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Joachim Lebovits

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| Joachim Lebovits. Stochastic Calculus with respect to Gaussian processes. 2014. hal-01052481v2

**HAL Id: hal-01052481**

**<https://inria.hal.science/hal-01052481v2>**

Preprint submitted on 5 May 2015 (v2), last revised 8 Feb 2018 (v3)

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# Stochastic Calculus with respect to Gaussian processes

Joachim LÉBOVITS\*<sup>†</sup>

April 28, 2015

## Abstract

Stochastic integration with respect to Gaussian processes has raised strong interest in recent years, motivated in particular by its applications in Internet traffic modeling, biomedicine and finance. The aim of this work is to define and develop a White Noise Theory-based anticipative stochastic calculus with respect to all Gaussian processes that have an integral representation over a real interval (that maybe infinite). This very rich, this class of Gaussian processes, which will be denoted  $\mathcal{G}$ , contains, among many others, Volterra processes (and thus fractional Brownian motion) and multifractional Brownian motions. It also contains Gaussian processes, the regularity of which varies along the time. The stochastic calculus provided in this work allows us to derive Itô formulas, a Tanaka Formula as well as to perform a first study of (both weighted and non weighted) local times of elements of  $\mathcal{G}$ , that includes providing occupation time formulas.

A systematic comparison of the integral *wrt* elements of  $\mathcal{G}$  we provide here, to the ones given by Malliavin calculus, in [AMN01], [MV05], [NT06], and by Itô stochastic calculus is also made. A complete comparison of the Itô and Tanaka formulas (*resp.* the results on Gaussian local times) we present here, is made with the ones proposed in [AMN01, MV05, NT06, KRT07, KR10, LN12, SV14] (*resp.* in [LN12]).

Moreover, one shows that the stochastic calculus *wrt* Gaussian processes provided in this work fully generalizes and extends the ones originally proposed in [MV05] and in [NT06], as well as it generalizes some results given in [AMN01]. In particular it generalizes results given by [MV05] by allowing one to consider highly irregular processes (even more irregular that those considered in [MV05] since their regularity may vary along the time in this work). It also extends the stochastic calculus for fractional Brownian motion provided in [EVdH03, BSØW04, Ben03a] and for multifractional Brownian motion in [LLV14, Leb13, LLVH14]. In addition, the stochastic calculus presented here offers alternative conditions to the ones required in [AMN01, MV05, NT06, KRT07, KR10, LN12] and [SV14], when one deals with stochastic calculus with respect to Gaussian processes in general.

**Keywords:** Gaussian processes and local times, stochastic analysis & White noise theory, Wick-Itô integrals, Itô and Tanaka formulas, varying regularity processes, fractional and multifractional Brownian motions.

**AMS Subject Classification:** 60G15; 60H40; 60H05; 60G22

## Contents

<b>1</b>	<b>Introduction</b>	<b>2</b>
<b>2</b>	<b>Background on White noise theory &amp; on operators <math>(M_H)_{H \in (0,1)}</math></b>	<b>9</b>
2.1	The spaces of stochastic test functions and stochastic distributions . . . . .	9
2.2	$(\mathcal{S})^*$ -process, $(\mathcal{S})^*$ -derivative and $(\mathcal{S})^*$ -integral . . . . .	11
2.3	S-transform and Wick product . . . . .	11
2.4	Family of operators $(M_H)_{H \in (0,1)}$ and a classical example of Gaussian processes in $\mathcal{G}$ . . .	12

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<b>3</b>	<b>Stochastic integral with respect to Gaussian process</b>	<b>13</b>
3.1	White Noise derivative of $G$ . . . . .	14
3.2	Generalized functionals of $G$ . . . . .	18
3.3	S-Transform of $G$ and $W^{(G)}$ . . . . .	19
3.4	Wiener integral with respect to $G$ . . . . .	19
3.5	The Wick-Itô integral with respect to Gaussian processes . . . . .	23
<b>4</b>	<b>Itô Formulas</b>	<b>27</b>
4.1	Itô Formula in $(L^2)$ for $C^{1,2}$ functions with sub-exponential growth . . . . .	28
4.2	Itô Formula for generalized functionals of $G$ . . . . .	33
4.2.1	Itô Formula in $(\mathcal{S})^*$ for generalized functionals of $G$ on an interval $[a, b]$ of $\mathcal{Z}_R^c$ . . .	34
4.2.2	Itô Formula in $(L^2)$ for certain generalized functionals of $G$ on any interval of $\mathcal{R}$ . . .	35
<b>5</b>	<b>Comparison with other stochastic integrals</b>	<b>37</b>
5.1	Comparison with Malliavin Calculus or divergence type integrals . . . . .	37
5.1.1	Comparison with divergence type integral of [AMN01] . . . . .	37
5.1.2	Comparison with the divergence type integral of [MV05] . . . . .	41
5.2	Comparison with Itô Integral . . . . .	42
<b>6</b>	<b>Tanaka formula &amp; Local times</b>	<b>46</b>
6.1	Tanaka Formula . . . . .	47
6.2	Local times of Gaussian processes in $\mathcal{G}$ . . . . .	48
<b>A</b>	<b>Appendix</b>	<b>55</b>
A.1	Bochner Integral . . . . .	55
A.2	Proof of Lemmas 6.4 and 6.5 . . . . .	56

## 1 Introduction

The purpose of this paper is to develop an anticipative stochastic calculus with respect to Gaussian process  $G := (G_t)_{t \in \mathcal{R}}$  that can be written under the form:

$$G_t = \int_{\mathbf{R}} g_t(u) dB_u, \quad (1.1)$$

where  $\mathbf{R}$  denotes the set of real numbers,  $\mathcal{R}$  denotes a closed interval of  $\mathbf{R}$  (that may be equal to  $\mathbf{R}$ ),  $B := (B_u)_{u \in \mathbf{R}}$  is Brownian motion on  $\mathbf{R}$  and  $(g_t)_{t \in \mathcal{R}}$  is a family of a square integrable functions<sup>1</sup> on  $\mathbf{R}$ . Denote  $\mathcal{G}$  the set of Gaussian processes that can be written under the form (1.1). The set  $\mathcal{G}$  obviously contains Volterra processes, as well as Gaussian Fredholm processes. For every positive real  $T$ , the process  $(V_t)_{t \in [0, T]}$  (*resp.*  $(F_t)_{t \in [0, T]}$ ) is said to be a Volterra (*resp.* a Gaussian Fredholm) process on  $[0, T]$ , if it can be written under the form:

$$V_t := \int_0^t K(t, s) dW_s; \quad \forall t \in [0, T], \quad (1.2)$$

$$\text{resp. } F_t := \int_0^T K_T(t, s) dW_s; \quad \forall t \in [0, T], \quad (1.3)$$

where  $(W_s)_{s \in [0, T]}$  is a Brownian motion and  $K$  (*resp.*  $K_T$ ) belongs to  $L^2([0, T]^2)$ . Note moreover that  $\mathcal{G}$  contains the Gaussian processes that can be written under the form:

$$H_t := \int_{-\infty}^t K(t, s) dB_s; \quad \forall t \in \mathbf{R}. \quad (1.4)$$

In view of building a white noise integral *wrt* any  $G$  in  $\mathcal{G}$ , one will have first to define a *Gaussian white noise* (*i.e.* a derivative of  $t \mapsto G_t$ ) as a Hida stochastic distribution. In addition to the

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<sup>1</sup>*i.e.* such that  $u \mapsto g_t(u)$  is measurable on  $\mathbf{R}$  and such that  $\int_{\mathbf{R}} |g_t(u)|^2 du < +\infty$ .

definition and the main properties of our stochastic integral with respect to Gaussian processes that belong to  $\mathcal{G}$ , as well as a comparison of our integral to the previous existing ones, our main results are:

- An Itô formula, that reads, for every  $T > 0$  and every  $C^{1,2}([0, T] \times \mathbf{R})$  function  $f$ , with sub exponential growth:

$$f(T, G_T) = f(0, 0) + \int_0^T \frac{\partial f}{\partial t}(t, G_t) dt + \int_0^T \frac{\partial f}{\partial x}(t, G_t) d^\circ G_t + \frac{1}{2} \int_0^T \frac{\partial^2 f}{\partial x^2}(t, G_t) dR_t,$$

where the equality holds in  $L^2(\Omega)$  and almost surely, where  $t \mapsto R_t$  denotes the variance function of  $G$ , which will be supposed to be a continuous function, of bounded variations; the meaning of the different terms will be explained below.

- Two occupation time formulas that read, for every positive real-valued Borel function  $\Phi$ :

$$\begin{aligned} \int_0^T \Phi(G_s(\omega)) ds &= \int_{\mathbf{R}} \ell_T^{(G)}(y)(\omega) \Phi(y) dy; \\ \int_0^T \Phi(G_s(\omega)) dR_s &= \int_{\mathbf{R}} \mathcal{L}_T^{(G)}(y)(\omega) \Phi(y) dy, \end{aligned}$$

where  $\ell_T^{(G)}(y)$  (*resp.*  $\mathcal{L}_T^{(G)}(y)$ ) denotes the non weighted (*resp.* the weighted) local time of  $G$  (which will be both defined precisely), at point  $y$ , up to time  $T$ . We will moreover establish that these two latter random variables are  $L^2$  random variables.

Itô stochastic calculus provides a non-anticipative stochastic integral with respect to Gaussian semimartingales. However, Itô's theory does not apply anymore when the Gaussian process considered is not a semimartingale. Two<sup>2</sup> main and parallel ways have been developed over the years to build a stochastic calculus with respect to Gaussian processes; the Itô integral *wrt* Brownian motion (for non anticipative integrands) being at the intersection of all these approaches. More precisely, one has the:

- *trajectorial or pathwise extensions*,
- *functional extensions*.

The trajectorial approach, initiated by [You36], provides generalizations of the Riemann–Stieltjes integral that are: the pathwise forward-type Riemann–Stieltjes integral (introduced in [Föl81]; see also [SV13] and references therein) and the pathwise generalized Lebesgue–Stieltjes integrals (introduced in [Zä98]). The reader interested in this approach, that also provides Itô formulas, will find in [FV10] a very complete overview. Let us also mention the stochastic calculus via regularization (see [RV07] and references therein), that is also a generalization of Itô integral. Since the *pathwise extensions* of Itô integral require, by their very definition, that the stochastic integral is built  $\omega$  by  $\omega$ , it will clearly appear that they are of a completely different nature from our definition of stochastic integral (that will be given in Definition 2 below). For this reason we will not compare, in this work, our approach to the *pathwise* ones.

Our main interest here consists in the *functional* approach. The *functional extensions* are rooted in the extension of Itô integral *wrt* Brownian motion to anticipative integrands built by Hitsuda in [Hit72, Hit78] and Skorohod in [Sko75]. This extension is therefore named Hitsuda-Skorohod integral. In [GT82] it was proved that the stochastic integral *wrt* Brownian motion and the adjoint of the derivative operator, on the Wiener space, coincide. This result led to many

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<sup>2</sup>The enlargement of filtration technique is a third method to extend Itô integral for non semimartingale (see [MY06] and references therein). However we will not discuss it in this paper since it is very rarely used in the literature.

developments in (anticipative) stochastic calculus with respect to Gaussian processes, the most significant of which is [AMN01]. This latter article provides, using Malliavin calculus, not only a divergence type integral with respect to continuous Volterra processes but also Itô formulas. In fact, all the *functional extensions* of Itô integral developed to build a stochastic integral *wrt* Gaussian processes so far, have been developed using the divergence type integral. One can divide these *functional extensions* into two groups, depending if the assumptions are made on the kernel  $K$  (first group) or on the covariance function  $R$  (second group). The first group is composed of [AMN01] and [MV05], while the second one is composed of [NT06, KRT07, KR10, LN12, SV14].

The stochastic calculus we propose in this work belongs to the first group since the set of assumptions we make will be about the kernel  $g$ ; however it does not use the divergence type integral. For this reason we will make, in this work, a systematic comparison of the stochastic integral we provide here with the ones provided in [AMN01] and in [MV05].

[MV05] provides a stochastic calculus, *wrt* a particular class of Volterra processes, that we will denote  $\mathcal{V}_\gamma$  - processes in the sequel, where  $\gamma : \mathbf{R}_+ \rightarrow \mathbf{R}$  is a deterministic function. For any deterministic function  $\gamma : \mathbf{R}_+ \rightarrow \mathbf{R}$ ,  $\mathcal{V}_\gamma$  - processes (or elements of  $\mathcal{V}_\gamma$ ) are defined in [MV05, Proposition 1] as being the processes, denoted  $\tilde{B}^\gamma := (\tilde{B}_t^\gamma)_{t \in [0, T]}$ , defined by setting:

$$\tilde{B}_t^\gamma := \int_0^t \varepsilon(t-s) dW_s; \quad \forall t \in [0, T], \quad (1.5)$$

with  $\gamma : \mathbf{R}_+ \rightarrow \mathbf{R}$  such that  $\gamma^2$  is of class  $C^2$  everywhere in  $\mathbf{R}_+$  except in 0; and such that  $(\gamma^2)'$  is non increasing. The map  $\varepsilon : \mathbf{R}_+^* \rightarrow \mathbf{R}$  is defined by setting  $\varepsilon := \sqrt{(\gamma^2)'}$ . While the set  $\mathcal{V}_\gamma$  does not contain fractional Brownian motion (fBm) (and therefore not multifractional Brownian motion either), it contains Gaussian processes that can be more irregular than any fBm. Moreover, it contains Gaussian processes that present longer-range negative interactions than any fBm with  $H < 1/2$ . More generally, than any Gaussian process defined by any given scale of almost-sure uniform continuity that is bounded below by the modulus of continuity of Brownian motion. However, by their very definition, the regularity of every  $\mathcal{V}_\gamma$  - process is constant along the time. Not only  $\mathcal{V}_\gamma$  is a subset of  $\mathcal{G}$ , but also the stochastic integral built in [MV05], *wrt* any element of  $\mathcal{V}_\gamma$  is a particular case of the Wick-Itô stochastic integral we provide in this work, as we will show further. This means that for every process  $\tilde{B}^\gamma$  in  $\mathcal{V}_\gamma$  and every stochastic process  $X$  such that the integral of  $X$  *wrt*  $\tilde{B}^\gamma$  exists in the sense defined in [MV05], our stochastic integral of  $X$  *wrt*  $\tilde{B}^\gamma$  exists. Moreover they are equal. This will allow us to deal with non continuous Gaussian processes, as it is the case in [MV05]. In addition, our set of Gaussian processes  $\mathcal{G}$  does not only contain Gaussian processes with fixed regularity (such as  $\mathcal{V}_\gamma$  - processes) but also Gaussian processes that have a regularity which varies along the time; and of course, the stochastic calculus provided in this paper applies to them. Besides, in [MV05] the variance function of elements in  $\mathcal{V}_\gamma$ , noted  $t \mapsto R_t$  in this paper, is assumed to be of class  $C^2$  on  $\mathbf{R}_+^*$ . We will not assume such a regularity here, for elements in  $\mathcal{G}$ . We will assume here that  $t \mapsto R_t$  is continuous and of bounded variations only in Sections 4 and 6 (in order to derive Itô formulas and results on local times of elements of  $\mathcal{G}$ ).

An another advantage of the stochastic integral we build in this work is that one does not have to extend it, once it is built, in order that the set of integrands is not empty (or not too small), as it is the case for divergence type integral. Indeed, as it has been noticed in [MV05, Remark 25 & p.407], it happens that the Gaussian process *wrt* which the divergence type integral is defined is not itself integrable (*e.g.* the fBm, for  $H \leq 1/4$  in [AMN01], or  $\tilde{B}^\gamma$ , when  $\gamma(r) \geq r^{1/4}$  in [MV05]); the same phenomenon happens also in [LN12] and in [SV14]. Moreover, in these two latter cases, the extended domain and the initial one are not comparable (see [LN12, p. 383]). This is a serious drawback, and especially when one wants to have at one's disposal an

Itô formula. [CN05] provides a method to extend the divergence type integral *wrt* fBm. Using the same ideas as the ones provided in [CN05], the authors of [MV05] extended the divergence type integral *wrt*  $\mathcal{V}_\gamma$  - processes as well. However, in these two particular cases, the authors had to require more regularity on the function  $f$ , to provide an Itô formula. Indeed, in both Itô formulas provided in these papers (that are [CN05, Lemma 4.3] and [MV05, Theorem 31]), the function with sub exponential growth, denoted  $f$ , is assumed to be of class  $C^\infty$ . Moreover, all its derivatives are assumed to have a sub-exponential growth. The stochastic integral we build in this work avoids this disadvantage and the Itô formula we provide here is, at our best knowledge, the most general one for general Gaussian processes that are not semimartingales. Note moreover that a general way to extend the divergence integral for Volterra processes, assuming it exists, has been provided in [LN05]. However Itô formula for extended divergence integral has not been provided in the same time for general Volterra processes, if one excepts [MV05]. Besides, while the stochastic calculus developed in [MV05] provided a Tanaka formula it seemed so far that the intrinsic method (*i.e.* the stochastic calculus we called of the second group above) is more suitable to provide ones with occupation formula for Gaussian local times (as it has been done in [LN12] for weighted local times). The stochastic calculus we provide here allows us to get both, Itô and Tanaka formulas, as well as occupation formulas for local times of any  $G$  in  $\mathcal{G}$ .

More importantly, in order to define the divergence integral of a continuous Gaussian process  $G$  in the way of [AMN01] and then of [MV05], it is essential to first know a representation of  $G$  on a compact set of the form  $[0, T]$ . In general, [HH93, Theorem 4.1] ensures that any Gaussian process may be written as a sum of two terms; one of them being  $\sum_{i=1}^N \int_0^t K_i(t, u) dW_i(u)$ , where  $N$  is possibly infinite and  $W$  is a Brownian motion. However it is not an easy task to obtain such a decomposition for a given process  $G$ . For instance, although a kernel is known for fBm, this is not the case of bifractional motion [HV03]. Likewise, writing the moving average and harmonizable multifractional Brownian motion in this form remains an open problem (see [LLVH14, Section 5] for more details). Moreover, as it has been noticed in [SV14, Ex. 3.3], Gaussian bridges in general are an example of Gaussian processes which do not admit “proper” Volterra representation (*i.e.* that can not be written under the form (1.2)). We will see that the stochastic calculus provided in this paper allows one to deal with this kind of processes<sup>3</sup>.

We therefore understands that there is no hope to use [AMN01] nor [MV05] in order to build a stochastic integral *wrt* Gaussian processes of the form (1.1), for which one does not know any integral representation on a compact set included in  $[0, T]$ . One then might consider to use the stochastic calculus *wrt* Gaussian processes developed in [NT06], [KRT07], [KR10], [LN12] or [SV14]. As we stated above, by the very definition of the stochastic calculus provided in [NT06, KRT07, KR10, LN12, SV14], one needs that the covariance function fulfills some requirements. However, it happens sometimes that one has to deal with Gaussian processes, given under the form (1.1), for which one does not know how to compute the covariance function, such as the one where  $g_t$  is defined by setting  $g_t(u) = \mathbf{1}_{[0,t]}(u) K_{h(t)}(t, u)$ , where  $h : [0, T] \rightarrow (0, 1)$  is a continuous deterministic function and where the family of Kernel  $(K_H)_{H \in (0,1)}$  is the one defined in [Nua06, (5.8)] in the case where  $H \in (1/2, 1)$  and in [Nua06, Proposition 5.1.3] in the case where  $H \in (0, 1/2)$ . Nevertheless, let us see briefly what has been done, and obtained, in these latter references.

In [NT06], a stochastic integral *wrt* continuous Gaussian processes is defined, again based on Malliavin calculus. However, the class of Gaussian processes *wrt* which a stochastic integral is

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<sup>3</sup>[SV14, Section 1] also noticed that Theorem 4.1 of [HH93] requires that the Gaussian process considered is purely non deterministic, which excludes some basic Gaussian processes such as the one given in [SV14, Example 3.1], for which the stochastic calculus we provide here also applies.

defined is a little bit restrictive since it requires, among other things, two conditions on the sums of the square of the covariance of the increments of the Gaussian process considered (see [NT06, (2), (3) & (4)]). As a consequence of this definition of stochastic integral, the Itô formula derived from [NT06] requires that  $f$  is a function of class  $C^7$ . This can be a serious drawback when one deals with not so much regular functions. Moreover, while the Wick product (which is a key tool in the White Noise theory) is used to define a stochastic integral in [NT06], the space of stochastic distributions (which is the natural set on which one can use Wick product) is not used at all. Therefore, one can not use the flexibility of stochastic distributions (*i.e.* White Noise theory) to obtain a less restrictive Itô formula, nor a Tanaka formula, nor an integral representation of local times *wrt*  $G$ . This latter is crucial to derive occupation time formulas, as we will show in this work.

The work provided in [KRT07] offers an alternative to the previous methods to build a stochastic integral *wrt* continuous Gaussian<sup>4</sup> processes, for which one knows the covariance function. Introducing the concept of covariance measure structure, the authors built and developed a stochastic calculus *wrt* “regular” processes (such as fBm with  $H \geq 1/2$ ). This work has been extended to the “singular” case in [KR10]. This approach is particularly suitable when the kernel is not explicitly known, under any of the representations (1.1) to (1.4), (like in the case of bifractional motion [HV03]). However, as we will show further, the conditions required to establish Itô formula in [KR10] are quite restrictive (since one of the requirements is that  $f$  is of class  $C^\infty$ ). The conditions required in [KRT07] are not so restrictive but they do not allow one to deal with “irregular” Gaussian processes, by the very essence of covariance measure structure. [LN12] provides an intrinsic stochastic calculus (requiring conditions only on the covariance function  $R$ ) for continuous Gaussian processes, based on divergence integral. In addition to Itô & Tanaka formulas, a proof that weighted local times of Gaussian processes considered are  $L^2$  random variables is provided. We extend most of these three results in this work.

Finally, [SV14] also provides a stochastic calculus, but *wrt* to Gaussian Fredholm processes (defined by (1.3)), that are not necessarily continuous. The authors do so by using a transfer principle. They make alternative assumptions to the ones we do in this work, to derive their Itô formula. If the assumptions made in [SV14] generalize the ones required in [LN12], no Tanaka nor local times properties are derived. Moreover, contrary to [LN12], that does not use any representation of the Gaussian process considered, [SV14] uses the operator  $K_T$  (in fact “its adjoint”, denoted  $K_T^*$ ) to build the stochastic integral *wrt* Gaussian process. This makes the method employed in [SV14] to build a stochastic calculus, in between intrinsic and “non intrinsic” (*i.e.* in between groups 1 and 2 described above). Moreover, even if one knows that  $G$  admits a Fredholm representation, finding explicitly the operator  $K_T$  or  $K_T^*$  (which is useful if one wants to have at least an idea of the domain of the divergence) seems not to be an easy task.

We want to build here a stochastic integral *wrt* Gaussian processes in  $\mathcal{G}$ , that is a functional extensions of Itô integral, and develop a stochastic calculus, with the biggest generality. Because we wanted that our stochastic calculus avoids all the drawbacks inherent in the previously mentioned methods, one had to find another way to do so. The resulting stochastic calculus is carried out within the framework of the White Noise Theory. Moreover, it is an extension to general Gaussian processes of the one built, in the recent years, in order to define a stochastic calculus *wrt* fBm in [EVdH03, BSØW04, Ben03a] and for mBm in [LLV14, Leb13, LLVH14]. In fact, the stochastic calculus with respect to Gaussian processes we provide here is the only one available when the Gaussian process can be written under form (1.1) but not under any of the form (1.2) to (1.4); or when the stochastic calculus provided in [KRT07, KR10, LN12] does not

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<sup>4</sup>as well as non Gaussian

apply. Moreover it provides new results on Gaussian local times as well as improvements of Itô formulas provided in [AMN01, CN05, MV05, NT06, KR10, LN12], by allowing one to have less restrictive hypotheses than the ones required in those references. It also improves the Itô formula given in [SV14], by first extending the class of processes for which one can provide an Itô formula; and second by offering alternative assumptions than the one required in [SV14]. It also offers extension of the results obtained for local times in [LN12]. Moreover, the stochastic integral we provide here avoids the drawback of having to be extended, as it might be the case in [AMN01, MV05, CN05, LN12, SV14] - to have a “reasonable” set of integrands, that includes in particular the Gaussian process *wrt* which one builds the stochastic integral. Besides, it is interesting to note that White Noise theory is a strong enough theory to allow one to develop a stochastic calculus that provides Itô, Tanaka and occupation times formulas within the same framework.

White Noise Theory is roughly speaking, the stochastic analogous of deterministic generalized functions (also known as tempered distributions). More precisely, White Noise Theory has been introduced by T.Hida in [Hid75]. The idea is to realize nonlinear functional on a Hilbert space as functions of white noise (which is defined as being the time derivative of Brownian motion). White Noise theory has now many application fields, such as quantum dynamics, quantum field theory, molecular biology, mathematical finance, among many others (see [HS04] for more details). One can find very good introductions (and more!) to White Noise Theory in [HKPS93, Kuo96, Si12] (see also references therein). One may also refer to [HOUZ10] for the study, in the white noise theory’s framework, of stochastic differential equations as well as stochastic partial differential equations. It is also notable that the stochastic integration developed from White Noise theory does not only allow us to solve, theoretically, certain stochastic differential equations (SDEs) but also lends itself well to numerical approximation of solutions of these SDEs (see [CLV14] for an example, in a mathematical finance context, of numerical approximations of solution of some linear SDEs, using the functional quantization).

## Gaussian Processes in $\mathcal{G}$ of “reference”

We will consider throughout this paper, a running example. The interest of this example is to give a concrete idea of the way the stochastic calculus we present in this paper works and to see in what extent it generalizes the stochastic calculus *wrt* Gaussian processes, provided in the literature that already exists. The Gaussian processes treated in this running example, which all belongs to  $\mathcal{G}$ , are: Brownian motion and Brownian bridge, fractional and multifractional Brownian motions as well as  $\mathcal{V}_\gamma$  - processes, that we have already presented above. For notational simplicity we will call, in the sequel of this paper, these processes the Gaussian processes of “reference”. Of course our aim, with this running example, is not to define a stochastic integral with respect to any of these Gaussian processes of “reference” since this has already been done in [EVdH03, BSØW04, Ben03a] for fBm, in [MV05] for  $\mathcal{V}_\gamma$  - processes and in [LLV14, Leb13, LLVH14] for mBm. Our aim is rather to show how the stochastic calculus we provide in this work, which is built on a very simple set of assumptions (given page 14), allows one to first recover easily the results provided in these previously cited articles and, second to improve these results in many places.

### *Fractional and Multifractional Brownian motions*

We only give here a presentation of fBm and mBm that is limited to the strict minimum. The reader interested in details about these two processes may refer, for fBm (*resp.* for mBm), to [Nua06] (*resp.* to [LLVH14]) and references therein. Introduced for the first time in [Kol40], and popularized in [MVN68], fBm is a centered Gaussian process, the covariance function of which is denoted  $R_H$  and is given by:

$$R_H(t, s) := \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t - s|^{2H}),$$

where  $H$  belongs to  $(0, 1)$ , and is usually called the Hurst exponent. Moreover, when  $H = 1/2$ , fBm reduces to standard Brownian motion. Among many other properties, fBm is able to match any prescribed constant local regularity and to model phenomena that presents long range dependence. These properties made this process very popular in many fields such as mathematical finance, Internet traffic modeling, image analysis and synthesis, physics and more. mBm, which is a Gaussian extension of fBm, was introduced in [PLV95] and in [BJR97] in order to match any prescribed non-constant deterministic local regularity and to decouple this property from long range dependence (this impossibility of doing so for fBm constitutes one of the drawbacks of this process). To obtain mBm, the idea is to replace the constant Hurst parameter  $H$  of fBm by a deterministic function  $t \mapsto h(t)$  ranging in  $(0, 1)$ . Several definitions of mBm exist and the reader interested in the evolution of these definitions may refer to [LLVH14] and references therein. We will only give here the definition of mBm given in [LLVH14, Definition 1.2], which is not only the most recent but also includes all previously known ones. A mBm on  $\mathbf{R}$ , with functional parameter  $h : \mathbf{R} \rightarrow (0, 1)$ , is a Gaussian process  $B^h := (B_t^h)_{t \in \mathbf{R}}$  defined, for all real  $t$ , by  $B_t^h := \mathbf{B}(t, h(t))$ , where  $\mathbf{B} := (\mathbf{B}(t, H))_{(t, H) \in \mathbf{R} \times (0, 1)}$  is fractional Brownian field on  $\mathbf{R} \times (0, 1)$  (which means that  $\mathbf{B}$  is a Gaussian field, such that, for every  $H$  in  $(0, 1)$ , the process  $(\mathbf{B}(t, H))_{t \in \mathbf{R}}$  is a fBm with Hurst parameter  $H$ ). In other words, a mBm is simply a “path” traced on a fractional Brownian field. Note also that when  $h$  is constant, mBm reduces to fBm. An explicit construction of an fBm and of a mBm, given under the form (1.1), will be presented at Example 2.8. Besides, one may refer to [ST94, AMN01, BDM10] to get precisions on the various integral representations of fBm and mBm that exist. The literature on Stochastic integration *wrt* fBm is extensive now. The reader interested in an exhaustive overview of the subject may refer to [Nua06, Cou07, Mis08] and references therein. One may also find treated in [Ben03a, BSØW04, EVdH03, Nua05], the particular case of stochastic integral *wrt* fBm, in the white noise theory framework.

More recent, the literature on Stochastic integration *wrt* mBm is less rich. Nevertheless, one may cite [BDM10] for a divergence type integral *wrt* to a Volterra-type mBm and [LLV14, Leb13] for a Wick-Itô multifractional integral (*i.e.* an integral *wrt* to normalized mBm, in the White Noise theory framework). Note moreover that [LLVH14] provides a general method of integration *wrt* to all classes of mBm, that does not only apply for divergence type integral and white noise theory integral but also for pathwise integral.

#### *Outline of the paper*

The remaining of this paper is organized as follows. In Section 2, we recall basic facts about white noise theory and about the family of operators  $(M_H)_{H \in (0, 1)}$ , which is instrumental for our running example. Let  $G := (G_t)_{t \in \mathcal{R}}$  be a Gaussian process that belongs to  $\mathcal{G}$ . In Section 3 one first defines the time derivative of  $G$ , in weak sense. We then define a stochastic integral *wrt*  $G$  and give its main properties. Various instances of Itô formula are given in Section 4, in both  $L^2(\Omega)$  and stochastic distributions senses. The main result, which is Theorem 4.4 that provides an Itô formula in  $L^2(\Omega)$ , is given in Subsection 4.1. This latter subsection ends with a complete comparison of Theorem 4.4 with all the Itô formulas for Gaussian processes, provided so far in the literature of *functional extensions* of itô integral (that are: [AMN01, Theorems 1 & 2], [MV05, Theorem 31], [NT06, Theorem 1], [KRT07, Corollary 8.13], [KR10, Proposition 11.7], [LN12, Theorem 3.2] and [SV14, Theorems 4.4 & 4.5]. Subsection 4.2 is devoted to the obtention of an Itô formula in the stochastic distributions sense. This result is a preparatory work for the study of local times provided in Section 6. In Section 5, we compare our stochastic integral with respect to elements of  $\mathcal{G}$ , to some of the ones that already exist. More precisely,

to the divergence type integrals, provided in [AMN01] and in [MV05], in Subsection 5.1 and to Itô integral in Subsection 5.2. In particular, we show therein how our integral fully generalizes the one built in [MV05]. Finally one provides in Section 6 a Tanaka formula, a definition of weighted and non weighted local times of Gaussian processes that belong to  $\mathcal{G}$ , as well as an integral representation of them. An occupation time formula for both, weighted and non weighted, local times of  $G$  is also established and constitutes the second main result of this paper. Finally we prove that both, weighted and non weighted local times of  $G$ , belong, as two parameters processes, to  $L^2(\mathbf{R} \times \Omega, \lambda \otimes \mu)$ , where  $\lambda$  denotes the Lebesgue measure on  $\mathbf{R}$ . We then compare the results we obtained to the ones provided in [LN12] for weighted local times.

## 2 Background on White noise theory & on operators $(M_H)_{H \in (0,1)}$

We recall in this section the standard set-up for classical white-noise theory. Readers interested in more details about White Noise Theory may refer *e.g.* to [HKPS93, Kuo96] and [Si12].

### 2.1 The spaces of stochastic test functions and stochastic distributions

Define  $\mathbf{N}$  (resp.  $\mathbf{N}^*$ ) the set of non negative integers (resp. positive integers). Let  $\mathcal{S}(\mathbf{R})$  be the Schwartz space endowed with its usual topology (*i.e.* a family of functions  $(f_n)_{n \in \mathbf{N}}$  of  $\mathcal{S}(\mathbf{R})^{\mathbf{N}}$  is said to converge to 0 if for all  $(p, q)$  in  $\mathbf{N}^2$  we have  $\lim_{n \rightarrow +\infty} \sup\{|x^p f_n^{(q)}(x)|; x \in \mathbf{R}\} = 0$ ).

Denote  $\mathcal{S}'(\mathbf{R})$  the space of tempered distributions, which is the dual space of  $\mathcal{S}(\mathbf{R})$ , and  $\widehat{F}$  or  $\mathcal{F}(F)$  the Fourier transform of any element  $F$  of  $\mathcal{S}'(\mathbf{R})$ . For every positive real  $p$ , denote  $L^p(\mathbf{R})$  the set of measurable functions  $f$  such that  $\int_{\mathbf{R}} |f(u)|^p du < +\infty$ . When  $f$  belongs to  $L^1(\mathbf{R})$ ,  $\widehat{f}$  is defined on  $\mathbf{R}$  by setting  $\widehat{f}(\xi) := \int_{\mathbf{R}} e^{-ix\xi} f(x) dx$ . Define the measurable space  $(\Omega, \mathcal{F})$  by setting  $\Omega := \mathcal{S}'(\mathbf{R})$  and  $\mathcal{F} := \mathcal{B}(\mathcal{S}'(\mathbf{R}))$ , where  $\mathcal{B}$  denotes the  $\sigma$ -algebra of Borel sets. The Bochner-Minlos theorem ensures that there exists a unique probability measure  $\mu$  on  $(\Omega, \mathcal{F})$  such that, for every  $f$  in  $\mathcal{S}(\mathbf{R})$ , the map  $\langle \cdot, f \rangle: (\Omega, \mathcal{F}) \rightarrow \mathbf{R}$  defined by  $\langle \cdot, f \rangle(\omega) = \langle \omega, f \rangle$  (where  $\langle \omega, f \rangle$  is by definition  $\omega(f)$ , *i.e.* the action of  $\omega$  on  $f$ ) is a centred Gaussian random variable with variance equal to  $\|f\|_{L^2(\mathbf{R})}^2$  under  $\mu$ . The map  $f \mapsto \langle \cdot, f \rangle$  being an isometry from  $(\mathcal{S}(\mathbf{R}), \langle \cdot, \cdot \rangle_{L^2(\mathbf{R})})$  to  $(L^2(\Omega, \mathcal{F}, \mu), \langle \cdot, \cdot \rangle_{L^2(\Omega, \mathcal{F}, \mu)})$ , it may be extended to  $L^2(\mathbf{R})$ . One may thus consider the centred Gaussian random variable  $\langle \cdot, f \rangle$ , for any  $f$  in  $L^2(\mathbf{R})$ . In particular, let  $t$  be in  $\mathbf{R}$ , the indicator function  $\mathbf{1}_{[0,t]}$  is defined by setting:

$$\mathbf{1}_{[0,t]}(s) := \begin{cases} 1 & \text{if } 0 \leq s \leq t, \\ -1 & \text{if } t \leq s \leq 0 \text{ except if } t = s = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then the process  $(\widetilde{B}_t)_{t \in \mathbf{R}}$ , where  $\widetilde{B}_t(\omega) := \widetilde{B}(t, \omega) := \langle \omega, \mathbf{1}_{[0,t]} \rangle$  is a standard Brownian motion with respect to  $\mu$ . It then admits a continuous version which will be denoted  $B$ . Define, for  $f$  in  $L^2(\mathbf{R})$ ,  $I_1(f)(\omega) := \langle \omega, f \rangle$ . Then  $I_1(f)(\omega) = \int_{\mathbf{R}} f(s) dB_s(\omega)$   $\mu$ -a.s., where  $\int_{\mathbf{R}} f(s) dB_s$  denotes the Wiener integral of  $f$ .

For every  $n$  in  $\mathbf{N}$ , let  $e_n(x) := (-1)^n \pi^{-1/4} (2^n n!)^{-1/2} e^{x^2/2} \frac{d^n}{dx^n} (e^{-x^2})$  be the  $n$ -th Hermite function. It is well known (see [Tha93]) that  $(e_k)_{k \in \mathbf{N}}$  is a family of functions of  $\mathcal{S}(\mathbf{R})$  that forms an orthonormal basis of  $L^2(\mathbf{R}, dt)$ . The following properties about the Hermite functions (the proof of which can be found in [Tha93]) will be useful.

**Theorem 2.1.** *There exist positive constants  $C$  and  $\gamma$  such that, for every  $k$  in  $\mathbf{N}$ ,*

$$|e_k(x)| \leq \begin{cases} C (k+1)^{-1/12} & \text{if } |x| \leq 2\sqrt{k+1}, \\ C e^{-\gamma x^2} & \text{if } |x| > 2\sqrt{k+1}. \end{cases}$$

Let  $(\|\cdot\|_p)_{p \in \mathbf{Z}}$  be the family norms defined by  $\|f\|_p^2 := \sum_{k=0}^{+\infty} (2k+2)^{2p} \langle f, e_k \rangle_{L^2(\mathbf{R})}^2$ , for all  $(p, f)$  in  $\mathbf{Z} \times L^2(\mathbf{R})$ . The operator  $A$ , defined on  $\mathcal{S}(\mathbf{R})$ , by setting  $A := -\frac{d^2}{dx^2} + x^2 + 1$ , admits the sequence  $(e_n)_{n \in \mathbf{N}}$  as eigenfunctions and the sequence  $(2n+2)_{n \in \mathbf{N}}$  as eigenvalues. It is then easily seen that  $A$  is invertible and that  $\|f\|_p = |A^p f|_0$ , for every  $f \in \mathcal{S}(\mathbf{R})$  and  $p$  in  $\mathbf{Z}$ . Besides, define, for  $p$  in  $\mathbf{N}$ , the spaces  $\mathcal{S}_p(\mathbf{R}) := \{f \in L^2(\mathbf{R}), \|f\|_p < +\infty\}$  and  $\mathcal{L}_p(\mathbf{R})$  as being the completion of  $L^2(\mathbf{R})$  with respect to the norm  $\|\cdot\|_{-p}$ . It is well known (see [Kuo96]) that the Schwartz space  $\mathcal{S}(\mathbf{R})$  is the projective limit of the sequence  $(\mathcal{S}_p(\mathbf{R}))_{p \in \mathbf{N}}$  and that the space  $\mathcal{S}'(\mathbf{R})$  of tempered distributions is the inductive limit of the sequence  $(\mathcal{L}_p(\mathbf{R}))_{p \in \mathbf{N}}$ . This means that we have the equalities  $\mathcal{S}(\mathbf{R}) = \bigcap_{p \in \mathbf{N}} \mathcal{S}_p(\mathbf{R})$  (resp.  $\mathcal{S}'(\mathbf{R}) = \bigcup_{p \in \mathbf{N}} \mathcal{L}_p(\mathbf{R})$ ) and that convergence in  $\mathcal{S}(\mathbf{R})$  (resp. in  $\mathcal{S}'(\mathbf{R})$ ) means convergence in  $\mathcal{S}_p(\mathbf{R})$  for every  $p$  in  $\mathbf{N}$  (resp. convergence in  $\mathcal{L}_p(\mathbf{R})$  for some  $p$  in  $\mathbf{N}$ ).

The space of stochastic test functions and distributions is built from Frechet spaces in a way analogous to the one  $\mathcal{S}'(\mathbf{R})$  is built from spaces  $(\mathcal{L}_p(\mathbf{R}))_{p \in \mathbf{N}}$ . More precisely, let  $(L^2)$  denote the space  $L^2(\Omega, \mathcal{G}, \mu)$ , where  $\mathcal{G}$  is the  $\sigma$ -field generated by  $\langle \cdot, f \rangle_{f \in L^2(\mathbf{R})}$ . According to Wiener-Itô's theorem, for every random variable  $\Phi$  in  $(L^2)$  there exists a unique sequence  $(f_n)_{n \in \mathbf{N}}$  of functions in  $\widehat{L}^2(\mathbf{R}^n)$  such that  $\Phi$  can be decomposed as  $\Phi = \sum_{n=0}^{+\infty} I_n(f_n)$ , where  $\widehat{L}^2(\mathbf{R}^n)$  denotes the set of all symmetric functions  $f$  in  $L^2(\mathbf{R}^n)$  and  $I_n(f)$  denotes the  $n$ -th multiple Wiener-Itô integral of  $f$  with the convention that  $I_0(f_0) = f_0$  for constants  $f_0$ . Moreover the equality  $\mathbf{E}[\Phi^2] = \sum_{n=0}^{+\infty} n! \|f_n\|_{L^2(\mathbf{R}^n)}^2$  holds, where  $\mathbf{E}$  denotes the expectation with respect to  $\mu$ . For any  $\Phi := \sum_{n=0}^{+\infty} I_n(f_n)$  satisfying the condition  $\sum_{n=0}^{+\infty} n! |A^{\otimes n} f_n|_0^2 < +\infty$ , define the element  $\Gamma(A)(\Phi)$  of  $(L^2)$  by  $\Gamma(A)(\Phi) := \sum_{n=0}^{+\infty} I_n(A^{\otimes n} f_n)$ , where  $A^{\otimes n}$  denotes the  $n$ -th tensor power of the operator  $A$  (see [Jan97, Appendix E] for more details about tensor products of operators). The operator  $\Gamma(A)$  is densely defined on  $(L^2)$ . It is invertible and its inverse  $\Gamma(A)^{-1}$  is bounded. We note, for  $\varphi$  in  $(L^2)$ ,  $\|\varphi\|_0^2 := \|\varphi\|_{(L^2)}^2$ . For  $n$  in  $\mathbf{N}$ , let  $\mathbb{D}\text{om}(\Gamma(A)^n)$  be the domain of the  $n$ -th iteration of  $\Gamma(A)$ . Define the family of norms  $(\|\cdot\|_p)_{p \in \mathbf{Z}}$  by:

$$\|\Phi\|_p := \|\Gamma(A)^p \Phi\|_0 = \|\Gamma(A)^p(\Phi)\|_{(L^2)}, \quad \forall p \in \mathbf{Z}, \quad \forall \Phi \in (L^2) \cap \mathbb{D}\text{om}(\Gamma(A)^p).$$

For  $p$  in  $\mathbf{N}$ , define  $(\mathcal{S}_p) := \{\Phi \in (L^2) : \Gamma(A)^p(\Phi) \text{ exists and belongs to } (L^2)\}$  and define  $(\mathcal{S}_{-p})$  as the completion of the space  $(L^2)$  with respect to the norm  $\|\cdot\|_{-p}$ . As in [Kuo96], we let  $(\mathcal{S})$  denote the projective limit of the sequence  $((\mathcal{S}_p))_{p \in \mathbf{N}}$  and  $(\mathcal{S})^*$  the inductive limit of the sequence  $((\mathcal{S}_{-p}))_{p \in \mathbf{N}}$ .

The space  $(\mathcal{S})$  is called the space of stochastic test functions and  $(\mathcal{S})^*$  the space of Hida distributions. One can show that, for any  $p$  in  $\mathbf{N}$ , the dual space  $(\mathcal{S}_p)^*$  of  $\mathcal{S}_p$  is  $(\mathcal{S}_{-p})$ . Thus we will only write  $(\mathcal{S}_{-p})$  in the sequel, to denote the space  $(\mathcal{S}_p)^*$ . Note also that  $(\mathcal{S})^*$  is the dual space of  $(\mathcal{S})$ . We will note  $\ll, \gg$  the duality bracket between  $(\mathcal{S})^*$  and  $(\mathcal{S})$ . If  $\phi, \Phi$  belong to  $(L^2)$ , then we have the equality  $\ll \Phi, \phi \gg = \langle \Phi, \phi \rangle_{(L^2)} = \mathbf{E}[\Phi \phi]$ . Besides, denote  $\langle, \rangle$  the duality bracket between  $\mathcal{S}'(\mathbf{R})$  and  $\mathcal{S}(\mathbf{R})$  and recall that every tempered distribution  $F$  can be written as  $F = \sum_{n=0}^{+\infty} \langle F, e_n \rangle e_n$ , where the convergence holds in  $\mathcal{S}'(\mathbf{R})$ . The next proposition, that will be used extensively in the sequel, is a consequence of the definition of  $(\mathcal{S})$  and  $(\mathcal{S})^*$ .

**Proposition 2.2.** *Let  $F$  be in  $\mathcal{S}'(\mathbf{R})$ . Define  $\langle \cdot, F \rangle := \sum_{n=0}^{+\infty} \langle F, e_n \rangle \langle \cdot, e_n \rangle$ . Then there exists  $p_0$  in  $\mathbf{N}$  such that  $\langle \cdot, F \rangle$  belongs to  $(\mathcal{S}_{-p_0})$ , and hence to  $(\mathcal{S})^*$ . Moreover we have  $\|\langle \cdot, F \rangle\|_{-p_0}^2 = |F|_{-p_0}^2$ . Conversely, define  $\Phi := \sum_{n=0}^{+\infty} b_n \langle \cdot, e_n \rangle$ , where  $(b_n)_{n \in \mathbf{N}}$  belongs to  $\mathbf{R}^{\mathbf{N}}$ . Then  $\Phi$  belongs to  $(\mathcal{S})^*$  if and only if there exists an integer  $p_0$  in  $\mathbf{N}$  such that  $\sum_{n=0}^{+\infty} b_n^2 (2n+2)^{-2p_0} < +\infty$ . In this latter case  $F := \sum_{n=0}^{+\infty} b_n e_n$  belongs to  $\mathcal{L}_{p_0}(\mathbf{R})$  and then to  $\mathcal{S}'(\mathbf{R})$ . It moreover verifies the equality  $|F|_{-p_0}^2 = \sum_{n=0}^{+\infty} b_n^2 (2n+2)^{-2p_0} = \|\Phi\|_{-p_0}^2$ .*

## 2.2 $(\mathcal{S})^*$ -process, $(\mathcal{S})^*$ -derivative and $(\mathcal{S})^*$ -integral

Denote  $\mathcal{B}(\mathbf{R})$  the Borelian  $\sigma$ -field on  $\mathbf{R}$  and  $m$  a measure on  $\mathcal{B}(\mathbf{R})$  such that  $(\mathbf{R}, \mathcal{B}(\mathbf{R}), m)$  is a sigma-finite measure space. Through this section,  $I$  denotes an element of  $\mathcal{B}(\mathbf{R})$ .

A measurable function  $\Phi : I \rightarrow (\mathcal{S})^*$  is called a stochastic distribution process, or an  $(\mathcal{S})^*$ -process. An  $(\mathcal{S})^*$ -process  $\Phi$  is said to be differentiable at  $t_0 \in I$  if  $\lim_{r \rightarrow 0} r^{-1} (\Phi_{t_0+r} - \Phi_{t_0})$  exists in  $(\mathcal{S})^*$ .

We note  $\frac{d\Phi_{t_0}}{dt}$  the  $(\mathcal{S})^*$ -derivative at  $t_0$  of the stochastic distribution process  $\Phi$ .  $\Phi$  is said to be differentiable over  $I$  if it is differentiable at every  $t_0$  of  $I$ . More generally, for every  $k$  in  $\mathbf{N}$ ,  $\Phi$  is said to be  $C^k$  in  $(\mathcal{S})^*$  if the process  $\Phi : I \rightarrow (\mathcal{S})^*$  is  $C^k$ .

It is also possible to define an  $(\mathcal{S})^*$ -valued integral in the following way (one may refer to [Kuo96, p.245-246] or [HP57, Def. 3.7.1 p.77] for more details).

**Theorem-Definition 2.1 (integral in  $(\mathcal{S})^*$ ).** *Assume that  $\Phi : I \rightarrow (\mathcal{S})^*$  is weakly in  $L^1(I, m)$ , i.e. assume that for all  $\varphi$  in  $(\mathcal{S})$ , the mapping  $u \mapsto \ll \Phi_u, \varphi \gg$ , from  $I$  to  $\mathbf{R}$ , belongs to  $L^1(I, m)$ . Then there exists a unique element in  $(\mathcal{S})^*$ , noted  $\int_I \Phi_u m(du)$ , such that, for all  $\varphi$  in  $(\mathcal{S})$ ,*

$$\ll \int_I \Phi(u) m(du), \varphi \gg = \int_I \ll \Phi_u, \varphi \gg m(du).$$

We say in this case that  $\Phi$  is  $(\mathcal{S})^*$ -integrable on  $I$  (with respect to the measure  $m$ ), in the *Pettis sense*. In the sequel, when we do not specify a name for the integral (*resp.* for the measure  $m$ ) of an  $(\mathcal{S})^*$ -integrable process  $\Phi$  on  $I$ , we always refer to the integral in Pettis' sense (*resp.* to the Lebesgue measure).

## 2.3 S-transform and Wick product

A presentation of S-transform and Wick product may be found in [Kuo96]. For  $f$  in  $L^2(\mathbf{R})$ , define the *Wick exponential* of  $\langle \cdot, f \rangle$ , noted  $: e^{\langle \cdot, f \rangle} :$ , as the  $(L^2)$  random variable equal to  $e^{\langle \cdot, f \rangle - \frac{1}{2} |f|_0^2}$ . The *S-transform* of an element  $\Phi$  of  $(\mathcal{S})^*$ , noted  $S(\Phi)$ , is defined as the function from  $\mathcal{S}(\mathbf{R})$  to  $\mathbf{R}$  given by  $S(\Phi)(\eta) := \ll \Phi, : e^{\langle \cdot, \eta \rangle} : \gg$  for any  $\eta$  in  $\mathcal{S}(\mathbf{R})$ . For any  $(\Phi, \Psi) \in (\mathcal{S})^* \times (\mathcal{S})^*$ , there exists a unique element of  $(\mathcal{S})^*$ , called the *Wick product* of  $\Phi$  and  $\Psi$ , and noted  $\Phi \diamond \Psi$ , such that  $S(\Phi \diamond \Psi)(\eta) = S(\Phi)(\eta) S(\Psi)(\eta)$  for every  $\eta$  in  $\mathcal{S}(\mathbf{R})$ . Note that, when  $\Phi$  belongs to  $(L^2)$ ,  $S\Phi(\eta)$  is nothing but  $\mathbf{E}[\Phi : e^{\langle \cdot, \eta \rangle} :] = e^{-\frac{1}{2} |\eta|_0^2} \mathbf{E}[\Phi e^{\langle \cdot, \eta \rangle}]$ . The following result, the proof of which can be found in [LLV14, Lemma 2.3.], will be intensively used in the sequel.

**Lemma 2.3.** *For any  $(p, q)$  in  $\mathbf{N}^2$  and  $(X, Y)$  in  $(\mathcal{S}_{-p}) \times (\mathcal{S}_{-q})$ ,*

$$|S(X \diamond Y)(\eta)| \leq \|X\|_{-p} \|Y\|_{-q} e^{|\eta|_{\max\{p, q\}}^2}.$$

Some useful properties of S transforms are listed in the proposition below. The proof of the results stated in this proposition can be found in [Kuo96, Chap 5].

**Proposition 2.4** (Some properties of S transforms). **1.** *When  $\Phi$  is deterministic then  $\Phi \diamond \Psi = \Phi \Psi$ , for all  $\Psi$  in  $(\mathcal{S})^*$ . Moreover, let  $(X_t)_{t \in \mathbf{R}}$  be a Gaussian process and let  $\mathcal{H}$  be the subspace of  $(L^2)$  defined by  $\mathcal{H} := \overline{\text{span}_{\mathbf{R}} \{X_t; t \in \mathbf{R}\}}^{(L^2)}$ . If  $U$  and  $V$  are two elements of  $\mathcal{H}$  then  $U \diamond V = UV - \mathbf{E}[UV]$ .*

**2.** *Let  $\Phi = \sum_{k=0}^{+\infty} a_k \langle \cdot, e_k \rangle$  and  $\Psi = \sum_{n=0}^{+\infty} I_n(f_n)$  be in  $(\mathcal{S})^*$ . Then their S-transform is given, for every  $\eta$  in  $\mathcal{S}(\mathbf{R})$ , by:*

$$S(\Phi)(\eta) = \sum_{k=0}^{+\infty} a_k \langle \eta, e_k \rangle_{L^2(\mathbf{R})}; \quad S(\Psi)(\eta) = \sum_{k=0}^{+\infty} \langle f_n, \eta^{\otimes n} \rangle$$

**3.** *For every  $(f, \eta, \xi)$  in  $L^2(\mathbf{R}) \times \mathcal{S}(\mathbf{R}) \times \mathbf{R}$ , we have the equality:*

$$S(e^{i\xi \langle \cdot, f \rangle})(\eta) = e^{\frac{1}{2} (|\eta|_0^2 + 2i\xi \langle f, \eta \rangle - \xi^2 |f|_0^2)}. \quad (2.1)$$

One may refer to [Jan97, Chap.3 and 16] for more details about Wick product. The following results on the S-transform will be used extensively in the sequel. See [Kuo96, p.39] and [HKPS93, p.280-281] for proofs. Denote  $\mathcal{F}(A; B)$  the set of  $B$ -valued functions defined on  $A$ .

**Lemma 2.5.** *The S-transform verifies the following properties:*

- (i) *The map  $S : \Phi \mapsto S(\Phi)$ , from  $(\mathcal{S})^*$  into  $\mathcal{F}(\mathcal{S}(\mathbf{R}); \mathbf{R})$ , is injective.*
- (ii) *Let  $\Phi : I \rightarrow (\mathcal{S})^*$  be an  $(\mathcal{S})^*$ -process. If  $\Phi$  is  $(\mathcal{S})^*$ -integrable over  $I$  wrt  $m$ , then one has, for all  $\eta$  in  $\mathcal{S}(\mathbf{R})$ ,  $S(\int_I \Phi(u) m(du))(\eta) = \int_I S(\Phi(u))(\eta) m(du)$ .*
- (iii) *Let  $\Phi : I \rightarrow (\mathcal{S})^*$  be an  $(\mathcal{S})^*$ -process differentiable at  $t \in I$ . Then, for every  $\eta$  in  $\mathcal{S}(\mathbf{R})$  the map  $u \mapsto [S\Phi(u)](\eta)$  is differentiable at  $t$  and verifies  $S[\frac{d\Phi}{dt}(t)](\eta) = \frac{d}{dt}[S[\Phi(t)](\eta)]$ .*

It is useful to have a criterion for integrability in  $(\mathcal{S})^*$  in term of conditions on the S-transform. This is the topic of the next theorems.

**Theorem 2.6.** [Kuo96, Theorem 13.5] *Let  $\Phi : I \rightarrow (\mathcal{S})^*$  be a stochastic distribution such that, for all  $\eta$  in  $\mathcal{S}(\mathbf{R})$ , the real-valued map  $t \mapsto S[\Phi(t)](\eta)$  is measurable and such that there exist a natural integer  $p$ , a real  $a$  and a function  $L$  in  $L^1(I, m)$  such that  $|S(\Phi(t))(\eta)| \leq L(t) e^{a|\eta|_p^2}$ , for all  $\eta$  of  $\mathcal{S}(\mathbf{R})$  and for almost every  $t$  of  $I$ . Then  $\Phi$  is  $(\mathcal{S})^*$ -integrable over  $I$ , wrt to  $m$ .*

We end this section with the following theorem that will be useful in the next section.

**Theorem 2.7.** [Ben03a, Theorem 2.17] *For any differentiable map  $F : I \rightarrow \mathcal{S}'(\mathbf{R})$ , the element  $\langle \cdot, F(t) \rangle$  is a differentiable stochastic distribution process which satisfies the equality:*

$$\frac{d}{dt} \langle \cdot, F(t) \rangle = \langle \cdot, \frac{dF}{dt}(t) \rangle.$$

## 2.4 Operators $(M_H)_{H \in (0,1)}$ and a classical example of Gaussian processes in $\mathcal{G}$

### Operators $(M_H)_{H \in (0,1)}$

The operator  $M_H$  will be useful in the sequel, not only to provide one with a representation of fBm and of mBm under the form (1.1), but also to verify that Assumptions  $(\mathcal{A})$ , we will make in Section 3, hold for both fBm and mBm. Let  $H$  belongs to  $(0, 1)$ ; following [EVdH03] and [LLV14, Section 2.2], define the  $L^2(\mathbf{R})$ -valued operator  $M_H$ , in the Fourier domain by:

$$\widehat{M_H(u)}(y) := \frac{\sqrt{2\pi}}{c_H} |y|^{1/2-H} \widehat{u}(y), \quad \forall y \in \mathbf{R}^*,$$

where  $c_x$  is defined, for every  $x$  in  $(0, 1)$  by:

$$c_x := \left( \frac{2\pi}{\Gamma(2x+1) \sin(\pi x)} \right)^{\frac{1}{2}}. \quad (2.2)$$

This operator is well defined on the homogeneous Sobolev space of order  $1/2 - H$ , noted  $L_H^2(\mathbf{R})$  and defined by setting:

$$L_H^2(\mathbf{R}) := \{u \in \mathcal{S}'(\mathbf{R}) : \widehat{u} = T_f; f \in L_{loc}^1(\mathbf{R}) \text{ and } \|u\|_H < +\infty\}, \quad (2.3)$$

where the norm  $\|\cdot\|_H$  derives from the inner product denoted  $\langle \cdot, \cdot \rangle_H$ , which is defined on  $L_H^2(\mathbf{R})$  by:

$$\langle u, v \rangle_H := \frac{1}{c_H^2} \int_{\mathbf{R}} |\xi|^{1-2H} \widehat{u}(\xi) \overline{\widehat{v}(\xi)} d\xi. \quad (2.4)$$

Since  $M_H$  is an isometry from  $(L_H^2(\mathbf{R}), \|\cdot\|_H)$  into  $(L^2(\mathbf{R}), \|\cdot\|_{L^2(\mathbf{R})})$ , one easily sees that, for every  $(H, t, s)$  in  $(0, 1) \times \mathbf{R}^2$ ,

$$\langle M_H(\mathbf{1}_{[0,t]}), M_H(\mathbf{1}_{[0,s]}) \rangle_{L^2(\mathbf{R})} = R_H(t, s). \quad (2.5)$$

We will say that an mBm is normalized when its covariance function, denoted  $R_h$ , verifies the equality:

$$R_h(t, s) = \frac{c_{h_{t,s}}^2}{c_{h(t)}c_{h(s)}} \left[ \frac{1}{2}(|t|^{2h_{t,s}} + |s|^{2h_{t,s}} - |t-s|^{2h_{t,s}}) \right], \quad (2.6)$$

where  $h_{t,s} := \frac{h(t)+h(s)}{2}$  and  $c_x$  has been defined in (2.2).

**Example 2.8** (Gaussian Processes in  $\mathcal{G}$  of “reference”: a running example). *Let  $H$  be real in  $(0, 1)$  and  $h : \mathbf{R} \rightarrow (0, 1)$  be a deterministic measurable function. Define the processes*

$$\begin{aligned} B &:= \{ \langle \cdot, \mathbf{1}_{[0,t]} \rangle; t \in \mathbf{R} \}; & \widehat{B} &:= \{ \langle \cdot, \mathbf{1}_{[0,t]} - t\mathbf{1}_{[0,1]} \rangle; t \in [0, 1] \}; \\ B^H &:= \{ \langle \cdot, M_H(\mathbf{1}_{[0,t]}) \rangle; t \in \mathbf{R} \}; & B^h &:= \{ \langle \cdot, M_{h(t)}(\mathbf{1}_{[0,t]}) \rangle; t \in \mathbf{R} \}; \\ \widetilde{B}^\gamma &:= \{ \langle \cdot, \mathbf{1}_{[0,t]} \varepsilon(t - \cdot) \rangle \text{ if } t \in \mathbf{R}_+^* \text{ \& } \widetilde{B}_0^\gamma := 0 \}, \end{aligned}$$

where  $\mathbf{1}_{[0,t]} \varepsilon(t - \cdot)$  denotes the map  $s \mapsto \mathbf{1}_{[0,t]}(s) \varepsilon(t - s)$ , defined on  $[0, t)$ . We know, thanks to Section 2.1, that  $B$  is a Brownian motion on  $\mathbf{R}$ . Moreover, since for any  $g_t$  that belongs to  $L^2(\mathbf{R})$ ,  $\langle \cdot, g_t \rangle \stackrel{\text{a.s.}}{=} \int_{\mathbf{R}} g_t(u) dB_u$ , it is clear, in view of (2.5), that  $B^H$  is a fBm of Hurst index  $H$ , that  $B^h$  is a normalized mBm of functional parameter  $h$ , that  $\widehat{B}$  is Brownian bridge on  $[0, 1]$  and that  $\widetilde{B}^\gamma$  is a  $\mathcal{V}_\gamma$ -process (defined in (1.5)).

A word on notation:  $B^H$  or  $B^{h(t)}$  will always denote an fBm with Hurst index  $H$  or  $h(t)$ , while  $B^h$  will stand for an mBm. Throughout this paper, unless otherwise specify, we will neither specify the value of  $H$  in  $(0, 1)$ , when we consider a fBm, nor the  $(0, 1)$ -valued function  $h$  when we consider a mBm, nor the function  $\gamma$  of a  $\mathcal{V}_\gamma$ -process. Moreover, since mBm is a generalization of fBm, which is itself a generalization of Brownian motion, one might want to treat only the case of mBm and then take constant function  $h$ , equal to  $H$  for fBm, and equal to  $1/2$  for Brownian motion. However we will not proceed that way in this paper and we will treat separately, from one hand the cases of fractional and multifractional Brownian motions and, from the other hand, the Brownian motion and bridge. The main reason is that we want to emphasize the minimum conditions that are really needed to ensure the existence and the main properties of stochastic integral *wrt* each of the Gaussian processes in  $\mathcal{G}$  of “reference” we consider. In addition, this choice will also allow us to point exactly the papers that our approach generalizes.

### 3 Stochastic integral with respect to Gaussian process

In this section and in the next one, we will follow the general organization of [Ben03a] and [LLV14], in order to indicate where the differences appear when we generalize, to all Gaussian processes of  $\mathcal{G}$ , the stochastic calculus *wrt* fBm (*resp.* mBm) that has been developed in [Ben03a] (*resp.* in [LLV14]).

The first part of this section is devoted to the definition of the time derivative, in the Stochastic distribution sense, of any element  $G := (G_t)_{t \in \mathcal{R}}$  of  $\mathcal{G}$ . Subsection 3.2 is devoted to the study of generalized functional of  $G$ , that will be useful in order to establish Tanaka formulas in Section 6. We compute the  $S$ -transforms of processes  $G$  and  $W^{(G)}$  in Subection 3.3. The Wiener integral *wrt*  $G$  is presented in Subsection 3.4, whereas the stochastic integral *wrt*  $G$  is built in Subsection 3.5.

We keep the same notations as in Section 2. In particular, the probability space  $(\Omega, \mathcal{F}, \mu)$ , described in the previous section is now fixed. Denote  $G := (G_t)_{t \in \mathcal{R}}$  the process defined, for every  $t$  in  $\mathcal{R}$ , by  $G_t := \langle \cdot, g_t \rangle$ , where  $(g_t)_{t \in \mathcal{R}}$  is a family of functions of  $L^2(\mathbf{R})$ . As we saw in Example 2.8,  $G$  is a Gaussian process which fulfills the equality  $G_t \stackrel{\text{a.s.}}{=} \int_{\mathbf{R}} g_t(u) dB_u$ . Denote

$(t, s) \mapsto R_{t,s}$  the covariance function of  $G$ . We hence have  $R_{t,s} := \mathbf{E}[G_t G_s] = \langle g_t, g_s \rangle_{L^2(\mathbf{R})}$ , for every  $(s, t)$  in  $\mathcal{R}^2$ . We will note in the sequel  $R_t$  instead of  $R_{t,t}$ . For the sake of notational simplicity we assume that  $G_0 \stackrel{a.s.}{=} 0$ . Moreover, when the Gaussian process  $G$  will admit a continuous modification, we will systematically use it and still call it  $G$ .

### 3.1 White Noise derivative of $G$

Define the map  $g : \mathcal{R} \rightarrow \mathcal{S}'(\mathbf{R})$  by setting  $g(t) := g_t$ . When  $g$  is differentiable at point  $t$ , one denotes  $g'_t$  its derivative. Denote  $\lambda$  the Lebesgue measure on  $\mathcal{R}$  and define  $L^1_{loc}(\mathcal{R}) := \{f : \mathcal{R} \rightarrow \mathbf{R} \text{ is measurable ; } f \in L^1((a, b)), \text{ for all finite interval } (a, b) \text{ s.t. } [a, b] \subset \mathcal{R}\}$ . In this section and in the next one (namely in Sections 3 and 4), we will make the following assumption:

$$(\mathcal{A}) \left\{ \begin{array}{l} \text{(i) The map } g \text{ is continuous on } \mathcal{R}, \\ \text{(ii) The map } g \text{ is differentiable } \lambda - \text{almost everywhere on } \mathcal{R}, \\ \text{(iii) There exists } q \text{ in } \mathbf{N}^* \text{ such that } t \mapsto |g'_t|_{-q} \text{ belongs to } L^1_{loc}(\mathcal{R}), \\ \text{(iv) For every } (a, b) \text{ in } \mathcal{R}^2 \text{ such that } a \leq b, \text{ one has, in } \mathcal{S}'(\mathbf{R}), \text{ the equality:} \end{array} \right. \quad (E_{a,b})$$

$$g_b - g_a = \int_a^b g'_u \, du.$$

Proposition 3.1 below will provide one with an easy way to check whether Assumption  $(\mathcal{A})$  holds or not. Besides, define the set  $\mathcal{R}_D$  by setting  $\mathcal{R}_D := \{t \in \mathcal{R}; g \text{ is differentiable at point } t\}$ .

Of course  $L^1_{loc}(\mathcal{R})$  contains in particular all measurable functions  $f : \mathcal{R}_D \rightarrow \mathbf{R}$  such that  $f \in L^1((a, b) \cap \mathcal{R}_D)$  (that we will denote  $L^1((a, b))$  in the sequel, by abuse of notation), for every finite interval  $(a, b)$  s.t.  $[a, b] \subset \mathcal{R}$ . For the sake of notational simplicity we will write  $\mathcal{A}_{(i)}$ , (resp.  $\mathcal{A}_{(ii)}$ ,  $\mathcal{A}_{(iii)}$  or  $\mathcal{A}_{(iv)}$ ), in the sequel, when one wants to refer to (i) (resp. to (ii), (iii) or (iv)) of Assumption  $(\mathcal{A})$ .

Making Assumptions  $\mathcal{A}_{(i)}$  and  $\mathcal{A}_{(ii)}$  seems reasonable since we want to “differentiate”, with respect to  $t$ , the Gaussian process  $G$ , which trajectories are, in general, not differentiable in the strong sense (e.g. the Brownian motion). The interest of Assumptions  $\mathcal{A}_{(iii)}$  and  $\mathcal{A}_{(iv)}$  will be explained when it will be needed for the first time; in Section 3.4, right after Definition 1.

**Remark 1. 1.** *A first consequence of Assumption  $(\mathcal{A})$  is that  $g$  is “weakly” locally absolutely continuous on  $\mathcal{R}$ ; that is that the map  $t \mapsto \langle g_t, \eta \rangle$  is absolutely continuous on every finite interval  $[a, b]$  of  $\mathcal{R}$ , for every  $\eta$  in  $\mathcal{S}(\mathbf{R})$ .*

**2.** *If  $g$  would have been a real-valued function, Assumption  $(\mathcal{A})$  would have been nothing but the local absolute continuity of  $g$  on  $\mathcal{R}$ . However,  $g$  is  $\mathcal{S}'(\mathbf{R})$ -valued. Thus, and even if a notion of absolute continuity (and more generally a notion of bounded variations) exists for  $\mathcal{S}'(\mathbf{R})$ -valued functions (see [HP57, Definitions 3.6.2 & 3.2.4]), the absolute continuity of  $g$  on an interval  $[a, b]$  of  $\mathcal{R}$  does not entail the differentiability of  $g$  in general (see an example that illustrates this fact in [HP57], right above Theorem 3.8.6).*

An easy way to see if Assumption  $(\mathcal{A})$  holds is to check if the sufficient condition provided in the following proposition, and that will be used a lot in the sequel, holds.

**Proposition 3.1.** *A sufficient condition for Assumption  $(\mathcal{A})$  to be verified is that:*

$$(\mathcal{D}) \left\{ \begin{array}{l} \text{(i) The map } g \text{ is continuous on } \mathcal{R} \text{ and differentiable on every finite interval } (a, b) \text{ s.t. } [a, b] \subset \mathcal{R}, \\ \text{(ii) There exists } q \text{ in } \mathbf{N}^* \text{ such that } t \mapsto |g'_t|_{-q} \text{ belongs to } L^1_{loc}(\mathcal{R}). \end{array} \right.$$

**Proof.** Indeed, these two conditions obviously entail that Assumptions  $\mathcal{A}_{(i)}$  to  $\mathcal{A}_{(iii)}$  hold. Moreover, these two conditions also entail Equality  $(E_{c,d})$ , for every  $[c, d]$  in  $\mathcal{R}_D$ .  $\mathcal{R}$  being a closed interval of  $\mathbf{R}$ , and in view of (ii) of Assumption  $(\mathcal{D})$ , there can be 0,1 or 2 points, at the maximum, that belong to  $\mathcal{R}$  but not to  $\mathcal{R}_D$ . Let us treat this latter case only and denote  $a$  and  $b$  these two points. The continuity of  $g$  at points  $a$  and  $b$ , from one hand, and the Lebesgue dominated convergence theorem, from the other hand, give us the equality  $(E_{a,b})$ .  $\square$

As previously, and for the sake of notational simplicity we will write  $\mathcal{D}_{(i)}$  (resp.  $\mathcal{D}_{(ii)}$ ) when one wants to refer to (i) or to (ii) of Assumption  $(\mathcal{D})$ .

Assumption  $\mathcal{A}_{(ii)}$ , Proposition 2.2 and Theorem 2.7 lead to the following definition of Gaussian white noise:

**Theorem-Definition 3.1 (Gaussian White Noise).** Define for every  $t$  in  $\mathcal{R}_D$ ,

$$W_t^{(G)} := \langle \cdot, g'_t \rangle, \quad (3.3)$$

where the equality holds in  $(\mathcal{S})^*$ . Then  $(W_t^{(G)})_{t \in \mathcal{R}_D}$  is a  $(\mathcal{S})^*$ -process and is the  $(\mathcal{S})^*$ -derivative of the process  $(G_t)_{t \in \mathcal{R}_D}$ . We will sometimes note  $\frac{dG_t}{dt}$  instead of  $W_t^{(G)}$ .

Using Proposition 2.2 one easily sees that (3.3) also reads, for every  $t$  in  $\mathcal{R}_D$ :

$$W_t^{(G)} = \sum_{k=0}^{+\infty} \langle g'_t, e_k \rangle \langle \cdot, e_k \rangle = \sum_{k=0}^{+\infty} \left( \frac{d}{dt} \langle g_t, e_k \rangle \right) \langle \cdot, e_k \rangle.$$

The regularity of the map  $t \mapsto \|W_t^{(G)}\|_{-p}$  is the topic of the following proposition.

**Proposition 3.2.** The map  $t \mapsto \|W_t^{(G)}\|_{-p}$  is continuous if and only if  $t \mapsto |g'_t|_{-p}$  is continuous.

**Proof.** Thanks to Proposition 2.2, one can write  $\|W_t^{(G)}\|_{-p} = |g'_t|_{-p}$ ,  $\forall (p, t)$  in  $\mathbf{N}^* \times \mathcal{R}_D$ .  $\square$

As the next example shows Assumptions  $(\mathcal{D})$  (and therefore Assumption  $(\mathcal{A})$ ) holds in the case of all Gaussian processes in  $\mathcal{G}$  of “reference”. Denote, for every  $n$  in  $(1/2, +\infty)$ ,

$$\mathcal{R}_n := \sum_{k=0}^{+\infty} (2k+2)^{-2n}. \quad (3.4)$$

**Example 3.3.** 1. (Brownian motion on  $\mathbf{R}$  & Brownian bridge on  $[0, 1]$ ). For the Brownian motion on  $\mathbf{R}$ , one has  $\mathcal{R} = \mathcal{R}_D = \mathbf{R}$ . Since  $g_t = \mathbf{1}_{[0,t]}$ , the map  $g$  clearly fulfills Assumption  $\mathcal{D}_{(i)}$  and one has  $g'_t = \delta_t$ , for every real  $t$ . Moreover, for every positive integer  $p$ , the map  $t \mapsto |g'_t|_{-p}$  is continuous and bounded on  $\mathbf{R}$ , which shows that  $\mathcal{D}_{(ii)}$  holds. Indeed we know, thanks to Theorem 2.1, that there exists  $C'$ , independent of  $t$  and  $p$ , such that :

$$\forall (p, t) \in \mathbf{N}^* \times \mathbf{R}, \quad |g'_t|_{-p}^2 = \sum_{k=0}^{+\infty} e_k^2(t) (2k+2)^{-2p} \leq C' \mathcal{R}_p.$$

(Brownian bridge on  $[0, 1]$ ) In this case,  $\mathcal{R} = \mathcal{R}_D = [0, 1]$  and  $g'_t = \delta_t - \mathbf{1}_{[0,1]}$ , for every  $t$  in  $\mathcal{R}$ . In view of what we have just stated above, in the case of Brownian motion, it is clear that the map  $g$  here fulfills Assumptions  $\mathcal{D}_{(i)}$  &  $\mathcal{D}_{(ii)}$  since the map  $t \mapsto |g'_t|_{-p}$  is continuous and bounded on  $[0, 1]$ , for every positive integer  $p$ . One moreover has, for a constant  $C'$ , which is independent of  $p$  and  $t$ :

$$\forall (p, t) \in \mathbf{N}^* \times [0, 1], \quad |g'_t|_{-p}^2 \leq C' \mathcal{R}_p.$$

2. (Fractional & Multifractional Brownian motions on  $\mathbf{R}$ ) In both these cases, one has  $\mathcal{R} = \mathcal{R}_D = \mathbf{R}$ . Thanks to [LLV14, Remark 4.3 and Proposition 4.10], we know that Assumption  $\mathcal{D}_{(i)}$ , as well as the fact that  $t \mapsto \|W_t^{(B^H)}\|_{-p}$  (resp.  $t \mapsto \|W_t^{(B^h)}\|_{-p}$ ) is continuous and bounded on any compact set of  $\mathbf{R}$ , are verified for every  $p \geq 2$  and  $H$  in  $(0, 1)$  (resp.  $h$  differentiable with locally bounded derivative).

3. (The process  $\tilde{B}^\gamma$ ) In this case,  $\mathcal{R} = \mathbf{R}_+$  and  $\mathcal{R}_D = \mathbf{R}_+^*$  (where we set  $\mathbf{R}_+^* = \mathbf{R} \setminus \{0\}$ ). Moreover Assumption  $(\mathcal{D})$  is also fulfilled, as Theorem 3.4 below shows.

The following theorem shows that  $\mathcal{V}_\gamma$ - processes fulfill Assumption  $(\mathcal{D})$  (and therefore Assumption  $(\mathcal{A})$ ). This result will be crucial to show, in Section 5.1.2, that the stochastic integral wrt  $\mathcal{V}_\gamma$ - processes, developed in [MV05], is a particular case of the one we build in this work. Let us first define the two maps  $E : \mathbf{R}_+ \rightarrow \mathbf{R}$  and  $\mathcal{E} : \mathbf{R}_+ \rightarrow \mathbf{R}$  by setting:

$$E(x) := \begin{cases} \int_0^x \varepsilon(u) du, & \text{if } x \in \mathbf{R}_+^*, \\ 0 & \text{if } x = 0, \end{cases} \quad \& \quad \mathcal{E}(x) := \int_0^x E(u) du.$$

**Theorem 3.4.** *The map  $\Phi : \mathbf{R}_+ \rightarrow \mathcal{S}'(\mathbf{R})$  defined by setting:*

$$\Phi(t) := \Phi_t := \begin{cases} \mathbf{1}_{[0,t)} \varepsilon(t - \cdot) & \text{if } t \in \mathbf{R}_+^*, \\ 0 & \text{if } x = 0, \end{cases}$$

fulfill Assumption  $(\mathcal{A})$ . More precisely, it is differentiable on  $\mathbf{R}_+^*$  and, for every positive real  $t$ , one has the equality:

$$\Phi'(t) := \frac{d}{dt}[\Phi(t)] = F_t - (G_t)' + (H_t)'', \quad (3.5)$$

where for every  $\Psi$  in  $\mathcal{S}'(\mathbf{R})$ ,  $\Psi'$  denotes the derivative of  $\Psi$ , in the sense of tempered distributions<sup>5</sup> and where  $F_t, G_t$  and  $H_t$  all belong to  $\mathcal{S}'(\mathbf{R})$  and are defined by setting,  $\forall t$  in  $\mathbf{R}_+^*$ :

- $F_t := \frac{\varepsilon(t-\cdot)}{t} \mathbf{1}_{[0,t)} + (\varepsilon(t) - \frac{E(t)}{t}) \delta_0$ ,
- $G_t := (E(t) - \frac{\mathcal{E}(t)}{t}) \delta_0 + u \mapsto \left( \frac{u \varepsilon(t-u) - E(t-u)}{t} \right) \mathbf{1}_{[0,t)}(u)$ ,
- $H_t := \frac{((t-\cdot)E(t-\cdot) - \mathcal{E}(t-\cdot))}{t} \mathbf{1}_{[0,t)}$ .

Furthermore, the map  $t \mapsto |\Phi'(t)|_{-q}$  belongs to  $L^2((0, b))$ , for every  $b > 0$  and every integer  $q \geq 3$ .

**Proof:** In view of Proposition 3.1, it is sufficient to show that Assumption  $(\mathcal{D})$  holds. Besides, it is clear that  $\Phi$  is well defined on  $\mathbf{R}_+$  since one has, for every  $t$  in  $\mathbf{R}_+$ , the equality:

$$\Phi_t = E(t) \delta_0 - (\mathbf{1}_{[0,t)} E(t - \cdot))'. \quad (3.6)$$

It is clear that function  $E$  (resp.  $\mathcal{E}$ ) is increasing, differentiable on  $\mathbf{R}_+^*$  and continuous on  $\mathbf{R}_+$  (resp. increasing and of class  $C^1$  on  $\mathbf{R}_+$ ). Equality (3.6) together with the properties of  $E$  entail that  $\Phi$  is continuous at  $t = 0$ . Let us now establish Equality (3.5).

For every  $t$  in  $\mathbf{R}_+^*$ ,  $\varphi$  in  $\mathcal{S}(\mathbf{R})$ , and  $r > 0$ , denote  $I_r := \langle \frac{\Phi_{t+r} - \Phi_t}{r}, \varphi \rangle$ . Using the change of variable formula gives us, after an easy computation:

$$\begin{aligned} I_r &= \frac{1}{r} \left( \int_0^{t+r} \varphi(u) \varepsilon(t+r-u) du - \int_0^t \varphi(u) \varepsilon(t-u) du \right) \\ &= \int_0^1 \frac{t}{r} [(1 + \frac{r}{t}) \varepsilon((t+r)(1-v)) \varphi(v(t+r)) - \varepsilon(t(1-v)) \varphi(vt)] dv \\ &= \int_0^1 \frac{t}{r} [\varepsilon((t+r)(1-v)) \varphi(v(t+r)) - \varepsilon(t(1-v)) \varphi(vt)] dv - \int_0^1 \varepsilon((t+r)(1-v)) \varphi(v(t+r)) dv \\ &=: I_r^{(1)} + I_r^{(2)}. \end{aligned}$$

<sup>5</sup>One therefore has  $\langle (G_t)', \varphi \rangle = - \langle G_t, \varphi' \rangle$ , for every  $(t, \varphi)$  in  $\mathbf{R}_+^* \times \mathcal{S}(\mathbf{R})$ .

For every  $r$  in  $(0, 1)$ , one has  $I_r^{(2)} = \int_0^{t+r} \mathbf{1}_{(0, t+r)}(u) \frac{\varepsilon(u)}{t} \varphi(t+r-u) du$ . Since  $t$  and  $\varepsilon$  are positive, Lebesgue's dominated convergence theorem applies and allows one to write that  $\lim_{r \rightarrow 0} I_r^{(2)} = \int_0^t \frac{\varepsilon(u)}{t} \varphi(t-u) du$  and thus<sup>6</sup>:

$$\lim_{r \rightarrow 0} I_r^{(2)} = \int_0^1 \varepsilon(t(1-v)) \varphi(vt) dv = \frac{1}{t} \int_0^t \varphi(u) \varepsilon(t-u) du. \quad (3.7)$$

Besides,  $I_r^{(1)}$  can be written under the following form:

$$\begin{aligned} I_r^{(1)} &= \int_0^1 t \varepsilon((t+r)(1-v)) \left( \frac{\varphi(v(t+r)) - \varphi(vt)}{r} \right) dv \\ &\quad + \int_0^1 \frac{t}{r} \varphi(vt) (\varepsilon((t+r)(1-v)) - \varepsilon(t(1-v))) dv =: J_r^{(1)} + J_r^{(2)}. \end{aligned}$$

The exact same method the the one we used to get (3.7) applies and allows one to write:

$$\lim_{r \rightarrow 0} J_r^{(1)} = \int_0^1 tv \varphi'(vt) \varepsilon(t(1-v)) dv = \frac{1}{t} \int_0^t u \varphi'(u) \varepsilon(t-u) du. \quad (3.8)$$

Having in mind that  $\varepsilon^2(r) = (\gamma^2)'(r)$ , an integration by parts in  $J_r^{(2)}$  yields:

$$\begin{aligned} J_r^{(2)} &= \varphi(0) \left( \frac{t}{t+r} \frac{(E(t+r) - E(t))}{r} - \frac{E(t)}{t+r} \right) - \frac{t}{t+r} \int_0^1 \varphi'(vt) E((t+r)(1-v)) dv \\ &\quad + \underbrace{\frac{t}{r} \int_0^1 \varphi'(vt) (E((t+r)(1-v)) - E(t(1-v))) dv}_{=: K_r}. \end{aligned}$$

It is then clear that:

$$\begin{aligned} \lim_{r \rightarrow 0} J_r^{(2)} &= \varphi(0) \left( \varepsilon(t) - \frac{E(t)}{t} \right) - \int_0^1 \varphi'(vt) E(t(1-v)) dv + \lim_{r \rightarrow 0} K_r \\ &= \varphi(0) \left( \varepsilon(t) - \frac{E(t)}{t} \right) - \frac{1}{t} \int_0^t \varphi'(u) E(t-u) du + \lim_{r \rightarrow 0} K_r. \end{aligned} \quad (3.9)$$

Thus, it only remains to determine  $\lim_{r \rightarrow 0} K_r$ . An integration by parts in  $K_r$  yields:

$$\begin{aligned} K_r &= \left[ \varphi'(vt) \left( \frac{\mathcal{E}(t(1-v))}{t} - \frac{\mathcal{E}((t+r)(1-v))}{t+r} \right) \right]_0^1 - t \int_0^1 \varphi''(vt) \left( \frac{\mathcal{E}(t(1-v))}{t} - \frac{\mathcal{E}((t+r)(1-v))}{t+r} \right) dv \\ &= \varphi'(0) \left( \frac{t}{t+r} \frac{\mathcal{E}(t+r) - \mathcal{E}(t)}{r} - \frac{\mathcal{E}(t)}{t+r} \right) - \frac{t}{t+r} \int_0^1 \varphi''(vt) \mathcal{E}(t(1-v)) dv \\ &\quad + \frac{t^2}{t+r} \int_0^1 \varphi''(vt) \left( \frac{\mathcal{E}((t+r)(1-v)) - \mathcal{E}(t(1-v))}{r} \right) dv. \end{aligned}$$

One then gets:

$$\begin{aligned} \lim_{r \rightarrow 0} K_r &= \varphi'(0) \left( E(t) - \frac{\mathcal{E}(t)}{t} \right) - \int_0^1 \varphi''(vt) \mathcal{E}(t(1-v)) dv + t \int_0^1 \varphi''(vt) (1-v) E(t(1-v)) dv \\ &= \varphi'(0) \left( E(t) - \frac{\mathcal{E}(t)}{t} \right) - \frac{1}{t} \int_0^t \varphi''(u) \mathcal{E}(t-u) du + \int_0^t \varphi''(u) \left( 1 - \frac{u}{t} \right) E(t-u) du. \end{aligned} \quad (3.10)$$

Finally, gathering Equalities (3.7) to (3.10) yields:

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<sup>6</sup>Note that one could also have used [MV05, Remark 3] and assume that  $(\gamma^2)'$  (and hence  $\varepsilon$ ) is non-increasing.

$$\begin{aligned}
\lim_{r \rightarrow 0} I_r &= \frac{1}{t} \int_0^t \varphi(u) \varepsilon(t-u) du + \frac{1}{t} \int_0^t \varphi'(u) (u \varepsilon(t-u) - E(t-u)) du \\
&\quad + \frac{1}{t} \int_0^t \varphi''(u) ((t-u) E(t-u) - \mathcal{E}(t-u)) du \\
&\quad + \varphi(0) \left( \varepsilon(t) - \frac{E(t)}{t} \right) + \varphi'(0) \left( E(t) - \frac{\mathcal{E}(t)}{t} \right), \quad (3.11)
\end{aligned}$$

which is nothing but (3.5). Let us now show that  $t \mapsto |\Phi'(t)|_{-q} \in L^1((0, b))$ , for every  $b$  in  $\mathbf{R}_+^*$  and every integer  $q \geq 3$ .

Let  $b$  be a positive real and  $q$  be an integer such that  $q \geq 3$ . It is sufficient to show that the map  $t \mapsto |\Phi'(t)|_{-q}^2$  belongs to  $L^1((0, b))$ . Using (3.5), one gets, for every integer  $k \geq 2$ ,

$$|\langle \Phi'(t), e_k \rangle| \leq \alpha_k \left( \varepsilon(t) + \frac{E(t)}{t} + M \right), \quad (3.12)$$

where  $\alpha_k := \|e_k\|_\infty + \|e'_k\|_\infty + \|e''_k\|_\infty$  and where  $M := (t+3) \sup_{s \in [0, t]} E(s) + \sup_{s \in [0, t]} \mathcal{E}(s) + \sup_{s \in (0, t]} \frac{\mathcal{E}(s)}{s}$ .

Using the relation  $e'_k(x) = \sqrt{\frac{k}{2}} e_{k-1}(x) - \sqrt{\frac{k+1}{2}} e_{k+1}(x)$  (see [Kuo96, p.354]) as well as Theorem 2.1, one easily obtains that  $\alpha_k \leq 48 (k+1)^2 \sum_{l=-2}^2 \|e_{k+l}\|_\infty^2 \leq 250 (k+1)^2$ , for every integer  $k \geq 2$ . It is then clear that there exists  $C > 0$  which does not depend on  $q$  nor  $t$  such that:

$$\begin{aligned}
|\Phi'(t)|_{-q}^2 &= \sum_{k=0}^{+\infty} |\langle \Phi'(t), e_k \rangle|^2 (2k+2)^{-2q} \leq \left( \varepsilon(t) + \frac{E(t)}{t} + M \right)^2 \sum_{k=0}^{+\infty} \alpha_k^2 (2k+2)^{-2q} \\
&\leq C \left( \varepsilon(t) + \frac{E(t)}{t} + M \right)^2 \sum_{k=0}^{+\infty} (2k+2)^{4-2q}. \quad (3.13)
\end{aligned}$$

It then remains to show that both  $\varepsilon^2$  and  $t \mapsto \frac{E(t)^2}{t^2}$  belong to  $L^1((0, b))$ . The first part is clear since  $\varepsilon^2 = (\gamma^2)'$ . Moreover, in a neighborhood of 0 one has<sup>7</sup>  $E(t) \leq 2t\varepsilon(t)$ , for  $t \neq 0$ . One therefore can write the following equality, which ends the proof:

$$\int_0^b \left( \frac{E(t)}{t} \right)^2 dt \leq 4 \int_0^b \varepsilon^2(t) dt = 4 \gamma^2(b) < +\infty. \quad \square$$

### 3.2 Generalized functionals of $G$

In the next section, we will derive various Itô formulas. It will be useful to obtain such formulas not only for functions but also for tempered distributions (in particular this will provide us with a Tanaka formula in Section 6). In that view, we define generalized functionals of  $G$ , using [Kuo96, Section 7.1]. We identify, here and in the sequel, any function  $f$  of  $L^1_{loc}(\mathbf{R})$  with its associated tempered distribution, denoted  $T_f$ , when it exists. In particular, one notes in this case:  $\langle f, \phi \rangle = \int_{\mathbf{R}} f(t) \phi(t) dt$ , for every  $\phi$  in  $\mathcal{S}(\mathbf{R})$ . In this latter case we say that the tempered distribution  $T := T_f$  is of function type. Define the sets  $\mathcal{Z}_R := \{t \in \mathcal{R}; R_t = 0\}$  and  $\mathcal{Z}_R^c := \{t \in \mathcal{R}; R_t > 0\}$ .

**Theorem-Definition 3.2.** *Let  $F$  be a tempered distribution. For every  $t$  in  $\mathcal{Z}_R^c$ , define*

$$F(G_t) := \frac{1}{\sqrt{2\pi R_t}} \sum_{k=0}^{+\infty} \frac{1}{k! R_t^k} \langle F, \xi_{t,k} \rangle I_k \left( g_t^{\otimes k} \right),$$

where, for every  $(x, k)$  in  $\mathbf{R} \times \mathbf{N}$ ,  $\xi_{t,k}(x) := \pi^{1/4} (k!)^{1/2} R_t^{k/2} \exp\{-\frac{x^2}{4R_t}\} e_k(x/(\sqrt{2R_t}))$ . Then for all real  $t$ ,  $F(G_t)$  is a Hida distribution, called generalized functional of  $G_t$ .

<sup>7</sup>See [MV05, p.396] for a proof of this fact.

**Proof.** This is an immediate consequence of [Kuo96, p.61-64], by taking  $f := g_t$ .  $\square$

**Remark 2.** As shown in [Ben03a], when  $F = f$  is of function type,  $F(G_t)$  coincides with  $f(G_t)$ .

The following theorem yields an estimate of  $\|F(G_t)\|_{-p}^2$  which will be useful in the sequel.

**Theorem 3.5.** Let  $p$  be in  $\mathbf{N}$ . Then there is a constant  $D_p$ , such that:

$$\forall F \in \mathcal{L}_p(\mathbf{R}), \forall t \in \mathcal{Z}_R^c, \quad \|F(G_t)\|_{-p}^2 \leq \max\{R_t^{-2p}; R_t^{2p}\} R_t^{-1/2} D_p |F|_{-p}^2. \quad (3.14)$$

**Proof.** This is a simple consequence of the following more general result: let  $f$  be a nonzero function in  $L^2(\mathbf{R})$ ,  $p \in \mathbf{N}$  and  $F \in \mathcal{L}_p(\mathbf{R})$ . There exists a constant  $D_p$ , independent of  $F$  and  $f$ , such that

$$\|F(\langle \cdot, f \rangle)\|_{-p}^2 \leq \max\{|f|_0^{-4p}; |f|_0^{4p}\} |f|_0^{-1} D_p |F|_{-p}^2.$$

The line of the proof is the same as in [Ben03a, Theorem 3.3] by replacing there  $t^{2H}$  by  $|f|_0^2$ .  $\square$

### 3.3 S-Transform of $G$ and $W^{(G)}$

The following theorem makes explicit the  $S$ -transforms of  $G$ , of the Gaussian white noise  $W^{(G)}$  and of generalized functionals of  $G$ . Denote  $\gamma$  the heat kernel density on  $\mathbf{R}_+ \times \mathbf{R}$  *i.e.*

$$\gamma(t, x) := \frac{1}{\sqrt{2\pi t}} \exp\left\{-\frac{x^2}{2t}\right\} \text{ if } t \neq 0 \text{ and } 0 \text{ if } t = 0. \quad (3.15)$$

The results provided in Theorem 3.6 below will be used a lot in the sequel, and from Section 3.4.

**Theorem 3.6.** For every  $\eta$  in  $\mathcal{S}(\mathbf{R})$  one has the following equalities:

- (i)  $S(G_t)(\eta) = \langle g_t, \eta \rangle_{L^2(\mathbf{R})}$ , for every  $t$  in  $\mathcal{R}$ ,
- (ii)  $S(W_t^{(G)})(\eta) = \langle g'_t, \eta \rangle = \frac{d}{dt}[\langle g_t, \eta \rangle_{L^2(\mathbf{R})}]$ , for every  $t$  in  $\mathcal{R}_D$ ;
- (iii) For  $p \in \mathbf{N}$ ,  $F \in \mathcal{L}_p(\mathbf{R})$ , and  $t$  in  $\mathcal{Z}_R^c$ ,  $S(F(G_t))(\eta) = \langle F, \gamma(R_t, \cdot - \langle g_t, \eta \rangle) \rangle$ .

Furthermore, there exists a constant  $D_p$ , independent of  $F, t$  and  $\eta$ , such that:

$$\forall t \in \mathcal{Z}_R^c, \quad |S(F(G_t))(\eta)|^2 \leq \max\{R_t^{-2p}; R_t^{2p}\} R_t^{-1/2} D_p |F|_{-p}^2 \exp\{|\eta|_p^2\}.$$

**Proof.** (i) Obvious in regard of point 2 of Proposition 2.4. Point (ii) is a straightforward consequence of (iii) in Lemma 2.5, and of (3.3). The equality in (iii) results from [Kuo96, Theorem 7.3 p.63] with  $f = g_t$ . The inequality results from (3.14) as in [Ben03a, Theorem 3.8].  $\square$

Before giving the general result on stochastic integral *wrt*  $G$  we deal, in the next subsection, with Wiener integral *wrt*  $G$ .

### 3.4 Wiener integral with respect to $G$

In all this subsection one denotes  $I$  a Borel set of  $\mathcal{R}$  and  $f : \mathcal{R} \rightarrow \mathbf{R}$  a deterministic and measurable function<sup>8</sup>. We want to define the integral of  $f$ , on  $I$ , with respect to  $G$ . Since the map  $s \mapsto G_s$  is (weakly) differentiable on  $I$ , one may think to define formally the Wiener integral *wrt*  $G$ , denoted  $\int_I f(s) d^\circ G_s$ , by setting:

$$\int_I f(s) d^\circ G_s := \int_I f(s) \frac{dG_s}{ds} ds = \int_I f(s) W_s^{(G)} ds, \quad (3.16)$$

assuming  $s \mapsto f(s) W_s^{(G)}$  is  $(\mathcal{S})^*$ -integrable on  $I$ . More precisely we have the following definition.

<sup>8</sup>This - of course! - includes the case of deterministic and measurable functions  $f : \mathcal{R}_D \rightarrow \mathbf{R}$

**Definition 1.** (Wiener integral with respect to  $G$ )

For any Borel set  $I$  of  $\mathcal{R}$  and any deterministic measurable function  $f : I \mapsto \mathbf{R}$  such that  $s \mapsto f(s) W_s^{(G)}$  is  $(\mathcal{S})^*$ -integrable on  $I$ , one says that  $\int_I f(s) d^\circ G_s$ , defined by (3.16), is the Wiener integral of  $f$  on  $I$ , with respect to  $G$ , if  $\int_I f(s) d^\circ G_s$  belongs to  $(L^2)$ .

Even if, in practice, there will often exist an integer  $q$  in  $\mathbf{N}$  such that the map  $t \mapsto |g'_t|_{-q}$  is bounded (as it was the case in Example 3.3), it seems more than reasonable to expect, even if  $t \mapsto |g'_t|_{-q}$  is not bounded, that, for any finite interval  $[a, b]$  of  $\mathcal{R}$ ,

$$\int_{(a,b)} 1 d^\circ G_s \text{ is well-defined and such that: } \int_{(a,b)} 1 d^\circ G_s = G_b - G_a, \text{ in } (\mathcal{S})^*. \quad (*)$$

Thanks to Equality (3.16) and Theorem 3.6, it is clear that  $(*)$  entails, among other consequences, that:

- The map  $s \mapsto \langle g'_s, \eta \rangle$  belongs to  $L^1_{loc}(\mathcal{R})$ , for every  $\eta$  in  $\mathcal{S}(\mathbf{R})$ , (3.17)

- $\int_{(a,b)} \langle g'_s, \eta \rangle ds = \langle g_b, \eta \rangle - \langle g_a, \eta \rangle$ , for every  $\eta$  in  $\mathcal{S}(\mathbf{R})$  and  $(a, b)$  in  $\mathcal{R}^2$ . (3.18)

Using Proposition 2.2, it is easy to establish, for all  $(F, \varphi, q)$  in  $\mathcal{S}'(\mathbf{R}) \times \mathcal{S}(\mathbf{R}) \times \mathbf{N}$ , the following inequality:

$$|\langle F, \varphi \rangle| \leq |F|_{-q} |\varphi|_q. \quad (3.19)$$

In view of (3.19), it appears that Assumption  $\mathcal{A}_{(iii)}$  is almost necessary to get  $(*)$ , if one deals with Pettis integrals, and necessary if one deals with Bochner integral<sup>9</sup>. Moreover, it is clear that (3.18) is nothing but Assumption  $\mathcal{A}_{(iv)}$ . Thus, the simple considerations given in  $(*)$ , as well as the ones given right above Remark 1 (about  $\mathcal{A}_{(i)}$  &  $\mathcal{A}_{(ii)}$ ), entail that Assumption  $(\mathcal{A})$  is almost minimal (*i.e.* necessary) to get a reasonable notion of Wiener integral. We will show, in the next sections of this work that these assumptions are also sufficient to provide us with a general non-anticipative stochastic integral.

**Remark 3.** By the very definition of the space  $\mathcal{S}'(\mathbf{R})$  as the inductive limit of the sequence  $(\mathcal{L}_p(\mathbf{R}))_{p \in \mathbf{N}}$ , Assumption  $\mathcal{A}_{(ii)}$  entails that, for every compact set  $\mathcal{K}$  of  $\mathcal{R}_D$ , there exists an integer  $q$  in  $\mathbf{N}$  such that  $|g'_t|_{-q} < +\infty$ , for every  $t$  in  $\mathcal{K}$ . Thus  $\mathcal{A}_{(iii)}$  appears to be only a slight reinforcement of  $\mathcal{A}_{(ii)}$

Denote  $\mathcal{E}(\mathcal{R})$  the set of step functions on  $\mathcal{R}$ . We have the following property.

**Proposition 3.7.** For any  $f$  in  $\mathcal{E}(\mathcal{R})$ ,  $\int_{\mathcal{R}} f(u) d^\circ G_u$  is a Wiener integral with respect to  $G$ . Moreover, let  $[a, b]$  be a finite interval of  $\mathcal{R}$ , then  $\int_a^b d^\circ G_u = G_b - G_a$  almost surely.

**Proof.** Fix  $\eta$  in  $\mathcal{S}(\mathbf{R})$ . From (ii) of Theorem 3.6,  $t \mapsto S(f(t) W_t^{(G)})(\eta)$  is measurable on  $\mathcal{R}$ . Moreover we have, thanks to Lemma 2.3,  $|S(f(t) W_t^{(G)})(\eta)| \leq |g'_t|_{-q} \sup_{t \in \mathcal{R}} |f(t)| e^{|\eta|^2 - q}$ , where  $q$  is the integer given by Assumption  $\mathcal{A}_{(iii)}$ . Theorem 2.6 then applies and entails that  $f$  is  $dG$ -integrable on  $(a, b)$ . Furthermore, thanks to Lemma 2.5, one has the equality:

$$S(\int_a^b dG_u)(\eta) = \int_a^b S(W_u^{(G)})(\eta) du = [S(G_u)(\eta)]_a^b = S(G_b)(\eta) - S(G_a)(\eta) = S(G_b - G_a)(\eta).$$

The equality, in  $(\mathcal{S}^*)$ , follows from the injectivity of the S-transform. Finally, since  $G_b - G_a$  belongs to  $(L^2)$ , the equality  $\int_a^b d^\circ G_u = G_b - G_a$  holds in  $(L^2)$  and hence almost surely.  $\square$

<sup>9</sup>See Appendix A.1 for precisions about Bochner integrals.

The following theorem gives a sufficient condition for an integral, of the form (3.16), to be a Wiener integral.

**Theorem 3.8.** *Assume that there exists  $q_0$  in  $\mathbf{N}$  such that the map  $s \mapsto f(s) |g'_s|_{-q_0}$  belongs to  $L^1(\mathcal{R})$ . Then  $Z := \int_{\mathcal{R}} f(s) d^\circ G_s$  is an element of  $(\mathcal{S})^*$ , which verifies the equality  $Z = \sum_{k=0}^{+\infty} (\int_{\mathcal{R}} f(s) \langle g'_s, e_k \rangle ds) \langle \cdot, e_k \rangle$ , in  $(\mathcal{S})^*$ . Moreover  $Z$  is a Gaussian random variable if and only if  $\sum_{k=0}^{+\infty} (\int_{\mathcal{R}} f(s) \langle g'_s, e_k \rangle ds)^2 < +\infty$ . In this latter case, one has:*

$$\mathbf{E}[Z^2] = \sum_{k=0}^{+\infty} \left( \int_{\mathcal{R}} f(s) \langle g'_s, e_k \rangle ds \right)^2.$$

**Remark 4.** *Denote  $\rightsquigarrow$  the equality in law and  $\mathcal{N}(0, \sigma^2)$  the law of an unidimensional centered Gaussian random variable with variance equal to  $\sigma^2$ . Let  $T > 0$  be such that  $[0, T] \subset \mathcal{R}$ . Taking  $f$  in  $L^1_{loc}(\mathcal{R})$  and  $f \mathbf{1}_{(0, T)}$  instead of  $f$  in the previous theorem, allows one to get a sufficient condition under which the process  $(Z_t)_{t \in [0, T]}$ , defined by setting  $Z_t := \int_0^t f(s) d^\circ G_s$ , is a Gaussian process, which fulfills:*

$$Z_t = \int_0^t f(s) d^\circ G_s \rightsquigarrow \mathcal{N} \left( 0, \sum_{k=0}^{+\infty} \left( \int_0^t f(s) \langle g'_s, e_k \rangle ds \right)^2 \right), \quad \forall t \in [0, T],$$

**Proof.** In order to show that equality  $\int_{\mathcal{R}} f(s) d^\circ G_s = \sum_{k=0}^{+\infty} (\int_{\mathcal{R}} f(s) \langle g'_s, e_k \rangle ds) \langle \cdot, e_k \rangle$  holds in  $(\mathcal{S})^*$ , let us establish points **a)**, **b)** and **c)** below.

**a)**  $s \mapsto f(s) W_s^{(G)}$  is  $(\mathcal{S})^*$ -integrable over  $\mathcal{R}$ .

One can use Theorem 2.6 since one has, for every  $(\eta, s)$  in  $\mathcal{S}(\mathbf{R}) \times \mathcal{R}_D$  and using Lemma 2.3,

$$|S(f(s) W_s^{(G)})(\eta)| \leq |f(s)| \|W_s^{(G)}\|_{-q_0} e^{|\eta|^2} \leq |f(s)| |g'_s|_{-q_0} e^{|\eta|^2}.$$

**b)**  $\Psi_f := \sum_{k=0}^{+\infty} (\int_{\mathcal{R}} f(s) \langle g'_s, e_k \rangle ds) \langle \cdot, e_k \rangle$  belongs to  $(\mathcal{S}_{-p_0})$ , as soon as  $p_0 \geq q_0 + 1$ .

Let  $p_0$  be in  $\mathbf{N}$  such that  $p_0 \geq q_0 + 1$ . Recall the definition of  $\mathcal{R}_n$  given in (3.4). Proposition 2.2 and (3.19) entail that

$$\|\Psi_f\|_{-p_0}^2 \leq \|s \mapsto f(s) |g'_s|_{-q_0}\|_{L^1(\mathcal{R})}^2 \mathcal{R}_{p_0 - q_0} < +\infty.$$

**c)**  $\Phi_f := \int_{\mathcal{R}} f(s) d^\circ G_s$  is equal to  $\Psi_f$  in  $(\mathcal{S})^*$ .

Define the  $(\mathcal{S})^*$ -process  $\tau$  and the family of  $(\mathcal{S})^*$ -processes  $(\tau_N)_{N \in \mathbf{N}}$  by setting, for every real  $s$ ,  $\tau(s) := \sum_{k=0}^{+\infty} f(s) \langle g'_s, e_k \rangle \langle \cdot, e_k \rangle$ , and  $\tau_N(s) := \sum_{k=0}^N f(s) \langle g'_s, e_k \rangle \langle \cdot, e_k \rangle$ . Obviously we have  $\Phi_f = \int_{\mathcal{R}} \tau(s) ds$ ,  $\Psi_f = \lim_{N \rightarrow +\infty} \int_{\mathcal{R}} \tau_N(s) ds$  in  $(\mathcal{S})^*$ . It then remains to show that

$\Phi_f = \lim_{N \rightarrow +\infty} \int_{\mathcal{R}} \tau_N(s) ds$  in  $(\mathcal{S})^*$ . For this purpose, we use Theorem A.2. Let  $(p_0, n)$  be a couple

of integers with  $p_0 \geq q_0 + 1$ . It is easily seen that  $\tau_N$  and  $\tau$  are weakly measurable on  $\mathcal{R}$  (see Definition 4) and, using the same upper-bound we used in **b)**, that  $\tau_N(s)$  and  $\tau(s)$  belong to  $(\mathcal{S}_{-p_0})$  for every real  $s$ . Moreover, using Proposition 2.2 and, again, the upper-bound we used in **b)**, it is clear that both functions  $s \mapsto \|\tau_N(s)\|_{-p_0}$  and  $s \mapsto \|\tau(s)\|_{-p_0}$  belong to  $L^1(\mathcal{R}, ds)$  since  $\|\tau_N(s)\|_{-p_0}^2 \leq \|\tau(s)\|_{-p_0}^2 \leq f^2(s) |g'_s|_{-q_0}^2 \mathcal{R}_{p_0 - q_0}$ . We hence have shown that both functions  $\tau_N(\cdot)$  and  $\tau(\cdot)$  are Bochner integrable on  $\mathcal{R}$ . Besides, for every  $(n, m)$  in  $\mathbf{N}^2$  with  $n \geq m$ , we have, thanks to the previous upper bound,

$$\begin{aligned} \int_{\mathcal{R}} \|\tau_n(s) - \tau_m(s)\|_{-p_0} ds &\leq \int_{\mathcal{R}} \left\| \sum_{k=m+1}^{+\infty} f(s) \langle g'_s, e_k \rangle \langle \cdot, e_k \rangle \right\|_{-p_0} ds \\ &\leq \mathcal{R}_{p_0 - q_0} \|s \mapsto f(s) |g'_s|_{-q_0}\|_{L^1(\mathcal{R})}, \end{aligned}$$

It is then clear that the left hand side of the previous inequality tends to 0 as  $(n, m)$  tends to  $(+\infty, +\infty)$ . Theorem A.2 (see Appendix A.1) applies and establishes **c)**. Finally,  $Z$  is the  $(L^2)$ -limit of a sequence of independent Gaussian variables if  $\sum_{k=0}^{+\infty} (\int_{\mathcal{R}} f(s) \langle g'_s, e_k \rangle ds)^2 < +\infty$ . The equality  $\mathbf{E}[Z^2] = \sum_{k=0}^{+\infty} (\int_{\mathcal{R}} f(s) \langle g'_s, e_k \rangle ds)^2$  then becomes obvious.  $\square$

**Example 3.9. 1.** If  $G$  is a Brownian motion  $B$ , then, using the expression of  $|\delta_t|_{-p}^2$  given in point 1 of Example 3.3, it is easy to see that  $\|s \mapsto |\delta_s|_{-p}^2\|_{L^2(\mathbf{R})}^2 = C' \mathcal{R}_p < +\infty$ , for every positive integer  $p$ . Since one moreover has the equality  $\sum_{k=0}^{+\infty} (\int_{\mathbf{R}} f(s) < \delta_s, e_k > ds)^2 = \|f\|_{L^2(\mathbf{R})}^2$ , Theorem 3.8 allows us to define the Wiener integral of  $f$ , in sense of Definition 1, for any function  $f$  in  $L^2(\mathbf{R})$ . This establishes that, our definition of Wiener integral wrt Brownian motion and the usual definition of Wiener integral coincide exactly.

Using Theorem 2.6, it is clear that  $I_f := \int_0^1 f(s) \diamond B_1 ds$  is well-defined in  $(\mathcal{S})^*$  when  $f$  belongs to  $L^2([0, 1])$ . In view of Point 1 of Proposition 2.4, it is clear that  $I_f$  is an  $(L^2)$  random variable. From the first part of this example one deduces that Theorem 3.8 allows us to define the Wiener integral of  $f$  wrt Brownian bridge, in sense of Definition 1, if and only if  $f$  belongs to  $L^2([0, 1])$ .

2. The case of Wiener integral wrt  $fBm$  has been treated in [LLV14, Section 4] (see in particular Proposition 4.31). It is shown that, for every  $H$  in  $(0, 1)$  the functions for which one can define a Wiener integral wrt  $fBm$ , in sense of Definition 1, are included in the Sobolev space  $L_H^2(\mathbf{R})$ , that has been defined in (2.3). Besides, in view of Example 3.3 and point 1. of this example, it is easy to see that, for every  $H$  in  $(0, 1)$ , from one hand  $|g'_s|_{-p}^2 \leq d_H \mathcal{R}_{p-2/3}$  (for a constant  $d_H$  that only depends on  $H$ ) and, from the other hand, that  $\sum_{k=0}^{+\infty} (\int_{\mathbf{R}} f(s) < g'_s, e_k > ds)^2 = \|f\|_{L_H^2}^2$ . Theorem 3.8 then allows us to extend [LLV14, Proposition 4.31] and to claim that the integral  $\int_{\mathbf{R}} f(s) d^\circ B_s^H$  is the Wiener integral of  $f$ , wrt  $B^H$ , for every function  $f$  in  $L^1(\mathbf{R}) \cap L_H^2(\mathbf{R})$ .

The case of Wiener integral wrt  $mBm$  has been treated in [LLV14, Sections 2.3 & 4] (see in particular Remark 4.30). It is shown that the functions for which one can define a Wiener integral w.r.t.  $fBm$ , in sense of Definition 1, are the functions included in  $\overline{\mathcal{E}(\mathbf{R})}^{<, >_h}$ , where  $<, >_h$  denotes the inner product, defined (in [LLV14, Sections 2.3 & Proposition 3.1.]) by setting  $< \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} >_h = R_h(t, s)$ , and where  $R_h$  has been defined in (2.6).

3. In the case of  $V_\gamma$  - processes, one can improve Proposition 14 of [MV05]. Indeed, denote  $\mathcal{H}$  the set of all functions for which [MV05] define a Wiener integral (see [MV05, Section 3]) and, for any  $\eta : [0, T] \rightarrow \mathbf{R}_+$ , continuous and increasing in a neighborhood of 0 and such that  $\lim_{0+} \eta = 0$ , define the set  $\mathbf{C}^\eta := \{f \in L^2([0, T]), \sup_{0 \leq r < s \leq T} |f(s) - f(r)|/\eta(s-r) < +\infty\}$ . In order to show that  $\mathcal{H}$  contains  $\mathbf{C}^\eta$ , [MV05, Proposition 14] has to require an additional assumption on  $\eta$ . No such an assumption is required here. In fact, it is very easy to show, using only Theorem 3.8 and Remark 4, that for every  $T > 0$ , the process  $(Z_t)_{t \in [0, T]}$ , defined by  $Z_t := \int_0^t f(s) d^\circ G_s$ , where  $f$  belongs to  $\mathbf{C}^\eta$ , is a Gaussian process. One just needs to see that, for every  $(f, s)$  in  $\mathbf{C}^\eta \times [0, T]$ :

$$|f(s)| |\Phi'(s)|_{-q_0} \leq (M + |f(r_0)|) (1 + |\Phi'(s)|_{-q_0})$$

where  $M := \sup_{0 \leq u \leq T} \eta(u) \sup_{0 \leq r < s \leq T} |f(s) - f(r)|/\eta(s-r)$ ,  $\Phi$  and  $\Phi'$  have been defined in Theorem 3.4 and  $r_0$  is any real in  $(0, T]$  such that  $|f(r_0)| < +\infty$ . Using Theorem 3.4, one concludes that  $s \mapsto |f(s)| |\Phi'(s)|_{-q_0}$  belongs to  $L^1((0, T))$  and one then uses Theorem 3.8

**Remark 5.** In fact one can extend the notion of Wiener integral wrt any  $G$  in  $\mathcal{G}$  in two ways. The first way, which is also the more general one, is given in Point 1. of Remark 17. The second way of doing it is the following: If the bilinear form  $<, >_R$ , defined on  $\mathcal{E}(\mathcal{R}) \times \mathcal{E}(\mathcal{R})$  by setting  $< \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} >_R := R(t, s)$  is an inner product; assuming there exists an isometry, denoted  $M : (\mathcal{E}(\mathcal{R}), <, >_R) \rightarrow (L^2(\mathbf{R}), <, >_{L^2(\mathbf{R})})$ , such that  $g_t := M(\mathbf{1}_{[0,t]})$ , then one can extend the notion of Wiener integral to any elements of  $\overline{\mathcal{E}(\mathcal{R})}^{<, >_R}$ . This latter space contains in general not only functions but also tempered distributions. This general method applies to  $fBm$  and  $mBm$  (see [LLV14, Section 3]), as well as to Volterra processes.

**Remark 6.** As it is explained in [SV14, Example 3.3.] the Brownian bridge admits several representations, among which are the orthogonal one, the Fredholm one and the canonical one. It is clear that both the orthogonal and the canonical representations of the Brownian bridge on  $[0, T]$  fulfill Assumption  $(\mathcal{D})$  on  $\mathcal{R} := [0, T]$  (since there exists  $q$  in  $\mathbf{N}^*$  such that  $t \mapsto |g'_t|_{-q}$  belongs to  $L^1([0, b], dt)$ , for every  $b$  in  $[0, T]$ ). This result can be extended to Gaussian bridges (see [GSV07] for more details about this latter notion), assuming the Gaussian process  $G := (G_t)_{t \in [0, T]}$  fulfills Assumption  $(\mathcal{A})$ .

### 3.5 The Wick-Itô integral with respect to Gaussian processes

We still assume in this section that Assumption  $(\mathcal{A})$  holds. We are now able to define, and give the main properties, of the Wick-Itô integral wrt  $G$ . We still denote  $I$  a Borel set of  $\mathcal{R}$  and let  $X := (X_t)_{t \in \mathcal{R}}$  be an  $(\mathcal{S})^*$ -valued process. Because the belonging to  $(\mathcal{S})^*$  is not stable by ordinary product, one can not generalize (3.16) to any  $(\mathcal{S})^*$ -valued process  $X$ , by simply setting:

$$\int_I X_s d^\circ G_s := \int_I X_s \frac{dG_s}{ds} ds = \int_I X_s W_s^{(G)} ds.$$

However, since the belonging to  $(\mathcal{S})^*$  is stable by Wick product one may extend the integral (3.16) to  $(\mathcal{S})^*$ -valued processes  $X$  in the following manner.

**Definition 2** (Wick-Itô integral wrt Gaussian process). *Let  $X : \mathcal{R} \rightarrow (\mathcal{S})^*$  be a process such that the process  $t \mapsto X_t \diamond W_t^{(G)}$  is  $(\mathcal{S})^*$ -integrable on  $\mathcal{R}$ . The process  $X$  is then said to be  $dG$ -integrable on  $\mathcal{R}$  (or integrable on  $\mathcal{R}$ ), wrt the Gaussian process  $G$ . The Wick-Itô integral of  $X$  wrt  $G$ , on  $\mathcal{R}$ , is defined by setting:*

$$\int_{\mathcal{R}} X_s d^\circ G_s := \int_{\mathcal{R}} X_s \diamond W_s^{(G)} ds. \quad (3.20)$$

For any Borel set  $I$  of  $\mathcal{R}$ , define  $\int_I X_s d^\circ G_s := \int_{\mathcal{R}} \mathbf{1}_I(s) X_s d^\circ G_s$ .

The Wick-Itô integral of an  $(\mathcal{S}^*)$ -valued process, wrt  $G$  is then an element of  $(\mathcal{S})^*$ . It is easy to see that Wick-Itô integration wrt  $G$ , is linear and that Definition 2 is coherent with Definition 1, of Wiener integral, we gave in the previous subsection. Moreover, and as it will be stated in Proposition 3.12 below, one of the advantages of Definition 2 is that our integral wrt  $G$  is centered, assuming it belongs to  $(L^2)$ . Used a lot in the sequel of this paper, the following condition ensures the integrability, on  $I$ , of an  $(\mathcal{S})^*$ -valued process  $X$ , wrt  $G$ .

Let  $I$  be a Borel set of  $\mathcal{R}$  and  $X : I \rightarrow (\mathcal{S})^*$  be an  $(\mathcal{S})^*$ -valued process. Denote the following condition:

$$(\mathcal{I}) \quad \begin{cases} (i) : t \mapsto S(X_t)(\eta) \text{ is measurable on } I, \text{ for all } \eta \text{ in } \mathcal{S}(\mathbf{R}) \\ (ii) : \exists (p, q) \in \mathbf{N}^2 \text{ such that the map } t \mapsto \|X_t\|_{-p} \|W_t^{(G)}\|_{-q} \text{ belongs to } L^1(I, dt). \end{cases}$$

When the processes  $X$  and  $G$  satisfy condition  $(\mathcal{I})$  on  $I$ , we will say that  $(X, G)$  satisfies  $(\mathcal{I})$  or  $(\mathcal{I}_{p,q})$ , when we want to specify the value of  $p$  and  $q$  in  $(ii)$  of Condition  $(\mathcal{I})$ . We will use the following theorem a lot in the sequel.

**Theorem 3.10.** *If  $(X, G)$  satisfies condition  $(\mathcal{I}_{p,q})$  on  $I$ , then  $\int_I X_s d^\circ G_s$  is well-defined and belongs to  $(\mathcal{S}_{-r})$  for every  $r \geq 2 + \max\{p; q\}$ . Moreover there exists a real constant  $C$ , independent of  $X$  and  $G$ , such that:*

$$\forall r \geq 2 + \max\{p; q\}; \quad \left\| \int_I X_t d^\circ G_t \right\|_{-r} \leq C \int_I \|X_t\|_{-p} \|W_t^{(G)}\|_{-q} dt. \quad (3.22)$$

**Proof.** For every  $\eta \in \mathcal{S}(\mathbf{R})$ , the measurability on  $I$  of  $t \mapsto S(X_t \diamond W_t^{(G)})(\eta)$  is clear since  $S(X_t \diamond W_t^{(G)})(\eta) = S(X_t)(\eta) \langle g'_t, \eta \rangle$ . Since Condition  $(\mathcal{I})$  is verified, one may write for every  $\eta$  in  $\mathcal{S}(\mathbf{R})$ , using Lemma 2.3:

$$|S(X_t \diamond W_t^{(G)})(\eta)| \leq e^{|\eta|_{\max\{p,q\}}^2} \|X_t\|_{-p} \|W_t^{(G)}\|_{-q}, \quad (3.23)$$

where  $p$  and  $q$  are given by condition  $(\mathcal{I})$ . It is now clear that Theorem 2.6 applies. The upper-bound (3.22), as well as the existence of  $r$  and  $C$ , results from [Kuo96, Theorem 13.5].  $\square$

The next corollary gives a sufficient condition for integrability, wrt  $G$ , of an  $(L^2)$ -valued process.

**Corollary 3.11.** *Let  $\mathcal{K}$  be a compact subset of  $\mathcal{R}_D$  and  $X : t \mapsto X_t$  be a process from  $\mathcal{K}$  into  $(L^2)$  such that  $t \mapsto S(X_t)(\eta)$  is measurable on  $\mathcal{K}$ , for all  $\eta$  in  $\mathcal{S}(\mathbf{R})$ , and such that  $t \mapsto \|X_t\|_0$  belongs to  $L^1(\mathcal{K})$ . If  $t \mapsto |g'_t|_{-q}$  is bounded on  $\mathcal{K}$ , then  $X$  is  $dG$ -integrable on  $\mathcal{K}$  and there exists an integer  $r \geq 2 + \max\{p; q\}$  and a constant  $C_{\mathcal{K}}$  such that,*

$$\forall r \geq 2 + \max\{p; q\}; \quad \left\| \int_{\mathcal{K}} X_t d^\circ G_t \right\|_{-r} \leq C_{\mathcal{K}} \int_{\mathcal{K}} \|X_t\|_0 dt.$$

**Proof.** In view of the proof of Theorem 3.10, the following inequality is sufficient to conclude.

$$|S(X_t \diamond W_t^{(G)})(\eta)| \leq \|X_t\|_0 e^{|\eta|_a^2} \sup_{t \in \mathcal{K}} |g'_t|_{-q}. \quad \square$$

We can now give the first properties of the Wick-Itô integral wrt  $G$ .

**Proposition 3.12.** (i) *Let  $I$  be a Borel subset of  $\mathcal{R}$  and  $X : I \rightarrow (\mathcal{S})^*$  a  $dG$ -integrable process over  $I$ . Assume that  $\int_I X_s d^\circ G_s$  belongs to  $(L^2)$ . Then  $\mathbf{E}[\int_I X_s d^\circ G_s] = 0$ .*

(ii) *Let  $[a, b] \subset \mathcal{R}$ . The  $(\mathcal{S}^*)$ -process  $\Psi$  defined by  $\Psi(t) := \int_a^t X_s d^\circ G_s$  is continuous on  $[a, b]$ , as soon as  $(X, G)$  satisfies condition  $(\mathcal{I})$  on  $[a, b]$ .*

**Proof.** (i) That  $S(\int_I X_s d^\circ G_s)(0) = \int_I S(X_s)(0) S(W_s^{(G)})(0) ds = 0$  is clear since  $S(W_s^{(G)})(0) = \langle g'_s, 0 \rangle = 0$ . Now, it sufficient to note that  $\mathbf{E}[U] = S(U)(0)$  for every r.v.  $U$  in  $(L^2)$ .

(ii) The integrability of  $X$  wrt  $G$  is proved by Theorem 3.10. Let  $t_0$  be fixed in  $(a, b)$ . In order to establish the continuity of  $\Psi$  in  $t_0$  we are going to use [Kuo96, Theorem 8.6]. By symmetry one may assume that  $t_0 \geq t$ . [Kuo96, Theorem 8.6] applies since we clearly have:

$$\begin{aligned} \text{a)} \quad & |S(\Psi(t) - \Psi(t_0))(\eta)| \leq e^{|\eta|_{\max\{p,q\}}^2} \int_t^{t_0} \|X_u\|_{-p} \|W_u^{(G)}\|_{-q} du \xrightarrow{t \rightarrow t_0} 0; \\ \text{b)} \quad & |S(\Psi(t))(\eta)| \leq e^{|\eta|_{\max\{p,q\}}^2} \int_{[a,b]} \|X_u\|_{-p} \|W_u^{(G)}\|_{-q} du. \end{aligned} \quad \square$$

**Proposition 3.13.** *Let  $(X, G)$  be a couple of processes that satisfies condition  $(\mathcal{I})$  on  $\mathcal{R}$ . Define, for every  $n$  in  $\mathbf{N}$ , the process  $G^{(n)} := (G_t^{(n)})_{t \in \mathcal{R}}$  by setting  $G_t^{(n)} := \langle \cdot, g_t^{(n)} \rangle$ , where  $g_t^{(n)}$  belongs to  $L^2(\mathbf{R})$  and let  $(X^{(n)})_{n \in \mathbf{N}} := \{(X_t^{(n)})_{t \in \mathcal{R}}; n \in \mathbf{N}\}$  denote a sequence of  $(\mathcal{S})^*$ -valued processes. Denote  $I$  a subset of  $\mathcal{R}$ . Let us denote the following conditions:*

- $a_1)$   $(X, G^{(n)})$  satisfies condition  $(\mathcal{I})$  on  $I$ , uniformly<sup>10</sup> in  $q$ .
- $a_2)$   $\exists (r, r_1) \in \mathbf{N} \times (0, +\infty]$  such that:  $\|G^{(n)} - G\|_{-r} \xrightarrow{n \rightarrow +\infty} 0$ , where the convergence holds both pointwisely on  $I$ , and in  $L^{r_1}(I)$ ,
- $a_3)$   $X$  is  $(\mathcal{S})^*$ -differentiable on  $I$  and there exist  $(a, l) \in \mathbf{N} \times \mathbf{R}$  and a function  $L \in L^{r_2}(I, dt)$  s.t.

$$\left| \frac{d}{ds} [S(X_s)(\eta)] \right| \leq L(s) e^{a|\eta|_l^2},$$

for all  $\eta$  of  $\mathcal{S}(\mathbf{R})$  and for a.e.  $s$  of  $I$ , where  $r_2 \in (0, +\infty]$  is such that  $r_1^{-1} + r_2^{-1} = 1$ ,

<sup>10</sup> i.e.  $\exists (p, q) \in \mathbf{N}^2$ , such that  $s \mapsto \|X_s\|_{-p} \|W_s^{(G^{(n)})}\|_{-q}$  belongs to  $L^1(I, ds)$ , for every  $n$  in  $\mathbf{N}$ .

$a'_2) \exists (r_1, r_2) \in (0, +\infty]^2$  with  $r_1^{-1} + r_2^{-1} = 1$ , such that:

$$(i) \|s \mapsto |g_s^{(n)} - g'_s|_{-q}\|_{L^{r_1}(I)} \xrightarrow{n \rightarrow +\infty} 0; \quad (ii) \quad s \mapsto \|X_s\|_{-p} \text{ belongs to } L^{r_2}(I).$$

If conditions  $(a_i)_{i \in \{1;2;3\}}$  or both conditions  $(a_1)$  and  $(a'_2)$  are fulfilled, then one has the convergence:

$$\int_I X_s d^\circ G_s^{(n)} \xrightarrow{n \rightarrow +\infty} \int_I X_s d^\circ G_s \quad \text{in } (\mathcal{S})^*.$$

Besides, denote the following conditions:

$b_1) (X^{(n)}, G)$  satisfies condition  $(\mathcal{I})$  on  $I$ , uniformly<sup>11</sup> in  $p$ .

$b_2) \exists r \in \mathbf{N}$  such that:  $\|X^{(n)} - X\|_{-r} \xrightarrow{n \rightarrow +\infty} 0$  pointwise.

$b_3) Both X and (X^{(n)})_{n \in \mathbf{N}}$  are  $(\mathcal{S})^*$ -differentiable on  $I$ . Moreover,  $\exists (l, a, r_2) \in \mathbf{N} \times \mathbf{R} \times (0, +\infty]$  and a function  $L$  in  $L^{r_2}(I, dt)$  such that, for every  $(n, \eta)$  in  $\mathbf{N} \times \mathcal{S}(\mathbf{R})$  and a.e.  $s \in I$ ,

$$\left| \frac{d}{ds} [S(X_s^{(n)})(\eta)] \right| + \left| \frac{d}{ds} [S(X_s)(\eta)] \right| \leq L(s) e^{a|\eta|^2},$$

$b_4) (i) \exists r' \in \mathbf{N}$ , s.t.  $\|s \mapsto \left\| \frac{d}{ds} [X_s - X_s^{(n)}] \right\|_{-r'}\|_{L^{r_2}(I)} \xrightarrow{n \rightarrow +\infty} 0$  and  $(ii) s \mapsto |g_s|_{-q}$  belongs to  $L^{r_1}(I)$ , where  $r_1 \in (0, +\infty]$  is such that  $r_1^{-1} + r_2^{-1} = 1$ .

$b'_2) \exists (r_1, r_2) \in (0, +\infty]^2$  with  $r_1^{-1} + r_2^{-1} = 1$ , such that:

$$(i) \|s \mapsto \|X_s - X_s^{(n)}\|_{-p}\|_{L^{r_2}(I)} \xrightarrow{n \rightarrow +\infty} 0 \text{ to } L^{r_2}(I) \quad (ii) \quad s \mapsto |g'_s|_{-q} \in L^{r_1}(I).$$

If conditions  $(b_i)_{i \in \{1;2;3;4\}}$  or both conditions  $(b_1)$  and  $(b'_2)$  are fulfilled, then one has the convergence:

$$\int_I X_s^{(n)} d^\circ G_s \xrightarrow{n \rightarrow +\infty} \int_I X_s d^\circ G_s \quad \text{in } (\mathcal{S})^*.$$

**Proof.** The scheme of the proof is ‘‘symmetric in  $G^{(n)}$  and  $X^{(n)}$ ’’; we will then only show the convergence  $\int_I X_s d^\circ G_s^{(n)} \xrightarrow{n \rightarrow +\infty} \int_I X_s d^\circ G_s$ . Denote  $A_n := \int_I X_s d^\circ G_s - \int_I X_s d^\circ G_s^{(n)}$ ; let us show that assumptions of [Kuo96, Theorem 8.6] are fulfilled. The existence of all following integrals come from Theorem 2.6.

• Case where  $a_1)$  &  $a'_2)$  are fulfilled:

$$|S(A_n)(\eta)|^{r_1 r_2} = \left( \left| \int_I S(X_s)(\eta) S(W_s^{(G^{(n)})} - W_s^{(G)})(\eta) ds \right| \right)^{r_1 r_2} \leq \left( \int_I \|X_s\|_{-p}^{r_2} ds \right) \left( \int_I |g_s^{(n)} - g'_s|_{-q}^{r_1} ds \right).$$

• Case where  $(a_i)_{i \in \{1;2;3\}}$  are fulfilled : For sake of notational simplicity we assume that  $I = [0, t]$ .

Using an integration by parts, one gets:

$$\begin{aligned} |S(A_n)(\eta)| &= \left| \int_I S(X_s)(\eta) S(W_s^{(G^{(n)})} - W_s^{(G)})(\eta) ds \right| = \left| \int_I S(X_s)(\eta) \langle (g^{(n)})'_s - g'_s, \eta \rangle ds \right| \\ &= \langle g_t^{(n)} - g_t, \eta \rangle S(X_t)(\eta) - \int_I \langle g_s^{(n)} - g_s, \eta \rangle \frac{d}{ds} [S(X_s)(\eta)] ds \\ &\leq e^{(1+a)|\eta|^2} (\|X_t\|_{-p} \|G_t^{(n)} - G_t\|_{-r} + \int_I \|G_s^{(n)} - G_s\|_{-r} L(s) ds). \end{aligned}$$

The Hölder inequality then allows one to establish the two conditions of [Kuo96, Theorem 8.6] and therefore achieves the proof.  $\square$

<sup>11</sup> i.e.  $\exists (p, q) \in \mathbf{N}^2$ , such that  $s \mapsto \|X_s^{(n)}\|_{-p} \|W_s^{(G)}\|_{-q}$  belongs to  $L^1(I, ds)$ , for every  $n$  in  $\mathbf{N}$ .

**Remark 7. 1.** *The advantage of condition  $(\mathcal{I})$  is that it allows us to make assumptions on both elements of the couple  $(X, G)$  instead of making assumptions only on  $X$  or only on  $G$ . This, for example, allows us to make more assumptions on  $X$  than  $G$  or the other way round. In this case, the more informations on the “regularity” of  $X$  (resp. of  $G$ ) one gets, the less informations one needs on the “regularity” of  $G$  (resp. of  $X$ ). We used this property in the proof of Corollary 3.11.*

**2.** *It is clear, in  $a_2)$  of (iii) of Proposition 3.13, that one can also choose the pointwise convergence in  $(L^2)$  or in probability instead of convergence in  $(\mathcal{S}_{-r})$ .*

**3.** *When  $G$  is an  $f$ Bm (resp. an  $m$ Bm), the Wick-Itô integral wrt  $G$  given by Definition 2 is nothing but the fractional (resp. multifractional) Wick-Itô integral defined in [EVdH03, BSØW04, Ben03a] (resp. in [LLV14, LLVH14, Leb13]).*

It is of interest to have also a criterion of integrability for generalized functionals of  $G$ . In that view, we set up the following notation: for  $p \in \mathbf{N}$ ,  $0 < a < b$ , we consider a map  $F : [a, b] \rightarrow \mathcal{L}_p(\mathbf{R})$  (hence  $F(t)$  is a tempered distribution for all  $t$ ). We then define  $F(t, G_t)(t) := F(t)(G_t)$ . Recall the following theorem (see [CG62, Lemmas 1 and 2 p.73-74]):

**Theorem 3.14.** *Let  $I$  be an interval of  $\mathbf{R}$ ,  $t \mapsto F(t)$  be a map from  $I$  into  $\mathcal{S}_{-p}(\mathbf{R})$ ,  $t \mapsto \varphi(t, \cdot)$  be a map from  $I$  into  $\mathcal{S}(\mathbf{R})$  and  $t_0 \in I$ . If both maps  $t \mapsto F(t)$  and  $t \mapsto \varphi(t, \cdot)$  are continuous (respectively differentiable) at  $t_0$ , then the function  $t \mapsto \langle F(t), \varphi(t, \cdot) \rangle$  is continuous (respectively differentiable) at  $t_0$ . In this latter case we have*

$$\frac{d\langle F(t), \varphi(t, \cdot) \rangle}{dt}(t_0) = \langle \frac{dF}{dt}(t_0), \varphi(t_0, \cdot) \rangle + \langle F(t_0), \frac{d}{dt}[t \mapsto \varphi(t, \cdot)](t_0) \rangle.$$

**Theorem 3.15.** *Let  $p \in \mathbf{N}$ ,  $[a, b]$  be an interval of  $\mathcal{Z}_R^c$  and let  $F : [a, b] \rightarrow \mathcal{L}_p(\mathbf{R})$  be a continuous map. If  $t \mapsto \max\{R_t^{-p-1/4}; R_t^{p-1/4}\}$  belongs to  $L^1([a, b])$  (resp. there exists an integer  $q$  such that  $t \mapsto |g'_t|_{-q} \max\{R_t^{-p-1/4}; R_t^{p-1/4}\}$  belongs to  $L^1([a, b])$ ), then the stochastic distribution process  $F(t, G_t)$  is  $(\mathcal{S})^*$ -integrable (resp.  $dG$ -integrable) on  $[a, b]$  (resp. on  $(a, b)$ ).*

**Proof.** We shall apply Theorem 2.6. The measurability of  $t \mapsto S(F(t, G_t))(\eta)$  results from Theorems 3.6 and 3.14, the continuity of  $F$  and the measurability of  $t \mapsto \langle F, \gamma(R_t, \cdot - \langle g_t, \eta \rangle^2) \rangle$ . Lemma 2.3 and (3.14) yield:

$$|S[F(t, G_t)](\eta)| \leq \max\{R_t^{-p}; R_t^p\} R_t^{-1/4} \sqrt{D_p} \max_{s \in [a, b]} |F(s)|_{-p} \exp\{|\eta|_p^2\},$$

which allows us to apply Theorem 2.6 to  $F(t, G_t)$ . The measurability of  $t \mapsto S(F(t, G_t) \diamond W_t^{(G)})(\eta)$  is obvious since  $S(F(t, G_t) \diamond W_t^{(G)})(\eta) = S(F(t, G_t))(\eta) \langle g'_t, \eta \rangle$ . Besides, denote  $r := \max\{q, p\}$ .  $F(t, G_t)$  and  $W_t^{(G)}$  both belong to  $(\mathcal{S}_{-r})$ , for all  $t$  in  $(a, b)$ . Using Lemma 2.3 and (3.14), and defining  $K_p := \sqrt{D_p} \max_{s \in [a, b]} |F(s)|_{-p}$ , we get for all  $t$  in  $(a, b)$ :

$$|S(F(t, G_t) \diamond W_t^{(G)})(\eta)| \leq \|F(t, G_t)\|_{-p} \|W_t^{(G)}\|_{-q} e^{|\eta|_r^2} \leq K_p |g'_t|_{-q} \max\{R_t^{-p+1/4}; R_t^{p+1/4}\} e^{|\eta|_r^2}.$$

Theorem 2.6 applies again to  $t \mapsto F(t, G_t) \diamond W_t^{(G)}$  and achieves the proof.  $\square$

**Remark 8.** *Of course conditions of Theorem 3.15 are obviously verified when the infimum of  $t \mapsto R_t$  on  $[a, b]$  is positive and when its supremum is upper-bounded on  $[a, b]$ . Moreover, in the particular case where these latter conditions hold, Theorem 3.15 entails that both quantities  $\int_a^b f(G_t) dt$  and  $\int_a^b f(G_t) dG_t$  exist in  $(\mathcal{S})^*$ , as soon as  $f$  is a function of polynomial growth.*

**Example 3.16** (Computation of  $\int_0^T G_t d^\circ G_t$ ). *Let  $T > 0$ . Assume that  $[0, T] \subset \mathcal{R}$  and that  $t \mapsto R_t$  is upper-bounded on  $[0, T]$ , then the following equality holds almost surely and in  $(L^2)$ .*

$$\int_0^T G_t d^\diamond G_t = \frac{1}{2} (G_T^2 - R_T) \quad (3.24)$$

Indeed, since  $|S(G_t \diamond W_t^{(G)})(\eta)| \leq \|G_t\|_0 \|W_t^{(G)}\|_{-q} e^{|\eta|_q^2} \leq (1 + \sup_{t \in [0, T]} R_t) |g'_t|_{-q} e^{|\eta|_q^2}$ . Indeed,

Assumption  $\mathcal{A}_{(iii)}$ , allows us to apply Theorem 2.6. This shows that both members of (3.24) belong to  $(\mathcal{S})^*$ . Besides, for every  $\eta$  in  $\mathcal{S}(\mathbf{R})$  we may write, using (ii) of Theorem 3.6,

$$\begin{aligned} S\left(\int_0^T G_t d^\diamond G_t\right)(\eta) &= \int_0^T S(G_t)(\eta) S(W_t^{(G)})(\eta) dt = \int_0^T \langle g_t, \eta \rangle \langle g'_t, \eta \rangle dt \\ &= \frac{1}{2} (S(G_T)(\eta))^2 = S\left(\frac{1}{2} G_T^2\right)(\eta), \end{aligned}$$

and then conclude that  $\int_0^T G_t d^\diamond G_t = \frac{1}{2} G_T^2$ . Point 1. of Proposition 2.4 establishes (3.24) in  $(\mathcal{S})^*$ . The equality in  $(L^2)$  is now clear since  $G_T^2 - R_T$  belongs to  $(L^2)$ .

**Remark 9. 1.** This result will be obtained as a direct consequence of Itô formulas provided in Section 4.

2. In the previous example, we could have replaced the assumption  $t \mapsto R_t$  is upper-bounded on  $[0, T]$  by  $\int_0^T R_t |g'_t|_{-q} dt < +\infty$ .

To end this section, we present a simple but classical stochastic differential equation, driven by a Gaussian process. We need first to generalize the definition of the *Wick exponential*, given at the beginning of Subsection 2.3, to the case where  $\Phi$  belongs to  $(\mathcal{S})^*$ . For any  $\Phi$  in  $(\mathcal{S})^*$  and  $k$  in  $\mathbf{N}^*$  let  $\Phi^{\diamond k}$  denotes the Wick product of  $\Phi$ , taken  $k$  times. For any  $\Phi$  in  $(\mathcal{S})^*$  such that the sum  $\sum_{k=0}^{+\infty} \frac{\Phi^{\diamond k}}{k!}$  converges in  $(\mathcal{S})^*$ , define the Wick-exponential of  $\Phi$ , and denote  $\exp^\diamond \Phi$ , the element of  $(\mathcal{S})^*$  defined by  $\exp^\diamond \Phi := \sum_{k=0}^{+\infty} \frac{\Phi^{\diamond k}}{k!}$ . For  $f$  in  $L^2(\mathbf{R})$  and  $\Phi := \langle \cdot, f \rangle$ , it is easy to verify that  $\exp^\diamond \Phi =: e^{\langle \cdot, f \rangle} :.$

**Example 3.17** (The Gaussian Wick exponential). Let  $\mathcal{R} = \mathbf{R}_+$  and let us consider the following Gaussian stochastic differential equation:

$$(E) : \begin{cases} dX_t = \alpha(t)X_t dt + \beta(t)X_t d^\diamond G_t \\ X_0 \in (\mathcal{S})^*, \end{cases}$$

where  $\alpha : \mathbf{R}_+ \rightarrow \mathbf{R}$  and  $\beta : \mathbf{R}_+ \rightarrow \mathbf{R}$  are two deterministic continuous functions. Of course (E) is a shorthand notation for  $X_t = X_0 + \int_0^t \alpha(s) X_s ds + \int_0^t \beta(s) X_s d^\diamond G_s$ . As in [HOUZ10], it is easy to guess the solution. Let us define the process  $Z$  by setting:

$$Z_t := X_0 \diamond \exp^\diamond \left( \int_0^t \alpha(s) ds + \int_0^t \beta(s) d^\diamond G_s \right), \quad t \in \mathbf{R}_+, \quad (3.25)$$

**Theorem 3.18.** The process  $Z$  defined by (3.25) is the unique solution, in  $(\mathcal{S})^*$ , of (E).

**Proof.** This is a straightforward application of [HOUZ10, Theorem 3.1.2].  $\square$

## 4 Itô Formulas

The main result of this section is Theorem 4.4, which provides an Itô Formula in  $(L^2)$ , for  $C^{1,2}$  functions, with sub-exponential growth. This latter result is given in Subsection 4.1, while the end of this subsection is devoted to a comparison between our Itô formula and all the previous ones, obtained for Gaussian processes in general, and provided in: [AMN01, Theorems 1 & 2], [MV05, Theorem 31], [NT06, Theorem 1], [KRT07, Corollary 8.13], [KR10, Proposition 11.7],

[LN12, Theorem 3.2] and in [SV14, Theorems 4.4 & 4.5]. In particular, it will be shown, in this same subsection, that the Itô formula we provide in Theorem 4.4 is the most general Itô formula for Gaussian processes of the form (1.1), established so far. The main result of Subsection 4.2 is an Itô Formula, in  $(L^2)$ -sense, for generalized functionals of  $G$  (Theorem 4.7). This latter will not only provide us with a Tanaka formula in Section 6, but will also give us a precise reason to define the weighted local time of  $G$ , the way we do it, in Section 6. In order to obtain such an Itô Formula, one needs first to establish an Itô formula in  $(\mathcal{S})^*$ . This is done in Subsection 4.2.1.

Let us first recall a few basic facts about *Lebesgue-Stieljes* & *Riemann-Stieljes* integrals, that will be used extensively in the remaining part of this work. Let  $[a, b]$  be an interval of  $\mathbf{R}$  and  $j : [a, b] \rightarrow \mathbf{R}$  be a function of bounded variation. Denote  $\alpha_j$  the signed measure such that  $j(t) = \alpha_j([a, t])$ , for every  $t$  in  $[a, b]$ . For any function  $f : [a, b] \rightarrow \mathbf{R}$ , denote  $\int_a^b f(s) dj(s)$  or  $\int_a^b f(s) d\alpha_j(s)$  the *Lebesgue-Stieljes integral* of  $f$  with respect to  $j$ , assuming it exists. In this latter case, we will write that  $f \in L^1(I, dj(t))$  or  $L^1(I, \alpha_j)$ . In the particular case where the function  $f$  is continuous on  $[a, b]$ , the *Lebesgue-Stieljes integral* of  $f$  exists and is also equal to the *Riemann-Stieljes integral* of  $f$ , which is denoted and defined by:

$$(R.S.) \int_a^b f(s) dj(s) := \lim_{\pi \rightarrow 0} \sum_{i=1}^n f(\xi_i^{(n)}) (j(x_i) - j(x_{i-1})), \quad (4.1)$$

where the convergence holds uniformly on all finite partitions  $\mathcal{P}_\pi^{(n)} := \{a := x_0 \leq x_1 \leq \dots \leq x_n := b\}$  of  $[a, b]$  such that  $\max_{1 \leq i \leq n} (x_i - x_{i-1}) \leq \pi$  and such that  $\xi_i^{(n)}$  belongs to  $[x_{i-1}, x_i]$ . The following result, will be used extensively in the sequel of this section.

**Lemma 4.1.** *Let  $[a, b]$  be a finite interval of  $\mathbf{R}$ ,  $I$  (resp.  $J$ ) an interval of  $\mathbf{R}_+$  (resp. of  $\mathbf{R}$ ) and let  $L : [a, b] \times I \times J$  be a  $C^1$ -function. Let  $f : [a, b] \rightarrow I$  and  $j : [a, b] \rightarrow J$  be two continuous functions of bounded variation on  $[a, b]$ . Then one has the following equality:*

$$\begin{aligned} L(b, f(b), j(b)) - L(a, f(a), j(a)) &= \int_a^b \frac{\partial L}{\partial u_1}(s, f(s), j(s)) ds + \int_a^b \frac{\partial L}{\partial u_2}(s, f(s), j(s)) df(s) \\ &+ \int_a^b \frac{\partial L}{\partial u_3}(s, f(s), j(s)) dj(s). \end{aligned} \quad (4.2)$$

**Proof.** All the integrands in the right hand side of (4.2) are continuous. Thus the *Lebesgue-Stieljes* integrals in the right hand side of (4.2) are also *Riemann-Stieljes* integrals. It is then easy to deduce (4.2), using (4.1).  $\square$

**Remark 10.** *In view of Theorem-Definition 2.1, it is clear that we can extend the notion of integral in  $(\mathcal{S})^*$  to the case where  $m$  is a signed measure (the notation remaining the same). We will therefore keep the same notations for this integral, whatever the measure  $m$  is (signed or positive).*

In the remaining of this paper, and unless otherwise specify, the measure  $m$  denote a measure, that may be  $\sigma$ -finite or signed.

#### 4.1 Itô Formula in $(L^2)$ for $C^{1,2}$ functions with sub-exponential growth

Let us begin with the following lemma, the proof of which is an immediate consequence of [Wid44, Theorems 1,2 p.88-89].

**Lemma 4.2.** Let  $T > 0$  and  $v : [0, T] \times \mathbf{R} \rightarrow \mathbf{R}$  be a continuous function such that there exists a couple  $(C_T, \lambda_T)$  of  $\mathbf{R} \times \mathbf{R}_+$  such that  $\max_{t \in [0, T]} |v(t, y)| \leq C_T e^{\lambda_T y^2}$  for all real  $y$ . Define; for every  $a > \lambda_T$ , the map  $J_v : \mathbf{R}_+ \times (0, 1/4a) \times \mathbf{R} \rightarrow \mathbf{R}$  by setting:

$$J_v(t, u_1, u_2) := \int_{\mathbf{R}} v(t, x) \gamma(u_1, x - u_2) dx. \quad (4.3)$$

Then  $J_v$  is well defined. Moreover  $\lim_{(t, u_1, u_2) \rightarrow (t_0, 0^+, l_0)} J_v(t, u_1, u_2) = v(t_0, l_0)$ ,  $\forall (t_0, l_0)$  in  $[0, T] \times \mathbf{R}$ .

It is easy to extend [Ben03b, Theorem 2.8] to the case of a Borel measure  $m$  instead of the Lebesgue measure. The next result, which constitutes this extension, is more suitable than Theorem 2.6, when one deals with  $L^2$ -valued integrands. The proof being obvious, is then left to the reader.

**Theorem 4.3.** Let  $m$  be a positive measure on  $(\mathcal{R}, \mathcal{B}(\mathcal{R}))$  and  $X : \mathcal{R} \rightarrow (L^2)$  be such that the function  $t \mapsto S(X_t)(\eta)$  is measurable, for all  $\eta$  in  $\mathcal{S}(\mathbf{R})$ , and such that  $t \mapsto \|X_t\|_0$  belongs to  $L^1(\mathcal{R}, m)$ . Then  $X$  is  $(\mathcal{S})^*$ -integrable over  $\mathcal{R}$  and verifies:

$$\left\| \int_{\mathcal{R}} X_t m(dt) \right\|_0 \leq \int_{\mathcal{R}} \|X_t\|_0 m(dt).$$

Through this subsection, we assume that  $T > 0$  and define  $\mathcal{R} := [0, T]$ . We can now give the main result of this section. Denote  $C^{1,2}([0, T] \times \mathbf{R}, \mathbf{R})$  the set of functions of two variables which belongs to  $C^1([0, T], \mathbf{R})$  as function of their first variable and to  $C^2(\mathbf{R}, \mathbf{R})$  as function of their second variable.

**Theorem 4.4.** Let  $T > 0$ . Let  $f$  be a  $C^{1,2}([0, T] \times \mathbf{R}, \mathbf{R})$  function. Furthermore, assume that there are constants  $C \geq 0$  and  $\lambda < (4 \max_{t \in [0, T]} R_t)^{-1}$  such that for all  $(t, x)$  in  $[0, T] \times \mathbf{R}$ ,

$$\max_{t \in [0, T]} \{ |f(t, x)|, |\frac{\partial f}{\partial t}(t, x)|, |\frac{\partial f}{\partial x}(t, x)|, |\frac{\partial^2 f}{\partial x^2}(t, x)| \} \leq C e^{\lambda x^2}. \quad (4.4)$$

Assume moreover that Assumption  $(\mathcal{A})$  holds and that the map  $t \mapsto R_t$  is both continuous and of bounded variations on  $[0, T]$ . Then the following equality holds in  $(L^2)$ :

$$f(T, G_T) = f(0, 0) + \int_0^T \frac{\partial f}{\partial t}(t, G_t) dt + \int_0^T \frac{\partial f}{\partial x}(t, G_t) d^\circ G_t + \frac{1}{2} \int_0^T \frac{\partial^2 f}{\partial x^2}(t, G_t) dR_t. \quad (4.5)$$

**Proof.** The general structure of this proof is similar to the proof of [Ben03b, Theorem 5.3]. However, one can not follow this latter completely since one does not assume (as it is the case for fBm) that  $\mathcal{Z}_R^T = \{0\}$ , where we set  $\mathcal{Z}_R^T := \mathcal{Z}_R \cap [0, T]$ . Equality (4.5) may be rewritten as:

$$\int_0^T \frac{\partial f}{\partial x}(t, G_t) d^\circ G_t = f(T, G_T) - f(0, 0) - \int_0^T \frac{\partial f}{\partial t}(t, G_t) dt - \frac{1}{2} \int_0^T \frac{\partial^2 f}{\partial x^2}(t, G_t) dR_t. \quad (4.6)$$

Thanks to (4.4) we may write, for every  $K$  in  $\{f, \frac{\partial f}{\partial t}, \frac{\partial f}{\partial x}, \frac{\partial^2 f}{\partial x^2}\}$  and  $t$  in  $[0, T]$ , that  $\mathbf{E}[K(t, G_t)^2] \leq C^2 (1 - 4\lambda \overline{R})^{-1/2} =: M^2$ , where we set  $\overline{R} := \sup\{R_t; t \in [0, T]\}$ . Moreover,  $t \mapsto \|K(t, G_t)\|_0$  belongs to  $L^1([0, T], dt)$  while  $t \mapsto \|\frac{\partial^2 f}{\partial x^2}(t, G_t)\|_0$  belongs to  $L^1([0, T], dR_t)$ . The measurability of the maps  $t \mapsto S(K(t, G_t))(\eta)$  will become clear thanks to (4.7). A simple application of Theorem 4.3 then yields that all members on the right hand side of (4.6) exist and are in  $(L^2)$ . Moreover, Lemma 2.3 provides the upper-bound

$$|S(\frac{\partial f}{\partial x}(t, G_t) \diamond W_t^{(G)})(\eta)| \leq M |g'_t|_{-q} e^{|\eta|_q^2},$$

for all  $(\eta, t)$  in  $\mathcal{S}(\mathbf{R}) \times [0, T]$ , where  $q$  is given by Assumption  $\mathcal{A}_{(iii)}$ . A straightforward application of Theorem 2.6 then shows that  $\int_0^T \frac{\partial f}{\partial x}(t, G_t) d^\circ G_t$  belongs to  $(\mathcal{S})^*$ . In order to prove the theorem, it then just remains to show that  $S$ -transform of both sides of (4.6) are equal. For this purpose, we first give an integral representation of  $S$ -transform of  $K(t, G_t)$ . Since  $\mathbf{E}[e^{\langle \cdot, \eta \rangle}] = 1$ , for every  $\eta$  in  $\mathcal{S}(\mathbf{R})$ , one can define a probability measure  $\mathbf{Q}_\eta$  on the space  $(\Omega, \mathcal{F})$  by setting  $\frac{d\mathbf{Q}_\eta}{d\mu} \stackrel{\text{def}}{=} e^{\langle \cdot, \eta \rangle}$ , where  $\frac{d\mathbf{Q}_\eta}{d\mu}$  denotes the Radon-Nikodym derivative of  $\mathbf{Q}_\eta$  with respect to  $\mu$ . To make computations easier we use the following obvious fact:  $\mathcal{L}_{X+\mathcal{S}(X)(\eta)}^{\mathbf{Q}} = \mathcal{L}_X^{\mathbf{Q}_\eta}$ , for every centered Gaussian random variable  $X$  and  $\eta$  in  $\mathcal{S}(\mathbf{R})$ , and where  $\mathcal{L}_Y^\rho$  denotes the law of a random variable  $Y$  under the probability measure  $\rho$ . In view of this fact, it is clear that  $G_t$  is a Gaussian variable with mean  $\langle g_t, \eta \rangle$  and variance  $R_t$ , under the probability measure  $\mathbf{Q}_\eta$ . One then gets, for every  $t$  in  $[0, T]$  and  $\eta$  in  $\mathcal{S}(\mathbf{R})$ :

$$S(K(t, G_t))(\eta) = \mathbf{E}_{\mathbf{Q}_\eta}[K(t, G_t)] = \int_{\mathbf{R}} K(t, u) R_t^{1/2} + \langle g_t, \eta \rangle \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du. \quad (4.7)$$

Denote  $\Gamma_R := [0, T] \setminus \mathcal{Z}_R^T$  and let  $\eta$  be in  $\mathcal{S}(\mathbf{R})$ . It is clear, in view of (4.7), that one has the equality:

$$S(K(t, G_t))(\eta) = \begin{cases} \int_{\mathbf{R}} K(t, v) \gamma(R_t, v - \langle g_t, \eta \rangle) dv, & \forall t \in \Gamma_R, \\ K(t, 0), & \forall t \in \mathcal{Z}_R^T, \end{cases} \quad (4.8)$$

Let  $a$  be a real in  $(\lambda, (4\bar{R})^{-1})$ . Thanks to Lemma 4.2, we know that the map  $J_K$  is well defined on  $\Sigma_a := [0, T] \times (0, 1/4a) \times \mathbf{R}$  and we clearly have:

$$S(K(t, G_t))(\eta) = J_K(t, R_t, \langle g_t, \eta \rangle), \quad \forall t \in \Gamma_R. \quad (4.10)$$

Moreover, it is clear that  $J_f$  is a  $C^1$ -function on  $\Sigma_a$ . Denote, for every  $\eta$  in  $\mathcal{S}(\mathbf{R})$ ,  $j_\eta : [0, T] \rightarrow \mathbf{R}$  the map defined by  $j_\eta(t) := \langle g_t, \eta \rangle$ . According to Point 1 of Remark 1,  $j_\eta$  is absolutely continuous on  $[0, T]$ . We first have the following result.

**Lemma 4.5.** *Let  $a$  be a real in  $(\lambda, (4\bar{R})^{-1})$  and  $J$  be the map, defined on  $\Sigma_a$  by (4.3). For every  $(t, \eta)$  in  $\Gamma_R \times \mathcal{S}(\mathbf{R})$ , one has the following equalities:*

$$\begin{aligned} J_{\frac{\partial^2 f}{\partial x^2}}(t, R_t, j_\eta(t)) &= 2 \frac{\partial J_f}{\partial u_1}(t, R_t, j_\eta(t)); & J_{\frac{\partial f}{\partial x}}(t, R_t, j_\eta(t)) &= \frac{\partial J_f}{\partial u_2}(t, R_t, j_\eta(t)); \\ \& J_{\frac{\partial f}{\partial t}}(t, R_t, j_\eta(t)) &= \frac{\partial J_f}{\partial t}(t, R_t, j_\eta(t)). \end{aligned}$$

**Proof of Lemma 4.5.** Using from one hand the equality  $\frac{\partial \gamma}{\partial t} = \frac{1}{2} \frac{\partial^2 \gamma}{\partial x^2}$ , valid on  $\mathbf{R}_+^* \times \mathbf{R}$ , and, from the other hand the theorem of differentiation under the integral sign, in a neighborhood of every  $(t, u_1, u_2)$  in  $\Sigma_a$ , provide equalities stated in Lemma 4.5 on each  $(t, u_1, u_2)$  of  $\Sigma_a$  and then allows us to conclude.  $\square$

Using Lemma 4.5, one gets, for every  $\eta$  in  $\mathcal{S}(\mathbf{R})$ ,

$$\mathcal{I}_\eta^{(1)} := \int_0^T S\left(\frac{\partial^2 f}{\partial x^2}(t, G_t)\right)(\eta) dR_t = \int_{\Gamma_R} J_{\frac{\partial^2 f}{\partial x^2}}(t, R_t, j_\eta(t)) dR_t = 2 \int_{\Gamma_R} \frac{\partial J_f}{\partial u_1}(t, R_t, j_\eta(t)) dR_t, \quad (4.11)$$

$$\mathcal{I}_\eta^{(2)} := \int_0^T S\left(\frac{\partial f}{\partial x}(t, G_t)\right)(\eta) S(W_t^{(G)})(\eta) dt = \int_{\Gamma_R} \frac{\partial J_f}{\partial u_2}(t, R_t, j_\eta(t)) dj_\eta(t), \quad (4.12)$$

$$\mathcal{I}_\eta^{(3)} := \int_0^T S\left(\frac{\partial f}{\partial t}(t, G_t)\right)(\eta) dt = \int_{\Gamma_R} \frac{\partial J_f}{\partial t}(t, R_t, j_\eta(t)) dt + \int_{\mathcal{Z}_R^T} \frac{\partial f}{\partial t}(t, 0) dt. \quad (4.13)$$

Thus, in order to end the proof, one just has to establish the following equality:

$$\begin{aligned}
S(f(T, G_T))(\eta) - S(f(0, 0))(\eta) &= \int_{\Gamma_R} \frac{\partial J_f}{\partial t}(t, R_t, j_\eta(t)) dt + \int_{\Gamma_R} \frac{\partial J_f}{\partial u_1}(t, R_t, j_\eta(t)) dR_t \\
&\quad + \int_{\Gamma_R} \frac{\partial J_f}{\partial u_2}(t, R_t, j_\eta(t)) dj_\eta(t) + \int_{\mathcal{Z}_R^T} \frac{\partial f}{\partial t}(t, 0) dt. \quad (4.14)
\end{aligned}$$

Since  $\Gamma_R$  is an open set of  $[0, T]$ , that does not contain 0, it can be written under the form

$$\Gamma_R = \bigsqcup_{i \in \mathbf{N}} (a_i, b_i) \sqcup (b, T], \quad (4.15)$$

where all the intervals in (4.15) are disjoint and where, by convention,  $(x, y) = (x, y] = \emptyset$ , for every reals  $x$  and  $y$  such that  $x \geq y$ . Note moreover that every element of  $\{a_i, b_i, i \in \mathbf{N}\}$  (as well as  $b$ , if  $(b, T] \neq \emptyset$ ) belongs to  $\mathcal{Z}_R^T$ . We need to distinguish between two cases:

**First case:**  $\exists (a', b') \in (0, T)^2$  with  $a' < b'$  s.t.  $(0, a')$  and  $(b', T]$  are both subsets of  $\Gamma_R$ .

Define  $a := \sup\{a' \in [0, T], \text{ s.t. } (0, a') \subset \Gamma_R\}$  and  $b := \inf\{b' \in [0, T], \text{ s.t. } (b', T] \subset \Gamma_R\}$ . Even if one has to consider a subset  $I$  of  $\mathbf{N}$ , one may assume, and we will in the sequel, that  $(a_i, b_i) \neq \emptyset$ , for every  $i$  in  $\mathbf{N}$ . One can find  $\rho$  in  $\mathbf{R}_+^*$  such that  $(\rho, a - \rho) \neq \emptyset$ ,  $(b + \rho, T - \rho) \neq \emptyset$ . Moreover, for every  $i$  in  $\mathbf{N}$ , one can find  $\rho_i$  in  $\mathbf{R}_+^*$  such that  $(a_i + \rho_i, b_i - \rho_i) \neq \emptyset$ . Since all these intervals belong to  $\Gamma_R$ , one can apply Lemma 4.1, with  $L = J_f$ , on each one of them. We then get, for every interval  $(x, y)$  in the set of intervals  $\Upsilon := \{(\rho, a - \rho), (b + \rho, T - \rho), (a_i + \rho_i, b_i - \rho_i); i \in \mathbf{N}\}$ ,

$$\begin{aligned}
&J_f(y, R_y, j_\eta(y)) - J_f(x, R_x, j_\eta(x)) \\
&= \int_x^y \frac{\partial J_f}{\partial t}(t, R_t, j_\eta(t)) dt + \int_x^y \frac{\partial J_f}{\partial u_1}(t, R_t, j_\eta(t)) dR_t + \int_x^y \frac{\partial J_f}{\partial u_2}(t, R_t, j_\eta(t)) dj_\eta(t). \quad (\Lambda_{x,y})
\end{aligned}$$

For any interval  $(x', y')$  which belongs to  $\{(0, a), (b, T), (a_i, b_i); i \in \mathbf{N}\}$ , there exists a sequence of elements  $(x_n, y_n)_{n \in \mathbf{N}}$  in  $\Upsilon^{\mathbf{N}}$  such that  $(x_n, y_n) \rightarrow (x', y')$ , as  $n \rightarrow +\infty$  and such that  $[x_n, y_n] \subset (x', y')$ . Lemma 4.2 then provides the convergence of the left hand side of  $(\Lambda_{x_n, y_n})$  to  $f(y', 0) - f(x', 0)$ , if  $(x', y')$  belongs to  $\{(0, a), (a_i, b_i); i \in \mathbf{N}\}$ , and to  $S(f(T, G_T))(\eta) - f(b, 0)$ , if  $(x', y') = (b, T)$ . Besides, the Lebesgue's dominated convergence theorem applies to each integrand of the right hand side of  $(\Lambda_{x_n, y_n})$ , since they are all continuous. This provides the convergence of the right hand side of  $(\Lambda_{x_n, y_n})$ , for any  $(x', y')$  in  $\{(0, a), (b, T), (a_i, b_i); i \in \mathbf{N}\}$ , to

$$\int_{x'}^{y'} \frac{\partial J_f}{\partial t}(t, R_t, j_\eta(t)) dt + \int_{x'}^{y'} \frac{\partial J_f}{\partial u_1}(t, R_t, j_\eta(t)) dR_t + \int_{x'}^{y'} \frac{\partial J_f}{\partial u_2}(t, R_t, j_\eta(t)) dj_\eta(t).$$

In view of (4.9) and (4.10), and making the summation of  $(\Lambda_{a_i, b_i})$ , over all  $i \in \mathbf{N}$ , we then get:

$$\begin{aligned}
&S(f(T, G_T))(\eta) - S(f(0, 0))(\eta) - (f(b, 0) - f(a, 0) - \sum_{i \in \mathbf{N}} (f(b_i, 0) - f(a_i, 0))) \\
&= \int_{\Gamma_R} \frac{\partial J_f}{\partial t}(t, R_t, j_\eta(t)) dt + \int_{\Gamma_R} \frac{\partial J_f}{\partial u_1}(t, R_t, j_\eta(t)) dR_t + \int_{\Gamma_R} \frac{\partial J_f}{\partial u_2}(t, R_t, j_\eta(t)) dj_\eta(t). \quad (4.16)
\end{aligned}$$

Denote  $\Delta := \int_{\mathcal{Z}_R^T} \frac{\partial f}{\partial t}(t, 0) dt$ , one has the equality:

$$\begin{aligned}
\Delta &= \int_{[0, T]} \frac{\partial f}{\partial t}(t, 0) dt - \int_{\Gamma_R} \frac{\partial f}{\partial t}(t, 0) dt = \int_{[0, T] \setminus (0, a] \sqcup (b, T]} \frac{\partial f}{\partial t}(t, 0) dt - \sum_{i \in \mathbf{N}} \int_{a_i}^{b_i} \frac{\partial f}{\partial t}(t, 0) dt \\
&= f(b, 0) - f(a, 0) - \sum_{i \in \mathbf{N}} (f(b_i, 0) - f(a_i, 0)). \quad (4.17)
\end{aligned}$$

Using (4.17), Equality (4.16) then reads:

$$\begin{aligned}
& S(f(T, G_T))(\eta) - S(f(0, 0))(\eta) - \int_{\mathcal{Z}_R^T} \frac{\partial f}{\partial t}(t, 0) dt \\
&= \int_{\Gamma_R} \frac{\partial J_f}{\partial t}(t, R_t, j_\eta(t)) dt + \int_{\Gamma_R} \frac{\partial J_f}{\partial u_1}(t, R_t, j_\eta(t)) dR_t + \int_{\Gamma_R} \frac{\partial J_f}{\partial u_2}(t, R_t, j_\eta(t)) dj_\eta(t), \quad (4.18)
\end{aligned}$$

which is nothing but (4.14) and therefore ends the proof in this case.

**Second case:** There is no  $(a', b')$  in  $(0, T)^2$  with  $a' < b'$  s.t.  $(0, a')$  &  $(b', T]$  are both subsets of  $\Gamma_R$ .

Since the cases of 0 and  $T$  can be treated in the same manner, we only treat here the case of  $T$ . We then assume that there is no  $b'$  in  $(0, T)$  such that  $(b', T] \subset \Gamma_R$ .

We need to distinguish between two cases. If there exists  $b'$  in  $(0, T)$  such that  $(b', T] \subset \mathcal{Z}_R^T$  then the problem can be reduced to establish (4.5) between 0 and  $\hat{b}$ , where  $\hat{b} := \inf\{b \in (0, T); [b', T] \subset \mathcal{Z}_R^T\}$ . Otherwise, one can find an increasing sequence  $(T_n)_{n \in \mathbf{N}}$  of  $\Gamma_R^{\mathbf{N}}$ , which converge to  $T$ . For every  $n$  in  $\mathbf{N}$ , denote  $(a_i^{(n)}, b_i^{(n)})$  the interval  $(a_i, b_i)$  of  $\Gamma_R$  which contains  $T_n$ .

For every integer  $n$ ,  $(a_i^{(n)}, T_n]$  is a non empty subset of  $\Gamma_R$ . Therefore, one can use first case to establish (4.5) between 0 and  $T_n$ . To establish the equality of S-transform of both sides of (4.5), (between 0 and  $T_n$ ), it then remains to apply, from one hand Lemma 4.2 to  $J_f(T_n, R_{T_n}, j_\eta(T_n))$  and, from the other hand, Lebesgue's dominated convergence theorem to the following integrals:

$$\begin{aligned}
\mathcal{I}_{\eta, n}^{(1)} &:= \int_0^{T_n} S\left(\frac{\partial^2 f}{\partial x^2}(t, G_t)\right)(\eta) dR_t = 2 \int_{\Gamma_R^{(n)}} \frac{\partial J_f}{\partial u_1}(t, R_t, j_\eta(t)) dR_t, \\
\mathcal{I}_{\eta, n}^{(2)} &:= \int_0^{T_n} S\left(\frac{\partial f}{\partial x}(t, G_t)\right)(\eta) S(W_t^{(G)}) (\eta) dt = \int_{\Gamma_R^{(n)}} \frac{\partial J_f}{\partial u_2}(t, R_t, j_\eta(t)) dj_\eta(t), \\
\mathcal{I}_{\eta, n}^{(3)} &:= \int_0^{T_n} S\left(\frac{\partial f}{\partial t}(t, G_t)\right)(\eta) dt = \int_{\Gamma_R^{(n)}} \frac{\partial J_f}{\partial t}(t, R_t, j_\eta(t)) dt + \int_{\mathcal{Z}_R^{T, (n)}} \frac{\partial f}{\partial t}(t, 0) dt,
\end{aligned}$$

where  $\Gamma_R^{(n)}$  denotes  $\Gamma_R \cap [0, T_n]$  and  $\mathcal{Z}_R^{T, (n)}$  denotes  $\mathcal{Z}_R^T \cap [0, T_n]$ . This result and the fact that  $J_f$  is a  $C^1$  function on  $\Sigma_a$  allows us to apply [Kuo96, Theorem 8.6] and thus to conclude.  $\square$

## Comparison with other Itô formulas for Gaussian processes

Since [AMN01], many Itô formula for Gaussian processes have been established. If one excepts Itô formula for Gaussian semimartingales, that are well known, all the Itô formulas provided, for Gaussian processes in general, in the literature of *functional extensions* so far (namely: [AMN01, Theorems 1 & 2], [MV05, Theorem 31], [NT06, Theorem 1], [KRT07, Corollary 8.13], [KR10, Proposition 11.7], [LN12, Theorem 3.2] and [SV14, Theorems 4.4 & 4.5]) are established using the divergence type integral.

If one excepts [SV14, Theorems 4.4 & 4.5], the assumptions made in all the Itô formulas provided in the references cited right above<sup>12</sup> imply that the variance function  $t \mapsto R_t$  is continuous and of bounded variations on  $[0, T]$ . If one excepts [KRT07, Corollary 8.13], the assumptions made in all the Itô formulas provided in the references cited above<sup>13</sup> imply that the function  $f$  is of class  $C^2$  and, together with all its derivatives, with sub-exponential growth (*i.e.* fulfills (4.4)). In view of this fact, it appears that the assumptions made in Theorem 4.4 are minimal. However,

<sup>12</sup>*i.e.* in [AMN01, Theorems 1 & 2], [MV05, Theorem 31], [NT06, Theorem 1], [KRT07, Corollary 8.13], [KR10, Proposition 11.7] and [LN12, Theorem 3.2].

<sup>13</sup>*i.e.* in [AMN01, Theorems 1 & 2], [MV05, Theorem 31], [NT06, Theorem 1], [KR10, Proposition 11.7], [LN12, Theorem 3.2] and [SV14, Theorems 4.4 & 4.5]

in order to see to what extent Theorem 4.4 generalizes Itô formulas for Gaussian processes that already exist, let us compare it to them, in details.

As we stated above, Theorem 4.4 (for  $f(t, x) := f(x)$ ) is less restrictive than any Itô formula provided in [AMN01, Theorems 1 & 2], [MV05, Theorem 31], [NT06, Theorem 1], [KR10, Proposition 11.7] and [LN12, Theorem 3.2] since they all require - at least - the same assumptions as the one made in Theorem 4.4. In addition,  $f$  is assumed to be of class  $C^\infty$  in [MV05, Theorem 31] and [KR10, Proposition 11.7], and of class  $C^7$  in [NT06, Theorem 1]. Moreover, the variance function  $t \mapsto R_t$  is assumed to: be of class  $C^2$  on  $\mathbf{R}_+^*$  in [MV05, Theorem 31], fulfill Assumptions (3) et (4) in [NT06, Theorem 1], fulfill Assumptions (A), (B) and (C) of [KR10, p.12-13] in [KR10, Proposition 11.7] and, in [LN12, Theorem 3.2], to verify the following conditions:

(i) for every  $s$  in  $[0, T]$ , the map  $t \mapsto R(t, s)$  is absolutely continuous on  $[0, T]$ ;

(ii) there exists  $\alpha > 1$  such that:  $\sup_{t \in [0, T]} \int_0^T \left| \frac{\partial R}{\partial s}(s, t) \right|^\alpha ds < +\infty$ .

Besides, in [AMN01, Theorems 1 & 2], and in addition to the assumptions made in Theorem 4.4, the kernel  $K$  has to fulfill Assumptions (K1) to (K3), in the singular case, and (K1) to (K4), in the regular case.

It now remains to discuss [KRT07, Corollary 8.13] and [SV14, Theorems 4.4 & 4.5]. In [SV14, Theorems 4.4 & 4.5], where  $G$  is assumed to be separable (see [SV14, Definition 2.1.]),  $t \mapsto R_t$  is assumed to be bounded, as well as of bounded variations on  $[0, T]$ , but not necessarily continuous. However, in order to establish Itô formulas, the authors of [SV14] have to require two additional assumptions on the covariance function  $R$ , that are:

(i) for every  $s$  in  $[0, T]$ , the map  $t \mapsto R(t, s)$  is of bounded variation on  $[0, T]$ ;

(ii)  $\sup_{t \in [0, T]} \int_0^T |R|(ds, t) < +\infty$ .

Finally,  $R$  is assumed to be continuous and of bounded variations on  $[0, T]$  in [KRT07, Corollary 8.13] and  $f$  is assumed to be of class  $C^2$  but not with sub-exponential growth; the second derivative of  $f$  therein is assumed to be bounded. However, since the stochastic calculus for Gaussian processes, developed in [KRT07] requires that the covariance function has a planar bounded variation, which corresponds to “regular” processes (such as fBm for  $H > 1/2$ ), one easily sees that the price to pay for relaxing the assumption on the growth of  $f$  is that one can not deal with irregular Gaussian processes. That is precisely to overcome this limitation on the regularity of  $G$  that [KR10], which requires much more than the growth condition we make on  $f$ , as we stated above, has been written.

In view of the arguments developed above, it appears that Theorem 4.4 is the most general *functional extension*-type Itô formula for Gaussian processes of the form (1.1) so far.

**Remark 11.** *Of course all the Gaussian processes in  $\mathcal{G}$  of “reference” fulfill assumptions of Theorem 4.4. In the case of mBm one needs to assume that  $h$  is a  $C^1$  function with its derivative bounded on  $\mathcal{R}_D$ . Note moreover that, applying Theorem 4.4 when  $G$  is:*

-a fBm allows one to recover [Ben03a, Theorem 4.1 & Rk. 4.6],

-a mBm allows one to recover [LLV14, Theorem 5.5],

-a  $\mathcal{V}_\gamma$  - process allows one to recover and extend, as we showed above, [MV05, Theorem 31].

## 4.2 Itô Formula for generalized functionals of $G$

A little bit more technical than the previous one, this subsection is of crucial importance for Section 6 and the study of Local times of  $G$ .

#### 4.2.1 Itô Formula in $(\mathcal{S})^*$ for generalized functionals of $G$ on an interval $[a, b]$ of $\mathcal{Z}_R^c$

Before establishing an Itô formula in  $(\mathcal{S})^*$ , let us first fix some notations. For a tempered distribution  $G$  and a positive integer  $n$ , let  $G^{(n)}$  denote the  $n^{\text{th}}$  distributional derivative of  $G$ . We also write  $G' := G^{(1)}$ . Hence, by definition, the equality  $\langle G', \varphi \rangle = -\langle G, \varphi' \rangle$  holds for all  $\varphi$  in  $\mathcal{S}(\mathbf{R})$ . For a map  $t \mapsto F(t)$  from  $[a, b]$  to  $\mathcal{L}_p(\mathbf{R})$  we will note  $\frac{\partial^n F}{\partial x^n}(t)$  the quantity  $(F(t))^{(n)}$ , that is the  $n^{\text{th}}$  derivative in  $\mathcal{S}'(\mathbf{R})$ , of the tempered distribution  $F(t)$ . Hence we may consider the map  $t \mapsto \frac{\partial^n F}{\partial x^n}(t)$  from  $[a, b]$  to  $\mathcal{S}'(\mathbf{R})$ . Moreover for any  $t_0$  in  $[a, b]$ , we will note  $\frac{\partial F}{\partial t}(t_0)$  the quantity  $\lim_{r \rightarrow 0} \frac{F(t_0+r) - F(t_0)}{r}$ , when it exists in  $\mathcal{L}_p(\mathbf{R})$ , for a certain integer  $p$ . When it exists,  $\frac{\partial F}{\partial t}(t_0)$  is a tempered distribution, which is said to be the derivative of the distribution  $F(t)$  with respect to  $t$  at point  $t = t_0$ . In line with Section 3.2, we then define, for  $t_0$  in  $[a, b]$  and a positive integer  $n$ , the following quantities:  $\frac{\partial^n F}{\partial x^n}(t_0, G_{t_0}) := (F(t_0))^{(n)}(G_{t_0})$  and  $\frac{\partial F}{\partial t}(t_0, G_{t_0}) := \left(\frac{\partial F}{\partial t}(t_0)\right)(G_{t_0})$ .

**Theorem 4.6.** *Let  $[a, b]$  be an interval of  $\mathcal{Z}_R^c$ ,  $p \in \mathbf{N}$ , and let  $F$  be an element of  $C^1([a, b], \mathcal{L}_p(\mathbf{R}))$  such that both maps  $\frac{\partial F}{\partial x}$  and  $\frac{\partial^2 F}{\partial x^2}$ , from  $[a, b]$  into  $\mathcal{L}_p(\mathbf{R})$ , are continuous. Assume moreover that  $(\mathcal{A})$  holds and that the map  $t \mapsto R_t$  is both continuous and of bounded variations on  $[a, b]$ . Then the following equality holds in  $(\mathcal{S})^*$ :*

$$F(b, G_b) - F(a, G_a) = \int_a^b \frac{\partial F}{\partial t}(s, G_s) ds + \int_a^b \frac{\partial F}{\partial x}(s, G_s) d^\circ G_s + \frac{1}{2} \int_a^b \frac{\partial^2 F}{\partial x^2}(s, G_s) dR_s. \quad (4.19)$$

**Proof.** Assumption  $(\mathcal{A}_{\text{iii}})$  and the fact the map  $t \mapsto R_t$  is upper and lower bounded on  $[a, b]$  allow one to apply Theorem 3.15 to  $\frac{\partial F}{\partial t}(s, G_s)$ ,  $\frac{\partial^2 F}{\partial x^2}(s, G_s)$  (with the Borel measure defined by  $t \mapsto R_t$ ), and to  $\frac{\partial F}{\partial x}(s, G_s)$ . This and Remark 8 entail that all the integrals on the right side of (4.19) exist. Thanks to Lemma 2.5 it is then sufficient to show the equality of the  $S$ -transforms of both sides of (4.19). Using Theorem 3.6 one then just has to establish, for every  $\eta$  in  $\mathcal{S}(\mathbf{R})$ , the following equality:

$$\begin{aligned} I_{a,b}(\eta) &= \int_a^b \langle \frac{\partial F}{\partial t}(s), \gamma(R_s, \cdot - \langle g_s, \eta \rangle) \rangle ds + \int_a^b \langle \frac{\partial F}{\partial x}(s), \gamma(R_s, \cdot - \langle g_s, \eta \rangle) \rangle \langle g'_s, \eta \rangle ds \\ &\quad + \int_a^b \langle \frac{\partial^2 F}{\partial x^2}(s), \gamma(R_s, \cdot - \langle g_s, \eta \rangle) \rangle dR_s, \end{aligned} \quad (4.20)$$

where we set  $I_{a,b}(\eta) := \langle F(b), \gamma(R_b, \cdot - \langle g_b, \eta \rangle) \rangle - \langle F(a), \gamma(R_a, \cdot - \langle g_a, \eta \rangle) \rangle$ . Define the map  $L : [a, b] \times \mathbf{R}_+ \times \mathbf{R}$  by setting:

$$L(u_1, u_2, u_3) := \langle F(u_1), \gamma(u_2, \cdot - u_3) \rangle.$$

The assumptions made on  $F$  entail that  $L$  is a  $C^1$ -function. Moreover, using the equality  $\frac{\partial \gamma}{\partial t} = \frac{1}{2} \frac{\partial^2 \gamma}{\partial x^2}$ , one gets:

$$\begin{aligned} \langle \frac{\partial F}{\partial x}(s), \gamma(R_s, \cdot - \langle g_s, \eta \rangle) \rangle &= \langle F(s), \frac{\partial \gamma}{\partial x}(R_s, \cdot - \langle g_s, \eta \rangle) \rangle, \\ \langle \frac{\partial^2 F}{\partial x^2}(s), \gamma(R_s, \cdot - \langle g_s, \eta \rangle) \rangle &= \frac{1}{2} \langle F(s), \frac{\partial \gamma}{\partial t}(R_s, \cdot - \langle g_s, \eta \rangle) \rangle. \end{aligned}$$

Since  $I_{a,b}(\eta) = L(b, R_b, \gamma(\langle g_b, \eta \rangle)) - L(a, R_a, \gamma(\langle g_a, \eta \rangle))$ , (4.20) also reads

$$\begin{aligned} L(b, f(b), j(b)) - L(a, f(a), j(a)) &= \int_a^b \frac{\partial L}{\partial u_1}(s, f(s), j_\eta(s)) ds + \int_a^b \frac{\partial L}{\partial u_2}(s, f(s), j_\eta(s)) df(s) \\ &\quad + \int_a^b \frac{\partial L}{\partial u_3}(s, f(s), j_\eta(s)) dj(s), \end{aligned} \quad (4.21)$$

with  $f(s) := R_s$  and  $j_\eta(s) := \langle g_s, \eta \rangle$ . According to Point 1 of Remark 1,  $j_\eta$  is absolutely continuous on  $[0, T]$ . Lemma 4.1 therefore applies and establishes (4.21), which achieves the proof.  $\square$

**Remark 12.** As Theorem 4.7 below will show, one can extend (4.19) to the case where  $[a, b] \cap \mathcal{Z}_R \neq \emptyset$ , if one makes additional assumptions on  $R$  and  $F$ . However, we will not extend (4.19) directly since one does not need such a general formula to get Theorem 4.7, and hence a Tanaka formula. Nevertheless, we think it is better to not specify (4.19), directly to  $F(\cdot)$  instead of  $F(\cdot, \cdot)$ , in Theorem 4.6, since this result can be extended to any interval  $[a, b]$  and then provide an Itô formula in  $(\mathcal{S})^*$ -sense.

#### 4.2.2 Itô Formula in $(L^2)$ for certain generalized functionals of $G$ on any interval of $\mathcal{R}$

Our goal is to get a Tanaka formula, as well as to perform a study of local times of  $G$ , in Section 6. For this purpose one needs both: to generalize (4.19) to the case where  $[a, b]$  contains elements of  $\mathcal{Z}_R$ , and to specify (4.19) to the case where  $F(t, x) = F(x)$ . However, Theorem 4.6 does not extend immediately to the case where  $[a, b]$  intersects the set  $\mathcal{Z}_R$  because the generalized functionals of  $G$  (see Theorem-Definition 3.2) are not defined in this situation. As in [Ben03a], we now extend our previous results to deal with this difficulty. Having in mind the definition of distributions of function type, given in Section 3.2, one has the following result.

**Theorem 4.7.** Let  $[a, b]$  be an interval of  $\mathcal{R}$ ,  $F$  be a distribution of function type that is continuous on  $\mathcal{Z}_R^{a,b} := [a, b] \cap \mathcal{Z}_R$ . Assume that the first distributional derivative of  $F$ , noted  $F'$ , is also of function type and that  $F$  and  $F'$  both have polynomial growth. Assume moreover that  $(\mathcal{A})$  holds, that the map  $t \mapsto R_t$  is both continuous and of bounded variations on  $[a, b]$  and that the following assumptions are satisfied:

- (i)  $\lambda(\mathcal{Z}_R^{a,b}) = \alpha_R(\mathcal{Z}_R^{a,b}) = 0$ ,
- (ii)  $t \mapsto R_t^{-1/2}$  belongs to  $L^1([a, b], dR_t)$ ,
- (iii)  $\exists q \in \mathbf{N}$  such that  $t \mapsto |g'_t|_{-q} R_t^{-1/2} \in L^1([a, b], dt)$ ,

where  $\lambda$  denote the Lebesgue measure on  $\mathcal{R}$  and where the signed measure  $\alpha_R$  has been defined at the beginning of Section 4. Then, the following equality holds in  $(L^2)$ :

$$F(G_b) - F(G_a) = \int_a^b F'(G_t) d^\circ G_t + \frac{1}{2} \int_a^b F''(G_t) dR_t. \quad (4.22)$$

**Proof.** Let us first show that  $t \mapsto F'(G_t) \diamond W_t^{(G)}$  is  $(\mathcal{S})^*$ -integrable on  $[a, b]$  and that  $t \mapsto F''(G_t)$  is  $(\mathcal{S})^*$ -integrable on  $[a, b]$  with respect to  $\alpha_R$ . Denote  $\Gamma_R^{a,b} := \{t \in [a, b]; R_t > 0\}$ .

- a)  $t \mapsto F'(G_t) \diamond W_t^{(G)}$  is  $(\mathcal{S})^*$ -integrable on  $(a, b)$ .

Denote  $\overline{R}_{a,b} := \sup_{t \in [a,b]} R_t$  and  $\widehat{D} := \sup \{ |e_k^2(u)| : k \in \mathbf{N}, u \in \mathbf{R} \}$ . We have the following upper bound, valid for all  $k$  in  $\mathbf{N}$  and  $t$  in  $\Gamma_R^{a,b}$ ,

$$| \langle F', \xi_{t,k} \rangle |^2 = | \langle F, \xi'_{t,k} \rangle |^2 = \pi^{1/2} k! R_t^k | \langle F, \frac{d}{dx} [e_k(x/\sqrt{2}R_t^{1/2}) \exp\{-x^2/4R_t\}] \rangle |^2,$$

where  $\xi'_{t,k}$  denotes  $\frac{d}{dx} x \mapsto \xi_{t,k}(x)$ . The right hand side of the previous equality can be decomposed into two parts. We get the upper bound of these two parts similiary. Thus, we only show here how to get the upper bond of

$$| \langle F, (2 R_t)^{-1/2} e'_k(x/\sqrt{2}R_t^{1/2}) \exp\{-x^2/4R_t\} \rangle | =: J_{t,k}.$$

Denote  $\widetilde{K}$  (*resp.*  $N$ ) the real (*resp.* the integer) such that  $|F(x)| \leq \widetilde{K}(1 + |x|^N)$ , for every real  $x$ . Using the relation  $e'_k(x) = \sqrt{\frac{k}{2}}e_{k-1}(x) - \sqrt{\frac{k+1}{2}}e_{k+1}(x)$  (see [Kuo96, p.354]), valid for all positive integer  $k$ , and the polynomial growth of  $F$ , one immediately gets:

$$\begin{aligned} J_{t,k} &\leq (2R_t)^{-1/2} \int_{\mathbf{R}} |F(x)| \sqrt{2(k+1)} \widehat{D} e^{-x^2/4R_t} dx \\ &\leq (1 + \sqrt{\widetilde{R}})^N \widehat{D} \widetilde{K} \int_{\mathbf{R}} (1 + |u|)^N e^{-u^2/4} du \sqrt{k+1}. \end{aligned}$$

It is now easy to obtain the next inequality, for a real  $D'$  which does not depend on  $k$  nor  $t$ :

$$\forall (t, k) \in \Gamma_R^{a,b} \times \mathbf{N}, \quad |\langle F', \xi_{t,k} \rangle|^2 \leq D'(k+1)! R_t^k. \quad (4.23)$$

Moreover, since the norm operator of  $A^{-1}$  is equal to  $1/2$  (see [Kuo96, p.17]), we get for a real  $D$  that does not depend on  $t$ , using Theorem-Definition 3.2 and (4.23), for all  $t$  in  $\Gamma_R^{a,b}$ :

$$\|F'(G_t)\|_{-1}^2 = \frac{1}{2\pi R_t} \sum_{k=0}^{+\infty} (k!)^{-1} R_t^{-2k} \langle F', \xi_{t,k} \rangle^2 |A^{-1}g_t|_0^{2k} \leq D R_t^{-1}. \quad (4.24)$$

In view of Lemma 2.3 and of (4.24), it is clear that there exists a real  $K$ , such that, for every  $\eta$  in  $\mathcal{S}(\mathbf{R})$  and  $t$  in  $\Gamma_R^{a,b} \cap \mathcal{D}_D$ ,

$$\left| S(F'(G_t) \diamond W_t^{(G)})(\eta) \right| \leq \|F'(G_t)\|_{-1} \|W_t^{(G)}\|_{-q} e^{|\eta|_q^2} \leq K |g'_t|_{-q} R_t^{-1/2} e^{|\eta|_q^2}. \quad (4.25)$$

One can then apply Theorem 2.6 and conclude that  $\int_a^b F'(G_t) d^\circ G_t$  exists in  $(\mathcal{S})^*$ .

b)  $t \mapsto F''(G_t)$  is  $(\mathcal{S})^*$ -integrable on  $[a, b]$ , with respect to  $\alpha_R$ .

Since  $F'$  also has polynomial growth we get, by replacing  $F$  by  $F'$  in (4.24), using Lemma 2.3, for every  $t$  in  $\Gamma_R^{a,b}$ ,

$$|S(F''(G_t))(\eta)| \leq K R_t^{-1/2} e^{|\eta|_1^2}.$$

Thanks to Assumption (i), we know that the right hand side of the previous inequality belongs to  $L^1([a, b] \setminus \mathcal{Z}_R^{a,b}, dR_t)$ . Theorem 2.6 then applies and ensures that  $\int_a^b F''(G_s) dR_s$  exists in  $(\mathcal{S})^*$ . To establish (4.22), it is then sufficient to show that the following equality holds in  $(L^2)$ :

$$F(G_b) - F(G_a) = \int_{\Gamma_R^{a,b}} F'(G_t) d^\circ G_t + \frac{1}{2} \int_{\Gamma_R^{a,b}} F''(G_t) dR_t. \quad (4.26)$$

Having in mind (iii) of Theorem 3.6, we proceed as in the proof of Theorem 4.4. Since  $\Gamma_R^{a,b}$  is an open set of  $[a, b]$ , it can be written under the form:

$$\Gamma_R^{a,b} = [a, c] \sqcup \bigsqcup_{i \in \mathbf{N}} (c_i, d_i) \sqcup (d, b), \quad (4.27)$$

where all the intervals in (4.27) are disjoint and where, by convention,  $(x, y) = (x, y] = \emptyset$ , for every reals  $x$  and  $y$  such that  $x \geq y$ . Note moreover that every element of  $\{c_i, d_i, i \in \mathbf{N}\}$  (as well as  $c$ , if  $[a, c] \neq \emptyset$ , and  $d$ , if  $(d, b] \neq \emptyset$ ) belongs to  $\mathcal{Z}_R^{a,b}$ . Exactly as in the proof of Theorem 4.4, one needs to distinguish between two cases:

**First case:**  $\exists (c', d') \in [a, b]^2$  with  $c' < d'$  s.t.  $[a, c')$  and  $(d', b]$  are both subsets of  $\Gamma_R^{a,b}$ .

**Second case:**  $\nexists (c', d') \in [a, b]^2$  with  $c' < d'$  s.t.  $[a, c')$  and  $(d', b]$  are both subsets of  $\Gamma_R^{a,b}$ .

In both cases, the same reasoning as the one we used in the first case of the proof of Theorem 4.4 (with Theorem 4.6 instead of Lemma 4.1 and Lemma 4.2 with  $v(t, \cdot) = F$ ) applies and allows us to conclude.  $\square$

**Remark 13. 1.** As it has been noticed in [Ben03a, Remark 4.6] (in the case of fBm), Equality (4.22) holds in  $(L^2)$  in the sense that both sides belong to  $(L^2)$ . If  $F''(G_s)$  belongs to  $(L^2)$  and under the additional assumption that  $\int_u^v \|F''(G_s)\|_0 dR_s < +\infty$ , then  $\int_u^v F''(G_s) dR_s$  is an  $(L^2)$ -valued Bochner integral (see Appendix A.1 for the definition of Bochner integral) and all members of both sides of (4.22) belong to  $(L^2)$ . Note moreover that the assumption  $t \mapsto R_t$  is bounded on  $[a, b]$  is sufficient but not necessary; however, if one wants to relieve assumption on  $R$ , one will have to weigh down the ones on  $F$ ,  $F'$  or on the process  $G$ .

2. Theorems 4.4 and 4.7 are the first Itô formulas provided, for Gaussian processes in general, obtained within the framework of White noise theory. To our best knowledge, only the cases of stochastic integrals wrt fractional (resp. multifractional) Brownian motions have been treated in [Ben03a, BSØW04] (resp. in [LLV14]), using White Noise Theory. In view of Point 1 of Remark 7, it is clear that Ito's formulas provided in this section also generalize the ones given in [Ben03a] and in [LLV14].

**Remark 14.** Once again, all the Gaussian processes of “reference” fulfill assumptions of Theorem 4.7. The only non obvious case is the one of  $V_\gamma$  - processes. In this latter case,  $t \mapsto R_t$  is absolutely continuous on  $[0, b]$  and therefore condition (iii) of Theorem 4.7 is fulfilled. Besides, because  $R$  is: absolutely continuous on  $[0, b]$ , non decreasing and such that  $\lim_{u \rightarrow 0^+} R'_u = +\infty$ , it is easy to show, using (3.13), that condition (i) of Theorem 4.7 entails condition (ii). Finally, one has the equality:

$$I := \int_0^{(2e)^{-1}} R_u^{-1/2} dR_u = \int_0^{(2e)^{-1}} R_u^{-1/2} R'_u du = 2 \int_0^{(2e)^{-1}} (R_u^{1/2})' du = 2 R_{1/2e}^{1/2}. \quad (4.28)$$

## 5 Comparison with other stochastic integrals

In this section we make, first in Subsection 5.1, a comparison of the Wick-Itô stochastic integral we developed above with the *functional extensions* of stochastic integrals developed in [AMN01, MV05] and then, in Subection 5.2, with the Itô integral.

### 5.1 Comparison with Malliavin Calculus or divergence type integrals

We start by making the comparison between our Wick-Itô integral and the divergence type integral developed in [AMN01]. We will then show, in Section 5.1.2, that the Wick-Itô integral fully generalizes the (extended) Skorohod integral developed in [MV05]. Let  $T > 0$  be fixed and let us take  $\mathcal{R} = [0, T]$ . Let  $G := (G_t)_{t \in [0, T]}$  be a Volterra process.

#### 5.1.1 Comparison with divergence type integral of [AMN01]

The goal of this section is to compare the Wick-Itô integral wrt  $G$  to the divergence integral wrt  $G$ , defined in [AMN01] and in [MV05] and studied in [AMN01, Nua05] and in [MV05]. In these two papers  $G$  is a continuous Volterra process, one therefore will assume (in this subsection only) that  $G$  is continuous on  $[0, T]$ .  $G$  being a Volterra process, it can be written, for any real  $t$  in  $[0, T]$ ,

$$G_t = \int_0^t K(t, s) dW_s,$$

where the kernel  $K(t, s)$ , defined on  $[0, T]^2$ , is such that  $K(t, s) = 0$  on the set  $[0, T]^2 \setminus \{(u, v) \in (0, T] \times [0, T] : v < u\}$  and verifies for any  $t \geq 0$ ,

$$\widehat{K}_t := \int_0^t K(t, s)^2 ds < \infty.$$

Denote  $L^2(\Omega, L^2([0, T]))$  the set of random process  $u$  such that  $\|u\|_{L^2(\Omega, L^2([0, T]))}^2 := \mathbf{E}[\int_0^T u_t^2 dt] < +\infty$ . The main result of this section is Theorem 5.3, which states that every process  $u$  which

belongs to  $L^2(\Omega, L^2([0, T]))$  and that belongs to the domain of the divergence of  $G$  is also Wick-Itô integrable *wrt*  $G$ , on  $[0, T]$ . Moreover, one has the equality  $\int_0^T u_s \delta G_s = \int_0^T u_s dG_s$ , where  $\int_0^T u_s \delta G_s$  denotes the divergence integral on  $[0, T]$ , associated to  $G$ , that has been defined in [AMN01].

In order to state rigorously this result we briefly recall some elements and notations of stochastic calculus of variations *wrt*  $G$  (for a presentation of Malliavin calculus, see *e.g.* [Bal03, Nua06]), as well as the approach of [AMN01] and [Nua05] for the construction of a stochastic integral *wrt* to Volterra processes. The real  $T > 0$  being fixed, one still note  $G$  the process  $(G_t)_{t \in [0, T]}$  since there is no risk of confusion.  $G$  being a centered Gaussian process, denote  $\mathcal{H}_T$  the reproducing kernel Hilbert space (R.K.H.S.) defined as the closure of the set  $\mathcal{E}_T := \text{span}\{\mathbf{1}_{[0, t]}, t \in [0, T]\}$ , with respect to the inner product  $\langle, \rangle_{\mathcal{H}_T}$ , that has been defined by setting  $\langle \mathbf{1}_{[0, t]}, \mathbf{1}_{[0, s]} \rangle_{\mathcal{H}_T} := R_{t, s}$ . Denote  $H_1$  the first Wiener chaos of  $G$  and  $G(\varphi)$  the image in  $H_1$  of an element  $\varphi$  of  $\mathcal{H}_T$  by the isometry, between  $\mathcal{H}_T$  and  $H_1$ , that associates  $\mathbf{1}_{[0, t]}$  to  $G_t$ .

**Remark 15.** *It is not always true that the bilinear form  $\langle, \rangle_{\mathcal{H}_T}$  defined by  $\langle \mathbf{1}_{[0, t]}, \mathbf{1}_{[0, s]} \rangle_{\mathcal{H}_T} := R_{t, s}$  is an inner product. For example, for the Brownian bridge  $\hat{B} := (\hat{B}_t)_{t \in [0, 1]}$  on  $[0, 1]$ , one has  $\|\mathbf{1}_{[0, 1]}\|_{\mathcal{H}_T} = 0$ . For this reason we will assume in the sequel that  $\langle, \rangle_{\mathcal{H}_T}$ , defined above, is really an inner product. The reader interested in details on Reproducing Kernels Hilbert Spaces may refer to [Jan97, Chap.8] as well as to [LLV14, Appendix B] in the case of  $mBm$ .*

Define  $\mathcal{S} := \{V = f(G(\varphi_1), G(\varphi_2), \dots, G(\varphi_n)), f \in C_b^\infty(\mathbf{R}^n), \varphi_i \in \mathcal{H}_T, i = 1, \dots, n\}$ . For an element  $V$  of  $\mathcal{S}$ , one defines the derivative operator  $D^G$  as:

$$D^G V := \sum_{i=1}^n \frac{\partial f}{\partial x_i}(G(\varphi_1), G(\varphi_2), \dots, G(\varphi_n)) \varphi_i.$$

The derivative operator  $D^G$  is a closable unbounded operator from  $L^2(\Omega)$  into  $L^2(\Omega; \mathcal{H}_T)$ . We note  $\mathbb{D}_G$  the closure of  $\mathcal{S}$  with respect to the norm defined by

$$\|V\|_{G, 1, 2} := (\mathbf{E}[V^2] + \mathbf{E}[\|D^G V\|_{L^2(\Omega; \mathcal{H}_T)}^2])^{\frac{1}{2}}.$$

We denote by  $\delta_G$ , and call divergence integral with respect to  $G$ , the adjoint of the derivative operator  $D^G$ . The domain of  $\delta^G$  in  $L^2$ , denoted  $\mathbb{D}om(\delta^G)$ , is the set of the elements  $u$  in  $L^2(\Omega; \mathcal{H}_T)$  such that there exists a constant  $c$  verifying:

$$|\mathbf{E}(\langle D^G V, u \rangle_{\mathcal{H}_T})| \leq c \|V\|_2,$$

for all  $V$  in  $\mathcal{S}$  and where  $\|\cdot\|_2$  denotes the norm in  $L^2(\Omega)$ . If  $u$  belongs to  $\mathbb{D}om(\delta^G)$ ,  $\delta^G(u)$  is the element of  $L^2(\Omega)$  defined by the duality relationship:

$$\mathbf{E}(V \delta^G(u)) = \mathbf{E}(\langle D^G V, u \rangle_{\mathcal{H}_T}), \text{ for all } V \text{ in } \mathbb{D}_G.$$

We will simply denote, in the sequel,  $D, \mathbb{D}, \delta$  and  $\|\cdot\|_{1, 2}$  when  $G$  is a Brownian motion. Define now the linear operator  $K_- : \mathcal{E}_T \rightarrow L^2([0, T])$  by  $K_-(\mathbf{1}_{[0, t]}) := K(t, \cdot)$  and denote  $\|\cdot\|_{\mathcal{H}_T}$  the norm on  $\mathcal{H}_T$  which derives from the inner product  $\langle, \rangle_{\mathcal{H}_T}$ . Since  $\|\varphi\|_{\mathcal{H}_T} = \|K_-(\varphi)\|_{L^2([0, T])}$ , for every  $\varphi$  in  $\mathcal{E}_T$ , it is clear that the operator  $K_-$  can be extended to a linear isometry, still denoted  $K_-$ , between  $(\mathcal{H}_T, \|\cdot\|_{\mathcal{H}_T})$  and a closed subset of  $L^2([0, T])$ . Besides, one can show, [AMN01, (12)], that

$$\mathbb{D}om(\delta_G) = (K_-)^{-1}(\mathbb{D}om(\delta)).$$

Moreover, for a process  $v$  in  $\mathbb{D}om(\delta_G)$  one has:

$$\delta_G(v) = \int_0^T (K_- v)(s) \delta W_s. \quad (5.1)$$

In other words,  $\delta_G(v)$ , the divergence integral of  $v$  wrt  $G$ , also noted  $\int_0^T v(s) \delta G_s$ , verifies the equality  $\int_0^T v(s) \delta G_s = \int_0^T (K_-v)(s) \delta W_s$ . In order to prove Theorem 5.3 below, one needs to define the adjoint of the operator  $K_-$ , that we will denote  $K_+$ , not only on the set  $\mathcal{E}_T$  but also on  $\mathcal{S}(\mathbf{R})$ . For this reason we recall the two following hypotheses, given in [Nua05] for fBm, that we will make in the sequel on the kernel  $K(t, s)$ .

H1)  $K(t, s)$  is continuously differentiable on  $\{0 < s < t \leq T\}$  and its partial derivative verify the following integrability condition:

$$\sup_{\varepsilon \leq t \leq T} \int_t^T \left| \frac{\partial K}{\partial r}(r, t) \right| (r - t) dr + \int_0^t \left| \frac{\partial K}{\partial t}(t, s) \right| (t - s) ds < \infty,$$

for any  $\varepsilon$  in  $(0, T)$ . Moreover,  $t \mapsto \int_0^{t \wedge b} \frac{\partial K}{\partial t}(t, s) (t \wedge b - s \vee a)_+ ds$  is continuous on  $(0, T]$ , for all  $0 \leq a \leq b$ .

H2) The function  $k(t) := \int_0^t K(t, s) ds$  is continuously differentiable on  $(0, T]$ .

We present here the arguments given in [Nua05, Section 2] for fBm about the operator  $K_+$ , but in a slightly different manner. Denote  $C_b^1(\mathbf{R})$  the set of differentiable functions which are bounded together with its derivatives. Hypotheses H1) and H2) allow us to define the operator  $K_+$  on  $\mathcal{E}_T \cup C_b^1(\mathbf{R})$  by setting, for every  $t$  in  $[0, T]$ ,

$$(K_+\varphi)(t) := k'(t) \varphi(t) + \int_0^t \frac{\partial K}{\partial t}(t, r) (\varphi(r) - \varphi(t)) dr.$$

In view of [Nua05, p.116], it is easy to check that we have, for any  $(\psi, \varphi)$  in  $\mathcal{E}_T \times \mathcal{E}_T$ , the equality

$$\langle K_+(\varphi), \psi \rangle_{L^2([0, T])} = \langle \varphi, K_-(\psi) \rangle_{L^2([0, T])}. \quad (5.2)$$

It is clear that one has, in this section,  $g_t := K_-(\mathbf{1}_{[0, t]})$ , for every  $t$  in  $[0, T]$ . It is established in [Nua05, Propostion 2] that  $g'_t$  exists and that  $g'_t = K_+(\cdot)(t)$  for every  $t$  in  $(0, T]$ . However it is not possible to establish that  $t \mapsto W_t^{(G)}$  is  $(\mathcal{S}^*)$ -integrable on  $[0, T]$  without any additional assumption. Moreover one needs to be able to establish that  $\int_0^T u_s dG_s$  exist for a reasonable class of processes  $u$ . Thus, following [Nua05, Proposition 7], we will assume in the sequel the following condition:

H3) The function  $C : t \mapsto |k'(t)| + \int_0^t \left| \frac{\partial K}{\partial t}(t, r) \right| (t - r) dr$  belongs to  $L^2([0, T])$ .

**Remark 16.** *It is clear that  $H_1)$ ,  $H_2)$  and  $H_3)$  entail that Assumptions  $\mathcal{D}_{(i)}$  and  $\mathcal{D}_{(ii)}$  hold. We will show, in the next subsection (Remark 18), that they are not always necessary.*

The following result will be useful in the proof of Theorem 5.3 below.

**Lemma 5.1.** *If Assumptions  $H_1)$ ,  $H_2)$  and  $H_3)$  hold, any process  $u$  in  $L^2(\Omega, L^2([0, T]))$  is Wick-Itô integrable with respect to  $G$ .*

**Proof.** The proof of this lemma, which consists on verifying that Condition  $(\mathcal{I})$  is verified with  $p = q = 2$ , can be found in [Nua05, Proposition 7].  $\square$

Since one has, for every  $(\eta, t)$  in  $\mathcal{S}(\mathbf{R}) \times [0, T]$ ,

$$|(K_+\eta)(t)| \leq |k'(t)| \|\eta\|_\infty + \|\eta'\|_\infty \int_0^t \left| \frac{\partial K}{\partial t}(t, r) \right| (t - r) dr,$$

Hypothesis H3) implies in particular that  $K_+(\eta)$  belongs to  $L^2([0, T])$ . Note moreover that, for every  $\eta$  in  $\mathcal{S}(\mathbf{R})$ ,

$$K_+(\eta)(t) = \langle \delta_t, K_+(\eta) \rangle = \frac{d}{dt} \langle K_-(\mathbf{1}_{[0,t]}), \eta \rangle_{L^2([0,T])}. \quad (5.3)$$

The following result, which is a consequence of results given in [Nua05, Section 2 and Proposition 7], will be essential in order to prove Theorem 5.3 below.

**Lemma 5.2.** *For any function  $\psi$  in  $\mathcal{H}_T \cap L^2([0, T])$  and  $\eta$  in  $\mathcal{S}(\mathbf{R})$ , one has:*

$$\langle K_+(\eta), \psi \rangle_{L^2([0,T])} = \langle \eta, K_-(\psi) \rangle_{L^2([0,T])}. \quad (5.4)$$

**Proof.** It is easy to check (5.4) directly in the case where  $\psi$  is in  $\mathcal{E}_T$  and  $\eta$  in  $\mathcal{S}(\mathbf{R})$ , using (5.3). The fact that, for every  $\psi \in \mathcal{E}_T$ ,  $\langle \psi, K_+(\cdot) \rangle_{L^2([0,T])}$  belongs to  $\mathcal{L}_p(\mathbf{R})$ , for every  $p$  in  $\mathbb{N}^*$  is also clear. Thus, for every  $p$  in  $\mathbb{N}^*$ , one easily sees that the map  $\Psi_p : \psi \mapsto \langle \psi, K_+(\cdot) \rangle_{L^2([0,T])}$  is uniformly continuous from  $(\mathcal{E}_T, \|\cdot\|_{\mathcal{H}_T})$  to  $(\mathcal{L}_p(\mathbf{R}), |\cdot|_{-p})$  and can then be extended uniquely to  $(\mathcal{H}_T, \|\cdot\|_{\mathcal{H}_T})$  (we will still denote  $\Psi_p$  this extension). The same argument can be applied to the map  $\Phi_p : \psi \mapsto \langle \cdot, K_-(\psi) \rangle_{L^2([0,T])}$ . The equality of  $\Psi_p$  and  $\Phi_p$  on  $\mathcal{H}_T$  from one hand, the equalities  $\Psi_p = \langle \psi, K_+(\cdot) \rangle_{L^2([0,T])}$  and  $\Phi_p = \langle \cdot, K_-(\psi) \rangle_{L^2([0,T])}$  on  $\mathcal{H}_T \cap L^2([0, T])$  from the other hand allow us to conclude.  $\square$

The main result of this section is the following.

**Theorem 5.3.** *Assume that  $H_1$ ,  $H_2$  and  $H_3$  hold. Let  $u$  be a process in  $L^2(\Omega, L^2([0, T]))$ . If  $u$  belongs to the domain of the divergence of  $G$ , then  $u$  is Wick-Itô integrable on  $[0, T]$  wrt  $G$ . Moreover one has the equality*

$$\int_0^T u_s \delta G_s = \int_0^T u_s d^\circ G_s. \quad (5.5)$$

**Proof.** The proof we give here is a generalization, to Volterra processes, of the proof provided, in the particular case of fBm, in [Nua05, Proposition 8]. We however write it down here for reader's convenience. The fact that  $\int_0^T u_s d^\circ G_s$  is well-defined has been established in Lemma 5.1. Besides, for every fixed  $\eta$  in  $\mathcal{S}(\mathbf{R})$ , one has:

$$\mathcal{L}_1 := S\left(\int_0^T u_s \delta G_s\right)(\eta) = S\left(\int_0^T (K_-u)(s) \delta W_s\right)(\eta) = \int_0^T S[(K_-u)(s)](\eta) \eta(s) ds.$$

Note that the last equality results from the fact that the Wick-Itô integral wrt Brownian motion generalizes the Hitsuda-Skorohod integral (see for example [HOUZ10, Theorem 2.5.9] or [Kuo96, (13.8)]). Using the previous equality, Fubini's theorem and Lemma 5.2, one gets:

$$\begin{aligned} \mathcal{L}_1 &= \int_0^T \mathbf{E}[(K_-u)(s) : e^{\langle \cdot, \eta \rangle} :] \eta(s) ds = \mathbf{E}[: e^{\langle \cdot, \eta \rangle} : \langle K_-u, \eta \rangle_{L^2([0,T])}] \\ &= \mathbf{E}[: e^{\langle \cdot, \eta \rangle} : \langle u, K_+(\eta) \rangle_{L^2([0,T])}] = \int_0^T \mathbf{E}[u_s : e^{\langle \cdot, \eta \rangle} :] K_+(\eta)(s) ds. \end{aligned}$$

It then remains to use (ii) of Theorem 3.6 as well as (5.3) to obtain:

$$\mathcal{L}_1 = \int_0^T S(u_s)(\eta) S(W_s^{(G)})(\eta) ds = S\left(\int_0^T u_s \diamond W_s^{(G)} ds\right)(\eta).$$

We hence have shown, for every  $\eta$  in  $\mathcal{S}(\mathbf{R})$ , the equality  $S(\int_0^T u_s \delta G_s)(\eta) = S(\int_0^T u_s d^\circ G_s)(\eta)$ . The injectivity of  $S$ -transform (see (i) of Lemma 2.5) allows us to conclude.  $\square$

**Example 5.4** (The case of fBm). *Let  $T > 0$ . For any  $H$  in  $(0, 1)$ , define  $B_t^H := \int_0^t K_H(t, s) dW_s$ , where the kernel  $K_H$  is defined in [Nua05, (13)].  $B^H$  is an fBm of Hurst index  $H$ . Moreover the process  $B^H$  fulfills<sup>14</sup> H1), H2) and H3). This implies in particular that the Wick-Itô integral  $\int_0^T B_s^H d^\circ B_s^H$  exists, for any  $H$  in  $(0, 1)$ . We know moreover, thanks to Example 3.16, that the equality  $\int_0^T B_s^H d^\circ B_s^H \stackrel{a.s.}{=} \frac{1}{2} ((B_T^H)^2 - T^{2H})$  is true for any  $H$  in  $(0, 1)$ . On the other hand, the divergence integral wrt  $B^H$  is only defined and developed in [AMN01, Section 8] or in [Nua06, Section 6] for  $H > 1/4$ , as we mentioned in the introduction. One moreover knows that  $B^H$  does not belong to  $\mathbb{D}om(\delta_{B^H})$  when  $H < 1/4$  and one has to use the extended divergence integral wrt fBm developed in [CN05].*

**Corollary 5.5.** *The set  $\{\int_0^T f(s) \delta G_s, f \in \mathcal{H}_T\}$  of Wiener divergence integral wrt  $G$ , coincide with the set  $\Theta_G := \overline{\{\int_0^T f(s) d^\circ G_s, f \in \mathcal{E}_T\}}^{(L^2)}$  of Wick-Itô Wiener integrals wrt  $G$ .*

**Proof.** The equality  $\{\int_0^T f(s) d^\circ G_s, f \in \mathcal{E}_T\} = \{\int_0^T f(s) \delta G_s, f \in \mathcal{E}_T\}$  is obvious, in view of (5.5). Besides, the equality  $\{\int_0^T f(s) \delta G_s, f \in \mathcal{H}_T\} = \overline{\{\int_0^T f(s) \delta G_s, f \in \mathcal{E}_T\}}^{(L^2)}$  results from Meyer inequalities (see [AMN01, (5)] for example).  $\square$

**Remark 17. 1.** *In view of the previous corollary, we see that one just has to extend the notion of Wiener integral given in Definition 1, and call Wiener integral wrt  $G$  in  $\mathcal{G}$ , any element of  $\Theta_G$ , if one wants that our set of Wiener integrals is the same that the one of [AMN01].*

**2.** *If Theorem 5.3 clearly states that the Wick-Itô integral has a bigger set of integrands than the divergence type integral developed in [AMN01], assuming they both belong to  $L^2(\Omega, L^2([0, T]))$ , one may wonder if this fact remains true outside  $L^2(\Omega, L^2([0, T]))$ . While this remains an open problem, here is what we can still say about it. The set  $\mathcal{H}_T$  may contains generalized functions (for example, one can see [Nua06, p.280] or [LLV14, Proposition 2.11] in the case where  $G$  is a fBm). When this happens (i.e. when, for almost every  $\omega$  in  $\Omega$ ,  $u(\omega)$  is a generalized function which belongs to  $\mathcal{H}_T$  and which is not a function),  $\int_0^T u_s \delta G_s$  has still a meaning and belongs to  $L^2(\Omega)$ . On the contrary,  $\int_0^T u_s d^\circ G_s$  can only be defined if  $s \mapsto u_s$  is a function (an  $(\mathcal{S}^*)$ -valued function but still a function). Define the space*

$$\Lambda := \{u \in L^2(\Omega; \mathcal{H}_T); u \text{ is Wick-Itô integrable wrt } G \text{ and such that } \int_0^T u_s d^\circ G_s \in L^2(\Omega)\}.$$

*A consequence of what we stated above is that the inclusion  $\mathbb{D}om(\delta_G) \subset \Lambda$  is not true. Note that the inclusion  $\Lambda \subset \mathbb{D}om(\delta_G)$  does not hold either. Indeed, if one considers again, as process  $G$ , the fBm  $B^H$ , as we did in Example 5.4, we know that  $B^H$  belongs to  $\Lambda$  for every  $H$  in  $(0, 1)$ , while  $B^H$  does not belong to  $\mathbb{D}om(B^H)$  if  $H$  is in  $(0, 1/4)$ . Finally, the only thing one can say in general is that we have the dense inclusion  $L^2(\Omega, L^2([0, T])) \cap \mathbb{D}om(\delta_G) \subset \Lambda$ .*

**3.** *The Itô formulas provided in the previous section also provide a way to compare divergence type integral with Wick-Itô integral. Besides, since a precise comparison between the Itô formulas we obtained in Section 4 and the ones obtained in [AMN01] has been made in the previous section, one does not discuss it again here.*

### 5.1.2 Comparison with the divergence type integral of [MV05]

The comparison between Wick-Itô stochastic integral and the one defined in [MV05] is easier, in view of Theorems 3.4 and 5.3. Indeed, one has the following result.

<sup>14</sup>One just has to consider  $B^H$  under the form given in Example 2.8.

**Theorem 5.6.** *For any process  $u$  such that the (extended) Skorohod integral wrt  $G$ , on  $[0, T]$ , defined in [MV05], exists, then  $u$  is also Wick-Itô integrable wrt  $G$ , on  $[0, T]$ . Moreover one has the equality*

$$[MV05] - \int_0^T u_s \delta G_s = \int_0^T u_s d^\circ G_s, \quad (5.6)$$

where [MV05] -  $\int_0^T u_s \delta G_s$  denotes the (extended) Skorohod integral of  $u$  wrt  $G$ , defined in [MV05].

**Remark 18. 1.** *Note that one does not have to make any additional assumptions (such as H1, H2 or H3) in this case.*

**2.** *As we stated in the introduction, this theorem as well as Theorem 3.4 show that the stochastic integral wrt to  $\mathcal{V}_\gamma$  - processes, built in [MV05], is a particular case of the Wick-Itô stochastic integral we provide here. This means that for every  $\mathcal{V}_\gamma$  - process  $\tilde{B}^\gamma$ , and every stochastic process  $X$ , such that the integral of  $X$  wrt  $\tilde{B}^\gamma$  exists in the sense defined in [MV05], the Wick-Itô stochastic integral of  $X$  wrt  $\tilde{B}^\gamma$  exists. Moreover they are equal.*

**Proof.** Using notations of Theorem 3.4, Lemma 2.3 and Cauchy-Schwarz inequality one gets, for every  $\eta$  in  $\mathcal{S}(\mathbf{R})$  and every integer  $q \geq 3$ :

$$\int_0^T |S(u_s)(\eta) S(W_s^{(G)})(\eta)| ds \leq \left( \int_0^T \|u_s\|_0^2 ds \right)^{1/2} \left( \int_0^T |\Phi'(s)|_{-q}^2 ds \right)^{1/2} e^{|\eta|_q^2}.$$

Since both quantities  $\|u\|_{L^2(\Omega, L^2([0, T]))}^2$  and  $\int_0^T |\Phi'(s)|_{-q}^2 ds$  are finite (by assumption for the first one and as a consequence of Theorem 3.4 for the second one), Theorem 2.6 applies and establishes the existence of  $\int_0^T u_s d^\circ G_s$ .

Besides, since in this case the extended domain of the [MV05] - Skorohod integral is, by its very definition (see [MV05, Def. 27]) a subset of  $L^2(\Omega, L^2([0, T]))$ , one can use the exact same proof as the one of Theorem 5.3; one just has to change therein  $K_-$  (resp.  $K_+$ ) by  $K_\gamma^*$  (resp.  $K_\gamma^{*,a}$ ) and note that the equality given in [MV05, Remark 12] has now the role played by Equality (5.4), in the proof of Theorem 5.3. The only thing which remains to be shown is that  $\mathcal{S}(\mathbf{R}) \subset \mathcal{H}'$ , where  $\mathcal{H}' := \{f \in L^2([0, T]), K_\gamma^{*,a} f \in L^2([0, T])\}$ . This latter inclusion results from [MV05, Proposition 15] (one just has to take therein  $\eta(s) := s^\alpha$  on  $\mathbf{R}_+^*$ , with  $\alpha \in (1/2, 1)$  and  $\eta(0) := 0$  and then show that  $\mathcal{S}(\mathbf{R}) \subset C^\eta$ , where  $C^\eta$  has been defined in Example 3.9.  $\square$

Note that the results provided in both Theorem 5.6 and Theorem 3.18 allow us to think that one could solve some linear stochastic evolution equations driven by infinite dimensional Gaussian processes. A first step would be to extend to elements of  $\mathcal{G}$ , what has been done in [MV07] for  $\mathcal{V}_\gamma$  - processes.

In addition to the results provided in this section about the differences between the Wick-Itô integral and the divergence type integrals developed in [AMN01] and in [MV05], every differences we pointed out in the introduction are valid. We will therefore not repeat them again here.

## 5.2 Comparison with Itô Integral

The goal of this section is to compare the Wick-Itô integral wrt  $G$  to the Itô integral wrt  $G$ , when  $G$  is a (Gaussian) semimartingale. In this subsection we still assume that  $\mathcal{R} = [0, T]$ . Since the line of reasoning we are following would be similar if  $t \mapsto G_t$  would not be continuous, we will assume, in this subsection, that  $G$  is continuous. Denote, for every  $t$  in  $[0, T]$ ,  $\mathcal{U}_t$  the complete<sup>15</sup>  $\sigma$ -field defined by  $\mathcal{U}_t := \sigma(\{G_s; 0 \leq s \leq t\})$  and denote  $\mathcal{U}$  the filtration  $(\mathcal{U}_t)_{t \in [0, T]}$ .

<sup>15</sup>If  $\mathcal{U}_t$  is not complete, we complete it and still denote it  $\mathcal{U}_t$ .

In this subsection one then assumes that  $G = (G_t)_{t \in [0, T]}$  is a continuous centered Gaussian  $\mathcal{U}$ -semimartingale of the form (1.1), which fulfills Assumption  $(\mathcal{A})$ . Let us give first the following result, that describes the structure of Gaussian semimartingales.

**Proposition 5.7.** *[Str83, Prop. 2 & Thm 1] The Gaussian  $\mathcal{U}$ -semimartingale  $G$  is a special  $\mathcal{U}$ -semimartingale: i.e. for almost every  $(t, \omega)$  in  $[0, T] \times \Omega$ , one can write:*

$$G_t = M_t + A_t, \quad (5.7)$$

where  $M := (M_t)_{t \in [0, T]}$  is a centered  $\mathcal{U}$ -martingale and  $A := (A_t)_{t \in [0, T]}$  is a centered  $\mathcal{U}$ -predictable process of bounded variations. Moreover,  $M$  and  $A$  both belong to the same Gaussian Hilbert space as  $G$ . In addition, the function of quadratic variation of  $G$ , denoted  $t \mapsto \langle G \rangle_t$ , is deterministic and  $M$  is bounded in  $L^p$ , for every positive real  $p$ .

Denote  $\mathcal{T} := (\mathcal{T}_t)_{t \in [0, T]}$  the filtration, defined by  $\mathcal{T}_t := \sigma(\{B_s; 0 \leq s \leq t\})$ , which we suppose complete (if it is not the case we complete it and still denote it  $\mathcal{T}_t$ ). Through this subsection, we will denote  $\mathcal{I}_G(X) := \int_0^T X_s dG_s$  (resp.  $\mathcal{J}_G(X) := \int_0^T X_s d^\circ G_s$ ) the Itô (resp. the Wick-Itô) integral of  $X$  wrt  $G$  on  $[0, T]$ , when it exists.

For any continuous martingale  $M := (M_t)_{t \in [0, T]}$ , bounded in  $(L^2)$  and such that  $M_0 = 0$ , denote  $L^2(M) := L^2([0, T] \times \Omega, \mathcal{P}, d\mu d\langle M \rangle_s)$  the space of progressively measurable processes  $K$  such that:

$$\|K\|_{L^2(M)}^2 := \mathbf{E}\left[\int_0^T K_s^2 d\langle M \rangle_s\right],$$

where  $\mathcal{P}$  denote the progressive  $\sigma$ -field with respect to  $\mathcal{U}$ . The following result will be used in order to establish Point (i) of Proposition 5.9 below.

**Lemma 5.8.** *Let  $G$  be a Gaussian martingale that fulfills Assumption  $(\mathcal{A})$  and let  $f$  be a  $C^{1,2}([0, T] \times \mathbf{R}, \mathbf{R})$  function. Denote  $F(t, x) := \frac{\partial f}{\partial x}(t, x)$  and define  $X_t := \frac{\partial f}{\partial x}(t, G_t)$ . If  $R$  and  $f$  both fulfill conditions of Theorem 4.4, then one has the following equality:*

$$\text{a.s.} \quad \int_0^T X_s dG_s = \int_0^T X_s d^\circ G_s.$$

In particular one has the equality:

$$\text{a.s.} \quad \int_0^T F(s, G_s) dG_s = \int_0^T F(s, G_s) d^\circ G_s. \quad (5.8)$$

**Proof of Lemma 5.8.**  $G$  being a Gaussian martingale, one gets  $\langle G \rangle_t = R_t$  almost surely, for every  $t$  in  $[0, T]$ . Since  $R$  and  $f$  both fulfill conditions of Theorem 4.4, one gets, using both Itô formulas (4.5) and [RY99, Theorem 3.3]):

$$\Delta_G(X) := \mathcal{I}_G(X) - \mathcal{J}_G(X) = 0,$$

which in particular implies that:  $\int_0^T F(s, G_s) dG_s = \int_0^T F(s, G_s) d^\circ G_s$ . □

The main result of this subsection is the following.

**Proposition 5.9. 1.** *Assume that  $G$  is Gaussian martingale, adapted to the filtration  $\mathcal{T}$  and such that  $G_{t_2} - G_{t_1}$  is independent of  $\mathcal{T}_{t_1}$ , for every  $0 \leq t_1 < t_2 \leq T$ . Then, for any  $X$  in  $L^2(G)$  such that  $X_t \in (L^2)$ , for every  $t$  in  $[0, T]$ , and such that  $(X, G)$  satisfies condition  $(\mathcal{I})$  (given in Section 3.5), then the map  $t \mapsto X_t$  is both  $dG$ -integrable and Itô-integrable, on  $[0, T]$ . Moreover we have the equality:*

$$\int_0^T X_s dG_s = \int_0^T X_s d^\circ G_s. \quad (5.9)$$

2. If the semimartingale  $G$  is not a Gaussian martingale, then Equality (5.9) does not hold in general, assuming both integral  $\int_0^T X_s dG_s$  and  $\int_0^T X_s d^\circ G_s$  do exist.

Note that the condition  $X_t \in (L^2)$ , for every  $t$  in  $[0, T]$  is only a slight reinforcement of the assumption  $X \in L^2(G)$ .

**Proof. 1.** Let  $G$  and  $X$  be of the form described in point 1. above. The existence of  $\mathcal{I}_G(X)$  is clear and the existence of  $\mathcal{J}_G(X)$  is obvious, in view of Theorem 3.10. The proof of the equality  $\mathcal{I}_G(X) = \mathcal{J}_G(X)$  is obtained by following exactly the same three steps as in the proof of [Kuo96, Theorem 13.12], in which the equality  $\mathcal{I}_B(X) = \mathcal{J}_B(X)$  is established ( $B$  being a Brownian motion). One then just has to substitute in there the process  $\varphi$  by  $X$ , to replace  $\mathbf{1}_{[t_1, t_2]}$  by  $g_{t_2} - g_{t_1}$  in the first step, and noticing that one can find, for any process  $X$  in  $L^2(G)$ , a sequence  $(X_n)_{n \in \mathbf{N}}$  of simple processes such that  $\lim_{n \rightarrow +\infty} \mathcal{I}_G(X_n) = \mathcal{I}_G(X)$ , where the convergence holds in  $(L^2)$ .

We will only write down here the first of these three steps; in order, first to make clear the differences with the case where  $G = B$ , and second, to translate the proof of [Kuo96, Theorem 13.12] in the notations we use in this paper.

Let  $(t_1, t_2)$  be in  $[0, T]^2$  such that  $0 \leq t_1 < t_2 \leq T$  and assume that  $X_t := X_{t_1} \mathbf{1}_{(t_1, t_2]}$ , where  $X_{t_1}$  is  $\mathcal{U}_1$ -mesurable. Let  $X_{t_1} = \sum_{n=0}^{+\infty} I_n(f_{t_1}^{(n)})$  be the chaos decomposition of  $X_{t_1}$ . By definition of Itô integral, and using the identity:

$$I_n(k) I_1(l) = I_{n+1}(k \otimes l) + n I_{n-1}(\langle k, l \rangle_{L^2(\mathbf{R})}),$$

where  $\widehat{\otimes}$  denotes the symmetric tensor product and which is valid for every:  $n$  in  $\mathbf{N}^*$ , symmetric function  $k$  in  $L^2(\mathbf{R}^n)$  and  $l$  in  $L^2(\mathbf{R})$ , one gets:

$$\begin{aligned} \mathcal{I}_G(X) &= X_{t_1}(G_{t_2} - G_{t_1}) = \sum_{n=0}^{+\infty} I_n(f_{t_1}^{(n)}) I_1(g_{t_2} - g_{t_1}) \\ &= \sum_{n=0}^{+\infty} (I_{n+1}(f_{t_1}^{(n)} \widehat{\otimes} (g_{t_2} - g_{t_1})) + n I_{n-1}(\langle f_{t_1}^{(n)}, g_{t_2} - g_{t_1} \rangle_{L^2(\mathbf{R})})). \end{aligned}$$

Lemma 3.11 of [Kuo96] applies here since  $\mathcal{U}$  is included in  $\mathcal{T}$ . One then knows that, for every  $n$  in  $\mathbf{N}^*$ ,  $f_{t_1}^{(n)}$  is equal to 0 almost everywhere on  $[0, T]^n \setminus [0, t_1]^n$ . Moreover, since  $G_{t_2} - G_{t_1}$  is independent of  $\mathcal{T}_{t_1}$ , it is clear that  $\langle f_{t_1}^{(n)}, g_{t_2} - g_{t_1} \rangle_{L^2(\mathbf{R})} = 0$ , for every  $n$  in  $\mathbf{N}^*$ . Using point 2 of Proposition 2.4, we get, for any  $\eta$  in  $\mathcal{S}(\mathbf{R})$ ,

$$\begin{aligned} S(\mathcal{I}_G(X))(\eta) &= S\left(\sum_{n=0}^{+\infty} (I_{n+1}(f_{t_1}^{(n)} \widehat{\otimes} (g_{t_2} - g_{t_1})))\right)(\eta) = \sum_{n=0}^{+\infty} \langle f_{t_1}^{(n)} \widehat{\otimes} (g_{t_2} - g_{t_1}), \eta^{\otimes(n+1)} \rangle \\ &= \sum_{n=0}^{+\infty} \langle f_{t_1}^{(n)}, \eta^{\otimes n} \rangle \langle g_{t_2} - g_{t_1}, \eta \rangle = S(X_{t_1})(\eta) S(G_{t_2} - G_{t_1})(\eta) = S(\mathcal{J}_G(X))(\eta). \end{aligned}$$

The injectivity of S-transform then allows us to write  $\mathcal{I}_G(X) = \mathcal{J}_G(X)$ .

2. In view of Proposition 5.7, three cases are possible for the structure of the semimartingale  $G$ . The case where  $G$  is a martingale has been treated in Point 1 below. Our goal here is to exhibit, when  $G$  is not a Gaussian martingale, some general and simple examples for which  $\mathcal{I}_G(X)$  and  $\mathcal{J}_G(X)$  both exist and are different. Let  $f$  be a  $C^{1,2}([0, T] \times \mathbf{R}, \mathbf{R})$  function. Assume that both  $R$  and  $f$  fulfill conditions of Theorem 4.4. Denote  $Y \equiv 0$  when the process  $Y := (Y_t)_{t \in [0, T]}$  is such that

$$Y_t(\omega) = 0, \quad \forall(\omega, t) \in \Omega' \times [0, T], \quad (5.10)$$

where  $\Omega'$  is measurable subset such that  $\mu(\Omega') = 1$ . We will denote  $Y \not\equiv 0$  when (5.10) is not satisfied.

(i) If  $M \equiv 0$  in (5.7). Define  $X_t := \frac{\partial f}{\partial x}(t, G_t)$ , for every  $t$  in  $[0, T]$ . In this case, both integrals  $\mathcal{I}_G(X)$  and  $\mathcal{J}_G(X)$  exist. Moreover, using both Itô formulas (*i.e.* (4.5) and, for instance, [RY99, Theorem 3.3]) we get:

$$\text{a.s.} \quad \Delta_G(X) := 2(\mathcal{I}_G(X) - \mathcal{J}_G(X)) = - \int_0^T \frac{\partial^2 f}{\partial x^2}(t, G_t) dR_t, \quad (5.11)$$

which will be different from 0, as soon as the right hand side term of (5.11) belongs to  $\mathbf{R}^*$ . Hence, for instance, for every  $T$  in  $\mathcal{Z}_R^c$ , on has  $\int_0^T X_s dG_s - \int_0^T X_s d^\circ G_s = 2^{-1} R_T > 0$ . This ends the case (i).

(ii) If  $A \not\equiv 0$  and  $M \not\equiv 0$  in (5.7). Assume that both  $M$  and  $A$  are continuous and that there exists a map  $t \mapsto g_t^{(1)}$  from  $[0, T]$  into  $L^2(\mathbf{R})$  such that  $g^{(1)}$  fulfills Assumption ( $\mathcal{A}$ ) and such that  $M_t = \langle \cdot, g_t^{(1)} \rangle$  almost surely, for every  $t$  in  $[0, T]$ . Let us compare  $\mathcal{I}_G(G)$  and  $\mathcal{J}_G(G)$ . The existence of  $\mathcal{I}_G(G)$  is clear. Moreover, using classical Itô formula, one gets:

$$\begin{aligned} \mathcal{I}_G(G) &= \int_0^T M_s dM_s + \int_0^T A_s dA_s + \int_0^T A_s dM_s + \int_0^T M_s dA_s \\ &= \int_0^T M_s dM_s + \int_0^T A_s dA_s + A_T M_T. \end{aligned}$$

The existence of  $\mathcal{J}_G(G)$  is clear in view of Example 3.16. Moreover, using again Example 3.16, the fact that  $M$  is bounded in  $(L^2)$  as well as an integration by parts, one gets:

$$\begin{aligned} \mathcal{J}_G(G) &= \int_0^T M_s d^\circ M_s + \int_0^T A_s d^\circ A_s + \int_0^T A_s d^\circ M_s + \int_0^T M_s d^\circ A_s \\ &= \int_0^T M_s d^\circ M_s + \int_0^T A_s d^\circ A_s + A_T \diamond M_T. \end{aligned}$$

Classical Itô formula, Example 3.16, (5.8) and finally, Propostion 5.7 and Point 1. of Proposition 2.4, yields:

$$\begin{aligned} \Theta_G(G) &:= \mathcal{I}_G(G) - \mathcal{J}_G(G) = \int_0^T A_s dA_s - \int_0^T A_s d^\circ A_s + A_T M_T - A_T \diamond M_T \\ &= 2^{-1} \mathbf{E}[A_T^2] + A_T M_T - A_T \diamond M_T = 2^{-1} \mathbf{E}[A_T^2] - \mathbf{E}[A_T M_T]. \end{aligned} \quad (5.12)$$

It is then easy to find a finite variation processes  $A$ , as well as a positive real  $T$ , and choose the map  $g^{(1)}$ , that defines the Gaussian martingale  $M$ , such that:  $2^{-1} \mathbf{E}[A_T^2] - \mathbf{E}[A_T M_T] \neq 0$ .  $\square$

**Remark 19. 1.** *In the particular case where there exists a function  $f : \mathbf{R} \rightarrow \mathbf{R}$ , which belongs to  $L^2(\mathbf{R})$ , such that  $G_t := \int_0^t f(u) dB_u$ , for every  $t$  in  $[0, T]$ , a.s., then all the assumptions of Propoistion 5.9 are reduced to  $X$  belongs to  $L^2(G)$ . Note also that the one recovers in particular, the result of [Kuo96, Theorem 13.12], that is  $\int_0^1 X_s d^\circ B_s = \int_0^1 X_s dB_s$ , for every  $X$  in  $L^2(B)$  (*i.e.* in the particular case where  $G = B$ ).*

**2.** *One may also remark, from what we stated in the previous sections, that the existence of  $\int_0^1 X_s d^\circ G_s$  does not imply the existence  $\int_0^1 X_s dG_s$ . Conversely the existence of  $\int_0^1 X_s dG_s$  does not imply the existence  $\int_0^1 X_s d^\circ G_s$ . Three natural questions then arise in this framework;*

(i) *If  $\int_0^T X_s dG_s$  exist, on which conditions on  $X$  the integral  $\int_0^T X_s d^\circ G_s$  will exist?*

(ii) If  $\int_0^T X_s d^\circ G_s$  exist, on which conditions on  $X$  the integral  $\int_0^T X_s dG_s$  will exist?

(iii) When both the integrals  $\int_0^T X_s d^\circ G_s$  and  $\int_0^T X_s dG_s$  do exist, what is the exact link between them?

In order to answer properly to these three questions one needs to use the operators  $D_{g_t}$  and  $D_{g_t}^*$ , defined in [Kuo96, Chap 9], and express both our Wick-Itô integral and the Itô integral using these operators. Since this would lead us too far from the goal of this present work, we will give the answer to these three questions in a future work.

3. In view of Lemma 5.8 (especially Equality (5.8)), it seems that Equality (5.9) remains true under weaker assumptions than the one proposed in Proposition 5.9. However, extend (5.9) under weaker assumptions is an open problem.

4. Of course one can limit our definition of Wick-Itô integral to Gaussian martingales only. Then, and as it is the case for Itô integral, one can extend the definition of Wick-Itô integral wrt  $G$  to the case where  $G$  is a Gaussian semimartingale, by simply setting:

$$\int_0^T X_s d^* G_s := \int_0^T X_s d^\circ M_s + \int_0^T X_s dA_s, \quad (5.13)$$

where  $M$  (resp.  $A$ ) denotes the martingale (resp. the bounded variation process) given by (5.7), and where  $M$  is assumed to be of the form (1.1) and fulfills Assumption ( $\mathcal{A}$ ).  $\int_0^T X_s d^* G_s$  will then be defined as soon as each member of the right hand side of (5.13) will exist. The Itô integral  $\int_0^T X_s dA_s$ , in the right hand side of (5.13), offers also the advantage, on  $\int_0^T X_s d^\circ A_s$  of being defined  $\omega$  by  $\omega$  since it is Stieljes integral.

## 6 Tanaka formula & Local times

The main results of this section are Theorem 6.1 and Theorem 6.6, which provide a Tanaka formula and two occupation time formulas; the first one is given in Subsection 6.1 while the second is given in Subsection 6.2. After having given the main result of Subsection 6.1, one then performs a complete comparison between our Tanaka formula and the ones provided so far, for general Gaussian processes (namely [LN12, Theorem 4.3] and, to a lesser extent, in [MV05, Theorem 36]). In particular, it will be shown that the Tanaka formula we provide here not only fully extends the one provided in [MV05, Theorem 36] but is also less restrictive, in many places, than the one provided by [LN12, Theorem 4.3]. We then apply our Tanaka formula to the Gaussian processes of “reference” and compare the resulting Tanaka formulas hence obtained with the Tanaka formulas established so far, for these particular Gaussian processes.

In Subsection 6.2 we provide a definition and a basic study of both (non weighted) and weighted local times of elements  $G$  in  $\mathcal{G}$ . After having provided an integral representation of each of these two processes, we establish, under classical assumptions, the belonging of both these local times of  $G$ , as two parameters processes, to  $L^2(\mathbf{R} \times \Omega, \lambda \otimes \mu)$ . The proof, that (non weighted) local time of  $G$  belongs to  $L^2(\mathbf{R} \times \Omega, \lambda \otimes \mu)$  we give here, is not only new and based on White noise analysis; it also offers the advantage to be easily adaptable to the weighted local time of  $G$ . We will then show in what extent the results we provide in this Subsection are new, and also give a complete comparison of our results to the one obtained in [LN12], for weighted local times (that are the only local times considered in this latter reference). Finally, we apply our result to the Gaussian processes of “reference” and show how our results extend the ones known for them, so far.

In addition to the references [GH80, MR06] and [RY99, Chapter VI], that provide a complete overview on local times, a summary on fractional Brownian motion’s local time (that includes

the concept of weighted fractional local time) can be found in [Cou07, Section 7.8] (see also references therein, in particular [CNT01]). Note that a study of fBm's local time in the white noise theory's framework can be found in [HØS05]. The non-weighted local times of particular mBms have been studied in [BDG06, BDG07] in the case of an harmonizable mBm while the weighted local time of a Volterra-type mBm has been studied in [BDM10]. This approach has been extended to multifractional Brownian sheets in [MWX08].

Through this section, let  $T$  denotes a positive real and recall that  $\mathcal{Z}_R^T := \mathcal{Z}_R \cap [0, T]$ .

## 6.1 Tanaka Formula

A consequence of Theorem 4.7 is the following Tanaka formula.

**Theorem 6.1** (Tanaka formula for  $G$ ). *Let  $T > 0$  be such that  $[0, T] \subset \mathcal{R}$  and  $c$  be real number. Assume that Assumption (A) holds and that the map  $t \mapsto R_t$  is both continuous and of bounded variations on  $[0, T]$  and such that:*

- (i)  $\lambda(\mathcal{Z}_R^T) = \alpha_R(\mathcal{Z}_R^T) = 0$ ,
- (ii)  $t \mapsto R_t^{-1/2} \in L^1([0, T], dR_t)$ ,
- (iii)  $\exists q \in \mathbf{N}$  such that  $t \mapsto |g'_t|_{-q} R_t^{-1/2} \in L^1([0, T], dt)$ .

Then, the following equality holds in  $(L^2)$ :

$$|G_t - c| = |c| + \int_0^T \text{sign}(G_t - c) d^\circ G_t + \int_0^T \delta_{\{c\}}(G_t) dR_t, \quad (6.1)$$

where the function  $\text{sign}$  is defined on  $\mathbf{R}$  by  $\text{sign}(x) := \mathbf{1}_{\mathbf{R}_+^*}(x) - \mathbf{1}_{\mathbf{R}_-}(x)$ .

**Proof.** This is a direct application of Theorem 4.7 with  $F : x \mapsto |x - c|$ .  $\square$

**Remark 20.** Thanks to (4.28), we clearly see that condition (ii) is satisfied if  $t \mapsto R_t$  is absolutely continuous.

## Comparison with Tanaka formula for general and particular Gaussian processes

As Equality (6.10) will show, one can write (6.1) with the (weighted) local time instead of the last term on the right hand side.

Because the approach used in [LN12] is more intrinsic (since all the assumptions are made exclusively on the variance function  $R$ ), it is not easy to compare conditions (H1), (H3a) & (H4), required by [LN12, Theorem 4.3], to conditions (i) to (iii) of Theorem 6.1. However, one can note the following facts. It is clear that one does not need here to make any assumption on the covariance function  $(t, s) \mapsto R_{t,s}$  but only on the variance function  $t \mapsto R_t$ . In [LN12]  $(t, s) \mapsto R_{t,s}$  is assumed to be continuous on  $[0, T]^2$ , while only  $t \mapsto R_t$  is assumed to be continuous on  $[0, T]$  here. Moreover, we do not need to assume that  $t \mapsto R_t$  is increasing, nor such that  $\mathcal{Z}_R^T = \{0\}$ . Besides, the fact that  $t \mapsto R_t$  is of bounded variations on  $[0, T]$  is sufficient here and one thus does not need to make Assumption (H3a). Finally we do not need any assumption of the type of [LN12, (H4)] to get a Tanaka formula.

Besides, in view of results of Theorem 5.6 and Remark 14, it is clear that Theorem 6.1 applies to all  $\mathcal{V}_\gamma$ -processes and, therefore, that Equality (6.1) (or if one prefers Equality (6.10) below) fully generalizes [MV05, Theorem 36] (with  $F : x \mapsto (x - c)_+$  instead of  $x \mapsto |x - c|$ ).

Note moreover that Theorem 6.1 also generalizes the classical Tanaka formula for Brownian motion (see [RY99, Theorem 1.5]), the one given for fBm in [Ben03a, Corollary 4.8] for any Hurst index  $H$  in  $(0, 1)$ , the one provided in [BDM10, Theorem 12] for Volterra type mBm (it is assumed in there that  $h$  takes values in  $(1/2, 1)$ ) and the one given in [LLV14, Theorem 6.1] for normalized mBm.

## 6.2 Local times of Gaussian processes in $\mathcal{G}$

In all this subsection one does not assume anymore that Assumptions ( $\mathcal{A}$ ) holds.  $G$  is just assumed to be of the form (1.1) and such that the map  $g$ , defined at the beginning of Section 3.1, is continuous on  $\mathcal{R}$ . One assumes, through this section, that  $[0, T] \subset \mathcal{R}$ . Let us first define the local time of  $G$ .

**Definition 3.** (*non weighted*) local time of  $G$

The (*non weighted*) local time of  $G$  at a point  $a \in \mathbf{R}$ , up to time  $T$ , denoted by  $\ell_T^{(G)}(a)$ , is defined by:

$$\ell_T^{(G)}(a) := \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2\varepsilon} \lambda(\{s \in [0, T]; G_s \in (a - \varepsilon, a + \varepsilon)\}),$$

where  $\lambda$  denotes the Lebesgue measure on  $\mathbf{R}$  and where the limit holds in  $(\mathcal{S})^*$ , when it exists.

Having in mind the definition of generalized functional of  $G$  (given in Theorem-Definition 3.2) one has the following result.

**Proposition 6.2.** [*Integral representation on  $[0, T]$  of non weighted local time*]

Assume that the variance map  $s \mapsto R_s$  is continuous on  $[0, T]$  and such that the Lebesgue measure of  $\mathcal{Z}_R^T$  is equal to 0, then:

1. The map  $s \mapsto \delta_a(G_s)$  is  $(\mathcal{S})^*$ -integrable on  $[0, T]$  for every  $a \in \mathbf{R}^*$ . The map  $s \mapsto \delta_0(G_s)$  is  $(\mathcal{S})^*$ -integrable on  $[0, T]$  if  $s \mapsto R_s^{-1/2}$  belongs to  $L^1([0, T], ds)$ .
2. The following equality holds in  $(\mathcal{S})^*$ :

$$\ell_T^{(G)}(a) = \int_0^T \delta_a(G_s) ds, \quad (6.2)$$

for every  $a \in \mathbf{R}^*$ ; and, also for  $a = 0$  if  $s \mapsto R_s^{-1/2} \in L^1([0, T], ds)$ .

**Remark 21.** The reason why we have to assume that the Lebesgue measure of  $\mathcal{Z}_R^T$  is equal to 0 lies in the fact that, when  $F$  belongs to  $\mathcal{S}'(\mathbf{R})$ ,  $F(G_t)$  is not defined if  $t$  belongs to  $\mathcal{Z}_R^T$  (Theorem-Definition 3.2). However, for any  $a$  in  $\mathbf{R}^*$ , if  $\delta_a(G_s)$  has a limit in  $(\mathcal{S})^*$ , as  $s$  tends to  $s_0 \in \mathcal{Z}_R^T$ , one may extend  $\delta_a(G_s)$  at point  $s_0$  by setting  $\delta_a(G_{s_0}) := 0$ . We can not say anything in general, when  $a = 0$ .

Before giving the proof of Proposition 6.2, let us state the following result, the proof of which is obvious, since  $t \mapsto R_t$  is a continuous function, and then left to the reader.

**Lemma 6.3.** One has the following result:

$$\forall \eta \in \mathcal{S}(\mathbf{R}), \forall a \in \mathbf{R}^*, \forall \varepsilon \in (0, \frac{|a|}{2}), \exists \gamma \in (0, \varepsilon), \text{ such that: } \forall (s, y) \in D(\mathcal{Z}_R^T, \gamma) \times D(a, \gamma),$$

$$-(y - \langle g_s, \eta \rangle)^2 < -a^2/16, \quad (6.3)$$

where we have set  $D(a, \gamma) := (a - \gamma, a + \gamma)$ ,  $D(\mathcal{Z}_R^T, \gamma) := \{u \in [0, T]; d(u, \mathcal{Z}_R^T) < \gamma\}$  and where  $d(A, x)$  denote the distance between the subset  $A$  of  $\mathbf{R}$  and the real  $x$ .

**Proof of Proposition 6.2.**

1. Thanks to [Kuo96, Theorem 7.3], we know that  $s \mapsto \delta_a(G_s)$  is an  $(\mathcal{S})^*$ -process, the  $S$ -transform of which is equal, for every  $(a, \eta, s) \in \mathbf{R} \times \mathcal{S}(\mathbf{R}) \times \Gamma_R$ , to:

$$S(\delta_a(G_s))(\eta) = \frac{1}{\sqrt{2\pi R_s}} \exp \left\{ \frac{-(a - \langle g_s, \eta \rangle)^2}{2R_s} \right\}, \quad (6.4)$$

where  $\Gamma_R$  has been defined in the proof of Theorem 4.4 and is equal to  $\{u \in [0, T]; R_u > 0\}$ . When  $a = 0$ , the right hand side of (6.4) is bounded by  $R_s^{-1/2}$ , which belongs, by assumption, to  $L^1([0, T], dt)$ . Theorem 2.6 then applies and establishes 1. in the case where  $a = 0$ .

Let  $a$  be in  $\mathbf{R}^*$ . The weak measurability of  $t \mapsto \delta_a(G_t)$  is clear. In view of [Kuo96, p. 246], one just has to show that  $t \mapsto S(\delta_a(G_t))(\eta)$  belongs to  $L^1([0, T], dt)$ , for every  $\eta$  in  $\mathcal{S}(\mathbf{R})$ . Let  $\eta$  be in  $\mathcal{S}(\mathbf{R})$  and  $\varepsilon$  be in  $(0, \frac{|a|}{2})$ . In view of (6.4) and thanks to Lemma 6.3, one knows that, for every  $s_0$  in  $\mathcal{Z}_R^T$ , there exists a neighborhood  $\mathcal{V}_{s_0}$  of  $s_0$  on which:

$$|S(\delta_a(G_s))(\eta)| \leq R_s^{-1/2} e^{-a^2/32R_s} =: l_a(s).$$

Since  $l_a$  belongs to  $L^1([0, T], dt)$  for every  $\eta$  in  $\mathcal{S}(\mathbf{R})$ , one has established **1.** in the case  $a \neq 0$  and therefore ends the proof of **1.**

**2.** Let  $\varepsilon > 0$ , and  $a$  be a real number. Define

$$I_\varepsilon(a) := \lambda(\{s \in [0, T]; G_s \in D(a, \varepsilon)\}); \quad \text{and} \quad \Phi_\varepsilon(a) \stackrel{(S)^*}{=} \int_{\mathbf{R}} \left( \int_0^T \mathbf{1}_{(a-\varepsilon, a+\varepsilon)}(y) \delta_y(G_s) ds \right) dy.$$

A simple application of Theorem 2.6 and Fubini's theorem shows that  $\Phi_\varepsilon(a)$  is well defined, for every  $a \in \mathbf{R}$ . Besides, since  $I_\varepsilon(a) = \int_0^T \mathbf{1}_{D(a, \varepsilon)}(G_s) ds$ , one has:

$$I_\varepsilon(a) = \int_0^T \left( \int_{\mathbf{R}} \mathbf{1}_{(a-\varepsilon, a+\varepsilon)}(y) \delta_y(G_s) dy \right) ds, \quad (6.5)$$

where we used the following lemma, the proof of which can be found in Appendix A.2.

**Lemma 6.4.** *Let  $j \in \mathcal{S}'(\mathbf{R})$  be of function type and let  $g$  be in  $L^2(\mathbf{R})$ . Define  $\Psi_j : \mathbf{R} \rightarrow (S)^*$  by setting  $\Psi_j(y) = j(y) \delta_y(\langle \cdot, g \rangle)$ . If  $\Psi_j$  is  $(S)^*$ -integrable on  $\mathbf{R}$ , then one has the equality*

$$\int_{\mathbf{R}} j(y) \delta_y(\langle \cdot, g \rangle) dy \stackrel{(S)^*}{=} j(\langle \cdot, g \rangle). \quad (6.6)$$

Moreover, if  $j(\langle \cdot, g \rangle)$  belongs to  $(L^2)$ , then (6.6) is also true in  $(L^2)$ .

A direct application of Fubini's Theorem and (6.5) show the equality of the  $S$ -transform of  $\Phi_\varepsilon(a)$  and  $I_\varepsilon(a)$ . The equality in  $(S)^*$  results of the injectivity of the  $S$ -transform. It then remains to show that  $\lim_{\varepsilon \rightarrow 0} \frac{I_\varepsilon(a)}{2\varepsilon}$  exists in  $(S)^*$ . Besides, let  $(\eta, a)$  be in  $\mathcal{S}(\mathbf{R}) \times \mathbf{R}$ . Let  $\varepsilon$  be in  $(0, 1)$  such that  $D(a, \varepsilon) \subset \mathbf{R}_+^*$  or  $D(a, \varepsilon) \subset \mathbf{R}_-^*$ , if  $a \in \mathbf{R}^*$ . Using (ii) of Lemma 2.5 and (6.4) we get:

$$S\left(\frac{I_\varepsilon(a)}{2\varepsilon}\right)(\eta) = \frac{1}{2\varepsilon} \int_{D(a, \varepsilon)} \left( \int_0^T \frac{1}{\sqrt{2\pi R_s}} \exp\left\{\frac{-1}{2R_s}(y - \langle g_s, \eta \rangle)^2\right\} ds \right) dy =: \frac{1}{2\varepsilon} \int_{D(a, \varepsilon)} \left( \int_0^T \rho_\eta(y, s) ds \right) dy. \quad (6.7)$$

We will use the following lemma, the proof of which can be found in Appendix A.2.

**Lemma 6.5.** *For every  $\eta$  in  $\mathcal{S}(\mathbf{R})$ , define the map  $G_\eta : \mathbf{R}^* \rightarrow \mathbf{R}$ , by setting  $G_\eta(y) := \int_0^T \rho_\eta(y, s) ds$ . Then  $G_\eta$  is continuous on  $\mathbf{R}^*$ . Moreover, if  $s \mapsto R_s^{-1/2}$  belongs to  $L^1([0, T] \setminus \mathcal{Z}_R^T, ds)$ , then one can extend  $G_\eta$  to  $\mathbf{R}$  by setting  $G_\eta(0) := \int_0^T \rho_\eta(0, s) ds$ . Furthermore, this extension is continuous on  $\mathbf{R}$ .*

The equality  $G_\eta(a) = S\left(\int_0^T \delta_a(G_s) ds\right)(\eta)$  and the continuity of  $G_\eta$ , on  $D(a, \varepsilon)$ , allows us to write  $\lim_{\varepsilon \rightarrow 0} \frac{1}{2\varepsilon} \int_{D(a, \varepsilon)} G_\eta(y) dy = S\left(\int_0^T \delta_a(G_s) ds\right)(\eta)$ . Thanks to (6.7), one finally gets  $\lim_{\varepsilon \rightarrow 0} S\left(\frac{I_\varepsilon(a)}{2\varepsilon}\right)(\eta) = S\left(\int_0^T \delta_a(G_s) ds\right)(\eta)$ . Moreover, using the same computations as in the proof of Lemma 6.5, it is easy to show that, for every  $a$  in  $\mathbf{R}^*$ ,

$$\forall \eta \in \mathcal{S}(\mathbf{R}), \forall y \in D(a, |a|/2), \quad |G_\eta(y)| \leq M_a e^{|\eta|_0},$$

where  $M_a := T(8(1+e)|a|^{-1} + \sup\{R_s^{-1/2} \exp(-a^2/32R_s)\})$ . One then gets:  $|S\left(\frac{I_\varepsilon(a)}{2\varepsilon}\right)(\eta)| \leq M_a e^{|\eta|_0}$ . [Kuo96, Theorem 8.6] then allows us to write  $\lim_{\varepsilon \rightarrow 0} \frac{I_\varepsilon(a)}{2\varepsilon} \stackrel{(S)^*}{=} \int_0^T \delta_a(G_s) ds$ , which ends the proof.  $\square$

**Remark 22.** We deduce from the proof of Proposition 6.2 that  $\ell_T^{(G)}(a)$  exists for every  $a$  in  $\mathbf{R}^*$ ; and also for  $a = 0$ , if  $s \mapsto R_s^{-1/2}$  belongs to  $L^1([0, T], ds)$ . Besides, the example of the Gaussian process  $(G_s)_{s \in \mathbf{R}}$ , defined by setting  $g_s := \sum_{k \geq 0} s(1+k)^{-1} e_k$ , shows that the condition  $s \mapsto R_s^{-1/2}$  belongs to  $L^1([0, T], ds)$  is required in order that the map  $s \mapsto \delta_0(G_s)$  is  $(\mathcal{S})^*$ -integrable on  $[0, T]$ .

**Theorem-Definition 6.1.** [Weighted local time of  $G$  and its integral representation in  $(\mathcal{S})^*$ ] Assume that the map  $s \mapsto R_s$  is continuous and of bounded variations on  $[0, T]$  and such that  $\alpha_R(\mathcal{Z}_R^T)$  is equal to 0. Then:

1. The map  $s \mapsto \delta_a(G_s)$  is  $(\mathcal{S})^*$ -integrable on  $[0, T]$  with respect to the measure  $\alpha_R$ , for every  $a \in \mathbf{R}^*$ . The map  $s \mapsto \delta_0(G_s)$  is  $(\mathcal{S})^*$ -integrable on  $[0, T]$  with respect to the measure  $\alpha_R$ , if  $s \mapsto R_s^{-1/2}$  belongs to  $L^1([0, T], dR_s)$ .
2. For every  $a \in \mathbf{R}$ , when  $s \mapsto \delta_a(G_s)$  is  $(\mathcal{S})^*$ -integrable on  $[0, T]$  with respect to the measure  $\alpha_R$ , one can define the weighted local time of  $G$  at point  $a$ , up to time  $T$ , denoted  $\mathcal{L}_T^{(G)}(a)$ , as being the  $(\mathcal{S})^*$ -process defined by setting:

$$\mathcal{L}_T^{(G)}(a) := \int_0^T \delta_a(G_s) dR_s, \quad (6.8)$$

where the equality holds in  $(\mathcal{S})^*$ .

**Proof.** Since the proof of Point 1 follows exactly the same steps as proof of Proposition 6.2 (one just has to replace the Lebesgue measure by  $\alpha_R$ ), it is left to the reader.  $\square$

The definition of weighted local time of  $G$  (Def.6.8) generalizes the one given, for fBm, in [Cou07, Definition 14]. Besides, assume that  $s \mapsto R_s$  is absolutely continuous on  $[0, T]$  and such that its derivative, denoted  $R'$ , is differentiable *a.e.* on  $[0, T]$ . If the following conditions are fulfilled:

- (i)  $s \mapsto R'_s \delta_a(G_s)$  is  $(\mathcal{S})^*$ -integrable on  $[0, T]$ ,
- (ii)  $s \mapsto R''_s \ell_s^{(G)}(a)$  is  $(\mathcal{S})^*$ -integrable on  $[0, T]$ ,
- (iii)  $s \mapsto R'_s \ell_s^{(G)}(a)$  has a limit in 0 and  $T$  (denoted respectively  $\ell_s^{(G)}(a) R'_s|_{s=0}$  and  $\ell_s^{(G)}(a) R'_s|_{s=T}$ ),

then an integration by parts yields:

$$\mathcal{L}_T^{(G)}(a) = \ell_s^{(G)}(a) R'_s|_{s=T} - \ell_s^{(G)}(a) R'_s|_{s=0} - \int_0^T \ell_s^{(G)}(a) R''_s ds, \quad (6.9)$$

where  $R''$  denotes the second derivative of  $R$ .

**Remark 23. 1.** In view of the definition of weighted local time, it is clear that (6.1) now reads:

$$|G_t - c| = |c| + \int_0^T \text{sign}(G_t - c) d^\circ G_t + \mathcal{L}_T^{(G)}(c). \quad (6.10)$$

As we stated at the end of Subsection 6.1, one recovers hence the Tanaka formula provided in [MV05, Theorem 36] and in [LN12, Theorem 4.3].

2. Equality (6.9) above generalizes, to Gaussian processes which belong to  $\mathcal{G}$ , the relation between weighted and non-weighted local times given, for fBm, in [Cou07, Definition 14].

The following theorem establishes that, under suitable assumptions, both weighted and non-weighted local times are  $(L^2)$  random variables and also provides an occupation time formula for both  $\ell^{(G)}$  and  $\mathcal{L}^{(G)}$ . Denote  $\mathcal{M}_b(\mathbf{R})$  the set of positive Borel functions defined on  $\mathbf{R}$ .

**Theorem 6.6.** Assume that the function  $s \mapsto R_s$  is continuous on  $[0, T]$ .

(i) Assume that  $\lambda_R(\mathcal{Z}_R^T) = 0$  and that  $s \mapsto R_s^{-1/2} \in L^1([0, T], ds)$ . If  $\mathbf{E}[\int_{\mathbf{R}} |\int_0^T e^{i\xi G_s} ds|^2 d\xi] < +\infty$ , then the map  $a \mapsto \ell_T^{(G)}(a)$  belongs to  $L^2(\lambda \otimes \mu)$ , where  $\lambda$  denotes the Lebesgue measure. Moreover one has the following equality, valid for  $\mu$ -a.e.  $\omega$  in  $\Omega$ ,

$$\forall \Phi \in \mathcal{M}_b(\mathbf{R}), \int_0^T \Phi(G_s(\omega)) ds = \int_{\mathbf{R}} \ell_T^{(G)}(y)(\omega) \Phi(y) dy. \quad (6.11)$$

(ii) Assume that  $\alpha_R(\mathcal{Z}_R^T) = 0$ . Assume moreover that  $s \mapsto R_s$  is of bounded variations on  $[0, T]$ , and such that  $s \mapsto R_s^{-1/2} \in L^1([0, T], dR_s)$ . If  $\mathbf{E}[\int_{\mathbf{R}} |\int_0^T e^{i\xi G_s} dR_s|^2 d\xi] < +\infty$ , then  $a \mapsto \mathcal{L}_T^{(G)}(a)$  belongs to  $L^2(\lambda \otimes \mu)$ . Moreover one has the following equality, valid for  $\mu$ -a.e.  $\omega$  in  $\Omega$ ,

$$\forall \Phi \in \mathcal{M}_b(\mathbf{R}), \int_0^T \Phi(G_s(\omega)) dR_s = \int_{\mathbf{R}} \mathcal{L}_T^{(G)}(y)(\omega) \Phi(y) dy. \quad (6.12)$$

**Remark 24. 1.** The first consequence is that, under assumptions of Theorem 6.6, both  $\ell_T^{(G)}(a)$  and  $\mathcal{L}_T^{(G)}(a)$  are  $(L^2)$  random variables for every real  $a$ . In a forthcoming paper we discuss the joint continuity of processes  $(T, a) \mapsto \ell_T^{(G)}(a)$  and  $(T, a) \mapsto \mathcal{L}_T^{(G)}(a)$ . In particular, under assumptions of local non-determinism<sup>16</sup>, it is shown that both map  $a \mapsto \ell_T^{(G)}(a)$  and  $a \mapsto \mathcal{L}_T^{(G)}(a)$  are continuous (the continuity at point 0 requires that  $t \mapsto R_t^{-1/2}$  belongs to  $L^1([0, T], dt)$  or to  $L^1([0, T], dR_t)$ ).

2. Since  $\mathbf{E}[X] = S(X)(0)$ , for every  $(L^2)$  random variable, it is easy to get, for almost every real  $a$ , the equalities:

$$\mathbf{E}[\ell_T^{(G)}(a)] = \int_0^T \frac{1}{\sqrt{2\pi R_s}} \exp\{-\frac{a^2}{2R_s}\} ds; \quad \& \quad \mathbf{E}[\mathcal{L}_T^{(G)}(a)] = \int_0^T \frac{1}{\sqrt{2\pi R_s}} \exp\{-\frac{a^2}{2R_s}\} dR_s,$$

when assumptions of Theorem 6.6 are fulfilled,

3. As we stated right after the proof of Theorem-Definition 6.1, the map  $t \mapsto R_t^{-1/2}$  belongs to  $L^1([0, T], dR_t)$  if  $t \mapsto R_t$  is absolutely continuous. In this latter case (6.12) holds under the single condition  $\mathbf{E}[\int_{\mathbf{R}} |\int_0^T e^{i\xi G_s} dR_s|^2 d\xi] < +\infty$ , which is equivalent to  $\int_0^T \int_0^T (\Delta(t, s))^{-1/2} dR_s dR_t < +\infty$ , where  $\Delta(t, s) := \mathbf{E}[(G_t - G_s)^2]$ . As it is shown in [GH80, (21.9) & (21.10)] any of these two latter conditions is necessary and sufficient to ensure the existence of a weighted local time, that is an  $(L^2)$  random variable. This shows that the assumptions we made on the variance function  $R$  are almost minimal.

4. The condition  $t \mapsto R_t^{-1/2} \in L^1([0, T], dt) \cap L^1([0, T], dR_t)$  is verified for all the Gaussian processes of “reference” (see Proposition 6.8 below for the case of  $\mathcal{V}_\gamma$ -processes).

Let us give the proof of Theorem 6.6 before discussing the results provided in it and before making the comparison of them with the results obtained in [LN12, §4].

**Proof.** Let us start with the proof of (i). Define, for every  $(\omega, a)$  in  $\Omega \times \mathbf{R}$ , the map denoted  $f_T^{(\omega)} : \mathbf{R} \rightarrow \mathbf{R}$  and, for every positive integer  $n$ , the family of maps denoted  $f_{T,n}^{(\omega)} : \mathbf{R} \rightarrow \mathbf{R}$  by setting:  $f_T^{(\omega)}(\xi) := \frac{1}{2\pi} \int_0^T e^{i\xi G_s(\omega)} ds$  and  $f_{T,n}^{(\omega)}(\xi) := e^{-\xi^2/n} f_T^{(\omega)}(\xi)$ . Recalling the notations about Fourier transform defined in Section 2.1, define also:

$$K_T^{(\omega)}(a) := \mathcal{F}(f_T^{(\omega)})(a); \quad \& \quad K_{T,n}^{(\omega)}(a) := \mathcal{F}(f_{T,n}^{(\omega)})(a). \quad (6.13)$$

<sup>16</sup>see [GH80, §3.24] and references therein for an overview of this notion

It is clear that  $\lim_{n \rightarrow +\infty} K_{T,n}^{(\omega)} = K_T^{(\omega)}$ , in  $L^2(\mathbf{R})$ , since  $\lim_{n \rightarrow +\infty} f_{T,n}^{(\omega)} = f_T^{(\omega)}$ , where the convergence holds in  $L^2(\mathbf{R})$ . Denotes  $K_{T,n}(a)$  (*resp.*  $K_T(a)$ ) the random variable  $\omega \mapsto K_{T,n}^{(\omega)}(a)$  (*resp.*  $\omega \mapsto K_T^{(\omega)}(a)$ ).

**Step 1:**  $\lim_{n \rightarrow +\infty} K_{T,n}(a) = \ell_T^{(G)}(a)$ , in  $(\mathcal{S})^*$ .

Let  $\omega$  in  $\Omega$ . Since  $f_{T,n}^{(\omega)}$  belongs to  $L^1(\mathbf{R}) \cap L^2(\mathbf{R})$ , one can write  $K_{T,n}^{(\omega)}(a) = \int_{\mathbf{R}} f_{T,n}^{(\omega)}(\xi) e^{-ia\xi} d\xi$ . For every  $\eta$  in  $\mathcal{S}(\mathbf{R})$ , using Lemma 2.5, we then have:

$$S(K_{T,n}(a))(\eta) = \int_{\mathbf{R}} \frac{1}{2\pi} e^{-ia\xi - \xi^2/n} \left( \int_0^T S(e^{i\xi G_s})(\eta) ds \right) d\xi.$$

Using Equality (2.1) and Fubini's theorem, that both obviously apply here, we get:

$$S(K_{T,n}(a))(\eta) = \frac{1}{\sqrt{2\pi}} \int_0^T \left( \int_{\mathbf{R}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\xi^2(\frac{2}{n} + R_s) + 2i\xi(a - \langle \eta, g_s \rangle))} d\xi \right) ds =: \frac{1}{\sqrt{2\pi}} \int_0^T J_{n,\eta}(s) ds.$$

Using the change of variable  $u := \xi \sqrt{\varepsilon_n + R_s}$ , where  $\varepsilon_n := \frac{2}{n}$ , in  $J_{n,\eta}$  yields:

$$S(K_{T,n}(a))(\eta) = \frac{1}{\sqrt{2\pi}} \int_0^T \frac{\exp\{-\frac{(a - \langle \eta, g_s \rangle)^2}{2(\varepsilon_n + R_s)}\}}{\sqrt{\varepsilon_n + R_s}} ds.$$

Using the same arguments that the one we used in the proof of Lemma 6.5 allows us to apply the Lebesgue dominated convergence. We therefore get, using (6.4) and then (6.2),

$$\begin{aligned} \lim_{n \rightarrow +\infty} S(K_{T,n}(a))(\eta) &= \int_0^T \frac{1}{\sqrt{2\pi R_s}} \exp\{-\frac{(a - \langle \eta, g_s \rangle)^2}{2R_s}\} ds = S\left(\int_0^T \delta_a(G_s) ds\right)(\eta), \\ &= S(\ell_T^{(G)}(a))(\eta). \end{aligned} \quad (6.14)$$

The domination used in the proof of Lemma 6.5 applies here also and prove the existence of a function  $L$ , which belongs to in  $L^1([0, T])$ , such that:

$$\forall (p, n, \eta, a) \in \mathbf{N} \times \mathbf{N}^* \times \mathcal{S}(\mathbf{R}) \times \mathbf{R}, \quad |S(K_{T,n}(a))(\eta)| \leq e^{|\eta|_p^2} \int_0^T L(u) du. \quad (6.15)$$

Formulas (6.14) and (6.15) allow us finally to apply [Kuo96, Theorem 8.6.] and we then establish that  $\lim_{n \rightarrow +\infty} K_{T,n}(a) = \ell_T^{(G)}(a)$ , in  $(\mathcal{S})^*$ .

**Step 2:** Let us show that there exists a strictly increasing function  $\varphi : \mathbf{N}^* \rightarrow \mathbf{N}^*$  such that, for almost every real  $a$ ,  $\lim_{n \rightarrow +\infty} K_{T,n}(a) = K_T(a)$ , in  $(L^2)$ .

From the first step we know that, for  $\mu$ -almost every  $\omega$ ,  $\lim_{n \rightarrow +\infty} \|K_{T,n}^{(\omega)} - K_T^{(\omega)}\|_{L^2(\mathbf{R})}^2 = 0$ .

From the other hand, one gets  $\sup_{n \in \mathbf{N}^*} \|K_{T,n}^{(\omega)}\|_{L^2(\mathbf{R})} \leq \|K_T^{(\omega)}\|_{L^2(\mathbf{R})}$ . Since, by assumption,  $\omega \mapsto$

$\|K_T^{(\omega)}\|_{L^2(\mathbf{R})}^2$  is an  $(L^2)$  random variable, the Lebesgue dominated convergence theorem applies.

Thus we get the convergence  $\lim_{n \rightarrow +\infty} \int_{\Omega} \|K_{T,n}^{(\omega)} - K_T^{(\omega)}\|_{L^2(\mathbf{R})}^2 d\omega = 0$ ; that may also be written

$\lim_{n \rightarrow +\infty} \int_{\mathbf{R}} \mathbf{E}[\|K_{T,n}(a) - K_T(a)\|^2] da = 0$ . This shows that there exists a strictly increasing function  $\varphi : \mathbf{N}^* \rightarrow \mathbf{N}^*$  such that for almost every real  $a$ ,  $\lim_{n \rightarrow +\infty} K_{T,\varphi(n)}(a) = K_T(a)$ , where the

limit holds in  $(L^2)$ . Finally, results of steps 1 and 2 yield, for almost every real  $a$ ,  $K_T(a) = \ell_T^{(G)}(a)$ .

We have then proved that the map  $a \mapsto \ell_T^{(G)}(a)$  belongs to  $L^2(\lambda \otimes \mu)$ .

**Step 3:** Proof of (6.11):

An application of the monotone class theorem allows us to establish (6.11) only for every  $\Phi$  in  $\mathcal{S}(\mathbf{R})$ . For any  $\Phi$  in  $\mathcal{S}(\mathbf{R})$ , denote  $\check{\Phi}$  the function defined by setting  $\check{\Phi}(\xi) := \Phi(-\xi)$  and, for any  $F$  in  $\mathcal{S}'(\mathbf{R})$ , denote  $\check{F}$  the tempered distribution defined by  $\langle \check{F}, \Phi \rangle = \langle F, \check{\Phi} \rangle$ , for every  $\Phi$  in  $\mathcal{S}(\mathbf{R})$ . For every  $\omega$  in  $\Omega$ , define  $\theta_T^{(\omega)} : \mathcal{S}(\mathbf{R}) \rightarrow \mathbf{R}$  by setting  $\theta_T^{(\omega)}(\Phi) := \int_0^T \Phi(G_s(\omega)) ds$ . Since  $\theta_T^{(\omega)}$  belongs to  $\mathcal{S}'(\mathbf{R})$ , one can compute its Fourier transform, denoted  $\mathcal{F}(\theta_T^{(\omega)})$ . We obtain, for almost every real  $u$ :

$$\mathcal{F}(\theta_T^{(\omega)})(u) = \int_0^T e^{-iuG_s(\omega)} ds = 2\pi \widetilde{f_T^{(\omega)}}(u) \quad (6.16)$$

Besides, since  $\theta_T^{(\omega)} = \frac{1}{2\pi} \mathcal{F}(\mathcal{F}(\theta_T^{(\omega)}))$  we get, using (6.16),  $\theta_T^{(\omega)} = \frac{1}{2\pi} \mathcal{F}(2\pi \widetilde{f_T^{(\omega)}}) = K_T^{(\omega)}$ ; and hence, for all  $\Phi$  in  $\mathcal{S}(\mathbf{R})$ ,

$$\theta_T^{(\omega)}(\Phi) = \langle K_T^{(\omega)}, \Phi \rangle. \quad (6.17)$$

Thanks to Step 2, we know that, for almost every real  $a$ ,  $K_T(a) = \ell_T^{(G)}(a)$ . The definition of  $\theta_T^{(\omega)}$  as well as (6.17) then yield:

$$\int_0^T \Phi(G_s(\omega)) ds = \int_{\mathbf{R}} \ell_T^{(G)}(a) \Phi(a) da.$$

The proof of (ii) is obvious in view of the one of (i), since one just has to take  $f_T^{(\omega)}(\xi) := \frac{1}{2\pi} \int_0^T e^{i\xi G_s(\omega)} dR_s$  and  $\theta_T^{(\omega)}(\Phi) := \int_0^T \Phi(G_s(\omega)) dR_s$ . It is therefore left to the reader.  $\square$

### Comparison with the existing results on Local times of general and particular Gaussian processes

Let us first recall that, for any Borel set  $\mathcal{B}$  of  $\mathbf{R}_+$ , the occupation measure of  $G$  on  $\mathcal{B}$ , denoted  $\nu_{\mathcal{B}}$ , is defined for all Borel sets  $\mathcal{D} \subset \mathbf{R}$  by setting  $\nu_{\mathcal{B}}(\mathcal{D}) := \lambda(\{s \in \mathcal{B}; G_s \in \mathcal{D}\})$ . When  $\nu_{\mathcal{B}}$  is absolutely continuous with respect to the Lebesgue measure on  $\mathbf{R}$ , one defines the local time of  $G$  on  $\mathcal{B}$ , noted  $\{L_G(\mathcal{B}, x); x \in \mathbf{R}\}$ , as being the Radon-Nikodym derivative of  $\nu_{\mathcal{B}}$ . Following [GH80, Section 2] we say in this case that  $G$  is (LT). When  $\mathcal{B} = [0, T]$ , we write  $\{L_G(T, x); x \in \mathbf{R}\}$  instead of  $\{L_G([0, T], x); x \in \mathbf{R}\}$ .

**Remark 25. 1.** *In Theorem 6.6, instead of giving our proof of (i), one may think to use [GH80, Theorem (21.9)]. In particular, we know thanks to [GH80, (21.10)] (or the equivalent condition [GH80, (22.3)]), that  $G$  is (LT) and that  $\{L_G(T, a); a \in \mathbf{R}\}$  belongs to  $L^2(\lambda \otimes \mu)$ . However we do not know, at this stage, that  $\{L_G(T, a); a \in \mathbf{R}\}$  and  $\{\ell_T^{(G)}(a); a \in \mathbf{R}\}$  are the same process (since one does not know, in particular that  $\ell_T^{(G)}(a)$  is a random variable, which belongs to  $L^2$ , for almost every real  $a$ ). Note moreover that [GH80, (21.10) or (22.3)] can not be used in the proof of (ii) of Theorem 6.6 either; at least without any additional assumption on the variance function  $t \mapsto R_t$ . In contrast, the proof we gave for (i) can be adapted for establishing proof of (ii).*

**2.** *Note that (6.12) allows one to get the occupation times formula given in [RY99, Corollary p. 224] in the case where  $G$ , which belongs to  $\mathcal{G}$ , is a continuous Gaussian semimartingale. It is also clear that (6.12) generalizes, to Gaussian non-semimartingale, the occupation times formula given in [RY99, Corollary p. 224].*

*In addition to the general references cited in the introduction of this section, [LN12, Section 4], is the only one recent paper in which local times of general Gaussian processes are studied at such a level of generality. The authors consider therein only what we have called here weighted*

local time, which they denote by  $L_t(x)$  and define, when it exist, as the density of the occupation measure:

$$m_t(\mathcal{B}) := \int_0^t \mathbf{1}_{\mathcal{B}}(G_s) dR_s, \quad \forall \mathcal{B} \in \mathcal{B}(R).$$

By the very definition of  $L_t(x)$ , they obtained the occupation formula (6.12). They then show that that  $L_t(x)$  is an  $(L^2)$  random variable, by giving its chaos decomposition.

In comparison with [LN12, Section 4], and as we already stated in Section 6.1, one first does not need to assume that  $(t, s) \mapsto R_{t,s}$  is continuous on  $[0, T]^2$ , but only that  $t \mapsto R_t$  and  $t \mapsto g_t$  are continuous on  $[0, T]$ . Besides, the fact that  $t \mapsto R_t$  is of bounded variations on  $[0, T]$  and such that  $\alpha_R(\mathcal{Z}_R^T) = 0$  is sufficient here and one thus does not need to make Assumption (H3a) of [LN12] (i.e. to assume that  $t \mapsto R_t$  is increasing and such that  $\mathcal{Z}_R^T = \{0\}$ ). Finally, and as we stated in Point 3 of Remark 24, the assumption  $\mathbf{E}[\int_{\mathbf{R}} |\int_0^T e^{i\xi G_s} dR_s|^2 d\xi] < +\infty$  is necessary and sufficient to ensure the existence of a weighted local time, that is an  $(L^2)$  random variable. It is then less restrictive than [LN12, (H4)]. Finally, Assumption [LN12, (H4)] is not required here but one needs to assume that  $s \mapsto R_s^{-1/2} \in L^1([0, T], dR_s)$ .

Let us end this section, and conclude this work, by studying the case of the Gaussian processes of “reference”.

**Example 6.7.** 1. If  $G$  is a Brownian motion or a Brownian bridge, then  $\ell_T^{(G)}(a)$ , given in Definition 3, always exists and belongs to  $L^2(\mathbf{R} \times \Omega, \lambda \otimes \mu)$ , as a two parameters process. Moreover, assumptions of Proposition 6.2, Theorem-Definition 6.1 and Theorem 6.6 are all fulfilled. Of course,  $\ell_T^{(B)}$  and  $\mathcal{L}_T^{(B)}$  both coincide and we recover classical results for Brownian motion local time (see [RY99, Chapter 6] for example) in all its generality.

2. (i) If  $G$  is a fBm  $B^H$ , then assumptions of Proposition 6.2, of Theorem-Definition 6.1 and Theorem 6.6 are all fulfilled, whatever the value of  $H$  in  $(0, 1)$  is. Thus  $\ell_T^{(G)}(a)$ , given in Definition 3, always exists and  $\{\ell_T^{(G)}(a)(\omega); (a, \omega)\}$  belongs to  $L^2(\mathbf{R} \times \Omega, \lambda \otimes \mu)$ . We therefore recover the results for non weighted local time given in [HØ02], as well as the occupation times formulas given, for both weighted and non weighted fractional local times, in [Cou07, Proposition 30 & Definition 14]. In particular, if we except the occupation time formula, the results we provide here on weighted local time when  $G = B^H$  (i.e. on  $\mathcal{L}_T^{(B^H)}$ ), are new.

(ii) To our best knowledge, there was no proof of existence of (weighted or non-weighted) local time of general mBm (i.e. in sense of [LLVH14, Definition 1.2], recalled in our introduction) in the literature so far. However, several results exist for certain classes of mBm. More precisely, in [BDG06, BDG07] the authors are interested in the mBm, which corresponds to  $Y_{(1,0)}$  in [ST06]. They establish, on each time interval  $[a, b] \subset (0, +\infty)$ , the existence of a square integrable local time (i.e. in  $L^2(\mathbf{R} \times \Omega, \lambda \otimes \mu)$ ), assuming  $\sup\{h(t); t \geq 0\} < \beta \wedge 1$ , where  $h$  is  $\beta$ -Hölder continuous. In [BDM10], a Volterra-type representation of mBm is studied. It is established, using the sufficient condition given in [GH80, Theorem 22.1], the existence of a square integrable local time, (assuming  $h$  is a differentiable and  $(1/2, 1)$ -valued function). As a consequence, the authors get (6.11). Starting from the Itô formula [AMN01, Theorem 2], they also get (6.12). All these results are recovered by Theorem 6.6. In particular, the results we provide on weighted local time when  $G = B^h$ , are new.

3. Finally, we provide new results on weighted and non weighted local times of  $\mathcal{V}_\gamma$  - processes in the Proposition 6.8 below.

**Proposition 6.8.** *Let  $B^\gamma$  be a  $\mathcal{V}_\gamma$  - process. Then both weighted and non-weighted local times of  $G$  (denoted respectively  $\ell_T^{(\gamma)}$  and  $\mathcal{L}_T^{(\gamma)}$ ) exist and belong to  $L^2(\mathbf{R} \times \Omega, \lambda \otimes \mu)$ , as two parameters processes. Moreover, for  $\mu$ -a.e.  $\omega$  in  $\Omega$  and all  $\Phi$  in  $\mathcal{M}_b(\mathbf{R})$ , one has the following equalities:*

$$\int_0^T \Phi(G_s(\omega)) ds = \int_{\mathbf{R}} \ell_T^{(\gamma)}(y)(\omega) \Phi(y) dy. \quad (6.18)$$

$$\int_0^T \Phi(G_s(\omega)) dR_s = \int_{\mathbf{R}} \mathcal{L}_T^{(\gamma)}(y)(\omega) \Phi(y) dy. \quad (6.19)$$

**Proof.** It is sufficient to apply Proposition 6.2, Theorem-Definition 6.1 and Theorem 6.6. Since  $t \mapsto R_t$  is absolutely continuous (thanks to Remark 14), one sees, in view of Remark 24, that one just have to prove that:  $\mathbf{E}[\int_{\mathbf{R}} |\int_0^T e^{i\xi B_s^\gamma} dR_s|^2 d\xi] < +\infty$  &  $\mathbf{E}[\int_{\mathbf{R}} |\int_0^T e^{i\xi B_s^\gamma} ds|^2 d\xi] < +\infty$ . However, thanks to [GH80, (21.10) & (22.3)], one knows that this equivalent to show that:  $\int_0^T \int_0^T (\Delta(t, s))^{-1/2} dt ds < +\infty$  &  $\int_0^T \int_0^T (\Delta(t, s))^{-1/2} dR_s dR_t < +\infty$ , where  $\Delta(t, s) := \mathbf{E}[(B_t^\gamma - B_s^\gamma)^2]$ . Thanks to [MV05, Proposition 1.1], we know that:  $R_{|t-s|} \leq \Delta(t, s) \leq 2 R_{|t-s|}$ , for every  $(t, s)$  in  $[0, T]^2$ . We then use the same arguments as the one given in Remark 14, as well as the fact that  $R$  is increasing, to get:

$$\Theta_T := \int_0^T \int_0^T (\Delta(s, u))^{-1/2} du ds \leq 2 \int_0^T \left( \int_0^t R_u^{-1/2} du \right) dt \leq 8(1 + R_T) + \frac{T^2}{\sqrt{\alpha}},$$

where  $\alpha := \inf\{u \in (0, T], R'_u \leq 1\}$  (where we set  $\inf \emptyset = +\infty$ ). This achieves the proof.  $\square$

## Acknowledgments

I want to express my deep gratitude to Jacques Lévy Véhel for his advices and for the very stimulating discussions we had about this work. I also want to thank Professor T. Hida for his warm welcome at the University of Nagoya, where a part of this paper was written, as well as Professor L. Chen and the Institute for Mathematical Sciences of Singapore (NUS), where another part of this paper was written. This work is dedicated to the memory of Professor Marc Yor.

## Appendix

### A.1 Bochner Integral

The following notions about Bochner integral come from [HP57, p.72, 80 and 82] and [Kuo96, p.247].

**Definition 4.** *Bochner integral [Kuo96, p.247] Let  $I$  be a Borelian subset of  $\mathbf{R}$  endowed with the Lebesgue measure. One says that  $\Phi : I \rightarrow (\mathcal{S})^*$  is Bochner integrable on  $I$  if it satisfies the two following conditions:*

- 1  $\Phi$  is weakly measurable on  $I$  i.e  $u \mapsto \langle \Phi_u, \varphi \rangle$  is measurable on  $I$  for every  $\varphi$  in  $(\mathcal{S})$ .
- 2  $\exists p \in \mathbb{N}$  such that  $\Phi_u \in (\mathcal{S}_{-p})$  for almost every  $u \in I$  and  $u \mapsto \|\Phi_u\|_{-p}$  belongs to  $L^1(I)$ .

The Bochner-integral of  $\Phi$  on  $I$  is denoted  $\int_I \Phi_s ds$ .

**Proposition A.1.** *If  $\Phi : I \rightarrow (\mathcal{S})^*$  is Bochner-integrable on  $I$  then there exists an integer  $p$  such that  $\|\int_I \Phi_s ds\|_{-p} \leq \int_I \|\Phi_s\|_{-p} ds$ . Moreover  $\Phi$  is also Pettis-integrable on  $I$  and both integrals coincide on  $I$ .*

**Remark 26.** *The previous proposition shows that there is no risk of confusion by using the same notation for both Bochner and Pettis integrals.*

**Theorem A.2.** Let  $p \in \mathbb{N}$  and  $(\Phi^{(n)})_{n \in \mathbb{N}}$  be a sequence of processes from  $I$  to  $(\mathcal{S})^*$  such that  $\Phi_u^{(n)} \in (\mathcal{S}_{-p})$  for almost every  $u \in I$  and for every  $n$ . Assume moreover that  $\Phi^{(n)}$  is Bochner-integrable on  $I$ , for every  $n$ , and that  $\lim_{(n,m) \rightarrow (+\infty, +\infty)} \int_I \|\Phi_s^{(m)} - \Phi_s^{(n)}\|_{-p} ds = 0$ . Then there exists an  $(\mathcal{S})^*$ -process (almost surely  $(\mathcal{S}_{-p})$ -valued), denoted  $\Phi$ , defined and Bochner-integrable on  $I$ , such that

$$\lim_{n \rightarrow +\infty} \int_I \|\Phi_s - \Phi_s^{(n)}\|_{-p} ds = 0 \quad (\text{A.1})$$

Furthermore, if there exists an  $(\mathcal{S})^*$ -process, denoted  $\Psi$ , which verifies (A.1), then  $\Psi_s = \Phi_s$  for a.e.  $s$  in  $I$ . Finally one has  $\lim_{n \rightarrow +\infty} \int_I \Phi_s^{(n)} ds = \int_I \Phi_s ds$ , where the equality and the limit both hold in  $(\mathcal{S})^*$ .

## A.2 Proof of Lemmas 6.4 and 6.5

### Proof of Lemma 6.4.

The existence of the left hand side of (6.6) results from (6.4) and Theorem 2.6. Using (ii) of Lemma 2.5 and again (6.4), we get, for every  $\eta$  in  $\mathcal{S}(\mathbf{R})$ ,

$$\begin{aligned} S\left(\int_{\mathbf{R}} j(y) \delta_y(\langle \cdot, g \rangle) dy\right)(\eta) &= \int_{\mathbf{R}} j(y) S(\delta_y(\langle \cdot, g \rangle))(\eta) dy = \frac{1}{\sqrt{2\pi}|g|_0} \int_{\mathbf{R}} j(y) \exp\left\{\frac{-(y - \langle g, \eta \rangle)^2}{2|g|_0^2}\right\} dy \\ &= \frac{1}{\sqrt{2\pi}|g|_0} \int_{\mathbf{R}} j(y) \exp\left\{\frac{-(y - \langle g, \eta \rangle)^2}{2|g|_0^2}\right\} dy = S(j(\langle \cdot, g \rangle))(\eta), \end{aligned}$$

where the last equality results from [Kuo96, Theorem 7.3]. The conclusion comes from the injectivity of  $S$ -transform.  $\square$

### Proof of Lemma 6.5.

Let  $(a, \eta)$  in  $\mathbf{R}^* \times \mathcal{S}(\mathbf{R})$  and  $\varepsilon$  be in  $(0, \frac{|a|}{2})$ . Among the assumptions that are required to apply the theorem of continuity under sign  $\int$ , only the domination of  $\rho_\eta$  is not obvious and will therefore be established here. Let  $\gamma$  be chosen in  $(0, \varepsilon)$  such that (6.3) holds. Denote  $\Gamma_\gamma := \{u \in [0, T]; d(u, \mathcal{Z}_R^T) \geq \gamma\}$ ,  $m_\gamma := \sup\{R_u^{-1/2}; u \in [0, T], \text{ s.t. } d(u, \mathcal{Z}_R^T) \geq \gamma\}$  and, for every  $r$  in  $(0, T]$ , denote  $M_r := \sup\{R_s^{-1/2} e^{-\frac{a^2}{32R_s}}; s \in [0, r]\}$ . Using Lemma 6.3, one gets, for every  $(s, y)$  in  $D(\mathcal{Z}_R^T, \gamma) \times D(a, \gamma)$ ,

$$|\rho_\eta(y, s)| = |\rho_\eta(y, s)| \mathbf{1}_{\Gamma_\gamma}(s) + |\rho_\eta(y, s)| \mathbf{1}_{D(\mathcal{Z}_R^T, \gamma)}(s) \leq m_\gamma \mathbf{1}_{\Gamma_\gamma}(s) + M_\gamma \mathbf{1}_{D(\mathcal{Z}_R^T, \gamma)}(s) =: f(s).$$

Since  $f$  belongs to  $L^1([0, T], ds)$ , the domination is complete. The case  $a = 0$  is simpler. Indeed, one has:

$$\forall(\varepsilon, \eta) \in \mathbf{R}^* \times \mathcal{S}(\mathbf{R}), \quad \forall(y, s) \in D(0, \varepsilon) \times [0, T], \quad |\rho_\eta(y, s)| \leq R_s^{-1/2}.$$

Since  $s \mapsto R_s^{-1/2}$  belongs to  $L^1([0, T] \setminus \mathcal{Z}_R^T, ds)$  by assumption, the domination is now complete and the theorem of continuity under sign  $\int$  applies here and ends the proof.  $\square$

## References

- [AMN01] E. Alòs, O. Mazet, and D. Nualart. Stochastic calculus with respect to Gaussian processes. *Ann. Probab.*, 29(2):766–801, 2001.
- [Bal03] V. Bally. An elementary introduction to Malliavin calculus. *rapport de recherche no 4718, INRIA*, 2003.
- [BDG06] B. Boufoussi, M. Dozzi, and R. Guerbaz. On the local time of multifractional Brownian motion. *Stochastics*, 78(1):33–49, 2006.
- [BDG07] B. Boufoussi, M. Dozzi, and R. Guerbaz. Sample path properties of the local time of multifractional Brownian motion. *Bernoulli*, 13(3):849–867, 2007.
- [BDM10] B. Boufoussi, M. Dozzi, and R. Marty. Local time and Tanaka formula for a Volterra-type multifractional Gaussian process. *Bernoulli*, 16(4):1294–1311, 2010.
- [Ben03a] C. Bender. An Itô formula for generalized functionals of a fractional Brownian motion with arbitrary Hurst parameter. *Stochastic Processes and their Applications*, 104:81–106, 2003.
- [Ben03b] C. Bender. An  $S$ -transform approach to integration with respect to a fractional Brownian motion. *Bernoulli*, 9(6):955–983, 2003.

- [BJR97] A. Benassi, S. Jaffard, and D. Roux. Elliptic Gaussian random processes. *Rev. Mat. Iberoamericana*, 13(1):19–90, 1997.
- [BSØW04] F. Biagini, A. Sulem, B. Øksendal, and N.N. Wallner. An introduction to white-noise theory and Malliavin calculus for fractional Brownian motion. *Proc. Royal Society, special issue on stochastic analysis and applications*, pages 347–372, 2004.
- [CG62] I.M. Chilov and G.E. Gelfand. *Les distributions*, volume 2. Dunod, 1962.
- [CLV14] S. Corlay, J. Lebovits, and J. Lévy Véhel. Multifractional stochastic volatility models. *Math. Finance*, 24(2):364–402, 2014.
- [CN05] P. Cheridito and D. Nualart. Stochastic integral of divergence type with respect to fractional Brownian motion with Hurst parameter  $H \in (0, \frac{1}{2})$ . *Ann. Inst. H. Poincaré Probab. Statist.*, 41(6):1049–1081, 2005.
- [CNT01] L. Coutin, D. Nualart, and C. A. Tudor. Tanaka formula for the fractional Brownian motion. *Stochastic Process. Appl.*, 94(2):301–315, 2001.
- [Cou07] L. Coutin. An Introduction to (Stochastic) Calculus with Respect to Fractional Brownian Motion. *Séminaire de Probabilités XL*, 1899:3–65, 2007.
- [EVdH03] R.J. Elliott and J. Van der Hoek. A general fractional white noise theory and applications to finance. *Mathematical Finance*, 13(2):301–330, 2003.
- [Föll81] H. Föllmer. Calcul d’Itô sans probabilités. In *Seminar on Probability, XV (Univ. Strasbourg, Strasbourg, 1979/1980) (French)*, volume 850 of *Lecture Notes in Math.*, pages 143–150. Springer, Berlin, 1981.
- [FV10] P.K. Friz and N.B. Victoir. *Multidimensional Stochastic Processes as Rough Paths: Theory and Applications*. Cambridge University Press, 2010.
- [GH80] D. Geman and J. Horowitz. Occupation densities. *Ann. Probab.*, 8(1):1–67, 1980.
- [GSV07] D. Gasbarra, T. Sottinen, and E. Valkeila. Gaussian bridges. In *Stochastic analysis and applications*, volume 2 of *Abel Symp.*, pages 361–382. Springer, Berlin, 2007.
- [GT82] B. Gaveau and P. Trauber. L’intégrale stochastique comme opérateur de divergence dans l’espace fonctionnel. *J. Funct. Anal.*, 46(2):230–238, 1982.
- [HH93] T. Hida and M. Hitsuda. *Gaussian processes*, volume 120 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1993. Translated from the 1976 Japanese original by the authors.
- [Hid75] T. Hida. *Analysis of Brownian functionals*. Carleton Univ., Ottawa, Ont., 1975. Carleton Mathematical Lecture Notes, No. 13.
- [Hit72] M. Hitsuda. Formula for Brownian partial derivatives, 1972. Second Japan-USSR Symposium on Probability Theory, Kyoto, 111–114.
- [Hit78] M. Hitsuda. Formula for Brownian partial derivatives, 1978. Proceedings of Faculty of Integrated Arts and Sciences. Hiroshima University III-4, 1–15.
- [HKPS93] T. Hida, H. Kuo, J. Potthoff, and L. Streit. *White Noise. An Infinite Dimensional Calculus*, volume 253. Kluwer academic publishers, 1993.
- [HØ02] Y. Hu and B. Øksendal. Chaos expansion of local time of fractional Brownian motions. *Stochastic Anal. Appl.*, 20(4):815–837, 2002.
- [HØS05] Y. Hu, B. Øksendal, and D.M Salopek. Weighted local time for fractional Brownian motion and applications to finance. *Stoch. Anal. Appl.*, 23(1):15–30, 2005.
- [HOUZ10] H. Holden, B. Oksendal, J. Ubøe, and T. Zhang. *Stochastic Partial Differential Equations, A Modeling, White Noise Functional Approach*. Springer, second edition, 2010.
- [HP57] E. Hille and R.S. Phillips. *Functional Analysis and Semi-Groups*, volume 31. American Mathematical Society, 1957.
- [HS04] T. Hida and Si Si. *An innovation approach to random fields*. World Scientific Publishing Co. Inc., River Edge, NJ, 2004. Application of white noise theory.
- [HV03] C. Houdré and J. Villa. An example of infinite dimensional quasi-helix. In *Stochastic models (Mexico City, 2002)*, volume 336 of *Contemp. Math.*, pages 195–201. Amer. Math. Soc., Providence, RI, 2003.
- [Jan97] S. Janson. *Gaussian Hilbert Spaces*, volume 129 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1997.

- [Kol40] A. Kolmogorov. Wienerische Spiralen und einige andere interessante Kurven in Hilbertschem Raum. *C. R. (Dokl.) Acad. Sci. URSS*, 26:115–118, 1940.
- [KR10] I. Kruk and F. Russo. Malliavin Skorohod calculus and Paley-Wiener integral for covariance singular processes. *Preprint*, 2010.
- [KRT07] I. Kruk, F. Russo, and C. A. Tudor. Wiener integrals, Malliavin calculus and covariance measure structure. *J. Funct. Anal.*, 249(1):92–142, 2007.
- [Kuo96] H.H. Kuo. *White Noise Distribution Theory*. CRC-Press, 1996.
- [Leb13] J. Lebovits. From stochastic integral *w.r.t.* fractional Brownian motion to stochastic integral *w.r.t.* multifractional Brownian motion. *Ann. Univ. Buchar. Math. Ser.*, 4(LXII)(1):397–413, 2013.
- [LLV14] J. Lebovits and J. Lévy Véhel. White noise-based stochastic calculus with respect to multifractional Brownian motion. *Stochastics An International Journal of Probability and Stochastic Processes*, 86(1):87–124, 2014.
- [LLVH14] J. Lebovits, J. Lévy Véhel, and E. Herbin. Stochastic integration with respect to multifractional Brownian motion *via* tangent fractional Brownian motions. *Stochastic Process. Appl.*, 124(1):678–708, 2014.
- [LN05] J. A. León and D. Nualart. An extension of the divergence operator for Gaussian processes. *Stochastic Process. Appl.*, 115(3):481–492, 2005.
- [LN12] P. Lei and D. Nualart. Stochastic calculus for Gaussian processes and application to hitting times. *Commun. Stoch. Anal.*, 6(3):379–402, 2012.
- [Mis08] Y.S. Mishura. *Stochastic calculus for fractional Brownian motion and related processes*, volume 1929 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2008.
- [MR06] M.B. Marcus and J. Rosen. *Markov processes, Gaussian processes, and local times*, volume 100 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2006.
- [MV05] O. Mocioalca and F. Viens. Skorohod integration and stochastic calculus beyond the fractional Brownian scale. *J. Funct. Anal.*, 222(2):385–434, 2005.
- [MV07] O. Mocioalca and F. Viens. Space regularity of stochastic heat equations driven by irregular Gaussian processes. *Commun. Stoch. Anal.*, 1(2):209–229, 2007.
- [MVN68] B. Mandelbrot and J.W. Van Ness. Fractional Brownian motions, fractional noises and applications. *SIAM Rev.*, 10:422–437, 1968.
- [MWX08] M. Meerschaert, D. Wu, and Y. Xiao. Local times of multifractional Brownian sheets. *Bernoulli*, 14(3):865–898, 2008.
- [MY06] R. Mansuy and M. Yor. *Random times and enlargements of filtrations in a Brownian setting*. Springer-Verlag New York, Inc., 2006.
- [NT06] D. Nualart and M.S. Taqqu. Wick-Itô formula for Gaussian processes. *Stoch. Anal. Appl.*, 24(3):599–614, 2006.
- [Nua05] D. Nualart. A white noise approach to fractional Brownian motion. In *Stochastic analysis: classical and quantum*, pages 112–126. World Sci. Publ., Hackensack, NJ, 2005.
- [Nua06] D. Nualart. *The Malliavin Calculus and Related Topics*. Springer, 2006.
- [PLV95] R. Peltier and J. Lévy Véhel. Multifractional Brownian motion: definition and preliminary results, 1995. rapport de recherche de l’INRIA, n<sup>o</sup> 2645.
- [RV07] F. Russo and P. Vallois. Elements of stochastic calculus via regularization. In *Séminaire de Probabilités XL*, volume 1899 of *Lecture Notes in Math.*, pages 147–185. Springer, Berlin, 2007.
- [RY99] D. Revuz and M. Yor. *Continuous martingales and Brownian motion*, volume 293 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, third edition, 1999.
- [Si12] Si Si. *Introduction to Hida distributions*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2012.
- [Sko75] A. V. Skorohod. On a generalization of the stochastic integral. *Teor. Veroyatnost. i Primenen.*, 20(2):223–238, 1975.
- [ST94] G. Samorodnitsky and M.S. Taqqu. *Stable Non-Gaussian Random Processes, Stochastic Models with Infinite Variance*. Chapman and Hall/C.R.C., 1994.
- [ST06] S. Stoev and M. Taqqu. How rich is the class of multifractional Brownian motions? *Stochastic Processes and their Applications*, 116:200–221, 2006.

- [Str83] C. Stricker. Semimartingales gaussiennes—application au problème de l’innovation. *Z. Wahrsch. Verw. Gebiete*, 64(3):303–312, 1983.
- [SV13] T. Sottinen and L. Viitasaari. Pathwise integrals and Itô-Tanaka formula for Gaussian processes. *Preprint*, 2013. Preprint, arXiv:1307.3578.
- [SV14] T. Sottinen and L. Viitasaari. Stochastic Analysis of Gaussian Processes via Fredholm Representation. 2014. Preprint, arXiv:1410.2230.
- [Tha93] S. Thangavelu. *Lectures of Hermite and Laguerre expansions*. Princeton University Press, 1993.
- [Wid44] D.V. Widder. Positive temperatures on an infinite rod. *Trans. Amer. Math. Soc.*, 55:85–95, 1944.
- [You36] L. C. Young. An inequality of the Hölder type, connected with Stieltjes integration. *Acta Math.*, 67(1):251–282, 1936.
- [Zä98] M. Zähle. Integration with respect to fractal functions and stochastic calculus. I. *Probab. Theory Related Fields*, 111(3):333–374, 1998.