any edge  $uv \in E(G)$ , there is a bag  $X \supseteq \{a, u, v\}$  in  $\mathcal{X}'$  since  $\{a, u, v\}$  induces a clique in  $G'$ . So  $X \in \mathcal{X}'_a$ . Delete  $a$  in each bag of  $\mathcal{X}'_a$  and denote by  $\mathcal{X}^-$  the obtained set of bags. So  $(T_a, \mathcal{X}^-)$  is a tree decomposition of  $G$  with width at most  $k$  and size at most  $s$ .

Before doing the reduction from the 3-PARTITION problem to the problem of computing  $s_4$ , let us first recall its definition.

**Definition 1.** [3-PARTITION]

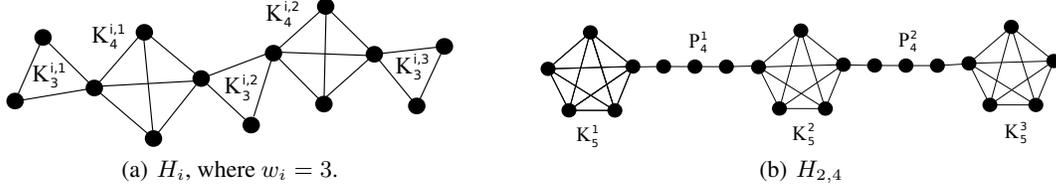
*Instance:* A multiset  $S$  of  $3m$  positive integers  $S = (w_1, \dots, w_{3m})$  and an integer  $b$ .

*Question:* Is there a partition of the set  $\{1, \dots, 3m\}$  into  $m$  sets  $S_1, \dots, S_m$  such that  $\sum_{i \in S_j} w_i = b$  for each  $j = 1, \dots, m$ ?

This problem is NP-complete even if  $|S_j| = 3$  for all  $j = 1, \dots, m$  [9].

Given an instance of 3-PARTITION, in the following, we construct a disconnected graph  $G(S, b)$  as in [8]. First, for each  $i \in \{1, \dots, 3m\}$ , we construct a connected graph  $H_i$  as follows. Take  $w_i$  copies of  $K_3$ , denoted by  $K_3^{i,q}$ ,  $q = 1, \dots, w_i$ , and  $w_i - 1$  copies of  $K_4$ , denoted by  $K_4^{i,q}$ ,  $q = 1, \dots, w_i - 1$  (the copies are mutually disjoint). Then for each  $q = 1, \dots, w_i - 1$ , we identify two different vertices of  $K_4^{i,q}$  with a vertex of  $K_3^{i,q}$  and with a vertex of  $K_3^{i,q+1}$ , respectively. This is done in such a way that each vertex of each  $K_3^{i,q}$  is identified with at most one vertex from other cliques. Informally the cliques form a 'chain' in which the cliques of size 3 and 4 alternate. See Figure 2(a) for an example of  $H_i$  where  $w_i = 3$ .

Second, we construct a graph  $H_{m,b}$  as follows. Take  $m + 1$  copies of  $K_5$ , denoted by  $K_5^1, \dots, K_5^{m+1}$ , and  $m$  copies of the path graph  $P_b$  of length  $b$  ( $P_b$  has  $b$  edges and  $b + 1$  vertices), denoted by  $P_b^1, \dots, P_b^m$ . (Again, the copies are taken to be mutually disjoint.) Now, for each  $j = 1, \dots, m$ , identify one of the endpoints of  $P_b^j$  with a vertex of



**Fig. 2.** Examples of gadgets in graph  $G(S, b)$ .

$K_5^j$ , and identify the other endpoint with a vertex of  $K_5^{j+1}$ . Moreover, do this in a way that ensures that, for each  $j$ , no vertex of  $K_5^j$  is identified with the endpoints of two different paths. See Figure 2(b) for an example of  $H_{2,4}$ .

Let  $G(S, b)$  be the graph obtained by taking the disjoint union of the graphs  $H_1, \dots, H_{3m}$  and the graph  $H_{m,b}$ . In the following, we prove that there is a tree decomposition of  $G(S, b)$  of width 4 and size at most  $s = 1 - 2m + 2 \sum_{i=1}^{3m} w_i$  if and only if there is a partition of the set  $\{1, \dots, 3m\}$  into  $m$  sets  $S_1, \dots, S_m$  such that  $\sum_{i \in S_j} w_i = b$  for each  $j = 1, \dots, m$  in the instance of 3-PARTITION.

In Lemma 2.2 of [8], a path decomposition of  $G(S, b)$  of width 4 and length  $1 - 2m + 2 \sum_{i=1}^{3m} w_i$  is constructed if there is a partition of the set  $\{1, \dots, 3m\}$  into  $m$  sets  $S_1, \dots, S_m$  such that  $\sum_{i \in S_j} w_i = b$  for each  $j = 1, \dots, m$  in the instance of 3-PARTITION. Obviously, this path decomposition is also a tree decomposition of  $G(S, b)$  of width 4 and size  $s$ . So we have the following lemma.

**Lemma 2.** *Given a multiset  $S$  of  $3m$  positive integers  $S = (w_1, \dots, w_{3m})$  and an integer  $b$ , if there is a partition of the set  $\{1, \dots, 3m\}$  into  $m$  sets  $S_1, \dots, S_m$  such that  $\sum_{i \in S_j} w_i = b$  for each  $j = 1, \dots, m$ , then  $G(S, b)$  has a tree decomposition of width at most 4 and size at most  $s = 1 - 2m + 2 \sum_{i=1}^{3m} w_i$ .*

Now we prove the other direction.

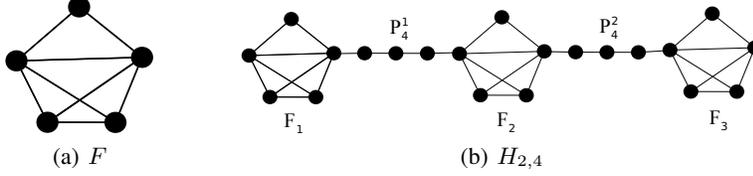
**Lemma 3.** *If  $G(S, b)$  has a tree decomposition  $(T, \mathcal{X})$  of width at most 4 and size at most  $s = 1 - 2m + 2 \sum_{i=1}^{3m} w_i$ , then there is a partition of the set  $\{1, \dots, 3m\}$  into  $m$  sets  $S_1, \dots, S_m$  such that  $\sum_{i \in S_j} w_i = b$  for each  $j = 1, \dots, m$ .*

*Proof.* Lemma 2.6 in [8] proved that if  $G(S, b)$  has a path decomposition  $(T, \mathcal{X})$  of width at most 4 and length at most  $1 - 2m + 2 \sum_{i=1}^{3m} w_i$ , then there is a partition of the set  $\{1, \dots, 3m\}$  into  $m$  sets  $S_1, \dots, S_m$  such that  $\sum_{i \in S_j} w_i = b$  for each  $j = 1, \dots, m$ . So it is enough to prove that any tree decomposition  $(T, \mathcal{X})$  of  $G(S, b)$  of width at most 4 and size at most  $s = 1 - 2m + 2 \sum_{i=1}^{3m} w_i$  is a path decomposition of  $G(S, b)$ .

As proved in Lemma 2.3 of [8], each bag in  $(T, \mathcal{X})$  contains exactly one of the cliques  $K_3^{i,q}, K_4^{i,q}, K_5^j$ . Indeed, each of these cliques has size at least 3. Moreover, any two of them share at most one vertex, and no two cliques of size 3 ( $K_3^{i,q}$ ) share a vertex. So each bag of  $(T, \mathcal{X})$  contains at most one of the cliques  $K_3^{i,q}, K_4^{i,q}, K_5^j$ . However, for any clique, there is a bag in  $(T, \mathcal{X})$  containing its vertices. Since  $s$  equals the number of the cliques  $K_3^{i,q}, K_4^{i,q}, K_5^j$ , each bag of  $(T, \mathcal{X})$  contains exactly one of them.

Moreover let us prove that any edge in  $K_4^{i,q}, K_5^j, P_b^j$  (i.e. both the two endpoints of the edge) is contained in exactly one bag. Since each bag in  $(T, \mathcal{X})$  contains exactly one of the cliques  $K_3^{i,q}, K_4^{i,q}, K_5^j$ , the two endpoints of any edge in the paths  $P_b^1, \dots, P_b^m$  are contained in a bag containing some  $K_3^{i,q}$ . (The bags containing a  $K_4^{i,q}$  (resp.,  $K_5^j$ ) cannot add another two vertices (resp., one vertex) since  $(T, \mathcal{X})$  is a tree decomposition of width at most 4.) Every bag containing some  $K_3^{i,q}$  contains at most one edge in the paths  $P_b^1, \dots, P_b^m$ , because the bag can add at most another two vertices and any  $K_3^{i,q}$  and  $P_b^j$  are disjoint. There are  $mb$  edges in the paths  $P_b^1, \dots, P_b^m$  and there are  $mb$  bags containing some  $K_3^{i,q}$ , so every bag containing a  $K_3^{i,q}$  contains exactly one edge in the paths  $P_b^1, \dots, P_b^m$ . So any edge in the paths  $P_b^1, \dots, P_b^m$  is contained in exactly one bag. Also each bag containing some  $K_3^{i,q}$  contains 5 vertices, so it does not contains any edge (i.e. both its two endpoints) in  $K_4^{i,q}$  or  $K_5^j$ . Therefore, any edge on  $K_4^{i,q}, K_5^j$  is contained in exactly one bag.

Now we prove that there are only two leaves in  $T$  and so  $T$  is a path. If a bag containing some  $K_3^{i,q}$  and an edge  $uv$  on some path  $P_b^j$  is a leaf bag in  $T$ , then its neighbor bag also contains  $u, v$  because both  $u$  and  $v$  are incident to



**Fig. 3.** Example of the new gadget in  $G(S, b)$ .

other edges in  $G(S, b)$ . This is a contradiction with any edge (its two endpoints) on  $P_b^j$  are contained only in one bag. So any bag containing some  $K_3^{i,q}$  is not a leaf bag in  $T$ . Similarly, we can prove that any bag containing any  $K_4^{i,q}$  or  $K_5^j$  for  $1 < j < m + 1$  is not a leaf bag in  $T$ . Thus there are only two bags containing  $K_5^1$  and  $K_5^{m+1}$  are leaves in  $T$ .

Then we get the following corollary.

**Corollary 1.** *It is NP-complete to compute  $s_4$  in the class of graphs of treewidth at most 4.*

Theorem 1 follows from Lemma 1 and Corollary 1.

We furthermore modify the reduction to prove theorem 2.

**Theorem 2.** *It is NP-complete to compute  $s_4$  in the class of planar graphs of treewidth at most 3.*

*Proof.* As in the previous reduction, we build a graph  $G(S, b)$  for an instance of 3-PARTITION; we keep the subgraphs  $H_i$  as they are and modify the graph  $H_{m,b}$  as follows. We replace the  $m+1$  copies of  $K_5$  by  $m+1$  copies of the graph  $F$  that consists of a  $K_4$  and a  $K_3$  sharing an edge as depicted in Figure 3(a). We denote the copies by  $F_1, F_2, \dots, F_{m+1}$ . The new graph  $G(S, b)$  we obtain is planar and has treewidth 3.

Lemma 2 is still true and for 3 to be correct we need to prove that if  $G(S, b)$  has a tree decomposition  $(T, \mathcal{X})$  of width at most 4 and size at most  $s = 1 - 2m + 2 \sum_{i=1}^{3m} w_i$ , then there is a bag of  $(T, \mathcal{X})$  containing  $F_i$ , for each  $F_i$ ,  $i \in 1, \dots, m + 1$ . Let us denote by  $K_3^i$  and  $K_4^i$  the two cliques sharing exactly one edge that form  $F_i$ . Each of these cliques, should appear in one bag. Note that among all the cliques of  $G(S, b)$ , the only cliques that can coexist in a bag are of the form  $K_3^i$  and  $K_4^i$  since the sum of the number of vertices of any other two cliques is more than 5. Let us suppose that there exists  $j \in 1, \dots, m + 1$  such that no bag of  $(T, \mathcal{X})$  contains  $F_j$ , i.e.  $K_3^j$  and  $K_4^j$  are not in the same bag. In this case the number of bags of  $(T, \mathcal{X})$  is at least the number of the cliques  $K_3^{i,q}, K_4^{i,q}, K_4^{i'}$  ( $i' \neq j$ ), plus the two bags containing  $K_3^j$  and  $K_4^j$ . This gives a size of at least  $2 - 2m + 2 \sum_{i=1}^{3m} w_i$  which is not possible.

### 3 Notations and preliminaries

In this section, we present the definitions and notations used throughout the report and some well-known facts about tree-decompositions.

#### 3.1 Notations

Given a graph  $G = (V, E)$ , for any  $S \subseteq V$ , For an integer  $c \geq 0$ , a graph  $G = (V, E)$  is  $c$ -connected if  $|V| > c$  and no subset  $V' \subseteq V$  with  $|V'| < c$  is a separator in  $G$ . A 2-connected component of  $G$  is a maximal 2-connected subgraph.

Let  $(T, \mathcal{X})$  be any tree-decomposition of  $G$ . Abusing the notations, we will identify a node  $t \in V(T)$  and its corresponding bag  $X_t \in \mathcal{X}$ . This means that, e.g., instead of saying  $t \in V(T)$  is adjacent to  $t' \in V(T)$  in  $T$ , we can also say that  $X_t \in \mathcal{X}$  is adjacent to  $X_{t'} \in \mathcal{X}$  in  $T$ . A bag  $B \in \mathcal{X}$  is called a leaf-bag if  $B$  has degree one in  $T$ . Let  $k \geq 1$  and  $G$  be a graph with  $tw(G) \leq k$ . A subset  $B \subseteq V(G)$  is a  $k$ -potential-leaf if there is a tree-decomposition  $(T, \mathcal{X})$  with width at most  $k$  and size  $s_k(G)$  such that  $B$  is a leaf bag of  $(T, \mathcal{X})$ . A subgraph  $H \subseteq V$  is a  $k$ -potential-leaf of  $G$  if  $V(H)$  is a  $k$ -potential-leaf of  $G$ . Note that a  $k$ -potential-leaf has size at most  $k + 1$ . Given a class of graphs  $\mathcal{C}$  and integer  $k \in \mathbb{N}^*$ , a set of graphs  $\mathcal{P}$  is called a complete set of  $k$ -potential-leaves of  $\mathcal{C}$ , if for any graph  $G \in \mathcal{C}$ , there exists a graph  $H \in \mathcal{P}$  such that  $H$  is a  $k$ -potential-leaf of  $G$ .

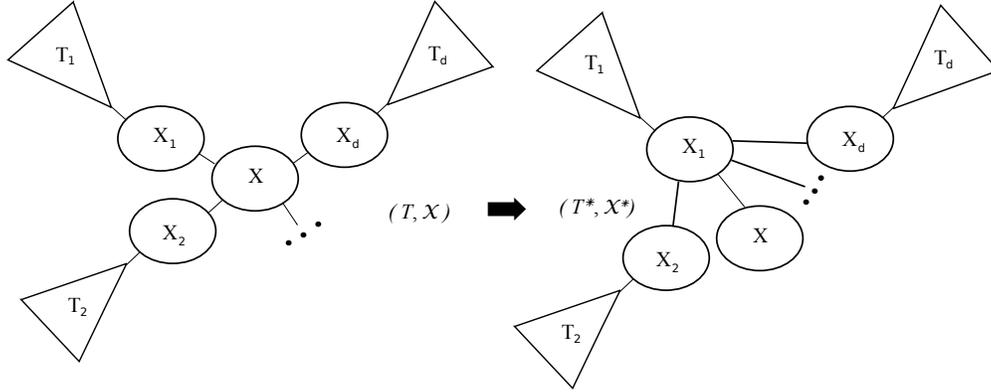
A tree-decomposition is *reduced* if no bag is contained in another one. It is straightforward that, in any leaf-bag  $B$  of reduced tree-decomposition, there is  $v \in V$  such that  $v$  appears only in  $B$  and so  $N[v] \subseteq B$ . Note that it implies that any reduced tree-decomposition has at most  $n - 1$  bags.

In the following we define two *transformation rules*, that take a tree-decomposition  $(T, \mathcal{X})$  of a graph  $G$ , and computes another one without increasing the width nor the size.

**Leaf.** Let  $X \in \mathcal{X}$  and  $N_T(X) = \{X_1, \dots, X_d\}$ . Assume that, for any  $1 < i \leq d$ ,  $X_i \cap X \subseteq X_1$ . Let  $(T^*, \mathcal{X}^*) = \text{Leaf}(X, X_1, (T, \mathcal{X}))$  denote the tree-decomposition of  $G$  obtained by replacing each edge  $X_i X \in E(T)$  by an edge  $X_i X_1$  for any  $1 < i \leq d$ . Note that  $X$  becomes a leaf-bag after the operation. See in Fig. 4.

**Reduce.** Let  $XX' \in E(T)$  with  $X \subseteq X'$ . Let  $(T^*, \mathcal{X}^*) = \text{Reduce}(X, X', (T, \mathcal{X}))$  denote the tree-decomposition of  $G$  obtained by deleting the bag  $X$  from the tree-decomposition  $\text{Leaf}(X, X', (T, \mathcal{X}))$ . Note that the size of the tree decomposition is decreased by one after the operation.

From any tree-decomposition of  $G$  with width  $k$  and size  $s$ , it is easy to obtain a reduced tree-decomposition of  $G$  with width at most  $k$  and size at most  $s - 1$  by applying the Reduce operation while it is possible (i.e., while a bag is contained in another one). In particular, any minimum size tree decomposition is reduced.



**Fig. 4.** In a tree decomposition  $(T, \mathcal{X})$ ,  $N_T(X) = \{X_1, \dots, X_d\}$  and for any  $1 < i \leq d$ ,  $X_i \cap X \subseteq X_1$ . For  $1 \leq i \leq d$ ,  $T_i \cup X_i$  induces the subtree containing  $X_i$  in  $T \setminus \{XX_i\}$ . Replace each edge  $X_i X \in E(T)$  by an edge  $X_i X_1$  for any  $1 < i \leq d$ . This gives a tree decomposition  $(T^*, \mathcal{X}^*) = \text{Leaf}(X, X_1, (T, \mathcal{X}))$ .  $X$  is a leaf-bag in  $(T^*, \mathcal{X}^*)$ .

We conclude this section by a general lemma on tree-decompositions. This lemma is known as folklore, we recall it for completeness.

**Lemma 4.** *Let  $(T, \mathcal{X})$  be a tree decomposition of a graph  $G$ . Let  $X \in \mathcal{X}$  and  $v, w \in X$ . If there exists a connected component in  $G \setminus X$  containing a neighbor of  $v$  and a neighbor of  $w$ , then there is a neighbor bag of  $X$  in  $(T, \mathcal{X})$  containing  $v$  and  $w$ .*

*Proof.* First, let us note that, for any connected subgraph  $H$  of  $G$ , the set of bags of  $T$  that contain a vertex of  $H$  induces a subtree of  $T$  (the proof is easy by induction on  $|V(H)|$ ).

Let  $C$  be a connected component in  $G \setminus X$  containing a neighbor of  $v$  and a neighbor of  $w$ . By above remark, let  $T'_C$  be the subtree of  $T$  induced by the bags that contain some vertex of  $C$ . Moreover, because no vertices of  $C$  are contained in the bag  $X$ , then  $T'_C$  is a subtree of  $T \setminus X$ . Let  $T_C$  be the connected component of  $T \setminus X$  that contains  $T'_C$ . Let  $Y \in V(T_C)$  be the bag of  $T_C$  which is a neighbor of  $X$  in  $T$ . Let  $x \in N(v) \cap C$  be a neighbor of  $v$  in  $C$ . Then there exists a bag  $Z \in \mathcal{X}$  in  $T_C$  containing both  $x$  and  $v$ . So both  $X$  and  $Z$  contain vertex  $v$ . Then the bag  $Y$ , which is on the path between  $X$  and  $Z$  in  $T$ , also contains  $v$ . Similarly, we can prove that  $w \in Y$ .

**Corollary 2.** *Let  $(T, \mathcal{X})$  be a tree decomposition of a 2-connected graph  $G$ . Let  $X \in \mathcal{X}$  and  $|X| \leq 2$ . Then there is a neighbor bag  $Y$  of  $X$  in  $(T, \mathcal{X})$  such that  $X \subseteq Y$ .*

*Proof.* Since  $G$  is 2-connected,  $|V(G)| \geq 3$ . So there exist at least another bag except  $X$  in  $\mathcal{X}$ .

If  $|X| = 1$ , let  $X = \{v\}$ . Then there is a neighbor bag  $Y$  of  $X$  containing  $v$ , since  $G$  is 2-connected and  $v$  is adjacent to some vertices in  $G$ . So  $X \subseteq Y$ .

Otherwise  $|X| = 2$  and let  $X = \{v, w\}$ . Let  $G_1$  be any connected component in  $G \setminus X$ . If  $v$  is not adjacent to any vertex in  $G_1$ , then  $\{w\}$  separates  $V(G_1)$  from  $\{v\}$ . It contradicts with the assumption that  $G$  is 2-connected. So any connected component in  $G \setminus X$  containing a neighbor of  $v$  and a neighbor of  $w$ . From Lemma 4, there is a neighbor bag  $Y$  of  $X$  containing  $v, w$ , i.e.  $X \subseteq Y$ .

### 3.2 General approach

In what follows, we propose polynomial-time algorithms to compute minimum-size tree-decompositions of graphs with small treewidth. Our algorithms mainly use the notion of potential-leaf.

Let  $k \geq 1$  and  $G = (V, E)$  be a graph with  $tw(G) \leq k$ . The key idea of our algorithms is to identify a finite complete set of potential-leaves. Then, our algorithms are recursive: given a graph  $G$  and a  $k$ -potential-leaf  $H$  from the complete set, we compute a minimum-size tree-decomposition of  $G$  by adding  $H$  to a minimum-size tree-decomposition of a smaller graph.

The next lemmas formalize the above paragraph.

**Lemma 5.** *Let  $k \geq 1$  and  $G = (V, E)$  be a graph with  $tw(G) \leq k$ . Let  $B \subseteq V$  be a  $k$ -potential-leaf of  $G$ . Let  $S \subset B$  be the set of vertices of  $B$  that have a neighbor in  $V \setminus B$ . Then  $s_k(G) = s_k(G_S \setminus (B \setminus S)) + 1$ .*

*Proof.* Let us first prove  $s_k(G) \leq s_k(G_S \setminus (B \setminus S)) + 1$ . Suppose that  $(T_S, \mathcal{X}_S)$  is a minimum size tree decomposition of width at most  $k$  of the graph  $G_S \setminus (B \setminus S)$ . Then there exists a bag  $X \in \mathcal{X}_S$  containing  $S$  because  $S$  induces a clique in the graph  $G_S \setminus (B \setminus S)$ . So add the bag  $B$  and make it adjacent to  $X$  in the tree decomposition  $(T_S, \mathcal{X}_S)$ . Then we obtain a tree decomposition of width at most  $k$  for graph  $G$  of size  $s_k(G_S \setminus (B \setminus S)) + 1$ .

Now we prove that  $s_k(G) \geq s_k(G_S \setminus (B \setminus S)) + 1$ . Let  $(T, \mathcal{X})$  be a minimum size tree decomposition of  $G$  of width at most  $k$  such that  $B$  is a leaf bag in it. Note that, if  $B = V$  then  $G_S \setminus (B \setminus S)$  is the empty graph. Let us assume that  $B \subset V$ . Then  $(T, \mathcal{X})$  is also a tree decomposition of  $G_S$ . Let  $B$  be adjacent to the bag  $Y$  in  $(T, \mathcal{X})$ . Then  $S \subset Y$  since each vertex in  $S$  is contained in another bag in  $(T, \mathcal{X})$ . Let  $(T', \mathcal{X}')$  be the tree decomposition obtained by deleting the vertices in  $B \setminus S$  in all the bags of  $(T, \mathcal{X})$ . Then  $B$  is changed to  $B' = S \in \mathcal{X}'$  and let  $Y$  be changed to  $Y' \in \mathcal{X}'$ . So  $B' \subseteq Y'$ . Then the tree decomposition  $Reduce(B', Y', (T', \mathcal{X}'))$  is a tree decomposition of  $G_S \setminus (B \setminus S)$  of size  $s_k(G) - 1$ . So  $s_k(G) - 1 \geq s_k(G_S \setminus (B \setminus S))$ .

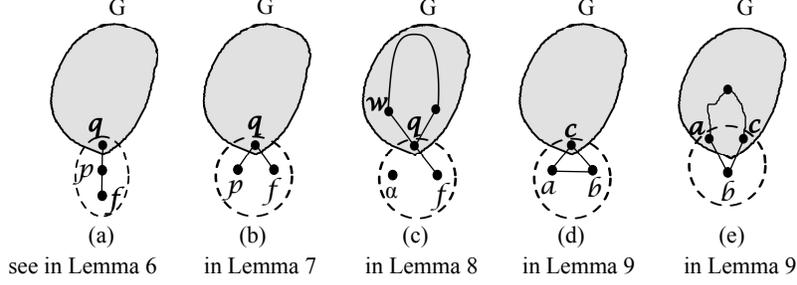
This lemma implies the following corollary:

**Corollary 3.** *Let  $k \in \mathbb{N}^*$  and  $\mathcal{C}$  be the class of graphs with treewidth at most  $k$ . If there is a  $g(n)$ -time algorithm  $\mathcal{A}_k$  that, for any  $n$ -vertex-graph  $G \in \mathcal{C}$ , computes a  $k$ -potential-leaf of  $G$ . Then  $s_k$  can be computed in  $O(g(n) \cdot n)$  time in the class of  $n$ -vertex graphs in  $\mathcal{C}$ . Moreover, a minimum size tree decomposition of width at most  $k$  can be constructed in the same time.*

*Proof.* Let  $G \in \mathcal{C}$  be a  $n$ -vertex-graph. Let us apply Algorithm  $\mathcal{A}_k$  to find a subgraph  $H$  of  $G$  in  $g(n)$  time, which is a  $k$ -potential-leaf of  $G$ . Let  $S \subset V(H)$  be the set of vertices having a neighbor in  $G \setminus H$  and  $G' = G_S \setminus (V(H) \setminus S)$ . Then, by Lemma 5,  $s_k(G) = s_k(G') + 1$ . Finally,  $|V(G')| \leq n - 1$  and  $G'$  has treewidth at most  $k$ . We then proceed recursively. So the total time complexity is  $O(g(n) \cdot n)$ . Moreover, for any minimum size  $(s_k(G'))$  tree decomposition  $(T', \mathcal{X}')$  of  $G'$  of width  $k$ , there is a bag  $X$  containing  $S$  since  $S$  induces a clique in  $G'$ . Add a new bag  $N = V(H)$  adjacent to  $X$  in  $(T', \mathcal{X}')$ . The obtained tree decomposition is a minimum size  $(s_k(G) = s_k(G') + 1)$  tree decomposition of  $G$  of width at most  $k$ .

## 4 Graphs with treewidth at most 2

In this section, we describe algorithm  $\mathcal{A}_2$  computes a 2-potential-leaf of a given graph. In particular, all graphs considered in this section have treewidth at most 2, i.e. partial 2-trees. Please see a complete set of 2-potential-leaf of graphs of treewidth at most 2 in Fig. 5. We are going to prove that any of the subgraphs in Fig. 5 is a potential-leaf and then that each non-empty graph of treewidth at most 2 contains one of them as a 2-potential-leaf.



**Fig. 5.** Complete set of 2-potential-leaves of graphs of treewidth at most 2.

**Lemma 6.** *Let  $G$  be a graph with treewidth at most 2 and  $p \in V(G)$  such that  $N(p) = \{f, q\}$  and  $f$  has degree one (see in Fig. 5(a)). Then  $\{f, p, q\}$  is a 2-potential-leaf of  $G$ .*

*Proof.* Let  $(T, \mathcal{X})$  be any tree-decomposition of  $G$  with width at most 2 and size at most  $s \geq 1$ . We show how to modify it to obtain a tree-decomposition with width at most 2 and size at most  $s$  and in which  $\{f, p, q\}$  is a leaf bag.

Since  $fp \in E(G)$ , there is a bag  $B$  in  $(T, \mathcal{X})$  containing both  $f$  and  $p$ . We may assume that  $B$  is the single bag containing  $f$  (otherwise, delete  $f$  from any other bag). Similarly, since  $pq \in E(G)$ , let  $X$  be a bag in  $(T, \mathcal{X})$  containing both  $p$  and  $q$ .

First, let us assume that  $X = B = \{f, p, q\}$ . In that case, we may assume that  $X$  is the single bag containing  $p$  (otherwise, delete  $p$  from any other bag). If  $X$  is a leaf bag, then the lemma is proved. Otherwise, let  $X_1, \dots, X_d$  be the neighbors of  $X$  in  $T$ . Since  $f$  and  $p$  appear only in  $X$ , then  $X \cap X_i \subseteq \{q\}$  for any  $1 \leq i \leq d$ . If there is  $1 \leq i \leq d$  such that  $q \in X_i$ , let us assume w.l.o.g., that  $q \in X_1$ . By definition of the operation *Leaf*, the tree-decomposition  $\text{Leaf}(X, X_1, (T, \mathcal{X}))$  has width at most 2, same size as  $(T, \mathcal{X})$ , and  $X$  is a leaf.

Second, consider the case when  $X \neq B$ . There are two cases to be considered. Either  $B = \{f, p\}$  or  $B = \{f, p, x\}$  with  $x \neq q$ . In the latter case, note that there is another bag  $B'$ , neighbor of  $B$ , that contains  $x$  unless  $x$  is an isolated vertex of  $G$ . In the former case or if  $x$  appears only in  $B$  (in which case,  $x$  is an isolated vertex), let  $B'$  be any neighbor of  $B$ . Let  $(T', \mathcal{X}')$  be obtained by deleting  $f, p$  in all bags of  $(T, \mathcal{X})$ . Then, contract the edge  $BB'$  in  $T'$ , i.e., remove  $B$  and make any neighbor of  $B$  adjacent to  $B'$ . Note that, in the resulting tree-decomposition of  $G \setminus \{f, p\}$ , there is a bag  $X'$  containing  $q$  and with  $|X'| \leq 2$  (the bag that results from  $X$ ). Finally, add a bag  $\{f, p, q\}$  adjacent to  $X'$  and, if node  $x$  was only in  $B$ , then add  $x$  to  $X'$ . The result is the desired tree-decomposition.

**Lemma 7.** *Let  $G$  be a graph with treewidth at most 2 and  $q \in V(G)$  such that  $q$  has at least two one-degree neighbors  $f$  and  $p$  (see in Fig. 5(b)). Then  $\{f, p, q\}$  is a 2-potential-leaf of  $G$ .*

*Proof.* Let  $(T, \mathcal{X})$  be any tree-decomposition of  $G$  with width at most 2 and size at most  $s \geq 1$ . We show how to modify it to obtain a tree-decomposition with width at most 2 and size at most  $s$  and in which  $\{f, p, q\}$  is a leaf bag.

Since  $fq \in E(G)$ , there is a bag  $B$  in  $(T, \mathcal{X})$  containing both  $f$  and  $q$ . We may assume that  $B$  is the single bag containing  $f$  (otherwise, delete  $f$  from any other bag). Similarly, since  $pq \in E(G)$ , let  $X$  be a bag in  $(T, \mathcal{X})$  containing both  $p$  and  $q$ . Again, we may assume that  $X$  is the single bag containing  $p$  (otherwise, delete  $p$  from any other bag).

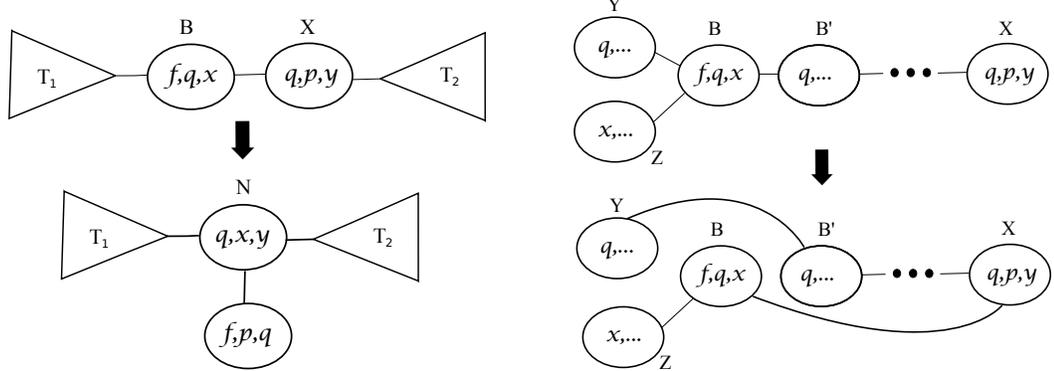
First, let us assume that  $X = B = \{f, p, q\}$ . If  $X$  is a leaf bag, then the lemma is proved. Otherwise, let  $X_1, \dots, X_d$  be the neighbors of  $X$  in  $T$ . Since  $f$  and  $p$  appear only in  $X$ , then  $X \cap X_i \subseteq \{q\}$  for any  $1 \leq i \leq d$ . If there is  $1 \leq i \leq d$  such that  $q \in X_i$ , let us assume w.l.o.g., that  $q \in X_1$ . By definition of the operation *Leaf*, the tree-decomposition  $\text{Leaf}(X, X_1, (T, \mathcal{X}))$  has width at most 2, same size as  $(T, \mathcal{X})$ , and  $X$  is a leaf.

Second, let us assume that  $X = \{f, q\}$  or  $B = \{p, q\}$ . In the former case, remove  $p$  from any bag and add  $p$  to  $X$ . In the latter case, remove  $f$  from any bag and add  $f$  to  $B$ . In both cases, we get a bag  $\{f, p, q\}$  as in the first case.

Otherwise, let  $B = \{f, q, x\}$ ,  $x \neq p$ , and  $X = \{p, q, y\}$ ,  $y \neq f$ .

- If  $B$  and  $X$  are adjacent in  $T$ , then add a new bag  $N = \{q, x, y\}$ ; remove  $B$  and  $X$  and make each of their neighbors adjacent to the new bag  $N$  and, finally, add a leaf-bag  $\{f, p, q\}$  adjacent to  $N$ . See in Fig. 6(a). The obtained tree-decomposition has the desired properties.

- Otherwise, if there is a neighbor  $B'$  of  $B$  with  $q, x \in B'$ , then remove  $B$ , make all neighbors of  $B$  adjacent to  $B'$  and finally add a leaf-bag  $\{f, p, q\}$  adjacent to  $X$ . The obtained tree-decomposition has the desired properties.
- Otherwise, let  $B'$  be the neighbor of  $B$  on the path between  $B$  and  $X$ . In this case,  $q \in B'$  and  $x \notin B'$ . Moreover,  $q$  does not belong to any neighbor of  $B$  that contains  $x$  and the other way around:  $x$  does not belong to any neighbor of  $B$  that contains  $q$ . For any neighbor  $Y$  of  $B$  with  $q \in Y$  (and hence  $x \notin Y$ ), replace the edge  $YB \in E(T)$  with the edge  $YB'$ . Finally, replace the edge  $BB' \in E(T)$  by the edge  $BX$ . See in Fig. 6(b). In the resulting tree-decomposition of  $G$ ,  $B$  and  $X$  are adjacent and we are back to the first item.



(a) In the tree decomposition  $(T, \mathcal{X})$ , let  $T_1 \cup B$  (resp.  $T_1 \cup B$ ) induce the subtree containing  $B$  (resp.  $X$ ) in  $T \setminus \{BX\}$ . Delete  $B$  and  $X$  and make each of their neighbors adjacent to the new bag  $N = \{q, x, y\}$ ; and add a leaf-bag  $\{f, p, q\}$  adjacent to  $N$ .

(b) To be simple and clear, we show only the induced path from  $B$  to  $X$  in  $T$  and two neighbors  $Y, Z \neq B'$  of  $B$ .  $Y$  contains  $q$  and  $Z$  contains  $x$ . Then we just change  $Y$  to be adjacent to  $B'$  instead of  $B$  and change  $B$  to be adjacent to  $X$  instead of  $B'$ .

**Fig. 6.** Explanation of proof of Lemma 7.

**Lemma 8.** Let  $G$  be a graph with treewidth at most 2 and  $q \in V(G)$  such that  $q$  has one neighbor  $f$  with degree 1 and for any vertex  $w \in N(q) \setminus \{f\}$ ,  $\{w, q\}$  belongs to a 2-connected component of  $G$ .

If  $G$  has an isolated vertex  $\alpha$ , then  $\{q, f, \alpha\}$  is a 2-potential-leaf; otherwise  $\{q, f\}$  is a 2-potential-leaf (see in Fig. 5(c)).

*Proof.* Let  $(T, \mathcal{X})$  be any tree-decomposition of  $G$  with width at most 2 and size at most  $s \geq 1$ . We show how to modify it to obtain a tree-decomposition with width at most 2 and size at most  $s$  and in which  $\{f, q, \alpha\}$  is a leaf bag if  $G$  has an isolated vertex  $\alpha$ ; and otherwise  $\{f, q\}$  is a leaf bag.

Since  $fq \in E(G)$ , there is a bag  $B$  in  $(T, \mathcal{X})$  containing both  $f$  and  $q$ . We may assume that  $B$  is the single bag containing  $f$  (otherwise, delete  $f$  from any other bag).

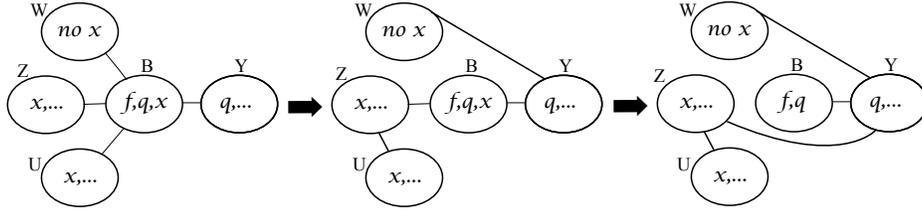
1. If  $B = \{f, q\}$ , then the intersection of  $B$  and any of its neighbor in  $T$  is empty or  $\{q\}$ . If there is a neighbor of  $B$  containing  $q$ , then let  $X$  be such a neighbor; otherwise let  $X$  be any neighbor of  $B$ . By definition of the operation *Leaf*, the tree-decomposition  $Leaf(B, X, (T, \mathcal{X}))$  has width at most 2, same size as  $(T, \mathcal{X})$ , and  $B$  is a leaf. If there are no isolated vertices, we are done. Otherwise, if there is an isolated vertex  $\alpha$  in  $G$ , then delete  $\alpha$  in all bags of the tree-decomposition  $Leaf(B, X, (T, \mathcal{X}))$  and add  $\alpha$  to bag  $B$ , i.e. make  $B = \{f, p, \alpha\}$ . The result is the desired tree decomposition.
2. Otherwise let  $B = \{f, q, x\}$ .
  - (a) If  $x$  is a neighbor of  $q$ , then  $x$  and  $q$  are in a 2-connected component of  $G$ . So there exists a connected component in  $G \setminus B$  containing a vertex adjacent to  $x$  and a vertex adjacent to  $q$ . From Lemma 4, there is

a neighbor  $X$  of  $B$  in  $(T, \mathcal{X})$  containing both  $x$  and  $q$ . Then by definition of the operation *Leaf*, the tree-decomposition  $Leaf(B, X, (T, \mathcal{X}))$  has width at most 2, same size as  $(T, \mathcal{X})$ , and  $B$  is a leaf. Then delete  $x$  in  $B$ , i.e.  $B = \{f, q\}$ . Finally, if  $\alpha$  is an isolated vertex of  $G$ , remove it to any other bag and add it to  $B$ . The result is the desired tree decomposition.

- (b) Otherwise  $x$  is not adjacent to  $q$ . If there is a neighbor  $X$  of  $B$  in  $(T, \mathcal{X})$  containing both  $x$  and  $q$ , then  $(T, \mathcal{X})$  is modified as in case 2a. Otherwise, any neighbor of  $B$  in  $(T, \mathcal{X})$  contains at most one of  $q$  and  $x$ .

If there is a neighbor of  $B$  in  $T$  containing  $q$ , then let  $Y$  be such a neighbor of  $B$ ; otherwise let  $Y$  be any neighbor of  $B$ . Delete the edges between  $B$  and all its neighbors not containing  $x$  except  $Y$  in  $(T, \mathcal{X})$  and make them adjacent to  $Y$ .

If there is no neighbor of  $B$  containing  $x$ , then  $x$  is an isolated vertex and we get a tree decomposition of the same size and width as  $(T, \mathcal{X})$ , in which there is a leaf bag  $B = \{f, q, x\}$ . It is a required tree decomposition. Otherwise let  $Z$  be a neighbor of  $B$  in  $(T, \mathcal{X})$  containing  $x$ , then delete the edges between  $B$  and all its neighbors containing  $x$  except  $Z$  in  $(T, \mathcal{X})$  and make them adjacent to  $Z$ . Now  $B$  has only two neighbors  $Y$  and  $Z$  and  $B \cap Y \subseteq \{q\}$ ,  $B \cap Z = \{x\}$  and  $Y \cap Z = \emptyset$ . Delete the edge between  $B$  and  $Z$  and make  $Z$  adjacent to  $Y$ . Delete  $x$  in  $B$ , i.e. make  $B = \{f, q\}$ . See the transformations in Fig. 7. Then we get a tree decomposition of the same size and width as  $(T, \mathcal{X})$ , in which  $B = \{f, q\}$  is a leaf bag. Again, if  $\alpha$  is an isolated vertex of  $G$ , remove it to any other bag and add it to  $B$ . The result is the desired tree decomposition.



**Fig. 7.** To be simple and clear, we show only the subtree induced by  $B, Y$  and another three neighbors  $Z, W, U$  of  $B$ .  $Y$  contains  $q$ ;  $Z, U$  both contain  $x$  and  $W$  does not contain  $x$ . First we make the bag not containing  $x$ , e.g.  $W$  adjacent to  $Y$  instead of  $B$ ; and make the bag containing  $x$  except  $Z$ , e.g.  $U$  adjacent to  $Z$  instead of  $B$ . Second, make  $Z$  adjacent to  $Y$  instead of  $B$  and delete  $x$  in  $B$ . Then  $B = \{f, q\}$  is a leaf-bag.

**Lemma 9.** Let  $G$  be a graph of treewidth at most 2. Let  $b \in V(G)$  with  $N(b) = \{a, c\}$ . If  $N(a) = \{b, c\}$  (see in Fig. 5(d)) or if there is a path, with at least one internal vertex, between  $a$  and  $c$  in  $G \setminus \{b\}$  (see in Fig. 5(e)), then  $\{a, b, c\}$  is a 2-potential-leaf of  $G$ .

*Proof.* Let  $G = (V, E)$  be a graph of treewidth at most 2. Let  $b \in V$  with exactly 2 neighbors  $a, c \in V$  satisfy the hypotheses of the lemma. If  $V = \{a, b, c\}$ , the result holds trivially, so let us assume that  $|V| \geq 4$ .

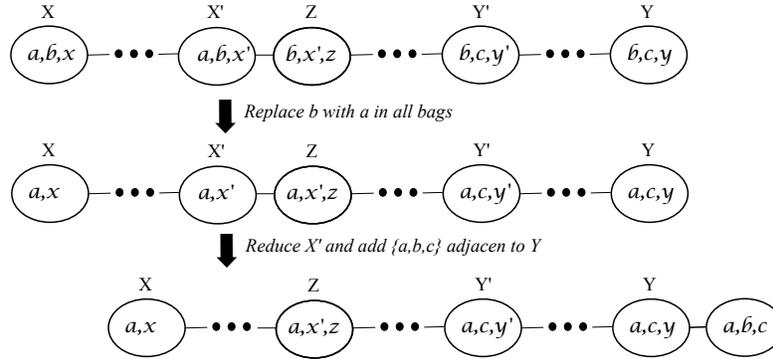
Let  $(T, \mathcal{X})$  be a reduced tree decomposition of width at most 2 of  $G$ . From  $(T, \mathcal{X})$ , we will compute a tree decomposition  $(T^*, \mathcal{X}^*)$  of  $G$  without increasing the width or the size and such that  $\{a, b, c\}$  is a leaf-bag of  $(T^*, \mathcal{X}^*)$ .

Let  $X$  be any bag of  $(T, \mathcal{X})$  containing  $\{a, b\}$  and  $Y$  be any bag containing  $\{b, c\}$ . The bags  $X, Y$  exist because  $ab, bc \in E$ . If  $X = \{a, b\}$ , then there exists a connected component in  $G \setminus X$  containing a neighbor of  $a$  and a neighbor of  $b$ . By Lemma 4, there is a neighbor of  $X$  in  $(T, \mathcal{X})$  that contains both  $a$  and  $b$ , contradicting the fact that  $(T, \mathcal{X})$  is reduced. So  $|X| = 3$  and, similarly,  $|Y| = 3$ .

- Let us first assume that  $X = Y = \{a, b, c\}$ . In particular, it is the case when  $N(a) = \{b, c\}$  since  $\{a, b, c\}$  induces a clique. We may assume that  $b$  only belongs to bag  $X$  (otherwise, remove  $b$  from any other bag).

If  $N(a) = \{b, c\}$ , then we can also assume that  $a$  only belongs to  $X$ . Let  $Z$  be any neighbor of  $B$  containing  $c$  if exists; otherwise let  $Z$  be any neighbor of  $B$  ( $Z$  exists since  $|V| \geq 4$ ). Otherwise, there exists a path  $P$  between  $a$  and  $c$  in  $G \setminus \{b\}$  with at least one internal vertex. In this latter case, there exists a connected component in  $G \setminus X$  containing a neighbor of  $a$  and a neighbor of  $c$ . So by Lemma 4, there is a neighbor bag  $Z$  of  $X$  in  $(T, \mathcal{X})$  containing both  $a$  and  $c$ . In both cases,  $Leaf(X, Z, (T, \mathcal{X}))$  is the desired tree-decomposition.

- Otherwise,  $X = \{a, b, x\}$  and  $Y = \{b, c, y\}$  with  $x \neq c$  and  $y \neq a$ ; and there exists a path  $P$  between  $a$  and  $c$  in  $G \setminus \{b\}$  with at least one internal vertex. Let  $Q$  be the path between  $X$  and  $Y$  in  $(T, \mathcal{X})$ . We may assume that  $b$  only belongs to the bags in  $Q$ , because otherwise  $b$  can be removed from any other bag.
  - If  $X$  is adjacent to  $Y$ , then by properties of tree-decomposition,  $X \cap Y$  separates  $a$  and  $c$ . Since  $\{b\}$  does not separate  $a$  and  $c$ ,  $X \cap Y = \{b, x\}$ , i.e.  $x = y$ . In this case,  $(T^*, \mathcal{X}^*)$  is obtained by making  $X = \{a, c, x\}$  and removing  $Y$  from  $(T, \mathcal{X})$ , then making all neighbors of  $Y$  adjacent to  $X$  and finally, adding a bag  $\{a, b, c\}$  adjacent to  $X$ .
  - Otherwise, let  $X'$  be the bag in the path  $Q$  containing  $a$ , which is closest to  $Y$ . Similarly, let  $Y'$  be the bag in the path  $Q$  containing  $c$ , which is closest to  $X$ . Finally, let  $Q'$  be the path from  $X'$  to  $Y'$  in  $T$  and note that  $b$  belongs to each bag in  $Q'$  and  $a$  and  $c$  do not belong to any internal bag in  $Q'$ . Then we may assume that  $b$  only belongs to the bags in  $Q'$ , because otherwise  $b$  can be removed from any other bag. If  $X'$  and  $Y'$  are adjacent in  $T$ , the proof is similar to the one in previous item. Otherwise, let  $Z$  be the neighbor of  $X'$  in  $Q'$ . By properties of tree-decompositions,  $X' \cap Z$  separates  $a$  and  $c$ . Since  $\{b\}$  does not separate  $a$  and  $c$ , let  $X' \cap Z = \{b, x'\}$ . Since  $Z \neq \{b, x'\}$  because  $(T, \mathcal{X})$  is reduced, then  $Z = \{b, x', z\}$  for some  $z \in V$ . Replace  $b$  with  $a$  in all the bags. By doing this  $(T, \mathcal{X})$  is changed to a tree decomposition  $(T^c, \mathcal{X}^c)$  of the graph  $G/ab$  obtained by contracting the edge  $ab$  in  $G$ . In  $(T^c, \mathcal{X}^c)$ , the bag  $X'$  has become  $X^c = \{a, x'\}$  and  $Z$  is changed to be  $Z^c = \{a, x', z\}$ . So  $X^c$  can be reduced in  $(T^c, \mathcal{X}^c)$ . Moreover  $Y$  is changed to  $Y^c = \{a, c, y\}$ . To conclude, let us add the bag  $\{a, b, c\}$  adjacent to  $Y^c$  in the tree decomposition  $Reduce(X^c, Z^c, (T^c, \mathcal{X}^c))$ . See in Fig. 8. The result is the desired tree-decomposition  $(T^*, \mathcal{X}^*)$  of  $G$ .



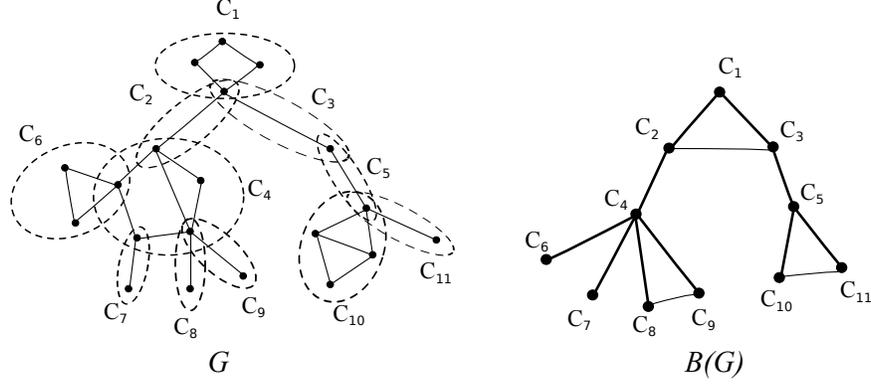
**Fig. 8.** To be simple and clear, we show only the path from  $X$  to  $Y$ . After the two transformations,  $\{a, b, c\}$  is a leaf-bag.

Before going further, let us introduce some notations. A *bridge* in a graph  $G = (V, E)$  is any subgraph induced by two adjacent vertices  $u$  and  $v$  of  $G$  (i.e.,  $uv \in E$ ) such that the number of connected components strictly increases when deleting the edge  $uv$ , but not the two vertices  $u, v$  in  $G$ , i.e.,  $G' = (V, E \setminus \{uv\})$  has strictly more connected components than  $G$ . A vertex  $v \in V$  is a *cut vertex* if  $\{v\}$  is a separator in  $G$ . A maximal connected subgraph without a cut vertex is called a *block*. Thus, every block of a graph  $G = (V, E)$  is either a 2-connected component of  $G$  or a bridge or an isolated vertex. Conversely, every such subgraph is a block. Different blocks of  $G$  intersect in at most one vertex, which is a cut vertex of  $G$ . Hence, every edge of  $G$  lies in a unique block, and  $G$  is the union of its blocks.

Let  $G = (V, E)$  be a connected graph and let  $r \in V$ . A spanning tree  $T$  of  $G$  is a BFS-tree of  $G$  if for any  $v \in V(G)$ , the distance from  $r$  to  $v$  in  $G$  is the same as the one in  $T$ . Let  $\mathcal{B} = \{C : C \text{ is a block of } G\}$ . The *block graph* of  $G$  is the graph  $B(G)$  whose vertices are the blocks of  $G$  and two block-vertices of  $B(G)$  are adjacent if the corresponding blocks intersect, that is,  $B(G) = (\mathcal{B}, \{C_1 C_2 : C_1, C_2 \in \mathcal{B} \text{ and } C_1 \cap C_2 \neq \emptyset\})$ . Note that  $B(G)$  is connected. Finally, a *block-tree* of  $G$  is any BFS-tree  $F$  (with any arbitrary root) of  $B(G)$ . See an example in Fig. 9.

There is a linear (in the number of edges) algorithm for computing all blocks in a given graph [10]. Also a BFS-tree can be found in linear (in the number of vertices plus the number of edges) time. So given a graph  $G = (V, E)$ , we can compute a block tree  $F$  of  $G$  in  $O(|V| + |E|)$  time.

Now we are ready to prove next theorem by using the Lemmas 6-9.



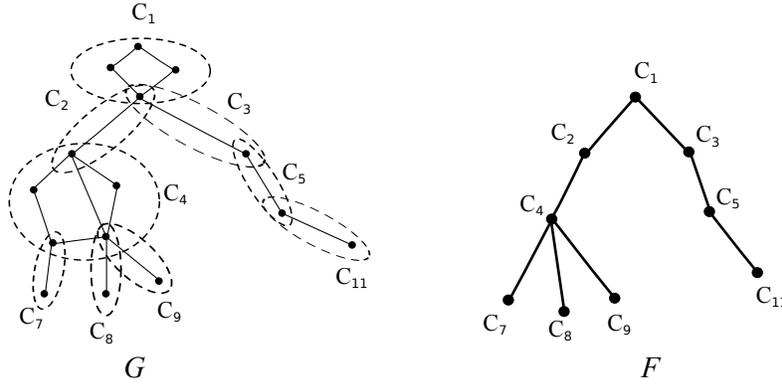
**Fig. 9.** Graph  $G$  is connected. For  $i = 1, \dots, 11$ , each  $C_i$  is a block of  $G$ .  $B(G)$  is the block graph of  $G$ . The BFS tree of  $B(G)$  with bold edges is a block tree of  $G$  with root  $C_1$ .

**Theorem 3.** *There is an algorithm that, for any  $n$ -vertex- $m$ -edge-graph  $G$  with treewidth at most 2, computes a 2-potential-leaf of  $G$  in time  $O(n + m)$ .*

*Proof.* If  $n \leq 3$ , then  $V(G)$  is a 2-potential-leaf of  $G$ . Let us assume that  $n \geq 4$ . First, let us compute the set of isolated vertices in  $G$ , which can be done in  $O(n)$  time. If  $G$  has only isolated vertices, then any three vertices induce a 2-potential-leaf of  $G$ . Otherwise, there is at least one edge in  $G$ .

Let  $G_1$  be any connected component of  $G$  containing at least one edge. If  $|V(G_1)| = 2$ , then from Lemma 9, either  $G$  has an isolated vertex  $\alpha$  and  $\{\alpha, u, v\}$  is a 2-potential-leaf or  $\{u, v\}$  is a 2-potential leaf.

Otherwise,  $|V(G_1)| \geq 3$ . Compute a block tree  $F$  of  $G_1$  rooted in an arbitrary block  $R$ . This can be done in time  $O(n + m)$ . Note that any node in  $F$  corresponds to either a 2-connected component of  $G$  or a bridge  $uv \in E(G)$ . Let  $C$  be a leaf block in  $F$ , which is furthest from  $R$  and  $|V(C)|$  is maximum. There are several cases to be considered.



**Fig. 10.** This graph  $G$  is an induced subgraph of the graph in Fig. 9. Its block tree  $F$ , with root  $C_1$ , has two blocks less than the one in Fig. 9 (the blocks  $C_6$  and  $C_{10}$ ). All leaf blocks,  $C_7, C_8, C_9, C_{11}$ , in  $F$  contains two vertices of  $G$ .

- let us first assume that  $C$  is a bridge in  $G$ , i.e.  $C$  consists of one edge  $fp \in E(G)$  and  $p$  is a cut vertex. Then  $f$  has degree one in  $G$  because  $C$  is a leaf block in  $F$ . Let  $P$  be the parent block of  $C$  in  $F$ . Then any child block  $A$  of  $P$  in  $F$  consists of one edge because  $C$  has the maximum number of vertices among all the children of  $P$ ; and  $A$  is a leaf block in  $F$  because  $C$  is a furthest leaf from the root block  $R$ .

If  $P$  has another child block except  $C$  in  $F$  containing the cut vertex  $p$ , then this child block also consists of one edge  $f'p \in E(G)$ , where  $f'$  has degree one in  $G$  because this child is also a leaf block in  $F$ . (For example, in Fig. 10, take  $C$  as  $C_8$ , which intersects  $C_9$  with a cut vertex.) From Lemma 7,  $\{f, p, f'\}$  is a 2-potential-leaf.

Otherwise  $P$  has only one child block  $C$  in  $F$  containing the cut vertex  $p$ . Then any vertex in  $N_G(p) \setminus \{f\}$  belongs to  $P$ . If  $P$  is also a bridge in  $G$ , i.e.,  $P$  consists of one edge  $pq \in E(G)$ , then  $p$  has degree 2 in  $G$ . (For example, in Fig. 10, take  $C$  as  $C_{11}$ , whose parent  $C_5$  is also a bridge in  $G$ .) From Lemma 6,  $\{f, p, q\}$  is a 2-potential-leaf of  $G$ . Otherwise,  $P$  is a 2-connected component of  $G$  and  $p \in V(G)$  satisfies the hypothesis of Lemma 8. (For example, in Fig. 10, take  $C$  as  $C_7$ , whose parent  $C_4$  is a 2-connected component of  $G$ .) Hence, either  $G$  has an isolated vertex  $\alpha$  and  $\{\alpha, f, p\}$  is a 2-potential-leaf or  $\{f, p\}$  is a 2-potential-leaf.

- Finally, let us assume that  $C$  is a 2-connected component of  $G$ . It is known that any graph with at least two vertices of treewidth  $k$  contains at least two vertices of degrees at most  $k$  [5]. There is no degree one vertex in  $C$  because  $C$  is 2-connected. So there exists two vertices with degree 2 in  $C$ . Since  $C$  is a leaf in  $F$ , there is only one cut vertex of  $G$  in  $C$ . So there exists a vertex  $b$  in  $C$  which has degree two in  $G$ . If  $|V(C)| \geq 4$ , then there exists a path between two neighbors  $a, c$  of  $b$  in  $G \setminus \{b\}$  containing at least one internal vertex. (For example, in Fig. 9, take  $C$  as  $C_{10}$ .) From Lemma 9,  $\{a, b, c\}$  is a 2-potential-leaf. Otherwise  $C$  is a triangle  $\{a, b, c\}$  with at least two vertices with degree 2 in  $G$ . Again from Lemma 9,  $\{a, b, c\}$  is a 2-potential-leaf.

So the total time complexity is  $O(n + m)$ .

**Corollary 4.**  $s_2$  can be computed in polynomial-time in general graphs. Moreover, a minimum size tree decomposition can be constructed in polynomial-time in the class of partial 2-trees.

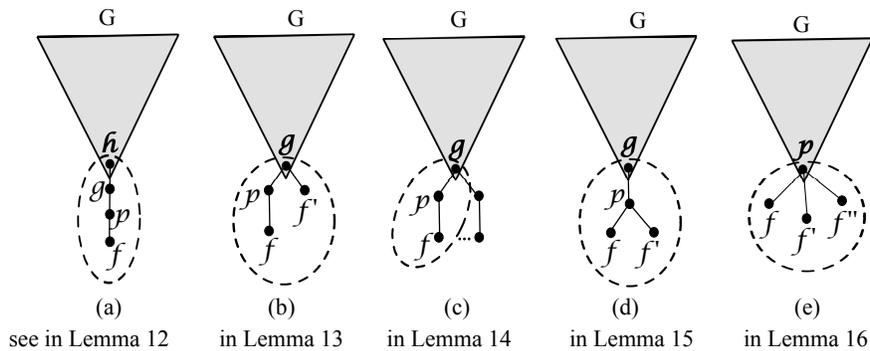
*Proof.* Let  $G$  be any graph. It can be checked in polynomial-time whether  $tw(G) \leq 2$  (e.g. see [17]). If  $tw(G) > 2$ , then  $s_2 = \infty$ . Otherwise  $tw(G) \leq 2$ , then the result follows from Theorem 3 and Corollary 3.

## 5 Minimum-size tree-decompositions of width at most 3

In this section, we study the computation of  $s_3$  in the class of trees and 2-connected outerplanar graphs.

### 5.1 computation of $s_3$ in trees

In this subsection, given a tree  $G$ , we show how to find a 3-potential-leaf in  $G$ . We characterize a complete set of 3-potential-leaves of trees in Fig. 11. We first prove that each of the subgraphs in Fig. 11 is a 3-potential-leaf and then that any tree with at least four vertices contains one of them.



**Fig. 11.** Complete set of 3-potential-leaves of trees.

**Lemma 10.** Let  $(T, \mathcal{X})$  be a tree decomposition of a tree  $G$ . Let  $X \in \mathcal{X}$  and  $N_T(X) = \{X_1, \dots, X_d\}$  for  $d \geq 1$ . Suppose that for any  $1 \leq i \leq d$ ,  $X_i \cap X \subseteq \{x\}$ . Then there is a tree decomposition  $(T', \mathcal{X}')$  of  $G$  of the same width and size as  $(T, \mathcal{X})$  such that  $X$  is a leaf bag.

*Proof.* If there is a bag  $X_i$  for  $1 \leq i \leq d$  containing  $x$ , then let  $B$  be  $X_i$ . Otherwise let  $B$  be any neighbor of  $X$ . By definition of the operation  $Leaf$ , the tree-decomposition  $Leaf(X, B, (T, \mathcal{X}))$  is a desired tree decomposition.

**Lemma 11.** Let  $G$  be a tree rooted at  $r \in V(G)$ . Let  $f$  be a leaf in  $G$ . Let  $p$  be the parent of  $f$  and let  $g$  be the parent of  $p$  in  $G$ . Let  $p$  have degree 2 in  $G$ . Let  $(T, \mathcal{X})$  be a tree decomposition of  $G$  of width at most 3 and size at most  $s \geq 1$ . If there is no bag in  $(T, \mathcal{X})$  containing all of  $f, p, g$ , then there is a tree decomposition  $(T', \mathcal{X}')$  of  $G$  of width at most 3 and size at most  $s$  such that  $\{f, p, g\} \in \mathcal{X}'$  is a leaf bag.

*Proof.* Since  $fp \in E(G)$ , there is a bag  $B$  in  $(T, \mathcal{X})$  containing both  $f$  and  $p$ . We may assume that  $B$  is the single bag containing  $f$  (otherwise, delete  $f$  from any other bag). Similarly, since  $pg \in E(G)$ , let  $X$  be a bag in  $(T, \mathcal{X})$  containing both  $p$  and  $g$ . Let  $P$  be the path in  $T$  from  $B$  to  $X$ . Then  $p$  is contained in all bags on  $P$  and we may assume that  $p$  is not contained in any other bags (otherwise, delete  $p$  from any other bag). Let  $B'$  be the neighbor of  $B$  on  $P$ . Then  $B \cap B' \supseteq \{p\}$ . Note that it is possible that  $B' = X$ .

If  $B = \{f, p\}$ , then make all other neighbors of  $B$  adjacent to  $B'$  and delete  $B$ . Add a bag  $\{f, p, g\}$  adjacent to  $X$ . The result is a desired tree decomposition  $(T', \mathcal{X}')$ .

Otherwise,  $B$  contains at least one vertex not in  $\{f, p\}$ . If  $B \cap B' = \{p\}$ , then  $\{p\}$  separates  $g$  from any vertex in  $B \setminus \{p\}$ . So  $B \setminus \{p\} = \{f\}$ , i.e.,  $B = \{f, p\}$ . It contradicts with the assumption.

So  $|B \cap B'| \geq 2$  and let  $\{p, x\} \subseteq B \cap B'$ . Then create a bag  $Z = (B \setminus \{f, p\}) \cup (B' \setminus \{p, x\})$ . (Note that  $x \in Z$  since  $x \in B$ .) So  $|Z| \leq 4$ . Make  $Z$  adjacent to all neighbors of  $B$  and all neighbors of  $B'$ ; and delete the two bags  $B$  and  $B'$ ; and delete  $f, p$  from all bags. Finally add another new bag  $N = \{f, p, g\}$  adjacent to some bag containing  $g$ . The obtained tree decomposition has width at most 3, same size as  $(T, \mathcal{X})$ , and a bag  $N = \{f, p, g\}$ .

**Lemma 12.** Let  $G$  be a tree rooted at  $r \in V(G)$  and  $|V(G)| \geq 4$ . Let  $f$  be a leaf in  $G$ . Let  $p$  be the parent of  $f$  and let  $g$  be the parent of  $p$  in  $G$ . Suppose that both  $p$  and  $g$  have degree 2. Let  $h$  be the parent of  $g$  (see in Fig. 11(a)), then  $H = G[\{f, p, g, h\}]$  is a 3-potential-leaf of  $G$ .

*Proof.* Let  $(T, \mathcal{X})$  be any reduced tree decomposition of width at most 3 and size at most  $s \geq 1$  of  $G$ . We show how to modify it to obtain a tree-decomposition with width at most 3 and size at most  $s$  and in which  $\{f, p, g, h\}$  is a leaf bag.

From Lemma 11, we can assume that there is a bag  $B$  in  $(T, \mathcal{X})$  containing all  $f, p, g$ . We may assume that  $B$  is the single bag containing  $f, p$  (otherwise, delete  $f, p$  from any other bag). Since  $gh \in E(G)$ , let  $Y$  be a bag in  $(T, \mathcal{X})$  containing both  $h$  and  $g$ .

1. If  $B = Y = \{f, p, g, h\}$ , then the intersection of  $B$  and any of its neighbor in  $T$  is contained in  $\{h\}$ . A desired tree decomposition can be obtained from Lemma 10.
2. If  $B = \{f, p, g\}$ , then the intersection of  $B$  and any of its neighbors in  $T$  is contained in  $\{g\}$ . From Lemma 10, there is a tree-decomposition  $(T', \mathcal{X}')$  of the same width and size as the ones of  $(T, \mathcal{X})$  such that  $B = \{f, p, g\}$  is a leaf. Then delete  $B$  in the tree-decomposition  $Leaf(B, B', (T, \mathcal{X}))$  and add a new bag  $N = \{f, p, g, h\}$  adjacent to  $Y$ . The obtained tree decomposition has the desired properties.
3. Otherwise,  $B = \{f, p, g, x\}$  where  $x \neq h$ . Then the intersection of  $B$  and any of its neighbor in  $T$  is contained in  $\{g, x\}$ . Let  $P$  be the path in  $T$  from  $B$  to  $Y$ . Then  $g$  is contained in all bags on  $P$ . Let  $B'$  be the neighbor of  $B$  on  $P$ . Note that it is possible that  $B' = Y$ . If  $B \cap B' = \{g\}$ , then  $\{g\}$  separates  $h$  from  $x$ . So  $x \in \{f, p\}$  i.e.  $B = \{f, p, g\}$ , a contradiction with the assumption. So we have  $B \cap B' = \{g, x\}$ . By definition of the operation  $Leaf$ , the tree-decomposition  $Leaf(B, B', (T, \mathcal{X}))$  has width at most 3, same size as  $(T, \mathcal{X})$ , and  $B = \{f, p, g, x\}$  is a leaf. Then delete  $B$  in the tree-decomposition  $Leaf(B, B', (T, \mathcal{X}))$  and add a new bag  $N = \{f, p, g, h\}$  adjacent to  $Y$ . The obtained tree decomposition has the desired properties since  $\{g, x\} \subseteq B'$  and  $\{g, h\} \subseteq Y$ .

**Lemma 13.** Let  $G$  be a tree rooted at  $r \in V(G)$  and  $|V(G)| \geq 4$ . Let  $f$  a leaf in  $G$ . Let  $p$  be the parent of  $f$  and let  $g$  be the parent of  $p$  in  $G$ . If  $p$  has degree 2 and  $g$  has a child  $f'$ , which is a leaf in  $G$  (see in Fig. 11(b)), then  $H = G[\{f, p, g, f'\}]$  is a 3-potential-leaf of  $G$ .

*Proof.* Let  $(T, \mathcal{X})$  be any reduced tree decomposition of width at most 3 and size at most  $s \geq 1$  of  $G$ . We show how to modify it to obtain a tree-decomposition with width at most 3 and size at most  $s$  and in which  $\{f, p, g, f'\}$  is a leaf bag.

From Lemma 11, we can assume that there is a bag  $B$  in  $(T, \mathcal{X})$  containing all  $f, p, g$ . We may assume that  $B$  is the single bag containing  $f, p$  (otherwise, delete  $f, p$  from any other bag). Since  $gf' \in E(G)$ , let  $Y$  be a bag in  $(T, \mathcal{X})$  containing both  $f$  and  $g$ . We may assume that  $Y$  is the single bag containing  $f'$  (otherwise, delete  $f'$  from any other bag).

- If  $B = Y = \{f, p, g, f'\}$ , then the intersection of  $B$  and any of its neighbor in  $T$  is contained in  $\{g\}$ . A desired tree decomposition can be obtained from Lemma 10.
- If  $B = \{f, p, g\}$ , then delete  $f'$  in  $Y$  and add  $f'$  in  $B$ , then we are in the previous case.
- Otherwise,  $B = \{f, p, g, x\}$  where  $x \neq f'$ . Then the intersection of  $B$  and any of its neighbor in  $T$  is contained in  $\{g, x\}$ . Let  $P$  be the path in  $T$  from  $B$  to  $Y$ . Then  $g$  is contained in all bags on  $P$ . Let  $B'$  be the neighbor of  $B$  on  $P$ . If  $B \cap B' = \{g, x\}$ , then by definition of the operation *Leaf*, the tree-decomposition  $Leaf(B, B', (T, \mathcal{X}))$  has width at most 3, same size as  $(T, \mathcal{X})$ , and  $B = \{f, p, g, x\}$  is a leaf. Then in the tree-decomposition  $Leaf(B, B', (T, \mathcal{X}))$ , delete  $f'$  in  $Y$  and remove  $x$  from  $B$  and add  $f'$  to  $B$ , i.e. make  $B = \{f, p, g, f'\}$ . The obtained tree decomposition has the desired properties since  $\{g, x\} \subseteq B'$ .  
Otherwise  $B \cap B' = \{g\}$ . Delete  $f'$  from the bag  $Y$  and add  $x$  in  $Y$ ; delete  $x$  from  $B$  and add  $f'$  in  $B$ , i.e., make  $B = \{f, p, g, f'\}$ ; finally make all neighbors of  $B$  except  $B'$  adjacent to  $Y$  since now  $\{g, x\} \subseteq Y$ . The result is the desired tree decomposition.

**Lemma 14.** *Let  $G$  be a tree rooted at  $r \in V(G)$  and  $|V(G)| \geq 3$ . Let  $f$  be one of the furthest leaves from  $r$ . Let  $p$  be the parent of  $f$  and let  $g$  be the parent of  $p$  in  $G$ . If  $g$  has degree at least 3 and any child of  $g$  has degree 2 in  $G$  (see in Fig. 11(c)), then  $H = G[\{f, p, g\}]$  is a 3-potential-leaf of  $G$ .*

*Proof.* Let  $(T, \mathcal{X})$  be any reduced tree decomposition of width at most 3 and size at most  $s \geq 1$  of  $G$ . We show how to modify it to obtain a tree-decomposition with width at most 3 and size at most  $s$  and in which  $\{f, p, g, f'\}$  is a leaf bag.

From Lemma 11, we can assume that there is a bag  $B$  in  $(T, \mathcal{X})$  containing all  $f, p, g$ . We may assume that  $B$  is the single bag containing  $f, p$  (otherwise, delete  $f, p$  from any other bag).

1. If  $B = \{f, p, g\}$ , then the intersection of  $B$  and any of its neighbor in  $T$  is contained in  $\{g\}$ . A desired tree decomposition can be obtained from Lemma 10.
2. Otherwise,  $B = \{f, p, g, x\}$ . Then the intersection of  $B$  and any of its neighbor in  $T$  is contained in  $\{g, x\}$ .
  - (a) If there is a neighbor  $B'$  of  $B$  such that  $B \cap B' = \{g, x\}$ , then by definition of the operation *Leaf*, the tree-decomposition  $Leaf(B, B', (T, \mathcal{X}))$  has width at most 3, same size as  $(T, \mathcal{X})$ , and  $B = \{f, p, g, x\}$  is a leaf. Then delete  $x$  in  $B$  in the tree-decomposition  $Leaf(B, B', (T, \mathcal{X}))$  since  $\{g, x\} \subseteq B'$ . The obtained tree decomposition has the desired properties.
  - (b) Otherwise any neighbor of  $B$  contains at most one of  $g$  and  $x$ . If  $x$  is not adjacent to  $g$ , then there is a connected component in  $G \setminus B$  containing a neighbor of  $g$  and a neighbor of  $x$ . From Lemma 4, there exists a neighbor bag of  $B$  in  $(T, \mathcal{X})$  containing  $g$  and  $x$ . It is a contradiction. So we have  $x$  is adjacent to  $g$  in this case.
    - i.  $x$  is a child of  $g$ . Then  $x$  has exactly one child  $y$ , which is a leaf in  $G$  since  $f$  is one of the furthest leaves from  $r$ . Since  $yx \in E(G)$ , there is a bag  $Y$  in  $(T, \mathcal{X})$  containing both  $y$  and  $x$ . We may assume that  $Y$  is the single bag containing  $y$  (otherwise, delete  $y$  from any other bag). Since  $\{g, x\} \subset B$  and any neighbor of  $B$  contains at most one of  $g$  and  $x$ , any bag except  $B$  contains at most one of  $g$  and  $x$ . Then  $g \notin Y$  because  $x \in Y$ . So  $y, x, g$  are not contained in one bag. From Lemma 11, we can modify  $(T, \mathcal{X})$  to obtain a tree-decomposition  $(T', \mathcal{X}')$  of width at most 3 and size at most  $s$  having a leaf bag  $X = \{y, x, g\}$ . Note that  $x$  (resp.  $y$ ) plays the same role as  $p$  (resp.  $f$ ) in  $G$ , i.e.,  $g, p, f$  and  $g, x, y$  are symmetric in  $G$ . Hence, the result is a desired tree decomposition.
    - ii.  $x$  is the parent of  $g$ . Let  $p'$  be another child of  $g$  and let  $f'$  be the child of  $p'$ , which is a leaf in  $G$ . Let  $B'$  be the bag in  $(T, \mathcal{X})$  containing both  $f'$  and  $p'$ . We may assume that  $B'$  is the single bag containing  $f'$  (otherwise, delete  $f'$  from any other bag). Let  $X'$  be a bag containing both  $p'$  and  $g$ . Then we have

$X' \neq B$  (because  $p' \notin B$ ). Since  $g \in X'$  any bag except  $B$  contains at most one of  $g$  and  $x$ , we have  $x \notin X'$ . In the following, we modify  $(T, \mathcal{X})$  to obtain a tree-decomposition  $(T', \mathcal{X}')$  with width at most 3 and size at most  $s$  having a bag  $\{f', p', g\}$ . Then we are in case 1, since  $g, p, f$  and  $g, p', f'$  are symmetric in  $G$ .

If  $B' = X' = \{f', p', g\}$  then, we are done. If  $B' = X' = \{f', p', g, x'\}$ . Then  $x'$  is not  $x$ , which is the parent of  $g$ , since  $x \notin X'$ . So we can do as in case 2a or case 2(b)i.

Otherwise,  $B' \neq X'$ . From Lemma 11, we can modify  $(T, \mathcal{X})$  to obtain a tree-decomposition with width at most 3 and size at most  $s$  having a leaf bag  $\{f', p', g\}$ .

**Lemma 15.** *Let  $G$  be a tree rooted at  $r \in V(G)$  and  $|V(G)| \geq 4$ . Let  $f$  a leaf in  $G$ . Let  $p$  be the parent of  $f$ . If  $p$  has exactly two children  $f, f'$  in  $G$  and let  $g$  be the parent of  $p$  in  $G$  (see in Fig. 11(d)), then  $H = G[\{f, f', p, g\}]$  is a 3-potential-leaf of  $G$ .*

*Proof.* Let  $(T, \mathcal{X})$  be any reduced tree decomposition of width at most 3 and size at most  $s \geq 1$  of  $G$ . We show how to modify it to obtain a tree-decomposition with width at most 3 and size at most  $s$  and in which  $\{f, f', p, g\}$  is a leaf bag.

Since  $fp \in E(G)$ , there is a bag  $B$  in  $(T, \mathcal{X})$  containing both  $f$  and  $p$ . We may assume that  $B$  is the single bag containing  $f$  (otherwise, delete  $f$  from any other bag). Similarly, let  $B'$  be the single bag in  $(T, \mathcal{X})$  containing both  $f'$  and  $p$ . Let  $X$  be a bag containing both  $p$  and  $g$ .

1. If  $B = B' = X = \{f, f', p, g\}$ , then we can assume that  $B$  is the single bag containing  $p$  (otherwise, delete  $p$  from any other bag). So the intersection of  $B$  and any of its neighbor in  $T$  is contained in  $\{g\}$ . Then a desired tree decomposition can be obtained from Lemma 10.
2. If  $B = B' = \{f, f', p\}$ , then the intersection of  $B$  and any of its neighbor in  $T$  is contained in  $\{p\}$ . Let  $Y$  be a neighbor of  $B$  in  $T$  containing  $p$ . By definition of the operation *Leaf*, the tree-decomposition  $Leaf(B, Y, (T, \mathcal{X}))$  has width at most 3, same size as  $(T, \mathcal{X})$ , and  $B = \{f, f', p\}$  is a leaf. Then delete  $B$  and add a new bag  $N = \{f, f', p, g\}$  adjacent to  $X$ . The result is a desired tree decomposition.
3. If  $B = B' = \{f, f', p, x\}$  and  $x \neq g$ , then the intersection of  $B$  and any of its neighbor in  $T$  is contained in  $\{p, x\}$ . Since  $x \notin \{f, f', g\}$ ,  $p$  is not adjacent to  $x$ . There is a connected component in  $G \setminus B$  containing a neighbor of  $p$  and a neighbor of  $x$ . From Lemma 4, there exists a neighbor bag of  $B$  in  $(T, \mathcal{X})$  containing  $p$  and  $x$ . Let  $Y$  be such a neighbor of  $B$  in  $T$ . By definition of the operation *Leaf*, the tree-decomposition  $Leaf(B, Y, (T, \mathcal{X}))$  has width at most 3, same size as  $(T, \mathcal{X})$ , and  $B = \{f, f', p, x\}$  is a leaf. Then delete  $x$  from  $B$  and we get a tree decomposition having a bag  $\{f, f', p\}$ . So we are in case 2.
4. If  $B \neq B'$  and  $|B| \leq 3$ , then delete  $f'$  in  $B$  and add  $f'$  in  $B'$ . Then we are in case 2 or 3. It is proved similarly if  $B \neq B'$  and  $|B'| \leq 3$ .
5. Otherwise  $B \neq B'$  and  $|B| = |B'| = 4$ . Let  $B = \{f, p, x, y\}$  and  $B' = \{f', p, x', y'\}$ . Let  $P$  be the path in  $T$  from  $B$  to  $B'$ . Then  $p$  is contained in all bags on  $P$ . Let  $Y$  be the neighbor of  $B$  on  $P$ . If  $B \cap Y = \{p\}$ , then  $\{p\}$  separates  $x$  from  $x'$ . But  $p$  is not a separator between any two vertices in  $V(G) \setminus \{f, f'\}$ . It is a contradiction. So w.l.o.g. we can assume that  $B \cap Y \supseteq \{p, x\}$ . Deleting  $f, f', p$  in all bags of  $(T, \mathcal{X})$ . Add a new bag  $Z = \{x, y\} \cup Y \setminus \{p, x\}$  adjacent to all neighbors of the two bags  $B, Y$  and delete  $B$  and  $Y$ . Finally add another new bag  $N = \{f, f', p, g\}$  adjacent to a bag containing  $g$ . The obtained tree decomposition has the desired properties.

**Lemma 16.** *Let  $G$  be a tree rooted at  $r \in V(G)$  and  $|V(G)| \geq 4$ . Let all children of  $p$  be leaves in  $G$  and  $p$  have at least three children  $f, f', f''$  (see in Fig. 11(e)). Then  $H = G[\{p, f, f', f''\}]$  is a 3-potential-leaf of  $G$ .*

*Proof.* Let  $(T, \mathcal{X})$  be any reduced tree decomposition of width at most 3 and size at most  $s \geq 1$  of  $G$ . We show how to modify it to obtain a tree-decomposition with width at most 3 and size at most  $s$  and in which  $\{p, f, f', f''\}$  is a leaf bag.

Since  $fp \in E(G)$ , there is a bag  $B$  in  $(T, \mathcal{X})$  containing both  $f$  and  $p$ . We may assume that  $B$  is the single bag containing  $f$  (otherwise, delete  $f$  from any other bag). Similarly, let  $B'$  (resp.  $B''$ ) be the single bag in  $(T, \mathcal{X})$  containing both  $f'$  (resp.  $f''$ ) and  $p$ .

1. If  $B = B' = B'' = \{f, f', f'', p\}$ , then the intersection of  $B$  and any of its neighbor in  $T$  is contained in  $\{p\}$ . A desired tree decomposition can be obtained from Lemma 10.

2. If  $B = B' = \{f, f', p\}$ , then delete  $f''$  in  $B''$  and add  $f''$  in  $B$ . Then we are in case 1. It can be proved similarly if  $B = B'' = \{f, f'', p\}$  or  $B' = B'' = \{f', f'', p\}$ .
3. If  $B = B' = \{f, f', p, x\}$  and  $x \neq f''$ , then the intersection of  $B$  and any of its neighbor in  $T$  is contained in  $\{p, x\}$ .

If  $x$  is a child of  $p$ , then  $x$  is also a leaf in  $G$  and  $x$  play the same role as  $f''$ . Then we are in case 1. So in the following we assume that  $x$  is not a child of  $p$ .

If  $x$  is not the parent of  $p$ , then  $p$  is not adjacent to  $x$ . So there is a connected component in  $G \setminus B$  containing a neighbor of  $p$  and a neighbor of  $x$ . From Lemma 4, there exists a neighbor bag of  $B$  in  $(T, \mathcal{X})$  containing  $p$  and  $x$ . Let  $Y$  be such a neighbor of  $B$  in  $T$ . By definition of the operation *Leaf*, the tree-decomposition  $Leaf(B, Y, (T, \mathcal{X}))$  has width at most 3, same size as  $(T, \mathcal{X})$ , and  $B = \{f, f', p, x\}$  is a leaf. Then delete  $x$  from  $B$  and we get a tree decomposition having a bag  $\{f, f', p\}$ . So we are in case 2.

Otherwise  $x$  is the parent of  $p$ . Let  $P$  be the path in  $T$  from  $B$  to  $B''$ . Then  $p$  is contained in all bags on  $P$ . Let  $Y$  be the neighbor of  $B$  on  $P$ . If  $B \cap Y = \{p, x\}$ , then by definition of the operation *Leaf*, the tree-decomposition  $Leaf(B, Y, (T, \mathcal{X}))$  has width at most 3, same size as  $(T, \mathcal{X})$ , and  $B = \{f, f', p, x\}$  is a leaf. Then deleting  $x$  from  $B$  we are in case 2. Otherwise,  $B \cap Y = \{p\}$ . So  $\{p\}$  separates  $x$  from all vertices in  $B'' \setminus \{p\}$ . Then all vertices in  $B'' \setminus \{p\}$  are children of  $p$  and so they are leaves in  $G$ . So we can assume that any vertex in  $B'' \setminus \{p\}$  are contained only in  $B''$  (otherwise we can delete it in any other bags). Then delete  $f, f'$  from  $B$  and add vertices of  $B'' \setminus \{f'', p\}$  in  $B$ ; and make  $B'' = \{f, f', f'', p\}$ . Then we are in case 1.

The cases  $B = B'' = \{f, f'', p, x\}$  and  $x \neq f'$  or  $B' = B'' = \{f', f'', p, x\}$  and  $x \neq f$  can be proved similarly.

4. Otherwise, none two of  $f, f', f''$  are contained in a same bag.

If  $|B| \leq 3$ , then delete  $f'$  in  $B'$  and add  $f'$  in  $B$ . Then we are in case 2 or 3. It is proved similarly if and  $|B'| \leq 3$  or  $|B''| \leq 3$ .

Otherwise  $|B| = |B'| = |B''| = 4$ . In the following, we are going to modify  $(T, \mathcal{X})$  to obtain a tree-decomposition with width at most 3 and size at most  $s$  having a bag  $X$  containing at least two of  $f, f', f''$  or  $f \in X$  and  $|X| \leq 3$ . Then we are in the above cases. Note that all children of  $p$  play the same role (they are all leaves) in  $G$ . So it is enough to get that  $X$  contains at least two children of  $p$  or that  $X$  contains one child of  $p$  and  $|X| \leq 3$ .

Let  $T_p$  be the subtree in  $T$  induced by all the bags containing  $p$ . If  $|V(T_p)| \leq 2$ , there exists one bag containing at least two children of  $p$  since  $p$  has at least three children. Then it is done. So we assume that  $|V(T_p)| \geq 3$ . There is a bag  $R \in V(T_p)$  containing both  $p$  and  $g$ . Root  $T_p$  at  $R$  and let  $L \in V(T_p)$  be one of the furthest leaf bag in  $T_p$  from  $R$ . If there is no child of  $p$  in  $L$ , then we can delete  $p$  in  $L$  and consider  $T_p \setminus \{L\}$ . So we can assume there is a vertex  $l \in L$ , which is a child of  $p$  in  $G$ . Let  $Y$  be the neighbor of  $L$  in  $T_p$ . If the intersection of  $L \cap Y = \{p\}$ , then  $p$  separate any vertex in  $L \setminus \{p\}$  and any vertex in  $Y \setminus \{p\}$ . So at least one of  $L, Y$ , denoted as  $X$ , contains only  $p$  and children of  $p$ . Then either  $X$  contains at least two children of  $p$  or  $X$  contains only one children and  $|X| = 2$ . So  $(T, \mathcal{X})$  and  $X$  satisfy the desired properties.

Otherwise,  $|L \cap Y| \geq 2$ . If  $Y$  has no other child except  $L$  in  $T_p$ , then  $Y \neq R$  since  $|V(T_p)| \geq 3$ . Let  $X = \{p, l\}$  if  $Y$  contains no child of  $p$ ; and  $X = \{p, l, l'\}$  if  $Y$  contains one child  $l'$  of  $p$ . Add a new bag  $Z = Y \cup L \setminus X$ . Since  $|Y \cap L| \geq 2$ ,  $|Y \cup L| \leq 6$ . Then  $|Z| \leq 4$ , since  $X \subseteq Y \cup L$  and  $|X| \geq 2$ . Make  $Z$  adjacent to all neighbors of  $Y, L$  in  $T$  and delete  $Y, L$ . Finally make  $X$  adjacent to  $R$ . The obtained tree decomposition and  $X$  have the desired properties.

Otherwise,  $Y$  has at least another child  $L'$  in  $T_p$ . Then  $L'$  is also a furthest leaf from  $R$  in  $T_p$ , since  $L$  is a furthest leaf from  $R$ . For the same reason as  $L$ , there is a vertex  $l' \in L'$ , which is a child of  $p$  in  $G$ . Let  $L = \{l, p, x, y\}$  and  $L' = \{l', p, x', y'\}$ . So the intersection of  $L$  (resp.  $L'$ ) and any of its neighbors except  $Y$  in  $T$  is contained in  $\{x, y\}$  (resp.  $\{x', y'\}$ ). Create a new bag  $N = \{x, y, x', y'\}$  adjacent to all neighbor of  $L, L'$  and delete  $L, L'$ . Finally add another bag  $X = \{p, l, l'\}$  adjacent to  $Y$ . The obtained tree decomposition and  $X$  have the desired properties.

From Lemmas 12- 16 and Corollary 3, we obtain the following result.

**Corollary 5.**  $s_3$  and a minimum size tree decomposition of width at most 3 can be computed in polynomial-time in the class of trees.

*Proof.* From Corollary 3, it is enough to prove we can find a 3-potential-leaf in any tree in polynomial time.

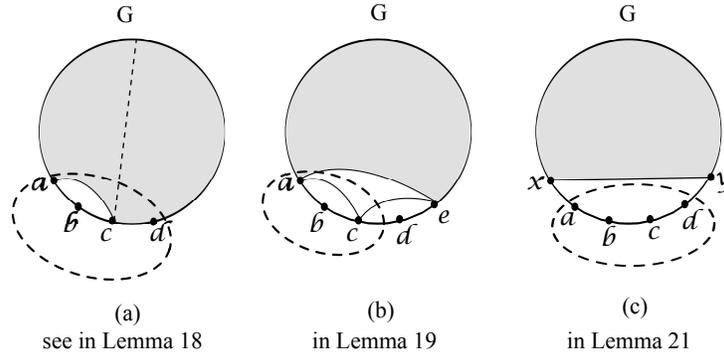
Let  $G$  be any tree. If  $|V(G)| \leq 4$ , then  $V(G)$  is a 3-potential-leaf. Let us assume that  $|V(G)| \geq 5$ . Root  $G$  at any vertex  $r$ . Let  $f$  be one of the furthest leaves from  $r$  in  $G$ . Let  $p, g, h$  be the first three vertices on the path from  $f$  to  $r$  in  $G$  (if exist), i.e.  $p$  is  $f$ 's parent;  $g$  is  $p$ 's parent; and  $h$  is  $g$ 's parent in  $G$ .

- If  $g, p$  both have only one child in  $G$ , then  $\{f, p, g, h\}$  is a 3-potential-leaf of  $G$  from Lemma 12;
- If  $p$  has only one child and  $g$  has a child  $f'$ , which is a leaf in  $G$ , then  $\{f, p, g, f'\}$  is a 3-potential-leaf of  $G$  from Lemma 13;
- If  $p$  has only one child and any child of  $g$  has exactly one child, then  $\{f, p, g\}$  is a 3-potential-leaf of  $G$  from Lemma 14;
- If  $p$  has only one child and there exist a child  $p'$  of  $g$ , which has exactly two children  $f_1, f_2$ , then  $\{f_1, f_2, p', g\}$  is a 3-potential-leaf of  $G$  from Lemma 15;
- If  $p$  has only one child and there exist a child  $p'$  of  $g$ , which has at least three children  $f_1, f_2, f_3$ , then  $\{f_1, f_2, f_3, p'\}$  is a 3-potential-leaf of  $G$  from Lemma 16;
- If  $p$  has exactly two children  $f, f'$ , then  $\{f, f', p, g\}$  is a 3-potential-leaf of  $G$  from Lemma 15;
- Otherwise  $p$  has at least three children  $f, f', f''$ , then  $\{f, f', f'', p\}$  is a 3-potential-leaf of  $G$  from Lemma 16.

In fact, the algorithm for trees can be extended to forests by consider their connected component, i.e., trees. The only difference is in Lemma 14 the 3-potential-leaf becomes  $\{f, p, g, \alpha\}$  if there is an isolated vertex  $\alpha$  in the given forest.

## 5.2 Computation of $s_3$ in 2-connected outerplanar graphs

In this subsection, given a 2-connected outerplanar graph  $G$ , we show how to find a 3-potential-leaf in  $G$ . See a complete set of 3-potential-leaves of 2-connected outerplanar graphs in Fig. 12. We first prove that each subgraph in the Fig. 12 is a 3-potential-leaf and then we show that any 2-connected outerplanar graphs contains one of them.



**Fig. 12.** Complete set of 3-potential-leaves of 2-connected outerplanar graphs.

The following fact is well known for 2-connected outerplanar graphs.

**Lemma 17.** [15] *A 2-connected outerplanar graph has the unique Hamilton cycle.*

In the rest of this subsection, let  $G$  be a 2-connected outerplanar graph and  $C$  be the Hamilton cycle in  $G$ .

**Definition 2.** *Any edge in  $E(G) \setminus E(C)$  is called a chord in  $G$ .*

The vertices  $v_1, \dots, v_j \in V(G)$ , for  $2 \leq j \leq |V(G)|$ , are *consecutive* in  $C$  ( we also say that they are consecutive in  $G$ ) if  $v_i v_{i+1} \in E(C)$  for  $1 \leq i \leq j - 1$ ; and  $v_1, \dots, v_j \in V(G)$  are also called the consecutive vertices from  $v_1$  to  $v_j$ .

**Lemma 18.** *Let  $a, b, c, d \in V(G)$  be consecutive in  $C$ . If  $\{a, b, c\}$  induces a clique and  $c$  has degree 3 in  $G$  (see in Fig. 12(a)), then  $H = G[\{a, b, c, d\}]$  is a 3-potential-leaf of  $G$ .*

*Proof.* Let  $(T, \mathcal{X})$  be any tree decomposition of width at most 3 and size at most  $s \geq 1$  of  $G$ . We show how to modify it to obtain a tree-decomposition with width at most 3 and size at most  $s$  and in which  $\{a, b, c, d\}$  is a leaf bag.

Since  $\{a, b, c\}$  induces a clique in  $G$ , there is a bag  $B$  containing all  $a, b, c$ . Let  $X$  be a bag in  $(T, \mathcal{X})$  containing both  $c$  and  $d$  (it exists since  $cd \in E(G)$ ). Note that  $b$  is not incident to any chords, i.e. has degree 2. (Because if  $by \in E(G)$  is a chord in  $G$ , then deleting all chords except  $ac, by$  in  $G$  and contracting the edges in  $C$  except  $ab, bc$  we get a  $K_4$ -minor in  $G$ . It is a contradiction with the fact that  $G$  is outerplanar.)

Replace  $b, c$  with  $a$  in all bags of  $(T, \mathcal{X})$ . Then  $(T, \mathcal{X})$  becomes a tree decomposition  $(T', \mathcal{X}')$  of the graph  $G'$  obtained by contracting the edges  $ab$  and  $bc$ . The bag  $X$  becomes  $X'$ , which contains both  $a$  and  $d$ ; and  $B$  becomes  $B' = \{a\}$  if  $B = \{a, b, c\}$  or  $B' = \{a, x\}$  if  $B = \{a, b, c, x\}$ . From Corollary 2, in both case there exists a neighbor  $Y$  of  $B'$  such that  $B' \subseteq Y$ . So  $B'$  can be reduced in  $(T', \mathcal{X}')$ . The tree decomposition  $Reduce(B', Y, (T', \mathcal{X}'))$  has one bag less than  $(T, \mathcal{X})$ . Finally, add a new bag  $N = \{a, b, c, d\}$  adjacent to  $X'$ , which contained both  $a$  and  $d$ , in the tree decomposition  $Reduce(B', Y, (T', \mathcal{X}'))$ . The result is a desired tree decomposition, because  $b, c$  are not adjacent to any vertices in  $V(G) \setminus N$ .

**Lemma 19.** *Let  $a, b, c, d, e \in V(G)$  be consecutive in  $C$ . If  $\{a, b, c\}$  and  $\{c, d, e\}$  induce two cliques respectively in  $G$  and  $ae \in E(G)$  (see in Fig. 12(b)), then  $H = G[\{a, b, c\}]$  is a 3-potential-leaf of  $G$ .*

*Proof.* Let  $(T, \mathcal{X})$  be any tree decomposition of width at most 3 and size at most  $s \geq 1$  of  $G$ . We show how to modify it to obtain a tree-decomposition with width at most 3 and size at most  $s$  and in which  $\{a, b, c\}$  is a leaf bag.

Since  $\{a, b, c\}$  (resp.  $\{c, d, e\}$ ) induces a clique in  $G$ , there is a bag  $X$  (resp.  $Y$ ) containing all  $a, b, c$  (resp.  $c, d, e$ ). Note that  $b, c, d$  are not adjacent to any vertices in  $V(G) \setminus \{a, b, c, d, e\}$ .

Delete  $b, c, d$  in all bags of  $(T, \mathcal{X})$ . Then  $(T, \mathcal{X})$  becomes a tree decomposition  $(T', \mathcal{X}')$  of the graph  $G' = G \setminus \{b, c, d\}$ . The bag  $X$  becomes  $X' = \{a\}$  if  $X = \{a, b, c\}$  or  $X' = \{a, x\}$  if  $X = \{a, b, c, x\}$ . From Corollary 2, in both case there exists a neighbor  $A$  of  $X'$  such that  $X' \subseteq A$ . So  $X'$  can be reduced in  $(T', \mathcal{X}')$ . Similarly, the bag  $Y$  becomes  $Y'$ , which can also be reduced in  $(T', \mathcal{X}')$ . After reducing the two bags  $X', Y'$  in  $(T', \mathcal{X}')$ , let the obtained tree decomposition be  $(T'', \mathcal{X}'')$ . Finally, add two new bags  $N_1 = \{a, b, c\}$  and  $N_2 = \{a, c, d, e\}$ ; make  $N_1$  adjacent to  $N_2$  and make  $N_2$  adjacent to a bag  $Z$  containing both  $a$  and  $e$  in the tree decomposition  $(T'', \mathcal{X}'')$ . ( $Z$  exists because  $ae \in E(G')$ .) The result is a desired tree decomposition.

**Lemma 20.** *Let  $C_l$  be a cycle of  $l \geq 4$  vertices. Let  $(T, \mathcal{X})$  be a tree decomposition of  $C_l$  of width at most 3. Then there exist either a bag containing all vertices of  $V(C_l)$  (only if  $l = 4$ ) or two bags  $X, Y \in \mathcal{X}$  such that  $X$  (resp.  $Y$ ) contains at least three consecutive vertices  $x_1, x_2, x_3$  (resp.  $y_1, y_2, y_3$ ) and the two edge sets  $\{x_1x_2, x_2x_3\} \cap \{y_1y_2, y_2y_3\} = \emptyset$ .*

*Proof.* The treewidth of any cycle is bigger than 1, so there exists a bag in any tree decomposition of a cycle (with at least 4 vertices) containing two vertices not consecutive, equivalently they are not adjacent in the cycle. We prove the lemma by induction on  $l$  in the following.

First let us prove that it is true for  $l = 4$ . Let  $a, b, c, d$  be the four consecutive vertices in  $C_4$ . Let  $(T, \mathcal{X})$  be a tree decomposition of width at most 3. Then there exists a bag containing  $a, c$  or  $b, d$ . W.l.o.g assume  $a, c$  are contained in one bag. So  $(T, \mathcal{X})$  is also a tree decomposition of  $H$ , obtained from  $C_4$  by adding the edge  $ac$ . The set  $\{a, b, c\}$  induces a clique in  $H$ . So there is a bag  $X$  containing  $a, b, c$ . For the same reason, there is a bag  $Y$  containing  $c, d, a$ . If  $X = Y$  then there is a bag containing all  $a, b, c, d$  of  $V(C_4)$ . Otherwise there are two bags  $X, Y$  such that  $X \supseteq \{a, b, c\}$  and  $Y \supseteq \{c, d, a\}$ . We see that  $\{ab, bc\} \cap \{cd, da\} = \emptyset$ . So the lemma is true for  $l = 4$ .

Now suppose it is true for  $l \leq n - 1$  and we prove it for  $l = n \geq 5$ . Note that since  $(T, \mathcal{X})$  has width 3 and  $l \geq 5$ , there is no bag containing all vertices of  $V(C_l)$ . So in the following we prove that there always exist two bags  $X, Y$  with the desired properties. Let  $v_1, \dots, v_n$  be the  $n$  consecutive vertices in  $C_n$ . Let  $(T, \mathcal{X})$  be a tree-decomposition of width at most 3 of  $C_n$ . Then there exists a bag containing two non-adjacent vertices  $v_i, v_j$  for  $1 \leq i < j \leq n$ . So  $(T, \mathcal{X})$  is also a tree decomposition of the graph  $H$ , obtained from  $C_n$  by adding the edge  $v_i v_j$ . The graph  $H$  is also the union of two subcycles  $C^1$  induced by  $\{v_i, \dots, v_j\}$  and  $C^2$  induced by  $\{v_j, \dots, v_n, \dots, v_i\}$ . Then  $\max\{|C^1|, |C^2|\} \leq n - 1$ . Let  $(T^1, \mathcal{X}^1)$  (resp.  $(T^2, \mathcal{X}^2)$ ) be the tree decomposition of  $C^1$  (resp.  $C^2$ ) obtained by deleting all vertices not in  $C^1$  (resp.  $C^2$ ) in the bags of  $(T, \mathcal{X})$ .

If  $|V(C^1)| = 3$  then there is a bag in  $(T^1, \mathcal{X}^1)$  containing  $V(C^1) = \{v_i, v_{i+1}, v_j = v_{i+2}\}$ . So  $v_i v_j \notin \{v_i v_{i+1}, v_{i+1} v_j\}$ .

If  $|V(C^1)| \geq 4$  then, by induction, there exist either a bag in  $(T^1, \mathcal{X}^1)$  containing all vertices of  $V(C^1) = \{v_i, v_{i+1}, v_{i+2}, v_j = v_{i+3}\}$  or two bags  $A, B$  in  $(T^1, \mathcal{X}^1)$  containing three consecutive vertices  $a_1, a_2, a_3$  and  $b_1, b_2, b_3$

respectively in  $C^1$ ; moreover,  $\{a_1a_2, a_2a_3\} \cap \{b_1b_2, b_2b_3\} = \emptyset$ . So we have either  $v_iv_j \notin \{a_1a_2, a_2a_3\}$  or  $v_iv_j \notin \{b_1b_2, b_2b_3\}$ .

In both cases ( $|V(C^1)| = 3$  and  $|V(C^1)| \geq 4$ ), there is at least one bag  $X$  in  $(T^1, X^1)$  containing three consecutive vertices in  $C^1$ , denoted as  $x_1, x_2, x_3$ , such that  $v_iv_j \notin \{x_1x_2, x_2x_3\}$ . So  $x_1, x_2, x_3$  are also consecutive in  $C$ . Similarly, there is at least one bag  $Y$  in  $(T^2, X^2)$  containing three consecutive vertices in  $C^2$ , denoted as  $y_1, y_2, y_3$ , such that  $v_iv_j \notin \{y_1y_2, y_2y_3\}$ . So  $y_1, y_2, y_3$  are also consecutive in  $C$ . Finally, we have  $\{x_1x_2, x_2x_3\} \cap \{y_1y_2, y_2y_3\} = \emptyset$  because  $E(C^1) \cap E(C^2) = \{v_iv_j\}$  and  $v_iv_j \notin \{x_1x_2, x_2x_3\}$ .

**Lemma 21.** *Let  $xy$  be a chord in  $G$ . Let  $C'$  be the set of all the consecutive vertices from  $x$  to  $y$  in  $C$  and  $|C'| \geq 4$ . If each vertex in  $C' \setminus \{x, y\}$  has degree 2 in  $G$ , then for any consecutive vertices  $a, b, c, d \in C'$  (see in Fig. 12(c)),  $H = G[\{a, b, c, d\}]$  is a 3-potential-leaf of  $G$ .*

*Proof.* Let  $(T, \mathcal{X})$  be any tree decomposition of width at most 3 and size at most  $s \geq 1$  of  $G$ . We show how to modify  $(T, \mathcal{X})$  to obtain a tree-decomposition of  $G$ , which has width at most 3, size at most  $s$  and a leaf bag  $\{a, b, c, d\}$ .

Note that the vertices of  $C'$  induce a cycle in  $G$ . Without confusion, we denote this cycle  $C'$ . Let  $(T', X')$  be the tree decomposition of  $C'$  obtained by deleting all vertices not in  $C'$  in the bags of  $(T, X)$ . From Lemma 20, there is either a bag containing all vertices in  $C'$  (only if  $|C'| = 4$ ); or two bags  $X, Y$  containing three consecutive vertices in  $C'$  respectively and the two corresponding edge sets do not intersect.

In the former case,  $V(C') = \{a, b, c, d\}$  and so  $(T, \mathcal{X})$  is also a tree decomposition of  $G \cup \{ac\}$ , from Lemma 18,  $\{a, b, c, d\}$  is a 3-potential-leaf of  $G$ .

In the latter case, let  $X \supseteq \{u, v, w\}$  and  $Y \supseteq \{u', v', w'\}$ , where  $u, v, w$  (resp.  $u', v', w'$ ) are consecutive in  $C'$ . Since  $\{uv, vw\} \cap \{u'v', v'w'\} = \emptyset$ , we have either  $xy \notin \{uv, vw\}$  or  $xy \notin \{u'v', v'w'\}$ . W.l.o.g. assume that  $xy \notin \{uv, vw\}$ . Then  $u, v, w$  are also consecutive in  $C$  and at least one of  $u, w$  has degree 2 in  $G$ . W.l.o.g. suppose  $w$  has degree 2 in  $G$ , i.e.  $w \notin \{x, y\}$  (since  $x, y$  have degree at least 3 in  $G$ ). Let  $z \in C'$  be the other neighbor (except  $v$ ) of  $w$  in  $C'$ . ( $z$  exists because  $w \notin \{x, y\}$ .)

$(T, \mathcal{X})$  is also a tree decomposition of  $G \cup \{uw\}$ , which is still an outerplanar graph by assumptions. Note that  $w$  has degree 3 in the graph  $G \cup \{uw\}$ . So from Lemma 18, we can modify  $(T, \mathcal{X})$  to obtain a tree-decomposition  $(T', \mathcal{X}')$  of  $G \cup \{uw\}$ , which has width at most 3, size at most  $s$  and a leaf bag  $L$  containing four consecutive vertices  $\{u, v, w, z\}$ . Note that  $(T', \mathcal{X}')$  is also a tree decomposition of  $G$ . So we get a tree decomposition where a leaf bag contains 4 consecutive vertices of  $C'$ . It remains to show how to modify it to obtain a tree decomposition with a leaf bag  $\{a, b, c, d\}$ .

Let  $B$  be the neighbor of  $L$  in  $T$ . Then  $u, z \in B$  since each of  $u, z$  is adjacent to some vertices in  $G \setminus L$ . We can assume that  $L$  is the single bag containing  $v, w$  in  $(T', \mathcal{X}')$ , because otherwise we can delete them in any other bags. Thus, deleting the bag  $L$  in  $(T', \mathcal{X}')$ , we get a tree decomposition  $(T_1, \mathcal{X}_1)$  of the graph  $G_1$ , which is the graph obtained by deleting  $v, w$  and adding an edge  $uz$  in  $G$ . So  $(T_1, \mathcal{X}_1)$  has width at most 3 and size at most  $s - 1$ . Note that the graph  $G_1$  is isomorphic to the graph  $G_2 \equiv G \cup \{ad\} \setminus \{b, c\}$  since  $z \in C'$ . So from the tree decomposition  $(T_1, \mathcal{X}_1)$  of  $G_1$  we can obtain a tree decomposition  $(T_2, \mathcal{X}_2)$  of  $G_2$  with the same width and size. Note that since  $ad \in E(G_2)$ , there is a bag  $Y$  containing both  $a$  and  $d$ . Finally, add a new bag  $N = \{a, b, c, d\}$  adjacent to  $Y$  in  $(T_2, \mathcal{X}_2)$ . The result is a desired tree decomposition.

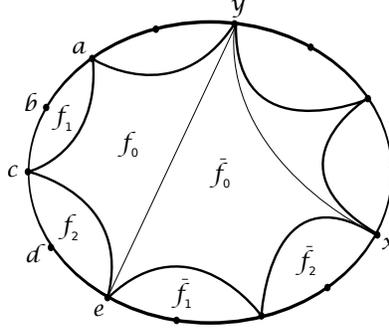
**Lemma 22.** *There is an algorithm that, for any 2-connected outerplanar graph  $G$ , computes a 3-potential-leaf of  $G$  in polynomial time.*

*Proof.* Let  $G$  be a 2-connected outerplanar graph and  $C$  be the unique Hamilton cycle of  $G$ . If  $|V(G)| \leq 4$ , then  $V(G)$  is a 3-potential-leaf of  $G$ . Otherwise,  $|V(G)| \geq 5$  and consider the outerplanar embedding of  $G$ .

- If there exists an inner face  $f$  with at most one chord of  $G$  and  $f$  has at least four vertices, then from Lemma 21, the set of any four consecutive vertices in  $f$ , which are also consecutive in  $C$ , is a 3-potential-leaf in  $G$ .
- If there is an inner face  $f = \{a, b, c\}$  with only one chord  $ac$  of  $G$  and  $c$  has degree 3, then let  $d$  be the other neighbor of  $c$  except  $b, a$ . From Lemma 18, the set of four consecutive vertices  $a, b, c, d$ , is a 3-potential-leaf in  $G$ .
- Otherwise, let  $\mathcal{F}$  be the set of all inner faces with only one chord of  $G$ . Then any face  $f \in \mathcal{F}$  has three vertices and both the two endpoints of the chord in  $f$  have degree at least 4, i.e., they are incident to some other chords except this one. We can prove by induction on  $|V(G)|$  that:

*Claim.* There exist two faces  $f_1, f_2 \in \mathcal{F}$  such that (1)  $f_1 = \{a, b, c\}$ ; (2)  $f_2 = \{c, d, e\}$ ; (3)  $a, b, c, d, e$  are consecutive in  $G$ ; (4) there is a face  $f_0$  containing both  $ac$  and  $ce$  and at most one chord, which is not in any face of  $\mathcal{F}$ . See in Fig. 13.

It is true when  $|V(G)| = 5$ . Assume that it is true for  $|V(G)| \leq n - 1$ . Now we prove it is true for  $|V(G)| = n$ . Note that  $\mathcal{F} \neq \emptyset$  if there is at least one chords in  $G$ , which is valid in this case. Let  $f \in \mathcal{F}$  have three consecutive vertices  $x, y, z$  and let  $xz \in E(G)$  be the single chord in  $f$ . Then the graph  $G \setminus y$  is a 2-connected outerplanar graph with  $n - 1$  vertices. From the assumption, we have the desired faces  $f'_0, f'_1, f'_2$  in  $G \setminus y$ . If  $xz$  is not an edge in any face of  $f'_1, f'_2$ , then these faces are also the desired faces in  $G$ . Otherwise, let  $xz$  be an edge of  $f'_1$  or  $f'_2 = \{x, z, t\}$ . Then  $z$  has degree 3 in  $G$ , i.e. it is not incident to any other chords except  $xz$ , since  $xt \in E(G)$ . So we are in second case above, which contradicts with the assumption.



**Fig. 13.**  $\mathcal{F}$  is the set of all inner faces with only one chord of  $G$ , such as  $f_1, f_2, \bar{f}_1, \bar{f}_2$ . The faces  $f_0, f_1, f_2$  satisfy the properties in Claim 5.2. But  $\bar{f}_0, \bar{f}_1, \bar{f}_2$  does not satisfy the properties since  $\bar{f}_0$  contains two edges  $ey, xy$  which are not in any face of  $\mathcal{F}$ .

In the following, let  $f_0, f_1, f_2$  be the faces as in Claim 5.2. If  $ae \in E(G)$ , then from Lemma 19,  $\{a, b, c\}$  is a 3-potential-leaf of  $G$ .

Otherwise, we can prove that any tree decomposition of  $G$  of width at most 3 can be modified to a tree decomposition of  $G \cup \{ae\}$  with the same width and size in the following. So  $\{a, b, c\}$  is a 3-potential-leaf of  $G$ .

Let  $(T, \mathcal{X})$  be a tree decomposition of width at most 3 and size at most  $s \geq 1$  of  $G$ . Let  $(T_0, \mathcal{X}_0)$  be the tree decomposition obtained by deleting all vertices not in  $f_0$ . Then  $(T_0, \mathcal{X}_0)$  is a tree decomposition of  $f_0$  (without confusion  $f_0$  is used to denote the face and the cycle induced by vertices in  $f_0$  as well). From Lemma 20, there is a bag containing three consecutive vertices  $u, v, w$  in  $f_0$  and  $uv, vw$  are edges of some faces in  $\mathcal{F}$ . (Note that  $u, v, w$  are not consecutive in  $C$ .) So  $(T, \mathcal{X})$  is also a tree decomposition of  $G \cup uv$ . The graph  $G \cup uv$  and the graph  $G \cup ae$  are isomorphic. So from  $(T, \mathcal{X})$  we can obtain a tree decomposition  $(T', \mathcal{X}')$  of  $G \cup ae$  with the same width and size. Then  $(T', \mathcal{X}')$  is the desired tree decomposition.

From Lemmas 22 and Corollary 3, we obtain the following result.

**Corollary 6.**  $s_3$  can be computed and a minimum size tree decomposition of width at most 3 can be constructed in polynomial-time in the class of 2-connected outerplanar graphs.

## 6 Conclusion

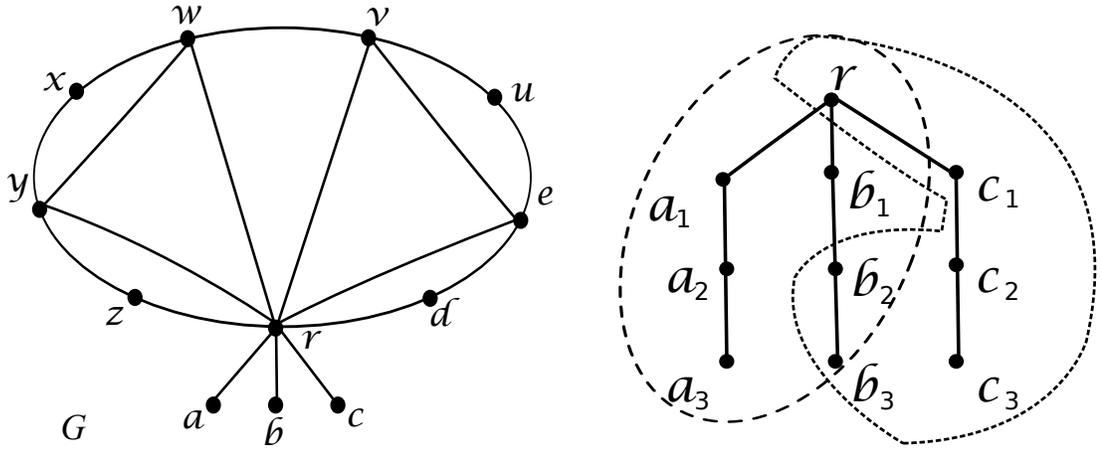
In this report, we gave preliminary results on the complexity of minimizing the size of tree-decompositions with given width. Table 1 summarizes our results as well as the remaining open questions.

We currently investigate the case of  $s_3$  in the class of connected graphs with treewidth 2 or 3 and we conjecture it is polynomially solvable. But it is more tricky than computing  $s_3$  in trees and 2-connected outerplanar graphs. It seems that a global view of the graph needs to be considered to decide whether a subgraph is a 3-potential-leaf of the graph.

	$s_1$	$s_2$	$s_3$	$s_4$	$s_k, k = \max\{tw + 1, 5\}$
Graphs of treewidth at most $tw = 1$	$P$ (trivial)	$P$	$P$	?	?
Graphs of treewidth at most $tw = 2$	-	$P$	?	?	?
Graphs of treewidth at most $tw = 3$	-	-	?	NP-hard	?
Graphs of treewidth at most $tw \geq 4$	-	-	-	NP-hard	NP-hard

**Table 1.** Summary of the complexity results.

See an example in Fig. 14(a). In this example,  $G$  is a connected outerplanar graph.  $\{r, a, b, c\}$  is not a 3-potential-leaf of  $G$ , but it is a 3-potential-leaf of  $G \setminus \{yw\}$ . Let  $G' \equiv G \setminus \{a, b, c\}$ . Then  $G'$  is 2-connected outerplanar. From the algorithm of computing  $s_3$  in 2-connected outerplanar graphs in subsection 5.2, we can compute that  $s_3(G') = 5$ . So if  $\{r, a, b, c\}$  is a potential-leaf of  $G$ , then  $s_3(G) = 6$ . But there exists a tree decomposition of  $G$  of width 3 and size 5, where the bags are  $\{a, r, z, y\}, \{r, y, x, w\}, \{b, r, w, v\}, \{r, v, u, e\}, \{c, r, d, e\}$ . So  $\{r, a, b, c\}$  is not a 3-potential-leaf of  $G$ . However, in the graph  $G'' \equiv G \setminus \{yw\}$ , we can prove that  $s_3(G'') = 5$  and there is a minimum size tree decomposition containing  $\{r, a, b, c\}$  as a leaf bag, i.e.  $\{r, a, b, c\}$  is 3-potential-leaf of  $G''$ . So the existence of the edge  $yw$ , not incident to any vertex in  $\{r, a, b, c\}$ , changes the behavior of  $\{r, a, b, c\}$ .



(a)  $\{r, a, b, c\}$  is not a 3-potential-leaf of  $G$ , but it is a 3-potential-leaf of  $G \setminus \{yw\}$ . The five bags  $\{a, r, z, y\}, \{r, y, x, w\}, \{b, r, w, v\}, \{r, v, u, e\}, \{c, r, d, e\}$  connecting as a path in this order forms a tree decomposition of  $G$ .

(b) In any minimum size tree decomposition of width 5 (and size 2) of this tree, there exists a bag inducing a forest, not a subtree. For example, in a tree decomposition of width 5 and size 2, one bag is  $\{r, a_1, a_2, a_3, b_1, b_2\}$  and the other one is  $\{r, b_2, b_3, c_1, c_2, c_3\}$ .

**Fig. 14.**

The problem of computing  $s_k$ , for  $k \geq 4$ , seems more intricate already in the case of trees. Indeed, our polynomial-time algorithms to compute  $s_k$ ,  $k \leq 3$ , in trees mainly rely on the fact that, for any tree  $T$ , there exists a minimum-size tree-decomposition of  $T$  with width at most 3, where each bag induces a connected subtree. This is unfortunately not true anymore in the case of tree-decomposition with width 5. As an example, consider the tree  $G$  (with 10 nodes) obtained from a star with three 3 leaves by subdividing twice each edge. See in Fig. 14(b).  $s_5(G) = 2$  and any minimum size tree decomposition has a bag  $X$  such that  $G[X]$  is disconnected.

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