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ESTIMATION FOR STOCHASTIC DAMPING HAMILTONIAN SYSTEMS UNDER PARTIAL OBSERVATION. III. DIFFUSION TERM.

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Abstract. This paper is the third part of our study started with Cattiaux, León and Prieur (2014 2013). For some ergodic hamiltonian systems we obtained a central limit theorem for a non-parametric estimator of the invariant density (Cattiaux et al. 2014) and of the drift term (Cattiaux et al. 2013), under partial observation (only the positions are observed). Here we obtain similarly a central limit theorem for a non-parametric estimator of the diffusion term.

1. Introduction

In this article we consider the estimation, using data sampled at high frequency, of the local variance or diffusion term $\sigma(\cdot,\cdot)$ in the system $\left(Z_t := (X_t, Y_t) \in \mathbb{R}^{2d}\,,\ t \geq 0\right)$ governed by the following Itô stochastic differential equation:

$$\begin{cases} dX_t &= Y_t dt \\ dY_t &= \sigma(X_t, Y_t)\, dW_t - (c(X_t, Y_t)Y_t + \nabla V(X_t))dt. \end{cases}$$

The function c is called the damping force and V the potential, σ is the diffusion term and W a standard brownian motion.

The problem of estimating the diffusion term, sometimes called volatility, in a model of diffusion has a somewhat long history and has a lot of motivations, in particular in the analysis of financial or neuronal data.

The beginning of the story takes place at the end of the eighties of the last century. The first and seminal articles were written by Dacunha-Castelle and Florens (1986), Florens (1989), Dohnal (1987) and Genon-Catalot and Jacod (1993). The method generally used is the central limit theorem for martingales. Recently an excellent survey introducing the subject and giving some important recent references was written by Podolskij and Vetter (2010). In that work the authors give some insights about the methods of proof of the limit theorems and recall also the existence of some goodness of fit tests useful in financial studies. This article also mentioned the names of those linked to the development in this

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area. They are, among other, Bibby and Sorensen (1995), Jacod and Protter (1998) and Barndorff-Nielsen et al. (2006). The second of the last cited works contains a deep study for the asymptotic behavior of discrete approximations of stochastic integrals, it is thus in tight relationship with the estimation of the diffusion term.

The present article is the continuation of two previous works by the authors: Cattiaux et al. (2014) and Cattiaux et al. (2013). In the first one we tackled the problem of estimating the invariant density of the system (1) and in the second one the estimation of the drift term $(x, y) \mapsto b(x, y) = -c(x, y)y + \nabla V(x)$ was studied. In both papers we assumed that the diffusion coefficient σ is constant, in order to control the mixing rate of the process (see the remarks at the end of the present paper for extensions to the non constant diffusion case).

Here we consider the estimation of the function σ, in particular we do no more assume necessarily that it is a constant. We observe the process in a high resolution grid *i.e.* Z_{ph_n}, $p = 1\dots, n$ with $h_n \xrightarrow[n\to+\infty]{} 0$. As for our previous works, we consider the case where only the position coordinates X_{ph_n} are observed (partial observations). This is of course the main technical difficulty. This situation leads us to define the estimator using the second order increments of process $\Delta_2 X(p, n) = X_{(p+1)h_n} - 2X_{ph_n} + X_{(p-1)h_n}$. This fact introduces some technicalities in the proof of each result.

In the first part of the article we consider the case of infill statistics $t = nh_n$ is fixed. Two situations are in view: firstly σ is a constant and we estimate σ^2 by using a normalization of $\Delta_2 X(p, n)$, secondly σ is no more constant and we estimate $\int_0^t \sigma^2(X_s, Y_s)ds$. In both cases we obtain a stable limit theorem with rate $\sqrt{n}$ for the estimators (for the definition of stable convergence in law see the next section).

This asymptotic convergence can be applied, for instance, for testing the null hypothesis $\mathcal{H}_0$: the matrix σ contains only non vanishing diagonal terms i.e. $\sigma_{ij} = 0$ for $i \neq j$.

In the second part we study the infinite horizon estimation $nh_n = t \xrightarrow[n\to+\infty]{} +\infty$. We assume that the rate of mixing of the process $(Z_t\,,\ t \geq 0)$ is sufficiently high. Whenever σ is a constant we obtain a central limit theorem (CLT) for the estimator of σ^2 with rate $\sqrt{n}$. However, in the case where σ is not a constant we get a new CLT but the rate now is $\sqrt{nh_n}$ and the asymptotic variance is the same as the one obtained for occupation time functionals.

The result in the infinite horizon can serve to test $\mathcal{H}_0 : \sigma(x, y) = \sigma$ against the sequence of alternatives $\mathcal{H}_1^n : \sigma_n = \sigma + c_n d(x, y)$, for some sequence c_n tending to zero as n tends to infinity, because of the difference in the convergence rate under the null and under the sequence of alternatives.

Estimation with partial observations has been considered previously in the literature. In Gloter (2006), the case of one dimensional diffusion V_t is studied. One observes only $S_t = \int_0^t V_s ds$, in a discrete uniform grid. The estimation is made for the parameters defining the variance and the drift. More recently for the same type of models, the problem of estimation was considered in Comte et al. (2009). In this last work, the study is non-parametric in nature, it deals with adaptive estimation, evaluating the

quadratic risk. The models in both these articles, contrary to models of type (1), do not allow the second equation to depend on the first coordinate. It can be written as
$$\begin{cases} dS_t &= V_t dt \\ dV_t &= \sigma(V_t)\, dW_t + b(V_t)dt \end{cases}.$$
The literature concerning the estimation for models of type (1) is rather scarce. However, two papers must be cited. Firstly Pokern et al. (2009) consider parameter estimation by using approximate likelihoods. The horizon of estimation is infinite and they assume $h_n \xrightarrow[n\to+\infty]{} 0$ and $nh_n \xrightarrow[n\to+\infty]{} +\infty$. Secondly Samson and Thieullen (2012) introduce, in the case of partial observations, an Euler contrast defined using the second coordinate only. However, we should point out that the present work, while dealing with non-parametric estimation, has a non-empty intersection with the one of Samson et al. (2012) when the diffusion term is constant.

Let us end this introduction with some comments about some possible generalizations. In the first place the methods that we use in this work can be adapted for considering the power variation type estimators defined as
$$V_F(n) = \sum_{p=0}^{(([\frac{t}{2h_n}]-1)h_n} F(\Delta_2 X(p,n)),$$
for F a smooth function, usually $F(x) = |x|^r$ the r^{th} power variation (see e.g., Jacod 2008). Secondly, it would be possible to study an estimator constructed through a Fourier transform method as the one defined in Malliavin and Mancino (2009).

2. Tools

2.1. **Stable convergence.**

In this article, the type of convergence we consider is the stable convergence, introduced by Renyi, whose definition is recalled below (see Definition 2.1). In this subsection all random variables or processes are defined on some probability space $(\Omega, \mathbb{F}, \mathbb{P})$.

Definition 2.1 (Definition 2.1 in Podolskij and Vetter (2010))**.** *Let Y_n be a sequence of random variables with values in a Polish space $(E, \mathcal{E})$. We say that Y_n converges stably with limit Y, written $Y_n \xrightarrow[n\to+\infty]{\mathcal{S}} Y$, where Y is defined on an extension $(\Omega', \mathbb{F}', \mathbb{P}')$ iff for any bounded, continuous function g and any bounded $\mathbb{F}$-measurable random variable Z it holds that*
$$\mathbb{E}(g(Y_n)Z) \to \mathbb{E}'(g(Y)Z)$$
as $n \to +\infty$.

If $\mathbb{F}$ is the σ-algebra generated by some random variable (or process) X, then it is enough to consider $Z = h(X)$ for some continuous and bounded h. It is thus clear that the stable convergence in this situation, is equivalent to the convergence in distribution of the sequence (Y_n, X) to (Y, X). It is also clear that convergence in Probability implies stable convergence. As shown in Podolskij and Vetter (2010), the converse holds true if Y is $\mathbb{F}$ measurable.

Notice that we may replace the assumption Z is bounded by $Z \in \mathbb{L}^1(\mathbb{P})$. This remark allows us to replace $\mathbb{P}$ by any $\mathbb{Q}$ which is absolutely continuous with respect to $\mathbb{P}$, i.e.

Proposition 2.2. *Assume that Y_n (defined on $(\Omega, \mathbb{F}, \mathbb{P})$) converges stably to Y. Let $\mathbb{Q}$ be a probability measure on Ω such that $\frac{d\mathbb{Q}}{d\mathbb{P}} = H$. Then Y_n (defined on $(\Omega, \mathbb{F}, \mathbb{Q})$) converges stably to the same Y (defined on $(\Omega', \mathbb{F}', \mathbb{Q}' = H\mathbb{P}')$).*

In particular, in the framework of our diffusion processes, this proposition combined with Girsanov transform theory will allow us to "kill" the drift.

2.2. **About the S.D.E.** (1). In all the paper we will assume (at least) that the coefficients in (1) satisfy:

- $\mathcal{H}_0$ the diffusion matrix σ is symmetric, smooth, bounded as well as its first and second partial derivatives and uniformly elliptic, i.e. $\forall x, y,\ \sigma(x, y) \geq \sigma_0\, Id$ (in the sense of quadratic forms) for a positive constant $\sigma_0 > 0$;
- $\mathcal{H}_1$ the potential V is lower bounded and continuously differentiable on $\mathbb{R}^d$;
- $\mathcal{H}_2$ the damping matrix c is continuously differentiable and for all $N > 0$: $\sup_{|x|\leq N,\, y\in\mathbb{R}^d} |c(x, y)| < +\infty$ and $\exists\, c_0,\ L > 0\ \ c^s(x, y) \geq c_0 Id$ for all $|x| > L$, $y \in \mathbb{R}^d$, c^s being the symmetrization of the matrix c.

Under these assumptions equation (1) admits an unique strong solution which is non explosive. In addition

Lemma 2.3 (Lemma 1.1 in Wu (2001)). *Assume $\mathcal{H}_0$, $\mathcal{H}_1$ and $\mathcal{H}_2$. Then, for every initial state $z = (x, y) \in \mathbb{R}^{2d}$, the s.d.e.* (1) *admits a unique strong solution $\mathbb{P}_z$ (a probability measure on Ω), which is non explosive. Moreover $\mathbb{P}_z << \mathbb{P}^0_z$ on $(\Omega, \mathcal{F}_t)$ for each $t > 0$, where $\mathbb{P}^0_z$ is the law of the solution of* (1) *associated to $c(x, y) = 0$ and $V = 0$, and with $(\mathcal{F}_t := \sigma(Z_s,\, 0 \leq s \leq t))_{t\geq 0}$.*

Remark 2.4. The formulation of $\mathcal{H}_0$ can be surprising. Let σ^* denote the transposed matrix of σ. Actually the law of the process depends on $\sigma\sigma^*$ (which is the second order term of the infinitesimal generator). If this symmetric matrix is smooth, it is well known that one can find a smooth symmetric square root of it, which is the choice we make for σ. As it will be clear in the sequel, our estimators are related to $\sigma\sigma^*$ (hence here σ^2). $\diamondsuit$

3. Finite horizon (infill) estimation.

We consider infill estimation, that is we observe the process on a finite time interval $[0, T]$, with a discretization step equal to h_n with $h_n \xrightarrow[n\to+\infty]{} 0$.

According to Lemma 2.3 and Proposition 2.2, *any $\mathbb{P}^0_z$ stably converging sequence Y_n is also $\mathbb{P}_z$ stably converging to the same limit.* Hence **in all this section we will assume that $\mathcal{H}_0$ is satisfied and that c and V are identically** 0**.** Any result obtained in this situation is thus true as soon as $\mathcal{H}_0$, $\mathcal{H}_1$ and $\mathcal{H}_2$ are satisfied.

3.1. **The case of a constant diffusion matrix.** We start with the definition of the "double" increment of the process.

Define for $1 \leq p \leq [\frac{T}{2h_n}] - 1 := p_n$ (here [.] denotes the integer part)

$$\Delta_2 X(p,n) = X_{(2p+1)h_n} - 2X_{2ph_n} + X_{(2p-1)h_n}. \tag{3.1}$$

Then

$$\begin{aligned} \sigma^{-1}\, \Delta_2 X(p,n) &= \int_{2ph_n}^{(2p+1)h_n} W_s ds - \int_{(2p-1)h_n}^{2ph_n} W_u du \\ &= \int_{2ph_n}^{(2p+1)h_n} (W_s - W_{2ph_n}) ds + \int_{(2p-1)h_n}^{2ph_n} (W_{2ph_n} - W_u) du. \end{aligned}$$

The right hand side is the sum of two independent centered normal random vectors, whose coordinates are independent, so that $\sqrt{\frac{3}{2h_n^3}}\, \sigma^{-1}\, \Delta_2 X(p,n)$ is a centered gaussian random vector with covariance matrix equal to Identity (recall that we assume that $\sigma = \sigma^*$).
Furthermore, all the $(\Delta_2 X(p,n))_{1 \leq p \leq p_n}$ are independent (thanks to our choice of the increments).

So we define our estimator $\hat{\sigma}_n^2$ of the matrix σ^2 as

$$\hat{\sigma}_n^2 = \frac{1}{[\frac{T}{2h_n}] - 1}\, \frac{3}{2h_n^3} \sum_{p=1}^{[\frac{T}{2h_n}]-1} \Delta_2 X(p,n) >< \Delta_2 X(p,n)\,, \tag{3.2}$$

where $A >< B$ denotes the (d,d) matrix obtained by taking the matrix product of the $(d,1)$ vector A with the transposed of the $(d,1)$ vector B, denoted by B^*.
Using what precedes we see that

$$\sigma^{-1}\, \hat{\sigma}_n^2\, \sigma^{-1} = \frac{1}{[\frac{T}{2h_n}] - 1} \sum_{p=1}^{[\frac{T}{2h_n}]-1} M(p,n)$$

where for each n the $M(p,n)$ are i.i.d. symmetric random matrices whose entries $M_{i,j}$ are all independent for $i \geq j$, satisfying $\mathbb{E}_z^0(M_{i,j}) = \delta_{i,j}$ and $\mathrm{Var}_z^0(M_{i,j}) = 1 + \delta_{i,j}$.

According to the law of large numbers and the Central Limit Theorem for triangular arrays of independent variables we have

Lemma 3.3 (convergence)**.** *Assume $c = 0$, $V = 0$ and $\mathcal{H}_\mathbf{0}$.*
Then if $h_n \xrightarrow[n \to +\infty]{} 0$, starting from any initial point $z = (x,y) \in \mathbb{R}^{2d}$, we have

$$\hat{\sigma}_n^2 \xrightarrow[n \to +\infty]{\mathbb{P}_z^0} \sigma^2\,,$$

and

$$\left(\left[\frac{T}{2h_n}\right] - 1\right)^{1/2} (\sigma^{-1}\, \hat{\sigma}_n^2\, \sigma^{-1} - Id) \xrightarrow[n \to +\infty]{\mathcal{D}} \mathcal{N}_{(d,d)},$$

where $\mathcal{N}_{(d,d)}$ is a (d,d) symmetric random matrix whose entries are centered gaussian random variables with $\mathrm{Var}(\mathcal{N}_{i,j}) = 1 + \delta_{i,j}$, *all the entries (i,j) for $i \geq j$ being independent.*

The consistence result is interesting since convergence in $\mathbb{P}_z^0$ probability implies convergence in $\mathbb{P}_z$ probability (i.e. for general c and V). The convergence in distribution however is not sufficient and has to be reinforced into a stable convergence.

This is the aim of what follows.

To this end we define the sequence of processes defined for $0 \leq t \leq T$,

$$\hat{\sigma}_n^2(t) = \frac{1}{[\frac{T}{2h_n}] - 1} \, \frac{3}{2h_n^3} \sum_{p=1}^{[\frac{t}{2h_n}]-1} \Delta_2 X(p,n) >< \Delta_2 X(p,n)\,, \tag{3.4}$$

where the empty sums are set equal to zero. We will prove

Theorem 3.5 (convergence in Skorohod's metric). *Assume $c = 0$, $V = 0$ and $\mathcal{H}_\mathbf{0}$.*
Then if $h_n \xrightarrow[n\to+\infty]{} 0$, starting from any initial point $z = (x,y) \in \mathbb{R}^{2d}$, we have

$$\left(W_{\cdot}, \left(\sqrt{\frac{T}{2h_n}}\, \sigma^{-1}\, \hat{\sigma}_n^2(\cdot)\, \sigma^{-1} \,-\, Id\right)\right) \xrightarrow[n\to+\infty]{\mathcal{D}([0,T])\times\mathcal{D}([0,T])} \left(W_{\cdot}, \widetilde{W_{\cdot}}\right)\,,$$

where $\left(\widetilde{W_t}\,,\ t\in[0,T]\right)$ is a (d,d) symmetric matrix valued random process whose entries are Wiener processes with variance $\mathrm{Var}_{i,j}(t) = (1+\delta_{i,j})\,(t/T)$, all the entries (i,j) for $i \geq j$ being independent. In addition $\widetilde{W_{\cdot}}$ is independent of $W_{\cdot}$.

According to the discussion on stable convergence we immediately deduce

Corollary 3.6 (stable convergence). *Under assumptions $\mathcal{H}_\mathbf{0}$, $\mathcal{H}_\mathbf{1}$ and $\mathcal{H}_\mathbf{2}$, if $h_n \xrightarrow[n\to+\infty]{} 0$, starting from any initial point $z = (x,y) \in \mathbb{R}^{2d}$, we have*

$$\sqrt{\frac{T}{2h_n}}\left(\sigma^{-1}\, \hat{\sigma}_n^2\, \sigma^{-1} - Id\right) \xrightarrow[n\to+\infty]{\mathcal{S}} \mathcal{N}_{(d,d)}\,,$$

where $\mathcal{N}_{(d,d)}$ is as in Lemma 3.3.

Proof of Theorem 3.5 : In the following, we fix $T = 1$ without loss of generality. Notice that we may also replace $\frac{T}{2h_n}$ by $[\frac{T}{2h_n}] - 1$ (using Slutsky's theorem if one wants).

The convergence of

$$t \mapsto Z_n(t) = \sqrt{\frac{1}{2h_n}}\, \sigma^{-1}\, \hat{\sigma}_n^2(t)\, \sigma^{-1}$$

to a matrix of Wiener processes is proved as for Donsker invariance principle. The only difference here is that instead of an i.i.d. sample we look at a triangular array of i.i.d. random vectors (on each row), but the proof in Billingsley (1999) applies in this situation. This result is sometimes called Donsker-Prohorov invariance principle. Writing W_t as the sum of its increments on the grid given by the intervals $[(2p-1)h_n, (2p+1)h_n]$ the convergence of the joint law of $(W_{\cdot}, Z_n(.))$ in $\mathcal{D}([0,1])$ is proved in exactly the same way. The final independence assumption is a simple covariance calculation. $\square$

3.2. **Estimation of the noise, general case.**

In this section, we do not assume anymore that the diffusion term σ is constant.
In the following, we want to estimate $\int_0^t \sigma^2(X_s, Y_s)ds$, for any $0 \le t \le T$.
To this end, we introduce the quadratic variation process defined for $n \in \mathbb{N}^*$ and $0 \le t \le T$ as

$$\mathcal{QV}_{h_n}(t) = \frac{1}{h_n^2} \sum_{p=1}^{\left[\frac{t}{2h_n}\right]-1} \Delta_2 X(p,n) >< \Delta_2 X(p,n)\,, \tag{3.7}$$

with $\Delta_2 X(p,n)$ defined in (3.1). The main result of this section is

Theorem 3.8. *Under assumptions $\mathcal{H}_\mathbf{0}$, $\mathcal{H}_\mathbf{1}$ and $\mathcal{H}_\mathbf{2}$, if $h_n \xrightarrow[n\to+\infty]{} 0$, starting from any initial point $z = (x,y) \in \mathbb{R}^{2d}$, we have for any $0 \le t \le T$*

$$\mathcal{QV}_{h_n}(t) \xrightarrow[n\to+\infty]{\mathbb{P}_z} \frac{1}{3} \int_0^t \sigma^2(X_s, Y_s)ds\,,$$

and

$$\sqrt{\frac{1}{h_n}} \left(\mathcal{QV}_{h_n}(t) - \frac{1}{3} \int_0^t \sigma^2(X_s, Y_s)ds \right) \xrightarrow[n\to+\infty]{\mathcal{S}} \frac{2}{3} \int_0^t \sigma(X_s, Y_s)\, d\widetilde{W}_s\, \sigma(X_s, Y_s)\,,$$

where $\left(\widetilde{W}_t\,,\ t \in [0,T]\right)$ is a symmetric matrix valued random process independent of the initial Wiener process $W_.$, whose entries $\widetilde{W}_.(i,j)$ are Wiener processes with variance $V_{i,j}(t) = (1 + \delta_{i,j})t$, these entries being all independent for $i \ge j$.

Recall that for the proof of this theorem, we only need to consider the case where $c = 0$ and $V = 0$.
In this case, the strong solution, with initial conditions $(X_0, Y_0) = (x,y) = z$, can be written as

$$Z_t = (X_t, Y_t) = \left(x + yt + \int_0^t Y_s ds, y + \int_0^t \sigma(X_s, Y_s) dW_s \right).$$

We thus have:

$$\Delta_2 X(p,n) = \int_{2ph_n}^{(2p+1)h_n} \big[\int_0^s \sigma(X_u, Y_u) dW_u \big] ds - \int_{(2p-1)h_n}^{2ph_n} \big[\int_0^s \sigma(X_u, Y_u) dW_u \big] ds\,.$$

Using Fubini's theorem for stochastic integrals, one gets:

$$\begin{aligned} \Delta_2 X(p,n) &= h_n \textstyle\int_0^{2ph_n} \sigma(X_u, Y_u) dW_u + \int_{2ph_n}^{(2p+1)h_n} ((2p+1)h_n - u) \sigma(X_u, Y_u) dW_u \\ &- h_n \textstyle\int_0^{(2p-1)h_n} \sigma(X_u, Y_u) dW_u - \int_{(2p-1)h_n}^{2ph_n} (2ph_n - u) \sigma(X_u, Y_u) dW_u\,, \end{aligned}$$

thus

$$\Delta_2 X(p,n) = \int_{(2p-1)h_n}^{(2p+1)h_n} (h_n - |u - 2ph_n|)\sigma(X_u, Y_u) dW_u \,. \tag{3.9}$$

If $p \neq q$ are two integers, denoting by $\Delta_2 X(p,n,i)$ the i^{th} coordinate of $\Delta_2 X(p,n)$, we immediately have, for all $i,j = 1, ..., d$,

$$\mathbb{E}_z^0 \left(\Delta_2 X(p,n,i)\, \Delta_2 X(q,n,j)\right) \,=\, 0\,. \tag{3.10}$$

As a warm up lap, we look at the convergence of the first moment of $\mathcal{QV}_{h_n}$.

Lemma 3.11 (preliminary result)**.** *Assume $c = 0$, $V = 0$ and $\mathcal{H}_0$. Then, if $h_n \xrightarrow[n\to+\infty]{} 0$, starting from any initial point $z = (x,y) \in \mathbb{R}^{2d}$, we have for any $0 \leq t \leq T$,*

$$\mathbb{E}_z^0 \mathcal{QV}_{h_n}(t) \xrightarrow[n\to+\infty]{} \frac{1}{3} \int_0^t \mathbb{E}_z^0\, \sigma^2(X_u, Y_u)\, du.$$

Recall that we assumed $\sigma = \sigma^*$, and of course look at the previous equality as an equality between real matrices.

Proof of Lemma 3.11: First, using Ito's isometry and Equality (3.9), one gets

$$\mathbb{E}_z^0 \left(\Delta_2 X(p,h_n) >< \Delta_2 X(p,h_n)\right) = \int_{(2p-1)h_n}^{(2p+1)h_n} (h_n - |u - 2ph_n|)^2\, \mathbb{E}_z^0\, \sigma^2(X_u, Y_u) du.$$

Since

$$\int_{(2p-1)h_n}^{(2p+1)h_n} (h_n - |u - 2ph_n|)^2\, du = \frac{2}{3}\, h_n^3\,,$$

we thus have

$$\frac{1}{h_n^2}\, \mathbb{E}_z^0 \left(\Delta_2 X(p,h_n) >< \Delta_2 X(p,h_n)\right) - \frac{1}{3} \int_{(2p-1)h_n}^{(2p+1)h_n} \mathbb{E}_z^0\, \sigma^2(X_u, Y_u)\, du =$$

$$= \frac{1}{h_n^2} \int_{(2p-1)h_n}^{(2p+1)h_n} (h_n - |u - 2ph_n|)^2\, \mathbb{E}_z^0 \left(\sigma^2(X_u, Y_u) - \sigma^2(X_{(2p-1)h_n}, Y_{(2p-1)h_n})\right)\, du \,+$$

$$+ \frac{1}{3} \int_{(2p-1)h_n}^{(2p+1)h_n} \mathbb{E}_z^0 \left(\sigma^2(X_{(2p-1)h_n}, Y_{(2p-1)h_n}) - \sigma^2(X_u, Y_u)\right)\, du\,.$$

Define on $\Omega \times [0,t]$, the sequence of random (matrices)

$$G_n(u) = \sum_{p=1}^{[\frac{t}{2h_n}]-1} \sigma^2(X_{(2p-1)h_n}, Y_{(2p-1)h_n})\, \mathbb{1}_{(2p-1)h_n \leq u < (2p+1)h_n}\,.$$

Since σ is continuous, G_n converges $\mathbb{P}_z^0 \otimes du$ almost everywhere to $\sigma^2(X_u, Y_u)$. In addition since σ is bounded, G_n is dominated by a constant which is $\mathbb{P}_z^0 \otimes du$ integrable

on $\Omega \times [0,t]$.
Hence using Lebesgue bounded convergence theorem, we get that

$$\int_0^t \mathbb{E}_z^0 \left(G_n(u) - \sigma^2(X_u, Y_u)\right)\, du \;\to\; 0\,.$$

Similarly, the variables

$$\sum_{p=1}^{[\frac{t}{2h_n}]-1} \frac{(h_n - |u-2ph_n|)^2}{h_n^2}\, \mathbb{1}_{(2p-1)h_n \le u < (2p+1)h_n} \left(\sigma^2(X_{(2p-1)h_n}, Y_{(2p-1)h_n}) - \sigma^2(X_u, Y_u)\right)$$

are bounded and converge almost everywhere to 0, so that their expectation also goes to 0. This completes the proof. □

Of course, a careful look at this proof shows that we did not use all the strength of $\mathcal{H}_0$, only the fact that σ is continuous and bounded. It is thus clearly possible to improve upon this result, using the same idea of introducing the skeleton Markov chain and controlling the errors.

Hence introduce

$$\Delta_2 H(p,h_n) = \int_{(2p-1)h_n}^{(2p+1)h_n} (h_n - |u-2ph_n|)\, \sigma(X_{(2p-1)h_n}, Y_{(2p-1)h_n})\, dW_u\,. \tag{3.12}$$

We may decompose

$$\begin{aligned} J_n^1 + J_n^2 + J_n^3 &= \mathcal{QV}_{h_n}(t) - \frac{1}{3}\int_0^t \sigma^2(X_u,Y_u)du \qquad \text{with} \qquad (3.13)\\ J_n^1 &= \mathcal{QV}_{h_n}(t) - \frac{1}{h_n^2}\sum_{p=1}^{\left[\frac{t}{2h_n}\right]-1} \Delta_2 H(p,h_n) >< \Delta_2 H(p,h_n)\,,\\ J_n^2 &= \left(\frac{1}{h_n^2}\sum_{p=1}^{\left[\frac{t}{2h_n}\right]-1} \Delta_2 H(p,h_n) >< \Delta_2 H(p,h_n)\right) - \frac{1}{3}\left(\int_0^t G_n(u)du\right)\\ J_n^3 &= \frac{1}{3}\left(\int_0^t G_n(u)du - \int_0^t \sigma^2(X_u,Y_u)du\right)\,. \end{aligned}$$

For $A = (A_{i,j})_{1\le i\le q, 1\le j\le r}$ a $q\times r$ real matrix, we define $|A|$ as $|A| = \max_{1\le i\le q, 1\le j\le r} |A_{i,j}|$.

We then have

Lemma 3.14. *Assume $c=0$, $V=0$, $(X_0,Y_0)=(x,y)\in\mathbb{R}^{2d}$ and $\mathcal{H}_0$. Then, there exist constants C depending on σ, its derivatives and the dimension only, such that for any $0\le t\le T$,*

$$\mathbb{E}_z^0\left(\left|\int_0^t G_n(u)du - \int_0^t \sigma^2(X_u,Y_u)du\right|\right) \le Ct\sqrt{h_n}\,, \tag{3.15}$$

and

$$\mathbb{E}_z^0\left(|\Delta_2 X(p,h_n) - \Delta_2 H(p,h_n)|^2\right) \leq C\, h_n^4 \,. \tag{3.16}$$

Proof. For the first part it is enough to show that

$$\int_0^t \mathbb{E}_z^0 \left|G_n(u) - \sigma^2(X_u,Y_u)\right| \, du \leq Ct\sqrt{h_n}\,.$$

But using the fact that σ and its first derivatives are continuous and bounded, there exists a constant C only depending on these quantities (but which may change from line to line), such that

$$\int_0^t \mathbb{E}_z^0 \left|G_n(u) - \sigma^2(X_u,Y_u)\right| \, du \,\leq\, Ct \sup_{|a-b|\leq 2h_n} \mathbb{E}_z^0(|Z_a - Z_b|) \,\leq\, Ct\sqrt{h_n}\,. \tag{3.17}$$

For the second part, we have

$$\mathbb{E}_z^0\left(|\Delta_2 X(p,h_n) - \Delta_2 H(p,h_n)|^2\right) =$$

$$= \mathbb{E}_z^0\left(\int_{(2p-1)h_n}^{(2p+1)h_n} (h_n - |u-2ph_n|)^2\, Trace((\sigma(X_{(2p-1)h_n},Y_{(2p-1)h_n}) - \sigma(X_u,Y_u))^2)\, du\right),$$

from which the result easily follows as before. □

We deduce immediately

Proposition 3.18. *Assume $c=0$, $V=0$ and $\mathcal{H}_0$. Then, if $h_n \xrightarrow[n\to+\infty]{} 0$, starting from any initial point $z=(x,y)\in\mathbb{R}^{2d}$, we have for any $0\leq t\leq T$, that J_n^1 and J_n^3 are converging to 0 in $\mathbb{L}^1(\mathbb{P}_z^0)$ (with rates h_n and $\sqrt{h_n}$), hence in $\mathbb{P}_z^0$ probability.*

Proof. The result for J_n^3 is contained in the previous Lemma. For J_n^1 we calculate $\mathbb{E}_z^0[|J_n^1|]$. The $(i,j)^{th}$ term of J_n^1 is given by

$$\frac{1}{h_n^2}\sum_{p=1}^{\left[\frac{t}{2h_n}\right]-1} (\Delta_2 X(p,h_n,i) - \Delta_2 H(p,h_n,i))(\Delta_2 X(p,h_n,j) - \Delta_2 H(p,h_n,j))\,,$$

so that, according to the previous Lemma and Cauchy-Schwarz inequality, we thus have $\mathbb{E}_z^0[|J_n^1|]\leq Ct\,h_n$. □

In order to prove the first part of Theorem 3.8, i.e. the convergence in Probability it remains to look at J_n^2. We have

$$J_n^2 = \frac{1}{h_n^2}\sum_{p=1}^{\left[\frac{t}{2h_n}\right]-1} \sigma(X_{(2p-1)h_n},Y_{(2p-1)h_n})\left(M(p,h_n) \,-\, \frac{2h_n^3}{3}\,Id\right)\sigma(X_{(2p-1)h_n},Y_{(2p-1)h_n})\,, \tag{3.19}$$

where

$$M(p,h_n) = \Delta_2 W(p,h_n) >< \Delta_2 W(p,h_n)$$

and

$$\Delta_2 W(p,h_n) = \int_{(2p-1)h_n}^{(2p+1)h_n} (h_n - |u-2ph_n|)\, dW_u \, .$$

As before we start with an estimation Lemma

Lemma 3.20. *Assume $c=0$, $V=0$, $(X_0,Y_0)=(x,y)\in\mathbb{R}^{2d}$ and $\mathcal{H}_0$. Then, there exist constants C depending on σ, its derivatives and the dimension only, such that*

$$\mathbb{E}^0_z\left(\left|M(p,h_n) \, - \, \frac{2h_n^3}{3}\, Id\right|^2\right) \leq C\, h_n^6 \, . \tag{3.21}$$

Proof. We shall look separately at the diagonal terms and the off diagonal terms of $M(p,h_n) \, - \, \frac{2h_n^3}{3}\, Id$.
The off diagonal terms are of the form $A_{i,j}(n)$ with

$$A_{i,j}(n) = \left(\int_{(2p-1)h_n}^{(2p+1)h_n} (h_n - |u-2ph_n|)\, dW^i_u\right) \left(\int_{(2p-1)h_n}^{(2p+1)h_n} (h_n - |u-2ph_n|)\, dW^j_u\right) \tag{3.22}$$

where W^i and W^j are independent linear Brownian motions. Introduce the martingales

$$U_i(s) = \int_{(2p-1)h_n}^{s} (h_n - |u-2ph_n|)\, dW^i_u$$

defined for $(2p-1)h_n \leq s \leq (2p+1)h_n$. Using Ito's formula $A_{i,j}(n)$ can be rewritten

$$\left(\int_{(2p-1)h_n}^{(2p+1)h_n} (U_j(u)-U_j((2p-1)h_n))\,(h_n - |u-2ph_n|)\, dW^i_u\right)$$
$$+ \left(\int_{(2p-1)h_n}^{(2p+1)h_n} (U_i(u)-U_i((2p-1)h_n))\,(h_n - |u-2ph_n|)\, dW^j_u\right)$$

so that

$$\begin{aligned}
\mathbb{E}^0_z\left(A^2_{i,j}(n)\right) &= 2\int_{(2p-1)h_n}^{(2p+1)h_n} (h_n - |u-2ph_n|)^2 \; \mathbb{E}^0_z[(U_i(u)-U_i((2p-1)h_n))^2]\, du \\
&= 2\int_{(2p-1)h_n}^{(2p+1)h_n} (h_n - |u-2ph_n|)^2 \left(\int_{(2p-1)h_n}^{u} (h_n - |s-2ph_n|)^2\, ds\right) du \\
&= c\, h_n^6 \, ,
\end{aligned}$$

where c is some universal constant, so that we get the result.

The diagonal terms can be written $A_{i,i}(n)$ with

$$
\begin{aligned}
A_{i,i}(n) &= \left(\int_{(2p-1)h_n}^{(2p+1)h_n} (h_n - |u-2ph_n|)\, dW_u^i\right)^2 - \frac{2}{3}\, h_n^3 \qquad (3.23)\\
&= 2\int_{(2p-1)h_n}^{(2p+1)h_n} (U_i(u) - U_i((2p-1)h_n))\,(h_n - |u-2ph_n|)\, dW_u^i\,,
\end{aligned}
$$

and we can conclude exactly as before. □

We can now state

Proposition 3.24. *Assume $c = 0$, $V = 0$ and $\mathcal{H}_0$. Then, if $h_n \xrightarrow[n\to+\infty]{} 0$, starting from any initial point $z = (x,y) \in \mathbb{R}^{2d}$, we have for any $0 \le t \le T$, that J_n^2 converges to 0 in $\mathbb{L}^2(\mathbb{P}_z^0)$ (with rate $\sqrt{h_n}$), hence in $\mathbb{P}_z^0$ probability.*

Proof. We look at each term $(J_n^2)_{ij}$ of the matrix J_n^2. Such a term can be written in the form

$$(J_n^2)_{ij} = \frac{1}{h_n^2} \sum_{p=1}^{\left[\frac{t}{2h_n}\right]-1} \sum_{l,k=1}^{d} a_{l,k,i,j}(X_{(2p-1)h_n}, Y_{(2p-1)h_n})\, A_{l,k}(p,n)$$

where the $a_{l,k,i,j}$'s are C_b^2 functions, and the $A_{l,k}(p,n)$ are defined in the proof of the previous Lemma (here we make explicit the dependence in p). Hence

$$h_n^4\,(J_n^2)_{ij}^2 =$$

$$= \sum_{p,q=1}^{\left[\frac{t}{2h_n}\right]-1} \sum_{l,k,i,j=1}^{d} b_{l,k,i,j}(X_{(2p-1)h_n}, Y_{(2p-1)h_n})\, c_{l,k,i,j}(X_{(2q-1)h_n}, Y_{(2q-1)h_n})\, A_{l,k}(p,n)\, A_{i,j}(q,n)$$

for some new functions $b_{l,k,i,j}$ and $c_{l,k,i,j}$. As we remarked in (3.10), the expectation of terms where $p \neq q$ is equal to 0, so that

$$
\begin{aligned}
\mathbb{E}_z^0\left[h_n^4\,(J_n^2)_{ij}^2\right] &\le C \sum_{l,k,i,j=1}^{d} \sum_{p=1}^{\left[\frac{t}{2h_n}\right]-1} \mathbb{E}_z^0\left[A_{l,k}(p,n)\, A_{i,j}(p,n)\right]\\
&\le C t\, h_n^5\,,
\end{aligned}
$$

according to the previous Lemma and Cauchy-Schwarz inequality, hence the result. □

We thus have obtained the first part of the main Theorem i.e.

Corollary 3.25 (consistence result)**.** *Under assumptions $\mathcal{H}_\mathbf{0}$, $\mathcal{H}_\mathbf{1}$ and $\mathcal{H}_\mathbf{2}$, if $h_n \xrightarrow[n\to+\infty]{} 0$, starting from any initial point $z = (x,y) \in \mathbb{R}^{2d}$, we have for any $0 \le t \le T$*

$$\mathcal{QV}_{h_n}(t) \xrightarrow[n\to+\infty]{\mathbb{P}_z} \frac{1}{3}\int_0^t \sigma^2(X_s, Y_s) ds.$$

We turn now to the second part of the main Theorem, i.e. the obtention of confidence intervals. Again we assume first that $c = 0$ and $V = 0$.

Since we will normalize by $\sqrt{h_n}$ we immediately see that the first "error" term $J_n^1/\sqrt{h_n}$ converges to 0 in $\mathbb{P}_z^0$ probability according to the rate of convergence we obtained in Proposition 3.18.
For the second error term J_n^3, the convergence rate in $\sqrt{h_n}$ is not sufficient to conclude. So we have to improve on it.

Lemma 3.26. *Assume $c = 0$, $V = 0$, $(X_0, Y_0) = (x, y) \in \mathbb{R}^{2d}$ and $\mathcal{H}_0$. Then, there exists some constant C depending on σ, its first two derivatives and the dimension only, such that for any $0 \le t \le T$,*

$$\mathbb{E}_z^0\left(\left|\int_0^t G_n(u)du - \int_0^t \sigma^2(X_u, Y_u)du\right|\right) \le C t\, h_n\,, \tag{3.27}$$

hence $\left(\int_0^t G_n(u)du - \int_0^t \sigma^2(X_u, Y_u)du\right)/\sqrt{h_n}$ goes to 0 in $\mathbb{P}_z^0$ probability.

Proof. To begin with

$$\sigma^2(X_u, Y_u) - G_n(u) = \sum_{p=1}^{[\frac{t}{2h_n}]-1} \left(\sigma^2(X_u, Y_u) - \sigma^2(X_{(2p-1)h_n}, Y_{(2p-1)h_n})\right) \mathbb{I}_{(2p-1)h_n \le u < (2p+1)h_n}\,.$$

Now look at each coordinate, and to simplify denote by f the coefficient σ_{ij}^2. It holds

$$\begin{aligned}
f(Z_u) - G_n^{ij}(u) &= \sum_{p=1}^{[\frac{t}{2h_n}]-1} \mathbb{I}_{(2p-1)h_n \le u < (2p+1)h_n} \int_{(2p-1)h_n}^u \\
&\quad \left(< \sigma(Z_s)\nabla_y f(Z_s),\, dW_s > + \frac{1}{2}\, Trace(\sigma\, D_y^2 f\, \sigma)(Z_s)\, ds + < Y_s, \nabla_x f(Z_s) > ds\right) \\
&= \sum_{p=1}^{[\frac{t}{2h_n}]-1} \mathbb{I}_{(2p-1)h_n \le u < (2p+1)h_n} \left(I^1(n,p,u) + I^2(n,p,u) + I^3(n,p,u) + I^4(n,p,u)\right)
\end{aligned}$$

With

$$\begin{aligned}
I^1(n,p,u) &= \int_{(2p-1)h_n}^u < \sigma(Z_s)\nabla_y f(Z_s) - \sigma(Z_{(2p-1)h_n})\nabla_y f(Z_{(2p-1)h_n}),\, dW_s >\,, \\
I^2(n,p,u) &= < \sigma(Z_{(2p-1)h_n})\nabla_y f(Z_{(2p-1)h_n}),\, W_u - W_{(2p-1)h_n} >\,, \\
I^3(n,p,u) &= \int_{(2p-1)h_n}^u \frac{1}{2}\, Trace(\sigma\, D_y^2 f\, \sigma)(Z_s)\, ds\,, \\
I^4(n,p,u) &= \int_{(2p-1)h_n}^u < Y_s, \nabla_x f(Z_s) > ds\,.
\end{aligned}$$

Notice that $|I^3(n,p,u)| \leq C(u-(2p-1)h_n)$ so that

$$\int_0^t \sum_{p=1}^{[\frac{t}{2h_n}]-1} \mathbb{1}_{(2p-1)h_n \leq u < (2p+1)h_n} \, |I^3(n,p,u)| du \leq Ct\, h_n \, .$$

Similarly $|I^4(n,p,u)| \leq C\, (\sup_{0\leq s\leq t} |Y_s|)(u-(2p-1)h_n)$ so that

$$\mathbb{E}_z^0 \left(\int_0^t \sum_{p=1}^{[\frac{t}{2h_n}]-1} \mathbb{1}_{(2p-1)h_n \leq u < (2p+1)h_n} \, |I^4(n,p,u)| du \right) \leq Ct\, h_n \; \mathbb{E}_z^0 \left(\sup_{0\leq s\leq t} |Y_s| \right) \leq Ct(1+t^{1/2})\, h_n$$

according to the Burkholder-Davis-Gundy inequality.

Now

$$\begin{aligned} (\mathbb{E}_z^0(|I^1(n,p,u)|))^2 &\leq E_z^0(|I^1(n,p,u)|^2)) \\ &= E_z^0 \left[\int_{(2p-1)h_n}^u |\sigma(Z_s)\nabla_y f(Z_s) - \sigma(Z_{(2p-1)h_n})\nabla_y f(Z_{(2p-1)h_n}|^2 \, ds \right] \\ &\leq C(u-(2p-1)h_n)\; \mathbb{E}_z^0(\sup_{|a-b|\leq 2h_n} |Z_a - Z_b|^2) \\ &\leq C\, h_n\, (u-(2p-1)h_n) \end{aligned}$$

using the fact that σ and its first two derivatives are bounded and (3.17). It follows that

$$\mathbb{E}_z^0 \left(\int_0^t \sum_{p=1}^{[\frac{t}{2h_n}]-1} \mathbb{1}_{(2p-1)h_n \leq u < (2p+1)h_n} \, |I^1(n,p,u)| du \right) \leq Ct\, h_n \, .$$

Finally

$$\left(\mathbb{E}_z^0 \left| \int_0^t \sum_{p=1}^{[\frac{t}{2h_n}]-1} \mathbb{1}_{(2p-1)h_n \leq u < (2p+1)h_n} \, I^2(n,p,u)\, du \right| \right)^2 \leq$$

$$\begin{aligned} &\leq \; \mathbb{E}_z^0 \left(\left| \int_0^t \sum_{p=1}^{[\frac{t}{2h_n}]-1} \mathbb{1}_{(2p-1)h_n \leq u < (2p+1)h_n} \, I^2(n,p,u)\, du \right|^2 \right) \\ &\leq \; 2\, \mathbb{E}_z^0 \left[\int_0^t \int_0^t \sum_{p,q=1}^{[\frac{t}{2h_n}]-1} \mathbb{1}_{(2p-1)h_n \leq u < (2p+1)h_n} \, \mathbb{1}_{(2q-1)h_n \leq s < (2q+1)h_n} \, \mathbb{1}_{s\leq u} \, I^2(n,p,u)\, I^2(n,q,s)\, ds\, du \right] \end{aligned}$$

As before, if $(2p-1)h_n \leq u < (2p+1)h_n$ and $(2q-1)h_n \leq s < (2q+1)h_n$,

$$\mathbb{E}_z^0(I^2(n,p,u)\, I^2(n,q,s)) = 0$$

as soon as $p \neq q$.
If $p = q$,

$$|\,\mathbb{E}^0_z(I^2(n,p,u)\,I^2(n,p,s))| \leq C\,\sqrt{u-(2p-1)h_n}\,\sqrt{s-(2p-1)h_n}\,,$$

so that for a fixed u between $(2p-1)h_n$ and $(2p+1)h_n$, s belongs to $[(2p-1)h_n, u]$ and

$$\int_{(2p-1)h_n}^{u} |\,\mathbb{E}^0_z(I^2(n,p,u)\,I^2(n,p,s))|\,ds \;\leq\; C\,h_n^{3/2}\,(u-(2p-1)h_n)^{1/2}\,.$$

Integrating with respect to du we finally get

$$\left(\mathbb{E}^0_z\left|\int_0^t \sum_{p=1}^{[\frac{t}{2h_n}]-1} \mathbb{1}_{(2p-1)h_n\leq u<(2p+1)h_n}\, I^2(n,p,u)\,du\right|\right)^2 \;\leq\; Ct\,h_n^2\,,$$

as expected. ☐

We turn now to the Central Limit Theorem for J^2_n defined in (3.19). We will prove

Proposition 3.28. *Assume $c(x,y)=0$, $V=0$ and $\mathcal{H}_0$. If $h_n \xrightarrow[n\to+\infty]{} 0$, starting from any initial point $z=(x,y)\in\mathbb{R}^{2d}$ and $\forall\, 0\leq t\leq T$,*

$$\sqrt{\frac{1}{h_n}}\,J^2_n(t) \xrightarrow[n\to+\infty]{\mathcal{S}} \frac{2}{3}\int_0^t \sigma(X_u,Y_u)\,d\widetilde{W}_u\,\sigma(X_u,Y_u)\,,$$

where $\left(\widetilde{W}_t\,,\ t\in[0,T]\right)$ is a symmetric matrix valued random process independent of the initial Wiener process $W_.$, whose entries $\widetilde{W}_.(i,j)$ are Wiener processes with variance $V_{i,j}(t)=(1+\delta_{i,j})t$, these entries being all independent for $i\geq j$.

Proof. Define

$$\xi_{n,p} = \frac{1}{h_n^2}\,\sigma(X_{(2p-1)h_n},Y_{(2p-1)h_n})\,\left(M(p,h_n)\,-\,\frac{2h_n^3}{3}\,Id\right)\,\sigma(X_{(2p-1)h_n},Y_{(2p-1)h_n})\,,$$

and $\mathcal{G}_{n,p}$ the σ-field generated by the $\xi_{n,j}$ for $j\leq p$. As we already saw

$$\mathbb{E}^0_z\,[\xi_{n,p}|\mathcal{G}_{n,p-1}] = 0$$

(here the null matrix), saying that for a fixed n the $\xi_{n,p}$ are martingale increments and $J^2_n(t)=\sum_p \xi_{n,p}$.
In order to prove the Proposition we can first show that for all $N\in\mathbb{N}$, all N-uple $t_1<...,t_N\leq t$,

$$\sqrt{\frac{1}{h_n}}\left(J^2_n(t),W_{t_1},...,W_{t_N}\right) \xrightarrow[n\to+\infty]{\mathcal{D}} \left(\int_0^t \sigma(X_u,Y_u)\,d\widetilde{W}_u\,\sigma(X_u,Y_u)\,,W_{t_1},...,W_{t_N}\right)\,,$$

and then apply the results we recalled on stable convergence as we did in the constant case. To get the previous convergence, one can use the Central Limit Theorem for triangular arrays of Lindeberg type, stated for instance in Dacunha-Castelle and Duflo (1983) Thm.

2.8.42.
Another possibility is to directly use Jacod's stable convergence theorem stated in Thm 2.6 of Podolskij et al. (2010). Actually in our situation, both theorems require exactly the same controls (this is not surprising), as soon as one verifies that the statement of Jacod's theorem extends to a multi-dimensional setting.
We choose the second solution, and use the notations in Podolskij et al. (2010) Thm 2.6, so that our $\xi_{n,p}/\sqrt{h_n}$ is equal to their X_{pn}.
Conditions (2.6) (martingale increments) and (2.10) (dependence on $W_{\cdot}$ only) in Podolskij et al. (2010) are satisfied. Condition (2.8) is also satisfied with $v_s = 0$ as we already remarked in the constant case. Here it amounts to see that

$$\mathbb{E}_z^0\left[A_{i,j}(n)(W^k_{(2p+1)h_n} - W^k_{(2p-1)h_n})|\mathcal{F}_{(2p-1)h_n}\right] = 0$$

for all triple (i,j,k) where the $A_{i,j}(n)$ are defined in (3.22) and (3.23), which is immediate. It thus remains to check the two conditions

$$\frac{1}{h_n}\sum_{p=1}^{[\frac{t}{2h_n}]-1} \mathbb{E}_z^0\left[({}^te_i\xi_{n,p}e_j)^2|\mathcal{F}_{(2p-1)h_n}\right] \xrightarrow[n\to+\infty]{\mathbb{P}_z^0} \int_0^t \theta^2_{ij}(X_u,Y_u)\,du\,, \tag{3.29}$$

for all $i,j = 1,...,d$ (e_l, $l = 1,...,d$ being the canonical basis) , and

$$\frac{1}{h_n}\sum_{p=1}^{[\frac{t}{2h_n}]-1} \mathbb{E}_z^0\left[|\xi_{n,p}|^2\,\mathbb{1}_{|\xi_{n,p}|>\varepsilon}|\mathcal{F}_{(2p-1)h_n}\right] \xrightarrow[n\to+\infty]{\mathbb{P}_z^0} 0 \quad \text{for all } \varepsilon>0\,, \tag{3.30}$$

where $|\xi|$ denotes the Hilbert-Schmidt norm of the matrix $|\xi|$.
We denote by $u_i = \sigma e_i$, and we use the notations of Lemma 3.20, $U_i(n,s) = \int_{(2p-1)h_n}^s (h_n - |u-2ph_n|)\,dW^i_u$ and simply $U_i(n) = U_i(n,(2p+1)h_n))$. Hence

$$A_{i,j}(n) = U_i(n)\,U_j(n) - \delta_{i,j}\,\frac{2h_n^3}{3}\,.$$

It follows

$$\begin{aligned} h_n^4\,\mathbb{E}_z^0\left[({}^te_i\xi_{n,p}e_j)^2|\mathcal{F}_{(2p-1)h_n}\right] &= \mathbb{E}_z^0\left[\left(\sum_{k,l} u_i^k\,A_{k,l}(n)\,u_j^l\right)^2|\mathcal{F}_{(2p-1)h_n}\right] \\ &= \mathbb{E}_z^0\left[\sum_{k,l,k',l'} u_i^k\,u_j^l\,u_i^{k'}\,u_j^{l'}\,A_{k,l}(n)\,A_{k',l'}(n)|\mathcal{F}_{(2p-1)h_n}\right] \\ &= \sum_{k,l,k',l'} u_i^k\,u_j^l\,u_i^{k'}\,u_j^{l'}\,\mathbb{E}_z^0\left[A_{k,l}(n)\,A_{k',l'}(n)|\mathcal{F}_{(2p-1)h_n}\right]\,. \end{aligned}$$

But all conditional expectations are vanishing except those for which $(k,l)=(k',l')$, in which case it is equal to

$$(1+\delta_{k,l})\left(\int_{(2p-1)h_n}^{(2p+1)h_n}(h_n-|u-2ph_n|)^2\,du\right)^2=\frac{4}{9}\,h_n^6\,(1+\delta_{k,l})\,.$$

Hence

$$\frac{1}{h_n}\sum_{p=1}^{[\frac{t}{2h_n}]-1}\mathbb{E}_z^0\left[({}^te_i\xi_{n,p}e_j)^2|\mathcal{F}_{(2p-1)h_n}\right]=\sum_{k,l=1}^{d}(1+\delta_{k,l})\frac{4h_n}{9}\sum_{p=1}^{[\frac{t}{2h_n}]-1}\sigma_{i,k}^2(Z_{(2p-1)h_n})\,\sigma_{j,l}^2(Z_{(2p-1)h_n})\,,$$

and converges to

$$\sum_{k,l=1}^{d}(1+\delta_{k,l})\frac{4}{9}\int_0^t\sigma_{i,k}^2(Z_u)\,\sigma_{j,l}^2(Z_u)\,du\,.$$

We get a similar result for $\frac{1}{h_n}\sum_{p=1}^{[\frac{t}{2h_n}]-1}\mathbb{E}_z^0\left[({}^te_i\xi_{n,p}e_j)({}^te_{i'}\xi_{n,p}e_{j'})|\mathcal{F}_{(2p-1)h_n}\right]$ for any pairs (i,j), (i',j'). It remains to remark that this increasing process is the one of

$$\frac{2}{3}\int_0^t\sigma(Z_u)\,d\widetilde{W}_u\,\sigma(Z_u)$$

where $\widetilde{W}_.$ is as in the statement of the proposition.

Finally, (3.30) is immediately checked, using the previous calculation, Cauchy-Schwarz inequality and Burkholder-Davis-Gundy inequality. □

To conclude the proof of the main theorem it is enough to apply Slutsky's theorem since all the error terms converge to 0 in Probability (recall that Slutsky's theorem also works with stable convergence).

4. Infinite-horizon estimation

In the previous section, we dealt with infill estimation. We now consider that we work with an infinite-horizon design. We aim at estimating the quantity $\mathbb{E}_\mu(\sigma^2(X_0,Y_0))$, where $(Z_t:=(X_t,Y_t)\in\mathbb{R}^2\,,\ t\geq 0)$ is still governed by (1) and μ is the invariant measure, supposed to exist. We thus have to introduce some new assumptions

- $\mathcal{H}_3$ There exists an (unique) invariant probability measure μ and the $\mathbb{P}_\mu$ stationary process $Z_.$ is α-mixing with rate τ i.e. (in our markovian situation) there exists a non-increasing function τ going to 0 at infinity such that for all $u\leq s$, all random variables F,G bounded by 1 s.t. F (resp. G) is $\mathcal{F}_u$ (resp. $\mathcal{G}_s$) measurable where $\mathcal{F}_u$ (resp. $\mathcal{G}_s$) is the σ-algebra generated by Z_v for $v\leq u$ (resp. $v\geq s$), one has
$$\mathrm{Cov}_\mu(f(Z_u)\,g(Z_s))\leq\tau(s-u)\,.$$

- $\mathcal{H}_4$ Define $b(x,y) := -\left(c(x,y)y + \nabla V(x)\right)$. There exists some $r \geq 4$ such that $\mathbb{E}_\mu(|b(Z_0)|^r) < +\infty$ and $\int_0^{+\infty} \tau^{1-(4/r)}(t)dt < +\infty$.

We shall come back later to these assumptions, indicating sufficient conditions for them to hold.

We introduce the following estimator

$$\mathcal{K}_n = \frac{3}{2}\frac{1}{(n-1)h_n^3}\sum_{p=1}^{n-1} \Delta_2 X(p,n) >< \Delta_2 X(p,n)\,, \tag{4.1}$$

where $\Delta_2 X(p,n)$ is the double increment of X defined in (3.1).

We now state the main result of this section:

Theorem 4.2. *Assume that $\mathcal{H}_0$ up to $\mathcal{H}_4$ are satisfied. Assume in addition that*

$$\int_1^{+\infty} t^{-1/2}\,\tau^{1/2}(t)dt < +\infty\,.$$

Let h_n be a sequence going to 0 such that $nh_n \to +\infty$ and $nh_n^3 \to 0$.
Then, in the stationary regime,

$$\sqrt{2nh_n}\left(\mathcal{K}_n - \mathbb{E}\sigma^2(X_0,Y_0)\right) \xrightarrow[n\to+\infty]{\mathcal{D}} \mathcal{N}\,, \tag{4.3}$$

where $\mathcal{N}$ is a symmetric random matrix, with centered gaussian entries satisfying

$$\mathrm{Cov}(\mathcal{N}_{i,j},\mathcal{N}_{k,l}) = \frac{1}{2}\int_0^{+\infty} \mathbb{E}_\mu(\bar\sigma^2_{i,j}(Z_0)\,\bar\sigma^2_{k,l}(Z_s) + \bar\sigma^2_{k,l}(Z_0)\bar\sigma^2_{i,j}(Z_s))\,ds\,.$$

where $\bar\sigma^2(z) = \sigma^2(z) - \mathbb{E}_\mu(\sigma^2(Z_0))$.

Remark 1. *In the case where σ is constant, this result is useless as the covariances are all vanishing.* $\diamondsuit$

Proof of Theorem 4.2:
From now on we assume that the assumptions $\mathcal{H}_0$ up to $\mathcal{H}_4$ are satisfied.

Of course since we are looking at the whole time interval up to infinity, it is no more possible to use Girsanov theory to reduce the problem to $c = V = 0$. Hence, arguing as for the statement of (3.9), and defining $b(x,y) := -\left(c(x,y)y + \nabla V(x)\right)$, we get:

$$\Delta_2 X(p,n) = \int_{(2p-1)h_n}^{(2p+1)h_n} (h_n - |u - 2ph_n|)\left(\sigma(Z_u)dW_u + b(Z_u)du\right).$$

We then define the semi-martingale $(H_t\,,\ (2p-1)h_n \leq t \leq (2p+1)h_n)$ by:

$$\begin{array}{lcl} dH_t & = & (h - |t-2ph_n|)\sigma(Z_t)dW_t + (h_n - |t-2ph_n|)b(Z_t)dt\,, \\ H_{(2p-1)h_n} & = & 0\,. \end{array}$$

so that $\Delta_2 X(p,n) = H_{(2p+1)h_n}$. Using Ito's formula we then have

$$\begin{aligned}
&(\Delta_2 X(p,n) >< \Delta_2 X(p,n))_{i,j} = \\
&\quad = \int_{(2p-1)h_n}^{(2p+1)h_n} (h_n - |u-2ph_n|)\left(H_u^i\,(\sigma(Z_u)dW_u)^j + H_u^j\,(\sigma(Z_u)dW_u)^i\right) + \qquad (4.4)\\
&\qquad + \int_{(2p-1)h_n}^{(2p+1)h_n} (h_n - |u-2ph_n|)\left(H_u^i\, b^j(Z_u) + H_u^j\, b^i(Z_u)\right)du \\
&\qquad + \int_{(2p-1)h_n}^{(2p+1)h_n} (h_n - |u-2ph_n|)^2\, \sigma_{i,j}^2(Z_u)\, du\,.
\end{aligned}$$

We have a simple but useful estimate, available for all $i = 1, ...d$, all $k \in \mathbb{N}$, all p and all u between $(2p-1)h_n$ and $(2p+1)h_n$

$$\mathbb{E}_\mu(|H_s^i|^{2k}) \leq C(k)(||\sigma||_\infty^{2k}\,(s-(2p-1)h_n)^k\, h_n^{2k} + (s-(2p-1)h_n)^{2k}\, h_n^{2k}\,(\mathbb{E}_\mu(|b(Z_0)|^{2k}))\,. \quad (4.5)$$

Indeed one can first use $(a+b)^{2k} \leq C(k)(a^{2k}+b^{2k})$, for positive numbers a, b which will be here the absolute values of the martingale part and of the bounded variation part. Then, if b_u is stationary and h_u bounded by h,

$$\mathbb{E}_\mu\left(\left(\int_0^t b_u\, h_u\, du\right)^m\right) \;\leq\; t^m\, h^m\, \mathbb{E}_\mu(b_0^m)\,,$$

which can be used with $m = 2k$, $t = (s-(2p-1)h_n)$, $b_u = b^i(Z_u)$, $h_u = (h_n - |u-2ph_n|) \leq 2h_n$. This gives the control for the bounded variation part. Finally using Burkholder-Davis-Gundy inequality, we are reduced to the same control for the martingale part, this time with $m = k$, $h_u = (h_n - |u-2ph_n|)^2 \leq 4h_n^2$ and $|b_u| \leq ||\sigma||_\infty^2$.

Now we can decompose

$$\mathcal{K}_n - \mathbb{E}_\mu \sigma^2(Z_0) = \overline{\mathcal{K}}_{n,1} + \overline{\mathcal{K}}_{n,2}$$

with

$$\overline{\mathcal{K}}_{n,1} = \frac{3}{2}\frac{1}{(n-1)h_n^3}\sum_{p=1}^{n-1}\left\{\Delta_2 X(p,n) >< \Delta_2 X(p,n) \;-\; \int_{(2p-1)h_n}^{(2p+1)h_n} (h_n - |s-2ph_n|)^2 \sigma^2(Z_s)ds\right\}$$

and

$$\overline{\mathcal{K}}_{n,2} = \frac{3}{2}\frac{1}{(n-1)h_n^3}\sum_{p=1}^{n-1}\int_{(2p-1)h_n}^{(2p+1)h_n} (h_n - |s-2ph_n|)^2\left\{\sigma^2(Z_s) - \mathbb{E}_\mu \sigma^2(Z_0)\right\}ds\,.$$

We shall look at both quantities separately, starting with $\overline{\mathcal{K}}_{n,2}$.

4.1. **Study of $\overline{\mathcal{K}}_{n,2}$.**

Lemma 4.6. *There exists some constant C only depending on the bounds of σ such that*

$$\mathbb{E}_\mu\left\{|\overline{\mathcal{K}}_{n,2}|^2\right\} \leq \frac{C}{n\,h_n}\int_0^{+\infty}\tau(t)\,dt\,.$$

Proof. $\frac{4}{9}\,\mathbb{E}_\mu\left\{\left(\overline{\mathcal{K}}_{n,2}\right)^2_{i,j}\right\} = \frac{1}{(n-1)^2h_n^6}$

$$\sum_{p,q=1}^{n-1}\int_{(2p-1)h_n}^{(2p+1)h_n}\int_{(2q-1)h_n}^{(2q+1)h_n}(h_n-|s-2ph_n|)^2(h_n-|u-2qh_n|)^2\,\mathbb{E}_\mu\left\{\overline{\sigma}^2{}_{i,j}(Z_s)\overline{\sigma}^2{}_{i,j}(Z_u)\right\}dsdu$$

$$\begin{aligned}
&\leq \frac{||\overline{\sigma}^2||^2_\infty}{(n-1)^2h_n^6}\sum_{p,q=1}^{n-1}\int_{(2p-1)h_n}^{(2p+1)h_n}\int_{(2q-1)h_n}^{(2q+1)h_n}(h_n-|s-2ph_n|)^2(h_n-|u-2qh_n|)^2\,\tau(|s-u|)\,duds\\
&\leq \frac{C||\overline{\sigma}^2||^2_\infty}{(n-1)}+\frac{C||\overline{\sigma}^2||^2_\infty}{(n-1)^2}\sum_{|p-q|\geq 2}\tau(2(|p-q|-1)h_n)\leq \frac{C||\overline{\sigma}^2||^2_\infty}{(n-1)}+\frac{C||\overline{\sigma}^2||^2_\infty}{(n-1)}\sum_{k=1}^{n-2}\tau(2k\,h_n)\,,\\
&\leq C||\overline{\sigma}^2||^2_\infty\left(\frac{1}{n-1}+\frac{1}{(n-1)h_n}\int_0^{+\infty}\tau(t)\,dt\right)\,,
\end{aligned}$$

with C some constant. We have used the fact that τ is non increasing for the final inequality. $\square$

The previous result indicates why the normalization $\sqrt{nh_n}$ has to be chosen. Now we decompose again

$$\overline{\mathcal{K}}_{n,2}=\overline{\mathcal{K}}_{n,21}+\overline{\mathcal{K}}_{n,22}$$

by decomposing

$$\sigma^2(Z_s)-\mathbb{E}_\mu\sigma^2(Z_0)=\sigma^2(Z_s)-\sigma^2(Z_{(2p-1)h_n})+\sigma^2(Z_{(2p-1)h_n})-\mathbb{E}_\mu\sigma^2(Z_0)\,.$$

We thus have

$$\begin{aligned}
\overline{\mathcal{K}}_{n,22} &= \frac{1}{2(n-1)h_n}\sum_{p=1}^{n-1}\int_{(2p-1)h_n}^{(2p+1)h_n}(\sigma^2(Z_{(2p-1)h_n})-\mathbb{E}_\mu\sigma^2(Z_0))\,ds\\
&= \frac{1}{2(n-1)h_n}\int_{h_n}^{(2n-1)h_n}(\sigma^2(Z_s)-\mathbb{E}_\mu\sigma^2(Z_0))\,ds\\
&\quad+\frac{1}{2(n-1)h_n}\sum_{p=1}^{n-1}\int_{(2p-1)h_n}^{(2p+1)h_n}(\sigma^2(Z_{(2p-1)h_n})-\sigma^2(Z_s))\,ds\\
&= \overline{\mathcal{K}}_{n,222}+\overline{\mathcal{K}}_{n,221}\,.
\end{aligned}$$

It follows that

$$\sqrt{2(n-1)h_n}\,\overline{\mathcal{K}}_{n,2}=\frac{1}{\sqrt{2(n-1)h_n}}\int_{h_n}^{(2n-1)h_n}(\sigma^2(Z_s)-\mathbb{E}_\mu\sigma^2(Z_0))\,ds+\sqrt{2(n-1)h_n}(\overline{\mathcal{K}}_{n,221}+\overline{\mathcal{K}}_{n,21})\,,$$

the first summand being the important term the two others being error terms. We shall show that these errors terms converge to 0 in $\mathbb{L}^2$. Indeed,

$$\begin{aligned}\mathbb{E}_\mu\left((n-1)^2 h_n^2\, (\overline{\mathcal{K}}_{n,221})^2_{i,j}\right) &\leq C \sum_{p,q=1}^{n-1} \int_{(2p-1)h_n}^{(2p+1)h_n} \int_{(2q-1)h_n}^{(2q+1)h_n} dsdu \\ &\qquad \mathbb{E}_\mu\left\{(\sigma^2_{i,j}(Z_s) - \sigma^2_{i,j}(Z_{(2p-1)h_n}))(\sigma^2_{i,j}(Z_u) - \sigma^2_{i,j}(Z_{(2q-1)h_n}))\right\}\end{aligned}$$

so that, as for the proof of lemma 4.6, what has to be done is to control

$$\mathrm{Cov}(\sigma^2_{i,j}(Z_s) - \sigma^2_{i,j}(Z_{(2p-1)h_n}), \sigma^2_{i,j}(Z_u) - \sigma^2_{i,j}(Z_{(2q-1)h_n}))\,.$$

The problem is that, if we use the α-mixing we will not improve upon the bound in the previous Lemma, since the uniform bound of these variables is still of order a constant. However for Markov diffusion processes one can show (see e.g. Cattiaux, Chafaï and Guillin (2012) lemma 4.2 and lemma 5.1, or Doukhan (1994) chapter 1, but the latter result also easily follows from the Riesz-Thorin interpolation theorem) the following:

Lemma 4.7. *Let F and G be as in the definition of the α-mixing except that they are not bounded. Assume that $F \in \mathbb{L}^r(\mathbb{E}_\mu)$ and $G \in \mathbb{L}^a(\mathbb{E}_\mu)$ for some r and a larger than or equal to 2. Then*

$$\mathrm{Cov}_\mu(F,G) \leq C\, \min\left(\tau^{\frac{r-2}{2r}}(s-u)\, ||F||_{\mathbb{L}^r}\, ||G||_{\mathbb{L}^2}\, ;\, \tau^{\frac{a-2}{2a}}(s-u)\, ||F||_{\mathbb{L}^2}\, ||G||_{\mathbb{L}^a}\right)\,,$$

for some constant C depending on a and r only. One also has

$$\mathrm{Cov}_\mu(F,G) \leq C\, \tau^{\frac{r-2}{2r}}((s-u)/2)\, \tau^{\frac{a-2}{2a}}((s-u)/2)\, ||F||_{\mathbb{L}^r}\, ||G||_{\mathbb{L}^a}\,,$$

for some constant C depending on a and r only.

Choosing $F = \sigma^2_{i,j}(Z_s) - \sigma^2_{i,j}(Z_{(2p-1)h_n})$ and $G = \sigma^2_{i,j}(Z_u) - \sigma^2_{i,j}(Z_{(2q-1)h_n})$, $r = a$, we see that what we have to do is to get a nice upper bound for $\mathbb{E}_\mu(|F|^r)$. But

$$|\sigma^2_{i,j}(Z_s) - \sigma^2_{i,j}(Z_{(2p-1)h_n})| \leq K\, |Z_s - Z_{(2p-1)h_n}|$$

where K only depends on σ and its first derivatives. Using Burkholder-Davis-Gundy inequality we thus have

$$\mathbb{E}_\mu(|F|^r) \quad\leq\quad C(h_n^{r/2} + h_n^r\; \mathbb{E}_\mu(|b(Z_0)|^r))\,.$$

It follows that, provided $\mathbb{E}_\mu(|b(Z_0)|^r) < +\infty$,

$$\mathrm{Cov}(\sigma^2_{i,j}(Z_s) - \sigma^2_{i,j}(Z_{(2p-1)h_n}), \sigma^2_{i,j}(Z_u) - \sigma^2_{i,j}(Z_{(2q-1)h_n})) \leq C\, h_n\, \tau^{1-(2/r)}((|p-q|-1)h_n)\,,$$

so that finally, as in the proof of Lemma 4.6 we get

$$\mathbb{E}_\mu\left((n-1)h_n\, (\overline{\mathcal{K}}_{n,221})^2_{i,j}\right) \leq C\, h_n\, \left(1 + \int_0^{+\infty} \tau^{1-(2/r)}(t)\, dt\right)\,. \tag{4.8}$$

Exactly in the same way we obtain the same result replacing $\overline{\mathcal{K}}_{n,221}$ by $\overline{\mathcal{K}}_{n,21}$.
It remains to look at

$$\frac{1}{\sqrt{2(n-1)h_n}} \int_{h_n}^{(2n-1)h_n} (\sigma^2(Z_s) - \mathbb{E}_\mu \sigma^2(Z_0))\, ds\,.$$

The asymptotic behavior of such additive functionals of stationary Markov processes has been extensively studied. For simplicity we refer to the recent Cattiaux et al. (2012) for an overview and a detailed bibliography. In particular, section 4 of this reference contains the following result (essentially due to Maxwell and Woodroofe), provided $\int_1^{+\infty} t^{-1/2}\,\tau^{1/2}(t)\,dt < +\infty$, the previous quantity converges in distribution to a centered gaussian random variable, as soon as nh_n goes to infinity. The calculation of the covariance matrix of these variables is done as in Cattiaux et al. (2012). We have thus obtained

Proposition 4.9. *Assume that $\mathcal{H}_0$ up to $\mathcal{H}_4$ are satisfied. Assume in addition that $\int_1^{+\infty} t^{-1/2}\,\tau^{1/2}(t)dt < +\infty$. Let h_n be a sequence going to 0 such that $nh_n \to +\infty$. Then, in the stationary regime, $\sqrt{2(n-1)h_n}\,\overline{\mathcal{K}}_{n,2}$ converges in distribution to a symmetric random matrix $\mathcal{N}$, with centered gaussian entries satisfying*

$$\mathrm{Cov}(\mathcal{N}_{i,j},\mathcal{N}_{k,l}) = \frac{1}{2}\int_0^{+\infty} \mathbb{E}_\mu(\bar\sigma^2_{i,j}(Z_0)\,\bar\sigma^2_{k,l}(Z_s) + \bar\sigma^2_{k,l}(Z_0)\bar\sigma^2_{i,j}(Z_s))\,ds\,.$$

4.2. Study of $\overline{\mathcal{K}}_{n,1}$.

Lemma 4.10. *Assume that for some $k \in \mathbb{N}^*$, $\mathbb{E}_\mu(|b(Z_0)|^{4k}) < +\infty$ and that $\int_0^{+\infty} \tau^{1-(1/k)}(t)dt < +\infty$.*
Then, there exists some constant $C(k)$ such that for all $i,j = 1,..,d$,

$$\mathrm{Var}_\mu\left[\left(\overline{\mathcal{K}}_{n,1}\right)_{i,j}\right] \leq \frac{C(k)}{n}\,.$$

Hence

$$\mathrm{Var}_\mu\left[\sqrt{nh_n}\,\left(\overline{\mathcal{K}}_{n,1}\right)_{i,j}\right] \to 0\,.$$

Proof. We write

$$\Delta_2 X(p,n) >< \Delta_2 X(p,n)\; - \int_{(2p-1)h_n}^{(2p+1)h_n} (h_n - |s-2ph_n|)^2\sigma^2(Z_s)ds = M_{p,n} + V_{p,n}$$

where $M_.$ (resp $V_.$) denotes the martingale (resp. bounded variation) part. As usual we use $\bar V$ for the centered $V - \mathbb{E}_\mu(V)$. Hence

$$\frac{4}{9}\,(n-1)^2\,h_n^6\;\mathrm{Var}_\mu\left[\left(\overline{\mathcal{K}}_{n,1}\right)_{i,j}\right] = \sum_{p,q=1}^{n-1} \mathbb{E}_\mu(M^{i,j}_{p,n}M^{i,j}_{q,n} + M^{i,j}_{p,n}\bar V^{i,j}_{q,n} + \bar V^{i,j}_{p,n}M^{i,j}_{q,n} + \bar V^{i,j}_{p,n}\bar V^{i,j}_{q,n})\,.$$

A lot of terms of this sum are vanishing, so that we get

$$\begin{aligned}\frac{4}{9}\,(n-1)^2\,h_n^6\ \mathrm{Var}_\mu\left[\left(\overline{\mathcal{K}}_{n,1}\right)_{i,j}\right] &= \sum_{p=1}^{n-1}\mathbb{E}_\mu((M_{p,n}^{i,j})^2+2\bar{V}_{p,n}^{i,j}M_{p,n}^{i,j}+(\bar{V}_{p,n}^{i,j})^2)\\ &\quad+\sum_{p>q=1}^{n-1}\mathbb{E}_\mu(\bar{V}_{p,n}^{i,j}M_{q,n}^{i,j}+2\,\bar{V}_{p,n}^{i,j}\bar{V}_{q,n}^{i,j})\end{aligned}$$

Using stationarity and (4.5) we get

$$\begin{aligned}\mathbb{E}_\mu((M_{p,n}^{i,j})^2) &= \int_0^{2h_n}(h_n-|u-h_n|)^2\ \mathbb{E}_\mu(\sigma_{i,i}^2(Z_s)(H_s^j)^2+\sigma_{j,j}^2(Z_s)(H_s^i)^2+2\sigma_{i,j}^2(Z_s)H_s^jH_s^i)\,ds\\ &\le C\,h_n^6\,(1+h_n\ \mathbb{E}_\mu(|b(Z_0)|^2))\,.\end{aligned}$$

Similarly

$$\begin{aligned}\mathbb{E}_\mu((V_{p,n}^{i,j})^2) &= \mathbb{E}_\mu\left[\left(\int_0^{2h_n}(h_n-|u-h_n|)\,(H_u^i\,b^j(Z_u)+H_u^j\,b^i(Z_u))du\right)^2\right]\\ &\le C\,h_n^3\int_0^{2h_n}\mathbb{E}_\mu(|b(Z_u)|^2)(H_u^i)^2)\,du\\ &\le C\,h_n^7\,(\mathbb{E}_\mu(|b(Z_0)|^4))^{1/2}\,(1+h_n(\mathbb{E}_\mu(|b(Z_0)|^4))^{1/2})\,.\end{aligned}$$

It follows that

$$\sum_{p=1}^{n-1}\mathbb{E}_\mu((M_{p,n}^{i,j})^2+2\bar{V}_{p,n}^{i,j}M_{p,n}^{i,j}+(\bar{V}_{p,n}^{i,j})^2)\le C\,(n-1)h_n^6\,.$$

Exactly in the same way one obtains that, for $k\in\mathbb{N}^*$, provided $\mathbb{E}_\mu(|b(Z_0)|^{2k})<+\infty$,

$$\mathbb{E}_\mu(|M_{p,n}^{i,j}|^{2k})\le C(k)\,h_n^{6k}$$

and provided $\mathbb{E}_\mu(|b(Z_0)|^{4k})<+\infty$,

$$\mathbb{E}_\mu(|V_{p,n}^{i,j}|^{2k})\le C(k)\,h_n^{7k}\,.$$

Again we shall use Lemma 4.7 to control

$$\mathbb{E}_\mu(\bar{V}_{p,n}^{i,j}M_{q,n}^{i,j})=\mathrm{Cov}_\mu(V_{p,n}^{i,j},M_{q,n}^{i,j})\ \text{ and }\ \mathbb{E}_\mu(\bar{V}_{p,n}^{i,j}\bar{V}_{q,n}^{i,j})=\mathrm{Cov}_\mu(V_{p,n}^{i,j},V_{q,n}^{i,j})\,,$$

and we obtain

$\mathrm{Cov}_\mu(V_{p,n}^{i,j},M_{q,n}^{i,j})\le C\,h_n^6\,\tau^{(k-1)/k}((p-q-1)/2)$ and $\mathrm{Cov}_\mu(V_{p,n}^{i,j},V_{q,n}^{i,j})\le C\,h_n^6\,\tau^{(k-1)/k}((p-q-1)/2)$ provided respectively $\mathbb{E}_\mu(|b(Z_0)|^{2k})<+\infty$ and $\mathbb{E}_\mu(|b(Z_0)|^{4k})<+\infty$.

We have thus obtained

$$\sum_{p>q=1}^{n-1}\mathbb{E}_\mu(\bar{V}_{p,n}^{i,j}M_{q,n}^{i,j}+2\,\bar{V}_{p,n}^{i,j}\bar{V}_{q,n}^{i,j})\le C\,(n-1)\,h_n^6\int_0^{+\infty}\tau^{(k-1)/k}(t)\,dt\,,$$

so that gathering all previous estimates we get the result. □

It remains to bound the expectation of $\left(\overline{\mathcal{K}}_{n,1}\right)_{i,j}$. But

$$\begin{aligned}
\mathbb{E}_\mu\left[\left(\overline{\mathcal{K}}_{n,1}\right)_{i,j}\right] &= \frac{3}{2(n-1)h_n^3}\sum_{p=1}^{n-1}\mathbb{E}_\mu\left[\int_{(2p-1)h_n}^{(2p+1)h_n}(h_n-|u-2ph_n|)\left(H_u^i b^j(Z_u)+H_u^j b^i(Z_u)\right)du\right]\\
&= \frac{3}{2(n-1)h_n^3}(A_{n,1}+A_{n,2})
\end{aligned}$$

with

$$A_{n,1}=\sum_{p=1}^{n-1}\mathbb{E}_\mu\left[\int_{(2p-1)h_n}^{(2p+1)h_n}(h_n-|u-2ph_n|)\left(H_u^i b^j(Z_{(2p-1)h_n})+H_u^j b^i(Z_{(2p-1)h_n})\right)du\right],$$

and

$$A_{n,2}=$$

$$=\sum_{p=1}^{n-1}\mathbb{E}_\mu\left[\int_{(2p-1)h_n}^{(2p+1)h_n}(h_n-|u-2ph_n|)\left(H_u^i(b^j(Z_u)-b^j(Z_{(2p-1)h_n}))+H_u^j(b^i(Z_u)-b^i(Z_{(2p-1)h_n}))\right)du\right].$$

$A_{n,2}$ can be studied exactly as we did before because $b^j(Z_u)-b^j(Z_{(2p-1)h_n})$ is centered. To be more precise, instead of calculating $A_{n,2}$ we look at the $\mathbb{L}^2$ norm of the random variable

$$\sum_{p=1}^{n-1}\int_{(2p-1)h_n}^{(2p+1)h_n}(h_n-|u-2ph_n|)\left(H_u^i(b^j(Z_u)-b^j(Z_{(2p-1)h_n}))+H_u^j(b^i(Z_u)-b^i(Z_{(2p-1)h_n}))\right)du$$

which is, thanks to the centering property, similar to the quantities we have studied in the proof of Lemma 4.10, that is we can use the mixing property for the *covariances*. It follows that $\sqrt{nh_n}\,\frac{A_{n,2}}{nh_n^3}$ goes to 0.

Finally, using the semi martingale decomposition of H_u,

$$\begin{aligned}
A_{n,1} &= \sum_{p=1}^{n-1}\int_{(2p-1)h_n}^{(2p+1)h_n}(h_n-|u-2ph_n|)\ \mathbb{E}_\mu\left(H_u^i b^j(Z_{(2p-1)h_n})+H_u^j b^i(Z_{(2p-1)h_n})\right)du\\
&= \sum_{p=1}^{n-1}\int_{(2p-1)h_n}^{(2p+1)h_n}\int_{(2p-1)h_n}^{u}(h_n-|u-2ph_n|)(h_n-|v-2ph_n|)\\
&\qquad\qquad \mathbb{E}_\mu\left(b^i(Z_v)b^j(Z_{(2p-1)h_n})+b^j(Z_v)b^i(Z_{(2p-1)h_n})\right)dv\,du
\end{aligned}$$

so that

$$|A_{n,1}|\le C\,nh_n^4\left(\mathbb{E}_\mu(|b(Z_0)|^2)\right)^2.$$

Hence $\sqrt{nh_n}\,\frac{A_{n,1}}{nh_n^3}$ goes to 0, provided $nh_n^3\to 0$. This completes the proof of the Theorem.
□

4.3. **The σ constant case.**
As we already remarked, if $\sigma(x,y)$ is constant, $\overline{\mathcal{K}}_{n,2}=0$. The good normalization is then $\sqrt{n}$. Indeed, in the previous proof we did not use the full strength of the bound

$$\mathbb{E}_\mu(|V_{p,n}^{i,j}|^{2k}) \leq C(k)\, h_n^{7k}\,,$$

furnishing some $h_n^{7/2}$ instead of a h_n^3 each time a bounded variation term appears. Hence all terms will go to 0 except the two remaining terms:

- $\sqrt{n}\,\frac{A_{n,1}}{nh_n^3} \leq C\,(\mathbb{E}_\mu(|b(Z_0)|^2))^2\,\sqrt{n}\,h_n$ for which we need $nh_n^2 \to 0$,
- and the remaining martingale term
$$\int_{(2p-1)h_n}^{(2p+1)h_n} (h_n - |u-2ph_n|) \int_{(2p-1)h_n}^{u} (h_n - |s-2ph_n|)(\sigma(Z_s)dW_s)^i\,(\sigma(Z_u)dW_u)^j$$
in (4.4).

But since σ is constant, this is exactly the martingale term we encountered in subsection 3.1. We thus have obtained

Theorem 4.11. *Assume that $\mathcal{H}_0$ up to $\mathcal{H}_4$ are satisfied and that σ is constant. Let h_n be a sequence going to 0 such that $nh_n \to +\infty$ and $nh_n^2 \to 0$. Then, in the stationary regime,*

$$\sqrt{n}\left(\mathcal{K}_n - \sigma^2\right) \xrightarrow[n\to+\infty]{\mathcal{D}} \sigma\,\mathcal{N}_{(d,d)}\,\sigma\,, \tag{4.12}$$

where $\mathcal{N}_{(d,d)}$ is as in Lemma 3.3.

4.4. **About $\mathcal{H}_3$ and $\mathcal{H}_4$.**
As we promised we come back to the conditions $\mathcal{H}_3$ and $\mathcal{H}_4$. Actually in full generality very few is known. All known results amounts to the existence of some Lyapunov function (see e.g. Wu (2001) Theorem 2.4) i.e. some non negative function ψ satisfying $-L\psi \geq \lambda\,\psi$ at infinity for some $\lambda > 0$. In this case τ has an exponential decay and the invariant measure exponential moments, so that $\mathcal{H}_3$ and $\mathcal{H}_4$ are satisfied provided b has some polynomial growth. General (and not really tractable) conditions for the existence of ψ are discussed in Wu (2001) sections 3 and 4. One can also relax the Lyapunov control as in Douc et al. (2009).

Tractable conditions are only known when σ is constant. They are recalled in Cattiaux et al. (2014) (see hypotheses $\mathcal{H}_1$ and $\mathcal{H}_2$ therein, based on Wu (2001) and Bakry et al. (2008)). Mainly, one has to assume that c and V have at most polynomial growth and that $< x, \nabla V(x) >$ is positive enough at infinity, for instance

$$< x, \nabla V(x) > \geq \lambda |x|$$

at infinity.

5. Examples and numerical simulation results

In this section we want to illustrate some of the main results of the paper. We start with the Itô stochastic differential equation defined by (1):

$$\begin{cases} dX_t &= Y_t dt \\ dY_t &= \sigma(X_t,Y_t)\, dW_t - (c(X_t,Y_t)Y_t + \nabla V(X_t))dt. \end{cases}$$

More precisely, we first consider an harmonic oscillator that is driven by a white noise forcing:

$$\begin{aligned} dX_t &= Y_t dt \\ dY_t &= \sigma\, dW_t - (\kappa Y_t + D X_t)dt \end{aligned} \tag{5.1}$$

with $\kappa > 0$ and $D > 0$. For this model we know that the stationary distribution is Gaussian, with mean zero and an explicit variance matrix given in e.g. Gardiner (1985).

For this example, the diffusion term is constant, equal to σ. Recall that the infill estimator with $T = 1$ is defined by (3.2):

$$\hat{\sigma}_n^2 = \frac{1}{[\frac{1}{2h_n}] - 1}\, \frac{3}{2h_n^3} \sum_{p=1}^{[\frac{1}{2h_n}]-1} \left(X_{(2p+1)h_n} - 2X_{2ph_n} + X_{(2p-1)h_n} \right)^2 .$$

As the model satisfies assumptions $\mathcal{H}_0$, $\mathcal{H}_1$ and $\mathcal{H}_2$, we know from Corollary 3.6 that if $h_n \xrightarrow[n\to+\infty]{} 0$, starting from any initial point $z = (x,y)$,

$$\sqrt{\frac{1}{2h_n}}\left(\hat{\sigma}_n^2 - \sigma^2\right) \xrightarrow[n\to+\infty]{\mathcal{S}} \mathcal{N}(0, 2\,\sigma^4)\,.$$

A 95% asymptotic confidence interval for σ^2 is thus defined as:

$$\mathbf{CI}_{95\%}(\sigma^2) = \left[\hat{\sigma}_n^2 - 1.96\,\sqrt{2}\hat{\sigma}_n^2\,\sqrt{2h_n}\,,\, \hat{\sigma}_n^2 + 1.96\,\sqrt{2}\hat{\sigma}_n^2\, \sqrt{2h_n}\right] .$$

In the following, we approximate the solution of (5.1) by an explicit Euler scheme. We choose $h_n = n^{-\gamma}$, $\gamma > 0$, $\kappa = 2$ and $D = 2$. Then, for different values of n and γ, we compute $M = 1000$ realizations of $\hat{\sigma}_n^2$. On these M realizations we compute the empirical relative mean squared error defined by $RMSE = \frac{1}{M}\sum_{j=1}^{M}\left(\frac{\hat{\sigma}_n^{2,j} - \sigma^2}{\sigma^2}\right)^2$, as far as the empirical coverage of the 95% confidence interval defined as $ECOV = \frac{1}{M}\sum_{j=1}^{M} \mathbf{1}_{\sigma^2 \in \mathbf{CI}^j_{95\%}(\sigma^2)}$. The results are summarized in Table 2 below.

σ	γ	n	RMSE	ECOV
1	0.5	100	0.47	0.85
1	0.5	1000	0.13	0.92
1	0.5	10^4	0.04	0.93
1	0.7	100	0.19	0.90
1	0.7	1000	0.03	0.94
1	0.7	10^4	0.006	0.95
2	0.5	100	2.03	0.86
2	0.5	1000	0.53	0.91
2	0.5	10^4	0.15	0.94
2	0.7	100	0.72	0.91
2	0.7	1000	0.13	0.94
2	0.7	10^4	0.02	0.95

TABLE 1. Infill estimation, empirical relative mean squared error (RMSE) and empirical coverage (ECOV) of the 95% confidence interval with $h_n = n^{-\gamma}$, $M = 1000$ realizations of the estimator, and for different values of n, γ and σ.

As expected, the more γ is high, the more fast is the convergence. The speed of convergence also depends (through a constant term in the asymptotic variance) on the unknown value of σ^2.

We now consider for the same model the infinite-horizon estimation. Model (5.1) satisfies assumptions $\mathcal{H}_0$ up to $\mathcal{H}_4$. Thus, if $h_n \xrightarrow[n\to+\infty]{} 0$, $nh_n \xrightarrow[n\to+\infty]{} +\infty$ and $nh_n^2 \xrightarrow[n\to+\infty]{} 0$, then through Theorem 4.11 we have:

$$\sqrt{n}\left(\mathcal{K}_n - \sigma^2\right) \xrightarrow[n\to+\infty]{\mathcal{D}} \mathcal{N}(0, 2\sigma^4)\,,$$

with $\mathcal{K}_n = \frac{3}{2}\,\frac{1}{(n-1)h_n^3}\,\sum_{p=1}^{n-1}\left(X_{(2p+1)h_n} - 2X_{2ph_n} + X_{(2p-1)h_n}\right)^2$. A 95% asymptotic confidence interval for σ^2 is thus defined as:

$$\mathbf{CI}_{95\%}(\sigma^2) = \left[\mathcal{K}_n - 1.96\,\frac{\sqrt{2}\mathcal{K}_n}{\sqrt{n}}\,,\, \mathcal{K}_n + 1.96\,\frac{\sqrt{2}\mathcal{K}_n}{\sqrt{n}}\right]\,.$$

In the following, we approximate the solution of (5.1) by an explicit Euler scheme. We choose $h_n = n^{-\gamma}$, $\gamma > 0$, $\kappa = 2$ and $D = 2$. Then, for different values of n and γ, we compute $M = 1000$ realizations of $\mathcal{K}_n$. On these M realizations we compute the empirical relative mean squared error defined by $RMSE = \frac{1}{M}\sum_{j=1}^{M}\left(\frac{\mathcal{K}_n^j - \sigma^2}{\sigma^2}\right)^2$, as far as the empirical coverage of the 95% confidence interval defined as $ECOV = \frac{1}{M}\sum_{j=1}^{M}\mathbf{1}_{\sigma^2\in\mathbf{CI}^j_{95\%}(\sigma^2)}$. The results are summarized in Table 2 below.

σ	γ	n	RMSE	ECOV
1	0.5	100	0.022	0.890
1	0.5	500	0.005	0.917
1	0.5	1000	0.002	0.923
1	0.7	100	0.019	0.942
1	0.7	500	0.004	0.947
1	0.7	1000	0.002	0.949
2	0.5	100	0.084	0.892
2	0.5	500	0.017	0.921
2	0.5	1000	0.008	0.933
2	0.7	100	0.085	0.926
2	0.7	500	0.018	0.936
2	0.7	1000	0.008	0.947

TABLE 2. Infinite-horizon estimation, empirical relative mean squared error (RMSE) and empirical coverage (ECOV) of the 95% confidence interval with $h_n = n^{-\gamma}$, $M = 1000$ realizations of the estimator, and for different values of n, γ and σ.

As expected, we observe that the rate of convergence does not depend on γ. The result of Theorem 4.11 has to be compared to the one in Theorem 2 in Samson and Thieullen (2012). In Samson and Thieullen (2012), the estimator is obtained by minimizing a contrast. More precisely, the authors in Samson and Thieullen (2012) define the contrast to minimize as:

$$\mathcal{L}_n(\sigma^2) = \sum_{p=1}^{n-2} \frac{3}{2} \frac{\left(X_{(p+1)h_n} - 2X_{ph_n} + X_{(p-1)h_n}\right)^2}{h_n^3 \sigma^2} + (n-2)\log(\sigma^2)\,,$$

and they obtain

$$\tilde{\sigma}_n^2 = \frac{3}{2}\frac{1}{n-2} \sum_{p=1}^{n-2} \frac{\left(X_{(p+1)h_n} - 2X_{ph_n} + X_{(p-1)h_n}\right)^2}{h_n^3}\,.$$

They obtain the same rate of convergence but with the asymptotic variance equal to $\frac{9}{4}\,\sigma^4$. Our definition (3.1) of the double increment of X, which is different from theirs, allows to recover the asymptotic variance $2\,\sigma^4$ they get for the case of complete observations. In the present paper, we do not study the optimality of the estimators. It is naturally a very interesting problem, which, for the model under study is still open.

We now consider a variant of Model (5.1) in which we consider a diffusion term which is non constant. We are only considering the infill estimation as it is not really tractable to construct a model with a non constant diffusion term satisfying assumptions $\mathcal{H}_3$ and $\mathcal{H}_4$, as already mentioned in Section 4.4.

More precisely, we consider the following model:

$$\begin{aligned} dX_t &= Y_t dt \\ dY_t &= \sigma \exp\left(\frac{1}{X_t^2+1}\right) dW_t - (\kappa Y_t + DX_t)dt. \end{aligned} \tag{5.2}$$

In that case, the infill estimator is defined as:

$$\mathcal{QV}_{h_n}(1) = \frac{1}{h_n^2} \sum_{p=1}^{\left[\frac{1}{2h_n}\right]-1} \left(X_{2p+1)h_n} - 2X_{2ph_n} + X_{(2p-1)h_n}\right)^2 .$$

Model (5.2) satisfies assumptions $\mathcal{H}_0$, $\mathcal{H}_1$ and $\mathcal{H}_2$. Thus, if $h_n \xrightarrow[n\to+\infty]{} 0$, we get from Theorem 3.8:

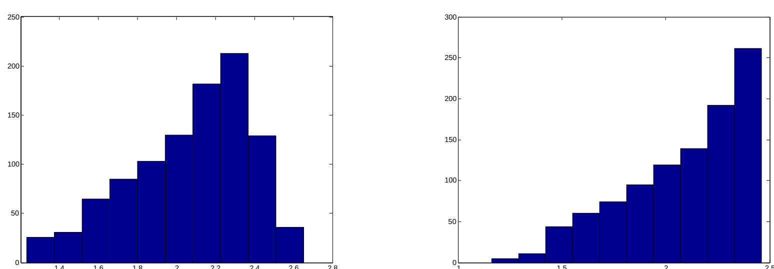
$$\sqrt{\frac{1}{h_n}} \left(\mathcal{QV}_{h_n}(1) - \frac{\sigma^2}{3} \int_0^1 \exp\left(\frac{2}{X_s^2+1}\right) ds \right) \xrightarrow[n\to+\infty]{\mathcal{S}} \frac{2\sigma^2}{3} \int_0^1 \exp\left(\frac{2}{X_s^2+1}\right) d\widetilde{W}_s \,,$$

where $\left(\widetilde{W}_t\,,\ t\in[0,T]\right)$ is a Wiener process independent of the initial Wiener process $W_{.}$, with variance equal to 2.

In the following, we choose $h_n = n^{-\gamma}$ with $\gamma = 0.7$. We compute $M = 1000$ realizations of the estimator $\mathcal{QV}_{h_n}(1)$ and $M = 1000$ realizations of the limit $\frac{\sigma^2}{3}\int_0^1 \exp\left(\frac{2}{X_s^2+1}\right) ds$. This integral is approximated by a quadrature formula with the rectangle rule.

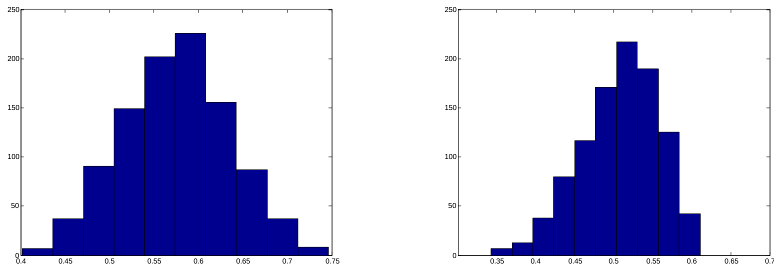
We consider two cases: for the first one, $n = 10^5$, $\kappa = 2$, $D = 2$ and $\sigma = 1$, for the second one $n = 10^4$, $\kappa = 2$, $D = 2$ and $\sigma = 0.5$ We compute for each case the empirical relative mean squared error (RMSE) and we draw (see Figures 1 and 2) both the histogram of the estimator and the one of the limit integral for the $M = 1000$ realizations.

We get for both cases $RMSE = 0.05$. As $\gamma = 0.7$ is the same for both cases, and as $n = 10^5$ for the first case and $n = 10^4$ for the second one, it proves that the asymptotic variance is smaller for the second case.



(a) (b)

FIGURE 1. Histograms on $M = 1000$ realizations of the estimator and of the limit integral, $n = 10^5$, $\sigma = 1$, $h_n = n^{-0.7}$.



(a) (b)

FIGURE 2. Histograms on $M = 1000$ realizations of the estimator and of the limit integral, $n = 10^4$, $\sigma = 2$, $h_n = n^{-0.7}$.

The histograms in Figures 1 and 2 are similar. However we note that we have a boundary effect for the upper tail in both cases, probably due to the approximation of the limit integral by a quadrature rule.

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