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# (Un)conditional consensus emergence under perturbed and decentralized feedback controls

(joint work with D. Kalise and M. Fornasier)



Mattia Bongini

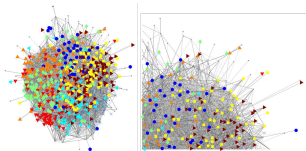
Technische Universität München,  
Department of Mathematics,  
Chair of Applied Numerical Analysis

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Netco 2014  
Tours  
June 23-27, 2014

# Introduction

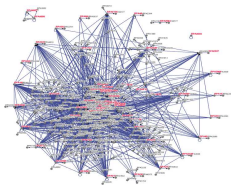
Large particle systems arise in many modern applications:



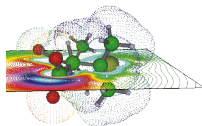
*Large Facebook “friendship” network*



*Image halftoning via variational dithering*



*Dynamical data analysis: *R. palustris*  
protein-protein interaction network*



*Computational chemistry: molecule  
simulation*

# A framework for social dynamics

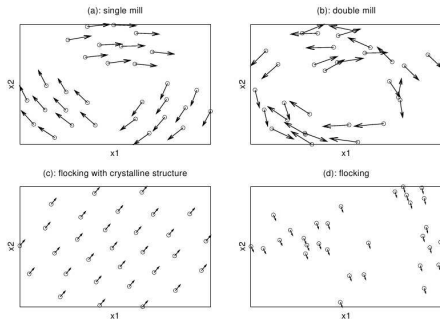
We consider large particle systems  
of form:

$$\begin{cases} \dot{x}_i = v_i, \\ \dot{v}_i = (H \star \mu_N)(x_i, v_i), \quad i = 1, \dots, N, \end{cases}$$

where  $\mu_N = \frac{1}{N} \sum_{j=1}^N \delta_{(x_j, v_j)}$ .

Several “social forces” encoded in  
the interaction kernel  $H$ :

- Repulsion-attraction
- Alignment
- ...



*Patterns related to different balance of social  
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# A framework for social dynamics

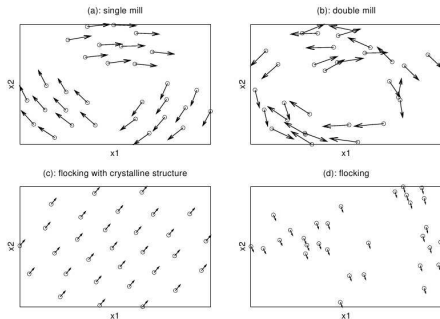
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*Patterns related to different balance of social  
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Understanding how superposition of re-iterated binary “social forces” yields global self-organization.

# Outline of the talk

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- we then explore a different strategy to enforce self-organization, namely **sparse control policies**, and compare it with the previous decentralized approach;
- finally, we shall see how sparse stabilization techniques can be exported successfully to different models.

## A framework for consensus emergence

The **Cucker-Smale model** is obtained by the choice of the interaction kernel  $H(x, v) = a(|x|)(-v)$ .

$$\begin{cases} \dot{x}_i = v_i \in \mathbb{R}^d \\ \dot{v}_i = \frac{1}{N} \sum_{j=1}^N a\left(\|x_i - x_j\|^2\right) (v_j - v_i) \in \mathbb{R}^d, \text{ for } i = 1, \dots, N, \end{cases}$$

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We say that a solution  $\{(x_i, v_i)\}_{i=1}^N$  **tends to consensus** if there exists a vector  $w \in \mathbb{R}^d$  such that

$$\lim_{t \rightarrow +\infty} \|v_i(t) - w\| = 0 \quad \text{for every } i = 1, \dots, N.$$



## Conditional consensus emergence for generic $a(\cdot)$

---

Consider the symmetric bilinear form

$$B(u, v) = \frac{1}{2N^2} \sum_{i,j} \langle u_i - u_j, v_i - v_j \rangle = \frac{1}{N} \sum_{i=1}^N \langle u_i, v_i \rangle - \langle \bar{u}, \bar{v} \rangle,$$

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$$X(t) = B(x(t), x(t)), \quad V(t) = B(v(t), v(t)).$$

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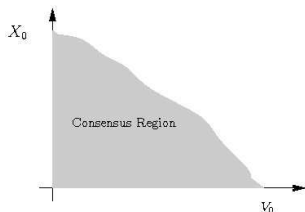
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## Theorem (Ha-Ha-Kim)

Let  $(x_0, v_0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$  be such that  $X_0 = B(x_0, x_0)$  and  $V_0 = B(v_0, v_0)$  satisfy

$$\int_{\sqrt{X_0}}^{\infty} a(\sqrt{2Nr}) dr \geq \sqrt{V_0}.$$

Then the solution with initial data  $(x_0, v_0)$  tends to consensus.



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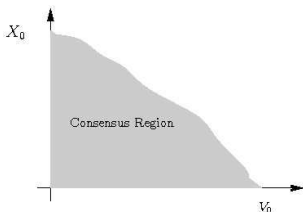
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Then the solution with initial data  $(x_0, v_0)$  tends to consensus.

Hence for  $a(t) := a_{\beta}(t)$  we have unconditional convergence for  $0 \leq \beta \leq \frac{1}{2}$ .





## Non-consensus events

---

If  $a(\cdot) = a_\beta(\cdot)$  and  $\beta > 1/2$  then the consensus condition is **not** satisfied by all  $(x_0, v_0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$ , as shown by Cucker and Smale.

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Consider  $\beta = 1$ ,  $N = 2$ ,  $d = 1$  and let  $x(t) = x_1(t) - x_2(t)$  and  $v(t) = v_1(t) - v_2(t)$  be the relative positions and velocities of the two agents on the line: rewrite the system as

$$\begin{cases} \dot{x} = v \\ \dot{v} = -\frac{v}{1+x^2} \end{cases}$$

with initial conditions  $x(0) = x_0$  and  $v(0) = v_0 > 0$ .



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By direct integration

$$v(t) = -\arctan x(t) + \arctan x_0 + v_0.$$

Hence, **if  $\arctan x_0 + v_0 > \pi/2 + \varepsilon$  we have**

$$v(t) > \pi/2 + \varepsilon - \arctan x(t) > \varepsilon, \quad \forall t \in \mathbb{R}_+.$$

## A possible way out...

---

However, in the work Caponigro-Fornasier-Piccoli-Trélat it is shown that the addition of an extra term of the form

$$\begin{cases} \dot{x}_i = v_i \\ \dot{v}_i = \frac{1}{N} \sum_{j=1}^N a(\|x_i - x_j\|^2) (v_j - v_i) + \gamma(\bar{v} - v_i), \text{ for } i = 1, \dots, N, \end{cases}$$

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where  $\gamma > 0$ , implies convergence to consensus unconditionally. This happens because each agent has an **additional** stirring to reach the **common** goal  $\bar{v}$ .

**But what happens if every agent has its own “local” goal?**

## Introducing decentralized controls

We study how the convergence to consensus is modified by the introduction of the following extra term

$$\begin{cases} \dot{x}_i = v_i \\ \dot{v}_i = \frac{1}{N} \sum_{j=1}^N a(\|x_i - x_j\|^2) (v_j - v_i) + \gamma(\bar{v}_i^R - v_i), \text{ for } i = 1, \dots, N, \end{cases}$$

where  $\gamma > 0$  and

$$\bar{v}_i^R = \frac{1}{\sum_{k=1}^N \chi_{[0,R]}(\|x_i - x_k\|)} \sum_{j=1}^N \chi_{[0,R]}(\|x_i - x_j\|) v_j,$$

is the local mean velocity calculated by the  $i$ -th agent taking into account only the agents inside a ball of radius  $R$  centered on him.



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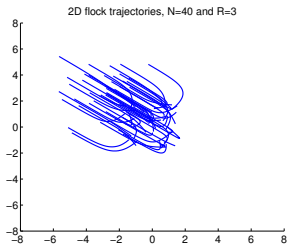
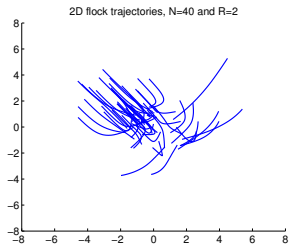
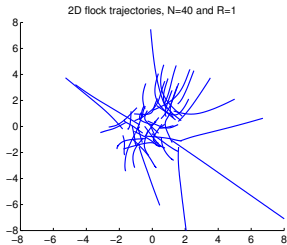
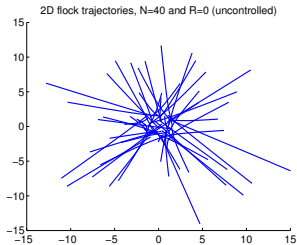
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is the local mean velocity calculated by the  $i$ -th agent taking into account only the agents inside a ball of radius  $R$  centered on him.

**GOAL:** Study the behavior of the system for  $R \rightarrow 0$  and  $R \rightarrow +\infty$ .

# Behavior of the modified system in function of $R$





## A different approach

---

To avoid the issues coming from a normalizing factor varying from agent to agent, in our study of the asymptotic behavior we replace it with a function  $\eta_R(t) \in L^\infty([0, +\infty))$  satisfying

$$\begin{aligned}\lim_{R \rightarrow 0} \eta_R(t) &= 1 \left( = \lim_{R \rightarrow 0} \sum_{k=1}^N \chi_{[0, R]}(\|x_i(t) - x_k(t)\|) \text{ for every } i \right), \\ \lim_{R \rightarrow +\infty} \eta_R(t) &= N \left( = \lim_{R \rightarrow +\infty} \sum_{k=1}^N \chi_{[0, R]}(\|x_i(t) - x_k(t)\|) \text{ for every } i \right),\end{aligned}$$



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like

$$\eta_R(t) = \max_{i=1, \dots, N} \left\{ \sum_{k=1}^N \chi_{[0, R]}(\|x_i(t) - x_k(t)\|) \right\}.$$



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Hence, we replace  $\bar{v}_i^R$  in the system with

$$\hat{v}_i^R = \frac{1}{\eta_R} \sum_{k=1}^N \chi_{[0, R]}(\|x_i - x_k\|) v_j.$$

## A sufficient condition

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### Theorem

Replace  $\bar{v}_i^R$  with  $\hat{v}_i^R$  in the model. Let  $(x_0, v_0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$  and define  $X_0 = B(x_0, x_0)$  and  $V_0 = B(v_0, v_0)$ . Then the solution with initial data  $(x_0, v_0)$  tends to consensus if one of the following is satisfied:

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- $\sqrt{2NX_0} \leq R$  and  $\int_{\sqrt{X_0}}^{\infty} a(\sqrt{2Nr})dr + \gamma \left( \frac{R}{\sqrt{2N}} - \sqrt{X_0} \right) \geq \sqrt{V_0}$ ;

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## Proof.

via the Ha-Ha-Kim Theorem. □

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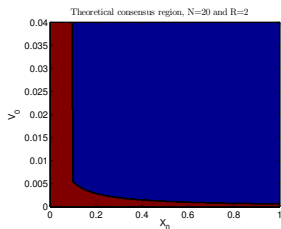
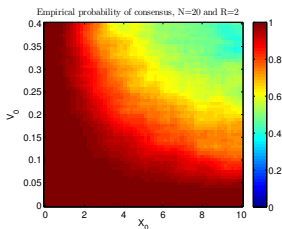
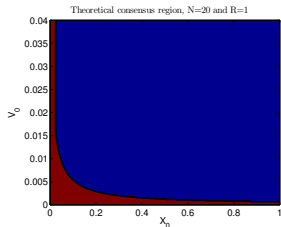
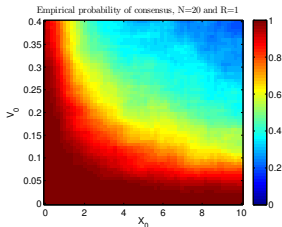
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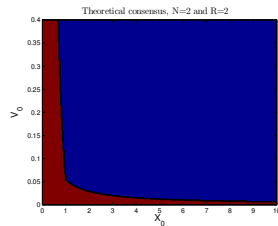
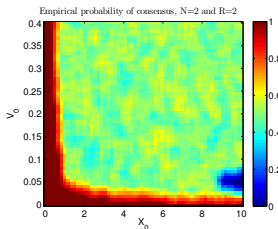
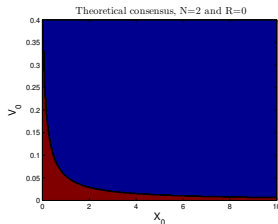
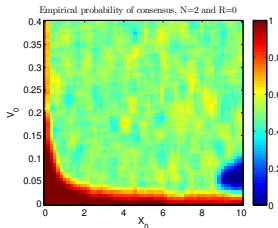
via the Ha-Ha-Kim Theorem. □

- Result generalizable to different forms of  $\eta_R$  and  $\chi_{[0,R]}$ .

# Variation of the empirical and theoretical consensus region in function of $R$



# Sharpness of the result for $N = 2$





## Efficiency of stabilization

---

- We have shown that an unconditional (w.r.t. the initial data) convergence to consensus is guaranteed only in the case  $R = \infty$ , i.e., the agents are completely “open-minded”.  
In this case, the Lyapunov functional  $V$  decreases at least as

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- **Question:** are there more economical and efficient strategies which guarantees unconditional stabilization?
- More precisely, we wish to make the quantity  $\frac{d}{dt}V(t)$  the smallest possible and use the minimal amount of intervention (e.g., by imposing sparsifying constraints).

## Self-organization vs organization by intervention

We introduce a control term in the Cucker-Smale model:

$$\begin{cases} \dot{x}_i = v_i \\ \dot{v}_i = \frac{1}{N} \sum_{j=1}^N a(\|x_j - x_i\|)(v_j - v_i) + u_i, \text{ for } i = 1, \dots, N. \end{cases}$$

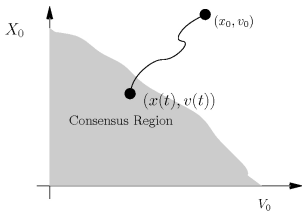
The admissible controls will be measurable functions  $u = (u_1, \dots, u_N) : [0, +\infty) \rightarrow \mathbb{R}^N$  such that  $\sum_{i=1}^N \|u_i(t)\| \leq M$  for every  $t > 0$ , for a given constant  $M$ .

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Our aim is then to find admissible controls steering the system to the consensus region and minimizing

$$\mathcal{R}(t, u) := \mathcal{R}(t) = \frac{d}{dt} V(t),$$

in a large class of controls.



# Greedy sparse control

---

## Theorem (Caponigro-Fornasier-Piccoli-Trélat)

For every initial condition  $(x_0, v_0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$  and  $M > 0$  there exist  $T > 0$  and a *sparse* control  $u : [0, T] \rightarrow (\mathbb{R}^d)^N$ , with  $\sum_{i=1}^N \|u_i(t)\| \leq M$  for every  $t \in [0, T]$  such that the associated solution tends to consensus.

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We can choose adaptively the control law explicitly as *one of the solutions* of the variational problem

$$\min B(v, u) + \frac{\mathcal{T}(x)}{N} \sum_{i=1}^N \|u_i\| \quad \text{subject to} \quad \sum_{i=1}^N \|u_i\| \leq M,$$

where

$$\mathcal{T}(x) = \int_{\sqrt{B(x,x)}}^{\infty} a(\sqrt{2Nr}) dr \quad (\text{threshold function by Ha-Ha-Kim})$$

# The 1-sparse feedback strategy

---

## Definition

For every  $t \geq 0$  we define the **1-sparse feedback control** as

$$u_i(t) = \begin{cases} -M \frac{v_i^\perp(t)}{\|v_i^\perp(t)\|} & \text{if } i = \hat{i}(t) \\ 0 & \text{if } i \neq \hat{i}(t) \end{cases}$$

where  $\hat{i}(t) \in \{1, \dots, N\}$  is the minimum index such that

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The control acts on the most “stubborn” agent. We may call it the “shepherd dog strategy”

# Dynamical systems driven by attraction and repulsion forces

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The **Cucker-Dong model**: for every  $1 \leq i \leq N$

$$\begin{cases} \dot{x}_i = v_i \in \mathbb{R}^d \\ \dot{v}_i = -b_i v_i + \sum_{j=1}^N a(\|x_i - x_j\|^2) (x_j - x_i) + \sum_{\substack{j=1 \\ j \neq i}}^N f(\|x_i - x_j\|^2) (x_i - x_j) \in \mathbb{R}^d \end{cases}$$

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- $b_i : [0, +\infty) \rightarrow [0, \Lambda]$  is the friction acting on the system,
- $a : [0, +\infty) \rightarrow [0, +\infty)$  is the rate of communication,
- $f : (0, +\infty) \rightarrow (0, +\infty)$  such that

$$\int_{\delta}^{+\infty} f(r) dr < \infty \text{ for every } \delta > 0, \quad \int_0^{+\infty} f(r) dr = +\infty$$

models the repulsion between agents.



## Example: Lennard-Jones potential

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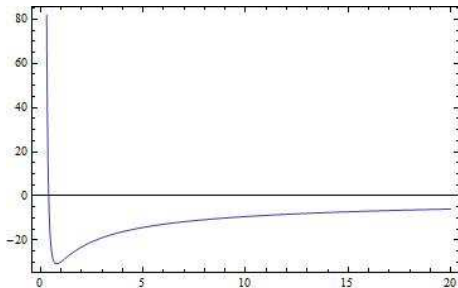
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*Difference  $f(r) - a(r)$  for Lennard-Jones potentials.*

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- the **total energy**  $E(t) := K(t) + P(t)$ .



# Conditional consensus emergence

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## Theorem (Cucker - Dong)

Consider a population of  $N$  agents modeled by a Cucker-Dong system with  $a(t) := a_\beta(t) = \frac{1}{(1+t^2)^\beta}$ ,  $\beta > 0$

$$\|x_i(0) - x_j(0)\| > 0 \text{ for all } i \neq j.$$

Then there exists a unique solution  $(x(t), v(t))$  of the system with initial state  $(x(0), v(0))$ . Moreover if one of the two following hypotheses holds:



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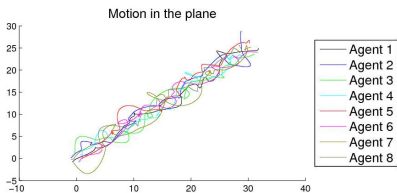
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then the population is *cohesive and collision-avoiding*, i.e., there exist two constants  $B_0$  and  $b_0 > 0$  such that, for every  $t \geq 0$

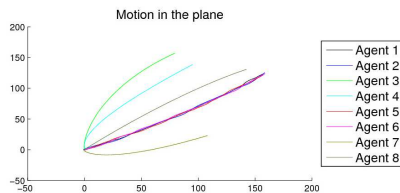
$$b_0 \leq \|x_i(t) - x_j(t)\| \leq B_0 \text{ for all } 1 \leq i \neq j \leq N.$$



# Non-consensus events need intervention

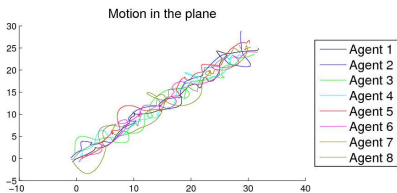


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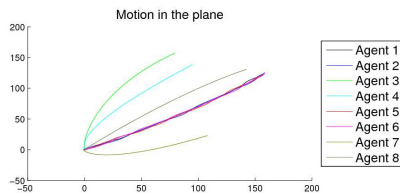


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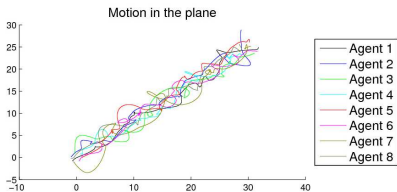
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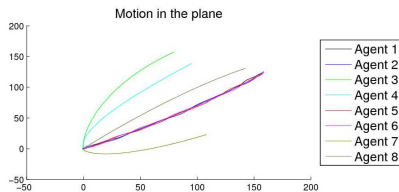
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- Assume we are in the case  $\beta > 1$  and  $E(0) \geq \vartheta$ . Can we again stabilize the society by external parsimonious intervention?
- We again introduce a control term inside the model

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under the sparsifying constraint  $\sum_{i=1}^N \|u_i(t)\| \leq M$  for  $M > 0$ .



# Sparse control strategy for the Cucker-Dong model

## Definition

For every  $t \geq 0$  we define the **1-sparse feedback control** as

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Define for every  $t > 0$  the set

$$K(t) := \left\{ u \in (\mathbb{R}^d)^N \mid \sum_{i=1}^N \|u_i\| \leq M \cdot \frac{E(t)}{E(0)} \right\},$$

and for every  $t > 0$  and  $q > 0$  the functional  $\mathcal{J}_{t,q} : (\mathbb{R}^d)^N \rightarrow \mathbb{R}$

$$\mathcal{J}_{t,q}(u) = \langle v(t), u \rangle + \frac{\left\| \frac{1}{N} \sum_{i=1}^N v_i(0) \right\|}{q} \sum_{i=1}^N \|u_i\|.$$

# Under the following hypotheses...

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## Theorem

Fix  $M > 0$ . Let  $(x_0, v_0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$  be such that the following hold:

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2.  $\|\frac{1}{N} \sum_{i=1}^N v_i(0)\| > 0$ ,
3.  $E(0) \geq \vartheta > E(0) \exp\left(-\frac{2\sqrt{3}}{9} \frac{M \|\frac{1}{N} \sum_{i=1}^N v_i(0)\|^3}{E(0)\sqrt{E(0)}\left(\Lambda\sqrt{E(0)} + \frac{M}{N}\right)}\right)$ .

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... we can prove the following.

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Theorem (continues...)

... then there exist  $T > 0$  and  $q > 0$  such that

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- there exists a solution of the controlled Cucker-Dong system associated to a control solving the variational problem

$$\tilde{u} \in \operatorname{argmin}_{u \in K(t)} \mathcal{J}_{t,q}(u) \text{ for every } t \leq T.$$

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 ⇒ we don't know if the conditions are necessary.

## A few info

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- **WWW:** <http://www-m15.ma.tum.de/>
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