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# Stability Analysis for Nonlinear Time-Delay Systems Applying Homogeneity

Efimov D., Polyakov A., Perruquetti W., Richard J.-P.

**Abstract**—Global delay independent stability is analyzed for nonlinear time-delay systems applying homogeneity theory. The results of [1] are extended to the case of non-zero degree of homogeneity. Several tools for stability analysis in time-delay systems using homogeneity are presented: in particular, it is shown that if a time-delay system is homogeneous with nonzero degree and it is globally asymptotically stable for some delay, then this property is preserved for any delay value, which is known as the independent of delay (IOD) stability. The results are illustrated by numerical experiments.

## I. INTRODUCTION

The theory of homogeneous dynamical systems has been well developed for continuous time-invariant differential equations [2], [3], [4], [5]. The main feature of a homogeneous nonlinear system is that its local behavior of trajectories is the same as global (local attractiveness implies global asymptotic stability, for example). In addition, the homogeneous stable/unstable systems admit homogeneous Lyapunov functions [5], [6], [7]. Since the subclass of nonlinear systems with a global symmetry is rather narrow, the concept of local homogeneity has been introduced [5], [8], [7]. Then, a local homogeneous approximation of nonlinear system can be computed representing locally the original system behavior. The advantage in this case is that the approximating dynamics is globally homogeneous and the equation complexity is decreased.

The dynamical systems subjected by a time-delay, whose models are presented by functional differential equations, find their applications in many areas of science and technology [9], [10]. Analysis of delay influence on the system stability is critical for many natural and human-developed systems [11], [12], [13]. Despite of variety of applications, most of them deal with linear time-delay models, which is originated by complexity of stability analysis for time-delay systems in general (design of a Lyapunov-Krasovskii functional or a Lyapunov-Razumikhin function is a complex problem), and

that constructive and computationally tractable conditions exist for linear systems only [14].

An attempt to apply the homogeneity theory for nonlinear functional differential equations in order to simplify their stability analysis has been performed in [15], [1]. In those works an extension of the homogeneity theory for time-delay systems has been proposed. Applications of the conventional homogeneity framework for analysis of time-delay systems (considering delay as a kind of perturbation, for instance) have been carried out even earlier in [16], [17], [18], [19].

The outline of this work is as follows. The preliminary definitions and the homogeneity for time-delay systems are given in Section 2. The property of scaling of solutions for some class of homogeneous time-delay systems is presented in Section 3 (the case of non-zero degree of homogeneity is treated comparing with [1]). An example is considered in Section 4.

## II. PRELIMINARIES

Consider an autonomous functional differential equation of retarded type [13]:

$$dx(t)/dt = f(x_t), \quad t \geq 0, \quad (1)$$

where  $x \in \mathbb{R}^n$  and  $x_t \in C_{[-\tau, 0]}$  is the state function,  $x_t(s) = x(t + s)$ ,  $-\tau \leq s \leq 0$  (we denote by  $C_{[a, b]}$ ,  $0 \leq a < b \leq +\infty$  the Banach space of continuous functions  $\phi : [a, b] \rightarrow \mathbb{R}^n$  with the uniform norm  $\|\phi\| = \sup_{a \leq s \leq b} |\phi(s)|$ , where  $|\cdot|$  is the standard Euclidean norm,  $C_{[a, b]}^1$  is used to denote the subset of  $C_{[a, b]}$  of continuously differentiable functions);  $f : C_{[-\tau, 0]} \rightarrow \mathbb{R}^n$  is a locally Lipschitz continuous function,  $f(0) = 0$ . The representation (1) includes pointwise or distributed time-delay systems with either constant or variable time delay  $\tau(t) \in [0, \tau]$ . We assume that solutions of the system (1) satisfy the initial functional condition  $x_0 \in C_{[-\tau, 0]}$ . It is known from the theory of functional differential equations [13] that under the above assumptions the system (1) has a unique solution  $x(t, x_0)$  satisfying the initial condition  $x_0$  and  $x_{t, x_0}(s) = x(t + s, x_0)$  for  $-\tau \leq s \leq 0$ , which is defined on some finite time interval  $[-\tau, T)$  (we will use the notation  $x(t)$  to reference  $x(t, x_0)$  if the origin of  $x_0$  is clear from the context).

The upper right-hand Dini derivative of a locally Lipschitz continuous functional  $V : C_{[-\tau, 0]} \rightarrow \mathbb{R}_+$  along the system (1) solutions is defined as follows for any  $\phi \in C_{[-\tau, 0]}$ :

$$D^+V(\phi) = \limsup_{h \rightarrow 0^+} \frac{1}{h} [V(\phi_h) - V(\phi)],$$

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where  $\phi_h \in C_{[-\tau,0]}$  for  $0 < h < \tau$  is given by

$$\phi_h = \begin{cases} \phi(\theta + h), & \theta \in [-\tau, -h) \\ \phi(0) + f(\phi)(\theta + h), & \theta \in [-h, 0]. \end{cases}$$

For a locally Lipschitz continuous function  $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$  the upper directional Dini derivative is defined as follows:

$$D^+V[x_t(0)]f(x_t) = \lim_{h \rightarrow 0^+} \sup \frac{V[x_t(0) + hf(x_t)] - V[x_t(0)]}{h}.$$

A continuous function  $\sigma : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  belongs to class  $\mathcal{K}$  if it is strictly increasing and  $\sigma(0) = 0$ ; it belongs to class  $\mathcal{K}_\infty$  if it is also radially unbounded. A continuous function  $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$  belongs to class  $\mathcal{KL}$  if  $\beta(\cdot, r) \in \mathcal{K}$  and  $\beta(r, \cdot)$  is a strictly decreasing to zero for any fixed  $r \in \mathbb{R}_+$ . The symbol  $\overline{1, m}$  is used to denote a sequence of integers  $1, \dots, m$ .

### A. Stability definitions

Let  $\Omega$  be an open subset of  $C_{[-\tau,0]}$ .

**Definition 1.** [20] The system (1) is said to be

- (a) stable if there is  $\sigma \in \mathcal{K}$  such that for any  $x_0 \in \Omega$ ,  $|x(t, x_0)| \leq \sigma(|x_0|)$  for all  $t \geq 0$ ;
- (b) asymptotically stable if it is stable and  $\lim_{t \rightarrow +\infty} |x(t, x_0)| = 0$  for any  $x_0 \in \Omega$ .

If  $\Omega = C_{[-\tau,0]}$ , then the corresponding properties are called global stability/asymptotic stability.

### B. Homogeneity

For any  $r_i > 0$ ,  $i = \overline{1, n}$  and  $\lambda > 0$ , define the dilation matrix  $\Lambda_r(\lambda) = \text{diag}\{\lambda^{r_i}\}_{i=1}^n$  and the vector of weights  $r = [r_1, \dots, r_n]^T$ .

For any  $r_i > 0$ ,  $i = \overline{1, n}$  and  $x \in \mathbb{R}^n$  the homogeneous norm can be defined as follows

$$|x|_r = \left( \sum_{i=1}^n |x_i|^{\rho/r_i} \right)^{1/\rho}, \quad \rho \geq \max_{1 \leq i \leq n} r_i.$$

For all  $x \in \mathbb{R}^n$ , its Euclidean norm  $|x|$  is related with the homogeneous one:

$$\underline{\sigma}_r(|x|_r) \leq |x| \leq \bar{\sigma}_r(|x|_r)$$

for some  $\underline{\sigma}_r, \bar{\sigma}_r \in \mathcal{K}_\infty$ . The homogeneous norm has an important property that is  $|\Lambda_r(\lambda)x|_r = \lambda|x|_r$  for all  $x \in \mathbb{R}^n$ . Define  $\mathbb{S}_r = \{x \in \mathbb{R}^n : |x|_r = 1\}$ .

For any  $r_i > 0$ ,  $i = \overline{1, n}$  and  $\phi \in C_{[a,b]}$ ,  $0 \leq a < b \leq +\infty$  the homogeneous norm can be defined as follows

$$\|\phi\|_r = \left( \sum_{i=1}^n \|\phi_i\|^{\rho/r_i} \right)^{1/\rho}, \quad \rho = \prod_{i=1}^n r_i.$$

There exist two functions  $\underline{\rho}_r, \bar{\rho}_r \in \mathcal{K}_\infty$  such that for all  $\phi \in C_{[a,b]}$

$$\underline{\rho}_r(\|\phi\|_r) \leq \|\phi\| \leq \bar{\rho}_r(\|\phi\|_r). \quad (2)$$

The homogeneous norm in the Banach space has the same important property that is  $\|\Lambda_r(\lambda)\phi\|_r = \lambda\|\phi\|_r$  for all  $\phi \in C_{[a,b]}$ , the corresponding unit sphere  $\mathcal{S}_r = \{\phi \in C_{[-\tau,0]} : \|\phi\|_r = 1\}$ . Define  $B_\rho^r = \{\phi \in C_{[-\tau,0]} : \|\phi\|_r \leq \rho\}$  as a closed ball of radius  $\rho > 0$  in  $C_{[-\tau,0]}$ .

**Definition 2.** [15] The function  $g : C_{[-\tau,0]} \rightarrow \mathbb{R}$  is called  $r$ -homogeneous ( $r_i > 0$ ,  $i = \overline{1, n}$ ), if for any  $\phi \in C_{[-\tau,0]}$  the relation

$$g(\Lambda_r(\lambda)\phi) = \lambda^d g(\phi)$$

holds for some  $d \in \mathbb{R}$  and all  $\lambda > 0$ .

The function  $f : C_{[-\tau,0]} \rightarrow \mathbb{R}^n$  is called  $r$ -homogeneous ( $r_i > 0$ ,  $i = \overline{1, n}$ ), if for any  $\phi \in C_{[-\tau,0]}$  the relation

$$f(\Lambda_r(\lambda)\phi) = \lambda^d \Lambda_r(\lambda) f(\phi)$$

holds for some  $d \geq -\min_{1 \leq i \leq n} r_i$  and all  $\lambda > 0$ .

In both cases, the constant  $d$  is called the degree of homogeneity.

The introduced notion of homogeneity in  $C_{[-\tau,0]}$  is reduced to the standard one in  $\mathbb{R}^n$  [6] under a vector argument substitution. An advantage of homogeneous systems described by nonlinear ordinary differential equations is that any of its solutions can be obtained from another solution under the dilation rescaling and a suitable time reparametrization. A similar property holds for some functional homogeneous systems.

**Proposition 1.** [1] Let  $x : \mathbb{R}_+ \rightarrow \mathbb{R}^n$  be a solution of the  $r$ -homogeneous system (1) with the degree  $d = 0$  for an initial condition  $x_0 \in C_{[-\tau,0]}$ . For any  $\lambda > 0$  define  $y(t) = \Lambda_r(\lambda)x(\lambda^d t)$  for all  $t \geq 0$ , then  $y(t)$  is also a solution of (1) with the initial condition  $y_0 = \Lambda_r(\lambda)x_0$ .

The Razumikhin approach for stability analysis of time-delay systems is based on Lyapunov-Razumikhin functions [11], [12], [13] defined on  $\mathbb{R}^n$ , which give a pointwise sufficient criteria for stability (not a functional one). This approach has been developed using homogeneous arguments as follows.

**Theorem 1.** [15], [1] Let the function  $f$  in (1) be  $r$ -homogeneous with degree  $d \geq -\min_{1 \leq i \leq n} r_i$  and there exist a locally Lipschitz continuous  $r$ -homogeneous Lyapunov-Razumikhin function  $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$  with degree  $\nu > \max\{0, -d\}$  such that

- (i) there exist functions  $\alpha, \gamma \in \mathcal{K}$  such that for all  $\varphi \in \mathcal{S}_r$

$$\max_{\theta \in [-\tau, 0]} V[\varphi(\theta)] < \gamma\{V[\varphi(0)]\} \Rightarrow$$

$$D^+V[\varphi(0)]f(\varphi) \leq -\alpha(|\varphi(0)|);$$

- (ii) there exists a function  $\gamma' \in \mathcal{K}$  such that  $\lambda s < \gamma'(\lambda s) \leq \lambda \gamma(s)$  for all  $s, \lambda \in \mathbb{R}_+ \setminus \{0\}$ .

Then the origin is globally asymptotically stable for the system (1).

The condition (i) imposed in Theorem 1 on the system (1) behavior is the conventional Razumikhin condition (except

that in the homogeneous case it can be verified on the sphere  $\mathcal{S}_r$  only). The constraint (ii) on existence of the function  $\gamma'$  is new.

### C. Local homogeneity

A disadvantage of the global homogeneity introduced so far is that such systems possess the same behavior “globally” [15], [1].

**Definition 3.** [15] The function  $g : C_{[-\tau,0]} \rightarrow \mathbb{R}$  is called  $(r, \lambda_0, g_0)$ -homogeneous ( $r_i > 0$ ,  $i = \overline{1, n}$ ;  $g_0 : C_{[-\tau,0]} \rightarrow \mathbb{R}$ ) if for any  $\phi \in \mathcal{S}_r$  the relation

$$\lim_{\lambda \rightarrow \lambda_0} \lambda^{-d_0} g(\Lambda_r(\lambda)\phi) - g_0(\phi) = 0$$

is satisfied (uniformly on  $\mathcal{S}_r$  for  $\lambda_0 \in \{0, +\infty\}$ ) for some  $d_0 \in \mathbb{R}$ .

The system (1) is called  $(r, \lambda_0, f_0)$ -homogeneous ( $r_i > 0$ ,  $i = \overline{1, n}$ ;  $f_0 : C_{[-\tau,0]} \rightarrow \mathbb{R}^n$ ) if for any  $\phi \in \mathcal{S}_r$  the relation

$$\lim_{\lambda \rightarrow \lambda_0} \lambda^{-d_0} \Lambda_r^{-1}(\lambda) f(\Lambda_r(\lambda)\phi) - f_0(\phi) = 0$$

is satisfied (uniformly on  $\mathcal{S}_r$  for  $\lambda_0 \in \{0, +\infty\}$ ) for some  $d_0 \geq -\min_{1 \leq i \leq n} r_i$ .

For a given  $\lambda_0$ ,  $g_0$  and  $f_0$  are called approximating functions.

For any  $0 < \lambda_0 < +\infty$  the following formulas give an example of  $r$ -homogeneous approximating functions  $g_0$  and  $f_0$ :

$$\begin{aligned} g_0(\phi) &= \|\phi\|_r^d \lambda_0^{-d_0} g(\Lambda_r(\lambda_0) \Lambda_r^{-1}(\|\phi\|_r) \phi), \quad d \geq 0, \\ f_0(\phi) &= \|\phi\|_r^d \lambda_0^{-d_0} \Lambda_r(\|\phi\|_r) \Lambda_r^{-1}(\lambda_0) f(\Lambda_r(\lambda_0) \Lambda_r^{-1}(\|\phi\|_r) \phi), \end{aligned}$$

$d \geq -\min_{1 \leq i \leq n} r_i$ . This property allows us to analyze local stability/instability of the system (1) on the basis of a simplified system

$$dy(t)/dt = f_0[y_\tau(t)], t \geq 0, \quad (3)$$

called the *local approximating dynamics* for (1).

**Theorem 2.** [15] Let the system (1) be  $(r, \lambda_0, f_0)$ -homogeneous for some  $r_i > 0$ ,  $i = \overline{1, n}$ , the function  $f_0$  be continuous and  $r$ -homogeneous with the degree  $d_0$ . Suppose there exists a locally Lipschitz continuous  $r$ -homogeneous Lyapunov-Razumikhin function  $V_0 : \mathbb{R}^n \rightarrow \mathbb{R}_+$  with the degree  $\nu_0 > \max\{0, -d_0\}$ ,

$$\alpha_1(|x|) \leq V_0(x) \leq \alpha_2(|x|)$$

for all  $x \in \mathbb{R}^n$  and some  $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$  such that:

(i) there exist functions  $\alpha, \gamma \in \mathcal{K}$  such that for all  $\varphi \in \mathcal{S}_r$

$$\max_{\theta \in [-\tau, 0]} V_0[\varphi(\theta)] < \gamma\{V_0[\varphi(0)]\} \Rightarrow$$

$$D^+ V_0[\varphi(0)] f_0(\varphi) \leq -\alpha(|\varphi(0)|);$$

(ii) there exists a function  $\gamma' \in \mathcal{K}$  such that  $\lambda s < \gamma'(\lambda s) \leq \lambda \gamma(s)$  for all  $s, \lambda \in \mathbb{R}_+ \setminus \{0\}$ .

Then (the functions  $\bar{\rho}_r$  and  $\underline{\rho}_r$  have been defined in (2))

1) if  $\lambda_0 = 0$ , then there exists  $0 < \bar{\lambda}_\varepsilon$  such that the system (1) is locally asymptotically stable at the origin with the domain of attraction containing the set

$$X_0 = \{\phi \in C_{[-\tau,0]} : \|\phi\| \leq \alpha_1^{-1} \circ \alpha_2 \circ \bar{\rho}_r(\bar{\lambda}_\varepsilon)\};$$

2) if  $\lambda_0 = +\infty$ , then there exists  $0 < \underline{\lambda}_\varepsilon < +\infty$  such that the system (1) is globally asymptotically stable with respect to forward invariant set

$$X_\infty = \{\phi \in C_{[-\tau,0]} : \|\phi\| \leq \alpha_1^{-1} \circ \alpha_2 \circ \underline{\rho}_r(\underline{\lambda}_\varepsilon)\};$$

3) if  $0 < \lambda_0 < +\infty$ , then there exist  $0 < \underline{\lambda}_\varepsilon \leq \lambda_0 \leq \bar{\lambda}_\varepsilon < +\infty$  such that the system (1) is asymptotically stable with respect to the forward invariant set  $X_\infty$  with region of attraction

$$\begin{aligned} X &= \{\phi \in C_{[-\tau,0]} : \alpha_1^{-1} \circ \alpha_2 \circ \underline{\rho}_r(\underline{\lambda}_\varepsilon) < \|\phi\| \\ &< \alpha_1^{-1} \circ \alpha_2 \circ \bar{\rho}_r(\bar{\lambda}_\varepsilon)\} \end{aligned}$$

provided that the set  $X$  is connected and nonempty.

In [1] analysis of the input-to-state stability property for the system (1) has been presented using the homogeneity theory.

## III. MAIN RESULTS

Proposition 1 establishes that all solutions of homogeneous time-delay systems of degree zero are interrelated, thus the local behavior implies global one, as for example in the following stable nonlinear system

$$\dot{x}(t) = -\varrho x(t) + \frac{x^2(t) + x^2(t - \tau)}{\max\{|x(t)|, |x(t - \tau)|\}}, \quad \varrho \geq 2,$$

where  $x(t) \in \mathbb{R}$ ,  $\tau > 0$  is a fixed time delay [1]. Some more complicated relations can also be established for the case when the degree is not zero.

**Proposition 2.** Let  $x(t, x_0)$  be a solution of the  $r$ -homogeneous system (1) with the degree  $d$  for an initial condition  $x_0 \in C_{[-\tau,0]}$ ,  $\tau \in (0, +\infty)$ . For any  $\lambda > 0$  the functional differential equation

$$dy(t)/dt = f(y_t), \quad t \geq 0 \quad (4)$$

with  $y_t \in C_{[-\lambda^{-d}\tau, 0]}$ , has a solution  $y(t, y_0) = \Lambda_r(\lambda)x(\lambda^d t, x_0)$  for all  $t \geq 0$  with the initial condition  $y_0 \in C_{[-\lambda^{-d}\tau, 0]}$ ,  $y_0(s) = \Lambda_r(\lambda)x_0(\lambda^d s)$  for  $s \in [-\lambda^{-d}\tau, 0]$ .

*Proof.* By definition  $x_{\lambda^d t}(s) = x(\lambda^d t + s) \in \mathbb{R}^n$  and  $y_t(s) = \Lambda_r(\lambda)x_{\lambda^d t}(s)$  for any  $-\lambda^{-d}\tau \leq s \leq 0$ , then

$$\begin{aligned} \dot{y}(t) &= \frac{d}{dt} (\Lambda_r(\lambda)x(\lambda^d t)) = \lambda^d \Lambda_r(\lambda) f(x_{\lambda^d t}) \\ &= f(\Lambda_r(\lambda)x_{\lambda^d t}) = f(y_t) \end{aligned}$$

and  $y(t)$  is a solution of (4).  $\square$

In order to better explain this result consider a particular case of the system (1) with the pointwise delay:

$$f(x_\tau) = F[x(t), x(t - \tau)],$$

then the homogeneity condition reads as  $F[\Lambda_r(\lambda)x, \Lambda_r(\lambda)z] = \lambda^d \Lambda_r(\lambda)F[x, z]$  for any  $x, z \in \mathbb{R}^n$  and  $\lambda \in (0, +\infty)$ . Select a solution  $x(t, x_0)$  of this system for the initial condition  $x_0 \in C_{[-\tau, 0]}$  for some  $\tau > 0$  as before (i.e.  $\dot{x}(t, x_0) = F[x(t, x_0), x(t - \tau, x_0)]$ ). Take some  $\lambda \in (0, +\infty)$  and define  $y_0 \in C_{[-\lambda^{-d}\tau, 0]}$ ,  $y_0(s) = \Lambda_r(\lambda)x_0(\lambda^d s)$  for  $s \in [-\lambda^{-d}\tau, 0]$ , then consider  $y(t, y_0) = \Lambda_r(\lambda)x(\lambda^d t, x_0)$ , taking its derivative with respect to time we obtain

$$\begin{aligned} \frac{dy(t, y_0)}{dt} &= \frac{d\Lambda_r(\lambda)x(\lambda^d t, x_0)}{dt} \\ &= \lambda^d \Lambda_r(\lambda)F[x(\lambda^d t, x_0), x(\lambda^d t - \tau, x_0)] \\ &= F[\Lambda_r(\lambda)x(\lambda^d t, x_0), \Lambda_r(\lambda)x(\lambda^d \{t - \lambda^{-d}\tau\}, x_0)] \\ &= F[y(t, y_0), y(t - \lambda^{-d}\tau, y_0)]. \end{aligned}$$

Therefore,  $y(t, y_0)$  is the system solution with initial conditions  $y_0$  for another delay  $\lambda^{-d}\tau$ .

For the case  $d = 0$  we recover the result of Proposition 1. In [1], using that result it has been shown for  $d = 0$  that local asymptotic stability implies global one (for the ordinary differential equations even more stronger conclusion can be obtained: local attractiveness implies global asymptotic stability [2]). In the present setting that result has the following correspondence.

**Lemma 1.** *Let the system (1) be  $r$ -homogeneous with degree  $d \neq 0$  and globally asymptotically stable for some delay  $0 < \tau_0 < +\infty$ , then it is globally asymptotically stable for any delay  $0 < \tau < +\infty$  (i.e. IOD).*

*Proof.* In this case for all  $x_0 \in C_{[-\tau_0, 0]}$  there is a function  $\sigma \in \mathcal{K}$  such that  $|x(t, x_0)|_r \leq \sigma(|x_0|_r)$  for all  $t \geq 0$  and  $\lim_{t \rightarrow +\infty} |x(t, x_0)|_r = 0$ . Take some  $\tau \in (0, +\infty)$  and select an initial condition  $y_0 \in C_{[-\tau, 0]}$ , then for  $\lambda = (\frac{\tau_0}{\tau})^{1/d}$  (this  $\lambda$  is well defined since  $d \neq 0$ ) there exists  $x_0 \in C_{[-\tau_0, 0]}$  such that  $y_0(s) = \Lambda_r(\lambda)x_0(\lambda^d s)$  for  $s \in [-\tau, 0]$ , and  $y(t, y_0) = \Lambda_r(\lambda)x(\lambda^d t, x_0)$  for all  $t \geq 0$  by Proposition 2. Thus  $\lim_{t \rightarrow +\infty} |y(t, y_0)|_r = \lambda \lim_{t \rightarrow +\infty} |x(\lambda^d t, x_0)|_r = \lambda \lim_{t \rightarrow +\infty} |x(t, x_0)|_r = 0$  and the solution  $y(t, y_0)$  is converging asymptotically to the origin. In addition,  $|y(t, y_0)|_r = \lambda |x(\lambda^d t, x_0)|_r \leq \lambda \sigma(|x_0|_r) = \lambda \sigma(\lambda^{-1}|y_0|_r)$  for all  $t \geq 0$ , which implies stability of the system (1) for the delay  $\tau$ . The proven convergence to the origin and stability give the global asymptotic stability of the system for an arbitrary delay  $\tau \in (0, +\infty)$ .  $\square$

It is well known fact for linear systems (homogeneous systems of degree  $d = 0$ ) that its stability for a sufficiently small delay does not imply stability for all  $\tau \in (0, +\infty)$ . For nonlinear homogeneous systems with degree  $d \neq 0$  this is not the case, according to the result of Lemma 1 if they are globally stable for some delay, they can preserve their stability for an arbitrary delay  $\tau \in (0, +\infty)$ . This is a surprising advantage of “nonlinear” time-delay systems.

Further let us consider several useful consequences of Proposition 2 and Lemma 1.

**Corollary 1.** *Let the system (1) be  $r$ -homogeneous with*

*degree  $d$  and asymptotically stable into the set  $\Omega = B_\rho^\tau = \{\phi \in C_{[-\tau, 0]} : \|\phi\|_r \leq \rho\}$  for some  $0 < \rho < +\infty$  for any value of delay  $0 \leq \tau < +\infty$ , then it is globally asymptotically stable IOD.*

The proofs of all the rest results are omitted due to space limitations.

**Corollary 2.** *Let the system (1) be  $r$ -homogeneous with degree  $d < 0$  and asymptotically stable into the set  $B_\rho^\tau$  for some  $0 < \rho' < +\infty$  for any value of delay  $0 \leq \tau \leq \tau_0$  with  $0 < \tau_0 < +\infty$ , then it is globally asymptotically stable IOD.*

**Corollary 3.** *Let the system (1) be  $r$ -homogeneous with degree  $d > 0$  and the set  $B_\rho^\tau$  for some  $0 < \rho < +\infty$  be uniformly globally asymptotically stable for any value of delay  $0 \leq \tau \leq \tau_0$ ,  $0 < \tau_0 < +\infty$ <sup>1</sup>, then (1) is globally asymptotically stable (at the origin) IOD.*

In addition, due to ISS property well inherited by homogeneous systems (see [21], [1] for details), under mild conditions its stability for  $\tau = 0$  implies the same property for sufficiently small delay. Similar results, connecting ISS and stability of the systems with respect to time delays, have been established in [22], [23].

**Lemma 2.** *Let  $f(x_\tau) = F[x(t), x(t - \tau)]$  in (1) and the system (1) be  $r$ -homogeneous with degree  $d > 0$  and globally asymptotically stable for  $\tau = 0$ , then for any  $\rho > 0$  there is  $0 < \tau_0 < +\infty$  such that (1) is locally asymptotically stable in  $B_\rho^{2\tau} \cap C_{[-2\tau, 0]}^1$  for any delay  $0 \leq \tau \leq \tau_0$ .*

If  $x_0 \in C_{[-2\tau, 0]}$  and there is no finite-time escape phenomenon in (1) on the interval  $t \in [0, 2\tau]$ , then next we can always consider solution of (1) with the corresponding new  $x'_0 \in C_{[-2\tau, 0]}^1$  for  $t \geq 2\tau$ , thus the result of Lemma 2 can be applied.

Using Theorem 2, these results can be used for local analysis of stability for not necessary homogeneous systems.

**Theorem 3.** *Let system (1) be  $(r, +\infty, f_0)$ -homogeneous with degree  $d_0 > 0$  and for the approximating system (3) the set  $B_\rho^\tau$  for some  $0 < \rho < +\infty$  be globally asymptotically stable for any value of delay  $0 \leq \tau \leq \tau_0$  with  $0 < \tau_0 < +\infty$ , then (1) has bounded trajectories IOD.*

Roughly speaking the result of Theorem 3 says that if the approximating dynamics (3) has bounded trajectories for some sufficiently small delays, then the original system (1) has bounded trajectories for any delay.

The results presented in this section open the door for a wealthy stability analysis of nonlinear time-delay systems. Let us consider an example for these results.

<sup>1</sup>In this case for any  $0 \leq \tau \leq \tau_0$ , any  $\varepsilon > 0$  and  $\kappa \geq 0$  there is  $0 \leq T_{\kappa, \tau}^\varepsilon < +\infty$  such that  $\|x_{t, x_0}\|_r \leq \rho + \varepsilon$  for all  $t \geq T_{\kappa, \tau}^\varepsilon$  for any  $x_0 \in B_\kappa^\tau$ , and  $|x(t, x_0)|_r \leq \sigma_\tau(|x_0|_r)$  for all  $t \geq 0$  for some function  $\sigma_\tau \in \mathcal{K}_\infty$  for all  $x_0 \in B_\rho^\tau$ .

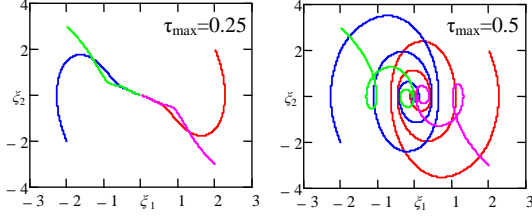


Figure 1. Trajectories of stabilized delayed double integrator

#### IV. EXAMPLE

In [3], for a chain of integrators

$$\begin{aligned}\dot{\xi}_i &= \xi_{i+1}, \quad i = 1, \dots, n-1, \\ \dot{\xi}_n &= u,\end{aligned}$$

where  $x = (\xi_1, \dots, \xi_n)^T \in \mathbb{R}^n$  is the state and  $u \in \mathbb{R}$  is the control, the following stabilizing controller has been proposed:

$$u = - \sum_{i=1}^n k_i |\xi_i|^{\alpha_i} \text{sign}(\xi_i),$$

where  $k_i$  form a Hurwitz polynomial and  $\alpha_i \in \mathbb{R}_+$ . If the powers  $\alpha_i$  are selected in a way providing homogeneity of the closed-loop system and if all of them are smaller and sufficiently close to 1, then the system is finite-time stable [3]. For the case  $n = 2$ ,  $\alpha_1 = \frac{\alpha}{2-\alpha}$ ,  $\alpha_2 = \alpha$  and an arbitrary value of  $\alpha \in (0, 1)$  a solution has been given in [24]. Let us assume that the state is available for measurements with delays  $\tau_i \in (0, \tau_{\max})$ ,  $0 < \tau_{\max} < +\infty$ :

$$u'(t) = - \sum_{i=1}^n k_i |\xi_i(t - \tau_i)|^{\alpha_i} \text{sign}[\xi_i(t - \tau_i)],$$

then applying the result of Lemma 2 to the closed-loop system we obtain that it is locally asymptotically stable in  $C_{[-2\tau_{\max}, 0]}^1$  for some  $0 < \tau_{\max} < +\infty$  if  $\alpha_i > 1$  are sufficiently close to 1. Indeed, in this case the homogeneity degree is *positive* (Corollary 2 cannot be applied) and we can use the same technique as in [3] to prove global asymptotic stability of the delay-free control (for the case  $n = 2$  the Lyapunov function  $V(x) = \frac{k_1}{\alpha_1+1} |\xi_1|^{\alpha_1+1} + \frac{1}{2} \xi_2^2$  can be used).

The results of computer simulation of this system for two different values of delays (for  $n = 2$ ,  $k_1 = 1$ ,  $k_2 = 2$ ,  $\alpha_1 = 1.5$ ,  $\alpha_2 = 1.2$ ) is shown in Fig. 1. As we can conclude, increasing the value of delay the system becomes more oscillatory and finally unstable for a sufficiently big value of  $\tau_{\max}$ .

#### V. CONCLUSIONS

The homogeneity theory extensions are obtained for time-delay systems. It is shown that for any degree of homogeneity the solutions of a homogeneous system are interrelated subject to the delay rescaling. Next, this fact is utilized in order to show that local uniform asymptotic stability of homogeneous systems implies global one, and that for

nonlinear homogeneous systems with non-zero degree global asymptotic stability for a delay endorses this property for an arbitrary large delay. Thus analysis of stability in nonlinear time-delay systems differs significantly from the linear case, but in a positive sense: being stable for a delay such systems save this property for any delay value, which is an interesting advantage of nonlinear time-delay systems. Efficiency of the proposed approach is demonstrated on example. Application of the developed results for analysis and design of control or estimation algorithms is a direction of future research.

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