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(Un)conditional consensus emergence under perturbed and decentralized feedback controls

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(Un)conditional consensus emergence under perturbed and decentralized feedback controls

(joint work with D. Kalise and M. Fornasier)



Mattia Bongini

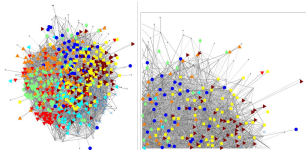
Technische Universität München,
Department of Mathematics,
Chair of Applied Numerical Analysis

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Netco 2014
Tours
June 23-27, 2014

Introduction

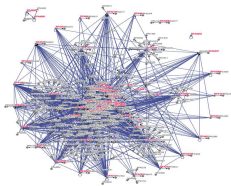
Large particle systems arise in many modern applications:



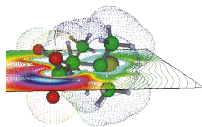
Large Facebook “friendship” network



Image halftoning via variational dithering



*Dynamical data analysis: *R. palustris*
protein-protein interaction network*



*Computational chemistry: molecule
simulation*

A framework for social dynamics

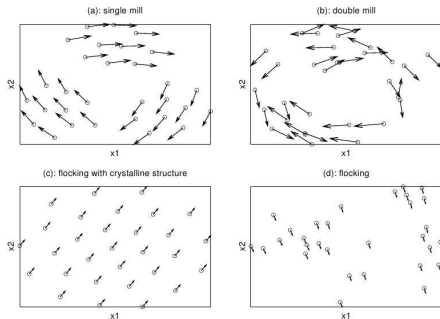
We consider large particle systems
of form:

$$\begin{cases} \dot{x}_i = v_i, \\ \dot{v}_i = (H \star \mu_N)(x_i, v_i), \quad i = 1, \dots, N, \end{cases}$$

where $\mu_N = \frac{1}{N} \sum_{j=1}^N \delta_{(x_j, v_j)}$.

Several “social forces” encoded in
the interaction kernel H :

- Repulsion-attraction
- Alignment
- ...



*Patterns related to different balance of social
forces.*

A framework for social dynamics

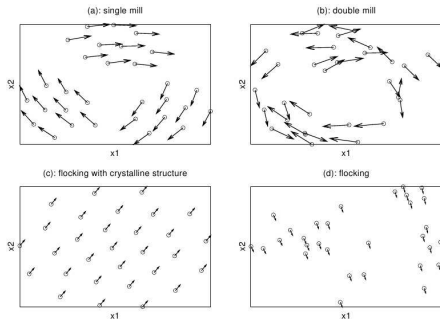
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*Patterns related to different balance of social
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Understanding how superposition of re-iterated binary “social forces” yields global self-organization.

Outline of the talk

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- we then explore a different strategy to enforce self-organization, namely **sparse control policies**, and compare it with the previous decentralized approach;
- finally, we shall see how sparse stabilization techniques can be exported successfully to different models.

A framework for consensus emergence

The **Cucker-Smale model** is obtained by the choice of the interaction kernel $H(x, v) = a(|x|)(-v)$.

$$\begin{cases} \dot{x}_i = v_i \in \mathbb{R}^d \\ \dot{v}_i = \frac{1}{N} \sum_{j=1}^N a\left(\|x_i - x_j\|^2\right) (v_j - v_i) \in \mathbb{R}^d, \text{ for } i = 1, \dots, N, \end{cases}$$

where $a(r) := a_\beta(r) = \frac{1}{(1+r^2)^\beta}$, $\beta > 0$ models the exchange of information between agents.

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We say that a solution $\{(x_i, v_i)\}_{i=1}^N$ **tends to consensus** if there exists a vector $w \in \mathbb{R}^d$ such that

$$\lim_{t \rightarrow +\infty} \|v_i(t) - w\| = 0 \quad \text{for every } i = 1, \dots, N.$$

Conditional consensus emergence for generic $a(\cdot)$

Consider the symmetric bilinear form

$$B(u, v) = \frac{1}{2N^2} \sum_{i,j} \langle u_i - u_j, v_i - v_j \rangle = \frac{1}{N} \sum_{i=1}^N \langle u_i, v_i \rangle - \langle \bar{u}, \bar{v} \rangle,$$

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$$X(t) = B(x(t), x(t)), \quad V(t) = B(v(t), v(t)).$$

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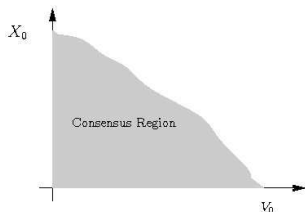
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Theorem (Ha-Ha-Kim)

Let $(x_0, v_0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$ be such that $X_0 = B(x_0, x_0)$ and $V_0 = B(v_0, v_0)$ satisfy

$$\int_{\sqrt{X_0}}^{\infty} a(\sqrt{2Nr}) dr \geq \sqrt{V_0}.$$

Then the solution with initial data (x_0, v_0) tends to consensus.



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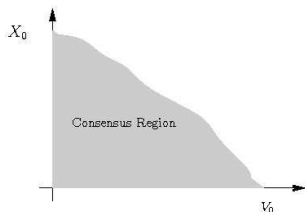
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Then the solution with initial data (x_0, v_0) tends to consensus.

Hence for $a(t) := a_{\beta}(t)$ we have unconditional convergence for $0 \leq \beta \leq \frac{1}{2}$.



Non-consensus events

If $a(\cdot) = a_\beta(\cdot)$ and $\beta > 1/2$ then the consensus condition is **not** satisfied by all $(x_0, v_0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$, as shown by Cucker and Smale.

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Consider $\beta = 1$, $N = 2$, $d = 1$ and let $x(t) = x_1(t) - x_2(t)$ and $v(t) = v_1(t) - v_2(t)$ be the relative positions and velocities of the two agents on the line: rewrite the system as

$$\begin{cases} \dot{x} = v \\ \dot{v} = -\frac{v}{1+x^2} \end{cases}$$

with initial conditions $x(0) = x_0$ and $v(0) = v_0 > 0$.

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By direct integration

$$v(t) = -\arctan x(t) + \arctan x_0 + v_0.$$

Hence, **if $\arctan x_0 + v_0 > \pi/2 + \varepsilon$ we have**

$$v(t) > \pi/2 + \varepsilon - \arctan x(t) > \varepsilon, \quad \forall t \in \mathbb{R}_+.$$



A possible way out...

However, in the work Caponigro-Fornasier-Piccoli-Trélat it is shown that the addition of an extra term of the form

$$\begin{cases} \dot{x}_i = v_i \\ \dot{v}_i = \frac{1}{N} \sum_{j=1}^N a(\|x_i - x_j\|^2) (v_j - v_i) + \gamma(\bar{v} - v_i), \text{ for } i = 1, \dots, N, \end{cases}$$

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where $\gamma > 0$, implies convergence to consensus unconditionally. This happens because each agent has an **additional** stirring to reach the **common** goal \bar{v} .

But what happens if every agent has its own “local” goal?

Introducing decentralized controls

We study how the convergence to consensus is modified by the introduction of the following extra term

$$\begin{cases} \dot{x}_i = v_i \\ \dot{v}_i = \frac{1}{N} \sum_{j=1}^N a(\|x_i - x_j\|^2) (v_j - v_i) + \gamma(\bar{v}_i^R - v_i), \text{ for } i = 1, \dots, N, \end{cases}$$

where $\gamma > 0$ and

$$\bar{v}_i^R = \frac{1}{\sum_{k=1}^N \chi_{[0,R]}(\|x_i - x_k\|)} \sum_{j=1}^N \chi_{[0,R]}(\|x_i - x_j\|) v_j,$$

is the local mean velocity calculated by the i -th agent taking into account only the agents inside a ball of radius R centered on him.

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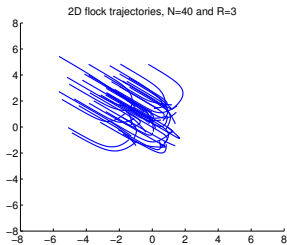
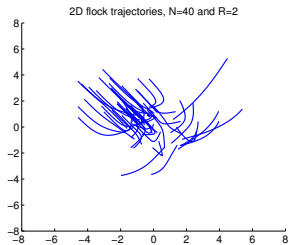
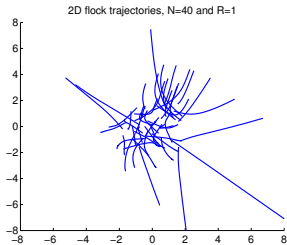
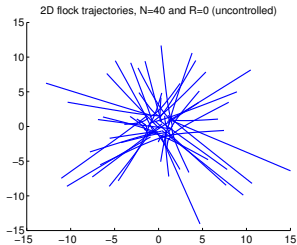
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GOAL: Study the behavior of the system for $R \rightarrow 0$ and $R \rightarrow +\infty$.

Behavior of the modified system in function of R



A different approach

To avoid the issues coming from a normalizing factor varying from agent to agent, in our study of the asymptotic behavior we replace it with a function $\eta_R(t) \in L^\infty([0, +\infty))$ satisfying

$$\lim_{R \rightarrow 0} \eta_R(t) = 1 \left(= \lim_{R \rightarrow 0} \sum_{k=1}^N \chi_{[0, R]}(\|x_i(t) - x_k(t)\|) \text{ for every } i \right),$$

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Hence, we replace \bar{v}_i^R in the system with

$$\hat{v}_i^R = \frac{1}{\eta_R} \sum_{k=1}^N \chi_{[0, R]}(\|x_i - x_k\|) v_j.$$

A sufficient condition

Theorem

Replace \bar{v}_i^R with \hat{v}_i^R in the model. Let $(x_0, v_0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$ and define $X_0 = B(x_0, x_0)$ and $V_0 = B(v_0, v_0)$. Then the solution with initial data (x_0, v_0) tends to consensus if one of the following is satisfied:

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- $\sqrt{2NX_0} \leq R$ and $\int_{\sqrt{X_0}}^{\infty} a(\sqrt{2Nr})dr + \gamma \left(\frac{R}{\sqrt{2N}} - \sqrt{X_0} \right) \geq \sqrt{V_0}$;

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Proof.

via the Ha-Ha-Kim Theorem. □

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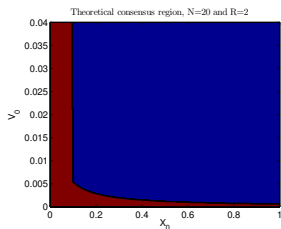
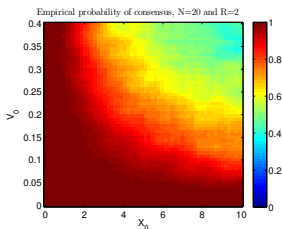
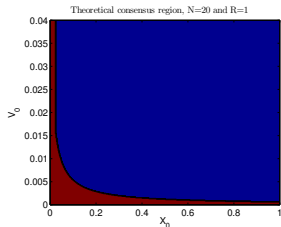
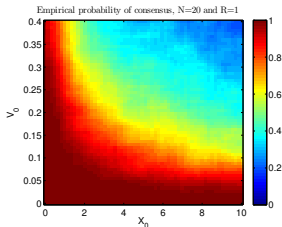
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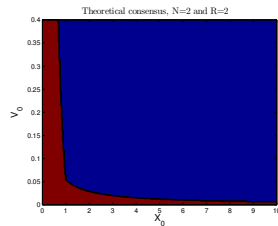
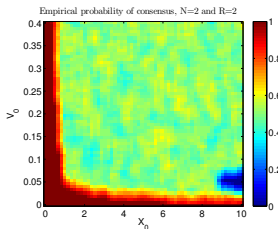
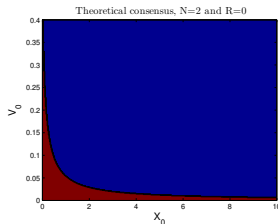
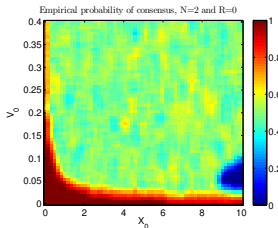
via the Ha-Ha-Kim Theorem. □

- Result generalizable to different forms of η_R and $\chi_{[0,R]}$.

Variation of the empirical and theoretical consensus region in function of R



Sharpness of the result for $N = 2$



Efficiency of stabilization

- We have shown that an unconditional (w.r.t. the initial data) convergence to consensus is guaranteed only in the case $R = \infty$, i.e., the agents are completely “open-minded”.
In this case, the Lyapunov functional V decreases at least as

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- **Question: are there more economical and efficient strategies which guarantees unconditional stabilization?**
- More precisely, we wish to make the quantity $\frac{d}{dt}V(t)$ the smallest possible and use the minimal amount of intervention (e.g., by imposing sparsifying constraints).

Self-organization vs organization by intervention

We introduce a control term in the Cucker-Smale model:

$$\begin{cases} \dot{x}_i = v_i \\ \dot{v}_i = \frac{1}{N} \sum_{j=1}^N a(\|x_j - x_i\|)(v_j - v_i) + u_i, \text{ for } i = 1, \dots, N. \end{cases}$$

The admissible controls will be measurable functions $u = (u_1, \dots, u_N) : [0, +\infty) \rightarrow \mathbb{R}^N$ such that $\sum_{i=1}^N \|u_i(t)\| \leq M$ for every $t > 0$, for a given constant M .

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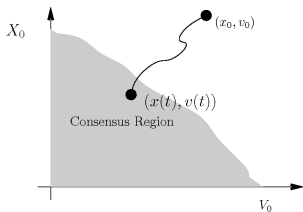
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Our aim is then to find admissible controls steering the system to the consensus region and minimizing

$$\mathcal{R}(t, u) := \mathcal{R}(t) = \frac{d}{dt} V(t),$$

in a large class of controls.



Greedy sparse control

Theorem (Caponigro-Fornasier-Piccoli-Trélat)

For every initial condition $(x_0, v_0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$ and $M > 0$ there exist $T > 0$ and a *sparse* control $u : [0, T] \rightarrow (\mathbb{R}^d)^N$, with $\sum_{i=1}^N \|u_i(t)\| \leq M$ for every $t \in [0, T]$ such that the associated solution tends to consensus.

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We can choose adaptively the control law explicitly as *one of the solutions* of the variational problem

$$\min B(v, u) + \frac{\mathcal{T}(x)}{N} \sum_{i=1}^N \|u_i\| \quad \text{subject to} \quad \sum_{i=1}^N \|u_i\| \leq M,$$

where

$$\mathcal{T}(x) = \int_{\sqrt{B(x,x)}}^{\infty} a(\sqrt{2Nr}) dr \quad (\text{threshold function by Ha-Ha-Kim})$$

The 1-sparse feedback strategy

Definition

For every $t \geq 0$ we define the **1-sparse feedback control** as

$$u_i(t) = \begin{cases} -M \frac{v_i^\perp(t)}{\|v_i^\perp(t)\|} & \text{if } i = \hat{i}(t) \\ 0 & \text{if } i \neq \hat{i}(t) \end{cases}$$

where $\hat{i}(t) \in \{1, \dots, N\}$ is the minimum index such that

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The control acts on the most “stubborn” agent. We may call it the “shepherd dog strategy”

Dynamical systems driven by attraction and repulsion forces

The **Cucker-Dong model**: for every $1 \leq i \leq N$

$$\begin{cases} \dot{x}_i = v_i \in \mathbb{R}^d \\ \dot{v}_i = -b_i v_i + \sum_{j=1}^N a(\|x_i - x_j\|^2) (x_j - x_i) + \sum_{\substack{j=1 \\ j \neq i}}^N f(\|x_i - x_j\|^2) (x_i - x_j) \in \mathbb{R}^d \end{cases}$$

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- $f : (0, +\infty) \rightarrow (0, +\infty)$ such that

$$\int_{\delta}^{+\infty} f(r) dr < \infty \text{ for every } \delta > 0, \quad \int_0^{+\infty} f(r) dr = +\infty$$

models the repulsion between agents.



Example: Lennard-Jones potential

- It is the potential of the Van der Waals force.

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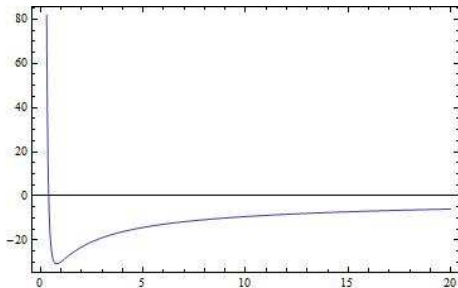
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Difference $f(r) - a(r)$ for Lennard-Jones potentials.

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We introduce

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- the **total energy** $E(t) := K(t) + P(t)$.



Conditional consensus emergence

Theorem (Cucker - Dong)

Consider a population of N agents modeled by a Cucker-Dong system with $a(t) := a_\beta(t) = \frac{1}{(1+t^2)^\beta}$, $\beta > 0$

$$\|x_i(0) - x_j(0)\| > 0 \text{ for all } i \neq j.$$

Then there exists a unique solution $(x(t), v(t))$ of the system with initial state $(x(0), v(0))$. Moreover if one of the two following hypotheses holds:



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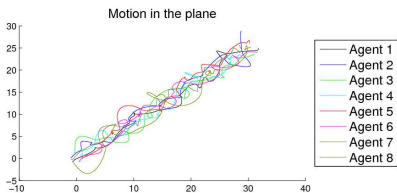
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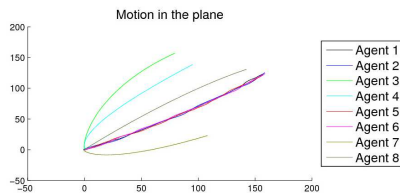
then the population is *cohesive and collision-avoiding*, i.e., there exist two constants B_0 and $b_0 > 0$ such that, for every $t \geq 0$

$$b_0 \leq \|x_i(t) - x_j(t)\| \leq B_0 \text{ for all } 1 \leq i \neq j \leq N.$$

Non-consensus events need intervention

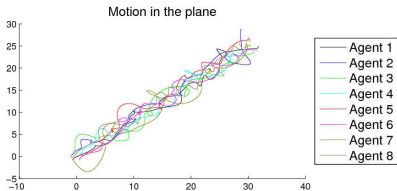


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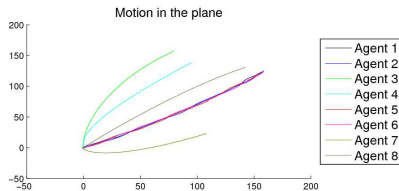


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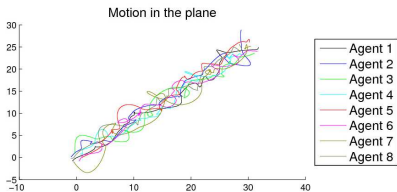
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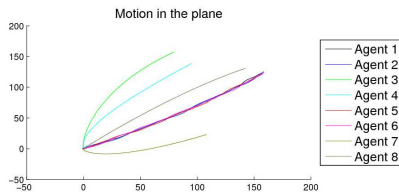
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- Assume we are in the case $\beta > 1$ and $E(0) \geq \vartheta$. Can we again stabilize the society by external parsimonious intervention?

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- Assume we are in the case $\beta > 1$ and $E(0) \geq \vartheta$. Can we again stabilize the society by external parsimonious intervention?
- We again introduce a control term inside the model

$$\begin{cases} \dot{x}_i = v_i \\ \dot{v}_i = -b_i v_i + \sum_{j=1}^N a(\|x_i - x_j\|^2) (x_j - x_i) + \sum_{\substack{j=1 \\ j \neq i}}^N f(\|x_i - x_j\|^2) (x_i - x_j) + u_i \end{cases}$$

under the sparsifying constraint $\sum_{i=1}^N \|u_i(t)\| \leq M$ for $M > 0$.



Sparse control strategy for the Cucker-Dong model

Definition

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Define for every $t > 0$ the set

$$K(t) := \left\{ u \in (\mathbb{R}^d)^N \mid \sum_{i=1}^N \|u_i\| \leq M \cdot \frac{E(t)}{E(0)} \right\},$$

and for every $t > 0$ and $q > 0$ the functional $\mathcal{J}_{t,q} : (\mathbb{R}^d)^N \rightarrow \mathbb{R}$

$$\mathcal{J}_{t,q}(u) = \langle v(t), u \rangle + \frac{\left\| \frac{1}{N} \sum_{i=1}^N v_i(0) \right\|}{q} \sum_{i=1}^N \|u_i\|.$$

Under the following hypotheses...

Theorem

Fix $M > 0$. Let $(x_0, v_0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$ be such that the following hold:

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2. $\|\frac{1}{N} \sum_{i=1}^N v_i(0)\| > 0$,
3. $E(0) \geq \vartheta > E(0) \exp\left(-\frac{2\sqrt{3}}{9} \frac{M \|\frac{1}{N} \sum_{i=1}^N v_i(0)\|^3}{E(0)\sqrt{E(0)}\left(\Lambda\sqrt{E(0)} + \frac{M}{N}\right)}\right)$.

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... we can prove the following.

Theorem (continues...)

... then there exist $T > 0$ and $q > 0$ such that

- the 1-sparse feedback control belongs to the set $\operatorname{argmin}_{u \in K(t)} \mathcal{J}_{t,q}(u)$ for every $t \leq T$;

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- there exists a solution of the controlled Cucker-Dong system associated to a control solving the variational problem

$$\tilde{u} \in \operatorname{argmin}_{u \in K(t)} \mathcal{J}_{t,q}(u) \text{ for every } t \leq T.$$

Summing up our results

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 ⇒ we don't know if the conditions are necessary.

A few info

- **WWW:** <http://www-m15.ma.tum.de/>
- **References:**
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