



HAL
open science

(Un)conditional consensus emergence under perturbed and decentralized feedback controls

Mattia Bongini, Dante Kalise, Massimo Fornasier

► **To cite this version:**

Mattia Bongini, Dante Kalise, Massimo Fornasier. (Un)conditional consensus emergence under perturbed and decentralized feedback controls. NETCO 2014 - New Trends in Optimal Control, Jun 2014, Tours, France. hal-01030745

HAL Id: hal-01030745

<https://inria.hal.science/hal-01030745>

Submitted on 22 Jul 2014

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

(Un)conditional consensus emergence under perturbed and decentralized feedback controls

(joint work with D. Kalise and M. Fornasier)



Mattia Bongini

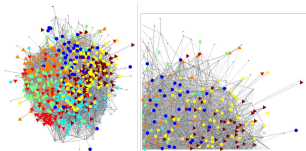
Technische Universität München,
Department of Mathematics,
Chair of Applied Numerical Analysis

mattia.bongini@ma.tum.de

Netco 2014
Tours
June 23-27, 2014

Introduction

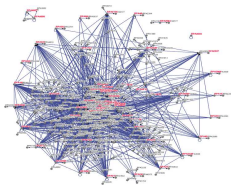
Large particle systems arise in many modern applications:



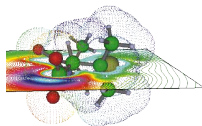
Large Facebook “friendship” network



Image halftoning via variational dithering



*Dynamical data analysis: *R. palustris*
protein-protein interaction network*



*Computational chemistry: molecule
simulation*

A framework for social dynamics

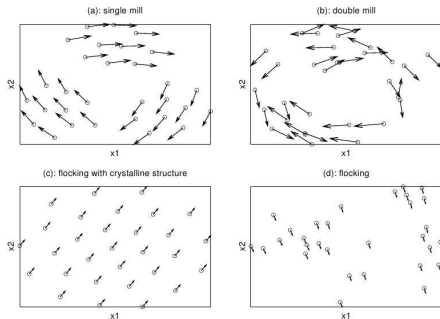
We consider large particle systems
of form:

$$\begin{cases} \dot{x}_i = v_i, \\ \dot{v}_i = (H \star \mu_N)(x_i, v_i), \quad i = 1, \dots, N, \end{cases}$$

where $\mu_N = \frac{1}{N} \sum_{j=1}^N \delta_{(x_j, v_j)}$.

Several “social forces” encoded in
the interaction kernel H :

- Repulsion-attraction
- Alignment
- ...



*Patterns related to different balance of social
forces.*

A framework for social dynamics

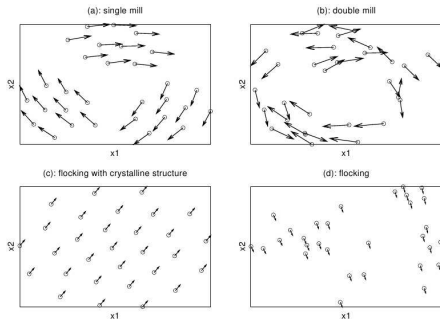
We consider large particle systems
of form:

$$\begin{cases} \dot{x}_i = v_i, \\ \dot{v}_i = (H \star \mu_N)(x_i, v_i), \quad i = 1, \dots, N, \end{cases}$$

where $\mu_N = \frac{1}{N} \sum_{j=1}^N \delta_{(x_j, v_j)}$.

Several “social forces” encoded in
the interaction kernel H :

- Repulsion-attraction
- Alignment
- ...



*Patterns related to different balance of social
forces.*

Understanding how superposition of re-iterated binary “social forces” yields global self-organization.

Outline of the talk

- Self-organization in *free* social interactions can fail due to an excess of narrow-mindedness of the social agents;

Outline of the talk

- Self-organization in *free* social interactions can fail due to an excess of narrow-mindedness of the social agents;
- we shall propose a **model of narrow-mindedness** and show a quantitative link between the degree of narrow-mindedness in the society and its convergence to global patterns;

Outline of the talk

- Self-organization in *free* social interactions can fail due to an excess of narrow-mindedness of the social agents;
- we shall propose a **model of narrow-mindedness** and show a quantitative link between the degree of narrow-mindedness in the society and its convergence to global patterns;
- we then explore a different strategy to enforce self-organization, namely **sparse control policies**, and compare it with the previous decentralized approach;

Outline of the talk

- Self-organization in *free* social interactions can fail due to an excess of narrow-mindedness of the social agents;
- we shall propose a **model of narrow-mindedness** and show a quantitative link between the degree of narrow-mindedness in the society and its convergence to global patterns;
- we then explore a different strategy to enforce self-organization, namely **sparse control policies**, and compare it with the previous decentralized approach;
- finally, we shall see how sparse stabilization techniques can be exported successfully to different models.

A framework for consensus emergence

The **Cucker-Smale model** is obtained by the choice of the interaction kernel $H(x, v) = a(|x|)(-v)$.

$$\begin{cases} \dot{x}_i = v_i \in \mathbb{R}^d \\ \dot{v}_i = \frac{1}{N} \sum_{j=1}^N a\left(\|x_i - x_j\|^2\right) (v_j - v_i) \in \mathbb{R}^d, \text{ for } i = 1, \dots, N, \end{cases}$$

where $a(r) := a_\beta(r) = \frac{1}{(1+r^2)^\beta}$, $\beta > 0$ models the exchange of information between agents.

A framework for consensus emergence

The **Cucker-Smale model** is obtained by the choice of the interaction kernel $H(x, v) = a(|x|)(-v)$.

$$\begin{cases} \dot{x}_i = v_i \in \mathbb{R}^d \\ \dot{v}_i = \frac{1}{N} \sum_{j=1}^N a\left(\|x_i - x_j\|^2\right) (v_j - v_i) \in \mathbb{R}^d, \text{ for } i = 1, \dots, N, \end{cases}$$

where $a(r) := a_\beta(r) = \frac{1}{(1+r^2)^\beta}$, $\beta > 0$ models the exchange of information between agents.

We say that a solution $\{(x_i, v_i)\}_{i=1}^N$ **tends to consensus** if there exists a vector $w \in \mathbb{R}^d$ such that

$$\lim_{t \rightarrow +\infty} \|v_i(t) - w\| = 0 \quad \text{for every } i = 1, \dots, N.$$

Conditional consensus emergence for generic $a(\cdot)$

Consider the symmetric bilinear form

$$B(u, v) = \frac{1}{2N^2} \sum_{i,j} \langle u_i - u_j, v_i - v_j \rangle = \frac{1}{N} \sum_{i=1}^N \langle u_i, v_i \rangle - \langle \bar{u}, \bar{v} \rangle,$$

and

$$X(t) = B(x(t), x(t)), \quad V(t) = B(v(t), v(t)).$$

Conditional consensus emergence for generic $a(\cdot)$

Consider the symmetric bilinear form

$$B(u, v) = \frac{1}{2N^2} \sum_{i,j} \langle u_i - u_j, v_i - v_j \rangle = \frac{1}{N} \sum_{i=1}^N \langle u_i, v_i \rangle - \langle \bar{u}, \bar{v} \rangle,$$

and

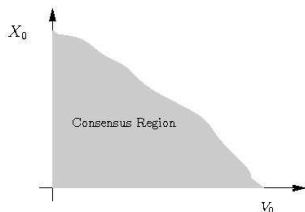
$$X(t) = B(x(t), x(t)), \quad V(t) = B(v(t), v(t)).$$

Theorem (Ha-Ha-Kim)

Let $(x_0, v_0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$ be such that $X_0 = B(x_0, x_0)$ and $V_0 = B(v_0, v_0)$ satisfy

$$\int_{\sqrt{X_0}}^{\infty} a(\sqrt{2Nr}) dr \geq \sqrt{V_0}.$$

Then the solution with initial data (x_0, v_0) tends to consensus.



Conditional consensus emergence for generic $a(\cdot)$

Consider the symmetric bilinear form

$$B(u, v) = \frac{1}{2N^2} \sum_{i,j} \langle u_i - u_j, v_i - v_j \rangle = \frac{1}{N} \sum_{i=1}^N \langle u_i, v_i \rangle - \langle \bar{u}, \bar{v} \rangle,$$

and

$$X(t) = B(x(t), x(t)), \quad V(t) = B(v(t), v(t)).$$

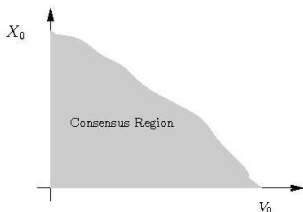
Theorem (Ha-Ha-Kim)

Let $(x_0, v_0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$ be such that $X_0 = B(x_0, x_0)$ and $V_0 = B(v_0, v_0)$ satisfy

$$\int_{\sqrt{X_0}}^{\infty} a(\sqrt{2Nr}) dr \geq \sqrt{V_0}.$$

Then the solution with initial data (x_0, v_0) tends to consensus.

Hence for $a(t) := a_{\beta}(t)$ we have unconditional convergence for $0 \leq \beta \leq \frac{1}{2}$.





Non-consensus events

If $a(\cdot) = a_\beta(\cdot)$ and $\beta > 1/2$ then the consensus condition is **not** satisfied by all $(x_0, v_0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$, as shown by Cucker and Smale.

Non-consensus events

If $a(\cdot) = a_\beta(\cdot)$ and $\beta > 1/2$ then the consensus condition is **not** satisfied by all $(x_0, v_0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$, as shown by Cucker and Smale.

Consider $\beta = 1$, $N = 2$, $d = 1$ and let $x(t) = x_1(t) - x_2(t)$ and $v(t) = v_1(t) - v_2(t)$ be the relative positions and velocities of the two agents on the line: rewrite the system as

$$\begin{cases} \dot{x} = v \\ \dot{v} = -\frac{v}{1+x^2} \end{cases}$$

with initial conditions $x(0) = x_0$ and $v(0) = v_0 > 0$.

Non-consensus events

If $a(\cdot) = a_\beta(\cdot)$ and $\beta > 1/2$ then the consensus condition is **not** satisfied by all $(x_0, v_0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$, as shown by Cucker and Smale.

Consider $\beta = 1$, $N = 2$, $d = 1$ and let $x(t) = x_1(t) - x_2(t)$ and $v(t) = v_1(t) - v_2(t)$ be the relative positions and velocities of the two agents on the line: rewrite the system as

$$\begin{cases} \dot{x} = v \\ \dot{v} = -\frac{v}{1+x^2} \end{cases}$$

with initial conditions $x(0) = x_0$ and $v(0) = v_0 > 0$.

By direct integration

$$v(t) = -\arctan x(t) + \arctan x_0 + v_0.$$

Hence, **if $\arctan x_0 + v_0 > \pi/2 + \varepsilon$ we have**

$$v(t) > \pi/2 + \varepsilon - \arctan x(t) > \varepsilon, \quad \forall t \in \mathbb{R}_+.$$



A possible way out...

However, in the work Caponigro-Fornasier-Piccoli-Trélat it is shown that the addition of an extra term of the form

$$\begin{cases} \dot{x}_i = v_i \\ \dot{v}_i = \frac{1}{N} \sum_{j=1}^N a(\|x_i - x_j\|^2) (v_j - v_i) + \gamma(\bar{v} - v_i), \text{ for } i = 1, \dots, N, \end{cases}$$

where $\gamma > 0$, implies convergence to consensus unconditionally.



A possible way out...

However, in the work Caponigro-Fornasier-Piccoli-Trélat it is shown that the addition of an extra term of the form

$$\begin{cases} \dot{x}_i = v_i \\ \dot{v}_i = \frac{1}{N} \sum_{j=1}^N a(\|x_i - x_j\|^2) (v_j - v_i) + \gamma(\bar{v} - v_i), \text{ for } i = 1, \dots, N, \end{cases}$$

where $\gamma > 0$, implies convergence to consensus unconditionally. This happens because each agent has an **additional** stirring to reach the **common** goal \bar{v} .

But what happens if every agent has its own “local” goal?

Introducing decentralized controls

We study how the convergence to consensus is modified by the introduction of the following extra term

$$\begin{cases} \dot{x}_i = v_i \\ \dot{v}_i = \frac{1}{N} \sum_{j=1}^N a(\|x_i - x_j\|^2) (v_j - v_i) + \gamma(\bar{v}_i^R - v_i), \text{ for } i = 1, \dots, N, \end{cases}$$

where $\gamma > 0$ and

$$\bar{v}_i^R = \frac{1}{\sum_{k=1}^N \chi_{[0,R]}(\|x_i - x_k\|)} \sum_{j=1}^N \chi_{[0,R]}(\|x_i - x_j\|) v_j,$$

is the local mean velocity calculated by the i -th agent taking into account only the agents inside a ball of radius R centered on him.

Introducing decentralized controls

We study how the convergence to consensus is modified by the introduction of the following extra term

$$\begin{cases} \dot{x}_i = v_i \\ \dot{v}_i = \frac{1}{N} \sum_{j=1}^N a(\|x_i - x_j\|^2) (v_j - v_i) + \gamma(\bar{v}_i^R - v_i), \text{ for } i = 1, \dots, N, \end{cases}$$

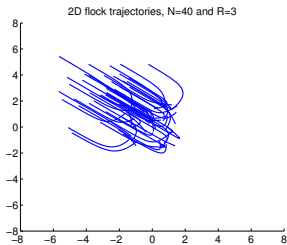
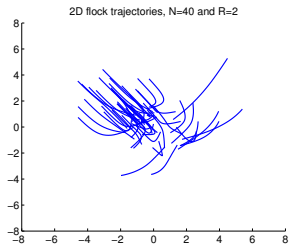
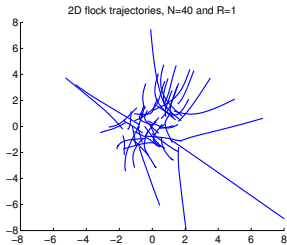
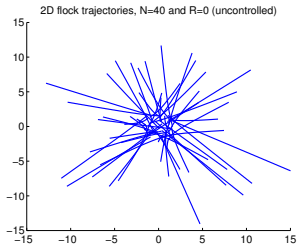
where $\gamma > 0$ and

$$\bar{v}_i^R = \frac{1}{\sum_{k=1}^N \chi_{[0,R]}(\|x_i - x_k\|)} \sum_{j=1}^N \chi_{[0,R]}(\|x_i - x_j\|) v_j,$$

is the local mean velocity calculated by the i -th agent taking into account only the agents inside a ball of radius R centered on him.

GOAL: Study the behavior of the system for $R \rightarrow 0$ and $R \rightarrow +\infty$.

Behavior of the modified system in function of R



A different approach

To avoid the issues coming from a normalizing factor varying from agent to agent, in our study of the asymptotic behavior we replace it with a function $\eta_R(t) \in L^\infty([0, +\infty))$ satisfying

$$\begin{aligned}\lim_{R \rightarrow 0} \eta_R(t) &= 1 \left(= \lim_{R \rightarrow 0} \sum_{k=1}^N \chi_{[0, R]}(\|x_i(t) - x_k(t)\|) \text{ for every } i \right), \\ \lim_{R \rightarrow +\infty} \eta_R(t) &= N \left(= \lim_{R \rightarrow +\infty} \sum_{k=1}^N \chi_{[0, R]}(\|x_i(t) - x_k(t)\|) \text{ for every } i \right),\end{aligned}$$



A different approach

To avoid the issues coming from a normalizing factor varying from agent to agent, in our study of the asymptotic behavior we replace it with a function $\eta_R(t) \in L^\infty([0, +\infty))$ satisfying

$$\lim_{R \rightarrow 0} \eta_R(t) = 1 \left(= \lim_{R \rightarrow 0} \sum_{k=1}^N \chi_{[0, R]}(\|x_i(t) - x_k(t)\|) \text{ for every } i \right),$$

$$\lim_{R \rightarrow +\infty} \eta_R(t) = N \left(= \lim_{R \rightarrow +\infty} \sum_{k=1}^N \chi_{[0, R]}(\|x_i(t) - x_k(t)\|) \text{ for every } i \right),$$

like

$$\eta_R(t) = \max_{i=1, \dots, N} \left\{ \sum_{k=1}^N \chi_{[0, R]}(\|x_i(t) - x_k(t)\|) \right\}.$$

A different approach

To avoid the issues coming from a normalizing factor varying from agent to agent, in our study of the asymptotic behavior we replace it with a function $\eta_R(t) \in L^\infty([0, +\infty))$ satisfying

$$\lim_{R \rightarrow 0} \eta_R(t) = 1 \left(= \lim_{R \rightarrow 0} \sum_{k=1}^N \chi_{[0, R]}(\|x_i(t) - x_k(t)\|) \text{ for every } i \right),$$

$$\lim_{R \rightarrow +\infty} \eta_R(t) = N \left(= \lim_{R \rightarrow +\infty} \sum_{k=1}^N \chi_{[0, R]}(\|x_i(t) - x_k(t)\|) \text{ for every } i \right),$$

like

$$\eta_R(t) = \max_{i=1, \dots, N} \left\{ \sum_{k=1}^N \chi_{[0, R]}(\|x_i(t) - x_k(t)\|) \right\}.$$

Hence, we replace \bar{v}_i^R in the system with

$$\hat{v}_i^R = \frac{1}{\eta_R} \sum_{k=1}^N \chi_{[0, R]}(\|x_i - x_k\|) v_j.$$

A sufficient condition

Theorem

Replace \bar{v}_i^R with \hat{v}_i^R in the model. Let $(x_0, v_0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$ and define $X_0 = B(x_0, x_0)$ and $V_0 = B(v_0, v_0)$. Then the solution with initial data (x_0, v_0) tends to consensus if one of the following is satisfied:

A sufficient condition

Theorem

Replace \bar{v}_i^R with \hat{v}_i^R in the model. Let $(x_0, v_0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$ and define $X_0 = B(x_0, x_0)$ and $V_0 = B(v_0, v_0)$. Then the solution with initial data (x_0, v_0) tends to consensus if one of the following is satisfied:

- $\sqrt{2NX_0} \leq R$ and $\int_{\sqrt{X_0}}^{\infty} a(\sqrt{2Nr})dr + \gamma \left(\frac{R}{\sqrt{2N}} - \sqrt{X_0} \right) \geq \sqrt{V_0}$;

A sufficient condition

Theorem

Replace \bar{v}_i^R with \hat{v}_i^R in the model. Let $(x_0, v_0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$ and define $X_0 = B(x_0, x_0)$ and $V_0 = B(v_0, v_0)$. Then the solution with initial data (x_0, v_0) tends to consensus if one of the following is satisfied:

- $\sqrt{2NX_0} \leq R$ and $\int_{\sqrt{X_0}}^{\infty} a(\sqrt{2Nr})dr + \gamma \left(\frac{R}{\sqrt{2N}} - \sqrt{X_0} \right) \geq \sqrt{V_0}$;
- $\sqrt{2NX_0} > R$ and $\int_{\sqrt{X_0}}^{\infty} a(\sqrt{2Nr})dr \geq \sqrt{V_0}$.

A sufficient condition

Theorem

Replace \bar{v}_i^R with \hat{v}_i^R in the model. Let $(x_0, v_0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$ and define $X_0 = B(x_0, x_0)$ and $V_0 = B(v_0, v_0)$. Then the solution with initial data (x_0, v_0) tends to consensus if one of the following is satisfied:

- $\sqrt{2NX_0} \leq R$ and $\int_{\sqrt{X_0}}^{\infty} a(\sqrt{2Nr})dr + \gamma \left(\frac{R}{\sqrt{2N}} - \sqrt{X_0} \right) \geq \sqrt{V_0}$;
- $\sqrt{2NX_0} > R$ and $\int_{\sqrt{X_0}}^{\infty} a(\sqrt{2Nr})dr \geq \sqrt{V_0}$.

Proof.

via the Ha-Ha-Kim Theorem. □

A sufficient condition

Theorem

Replace \bar{v}_i^R with \hat{v}_i^R in the model. Let $(x_0, v_0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$ and define $X_0 = B(x_0, x_0)$ and $V_0 = B(v_0, v_0)$. Then the solution with initial data (x_0, v_0) tends to consensus if one of the following is satisfied:

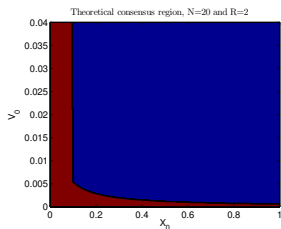
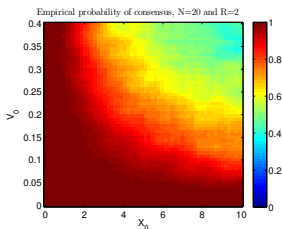
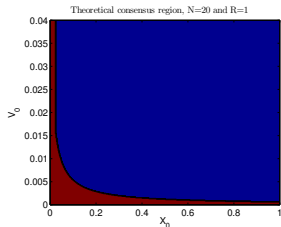
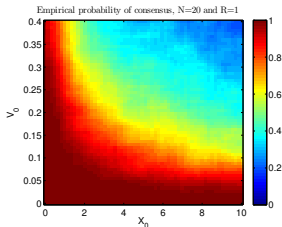
- $\sqrt{2NX_0} \leq R$ and $\int_{\sqrt{X_0}}^{\infty} a(\sqrt{2Nr})dr + \gamma \left(\frac{R}{\sqrt{2N}} - \sqrt{X_0} \right) \geq \sqrt{V_0}$;
- $\sqrt{2NX_0} > R$ and $\int_{\sqrt{X_0}}^{\infty} a(\sqrt{2Nr})dr \geq \sqrt{V_0}$.

Proof.

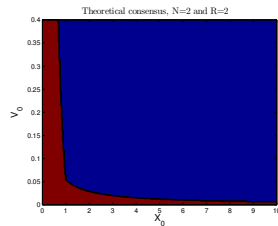
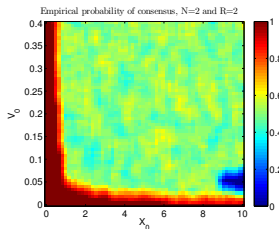
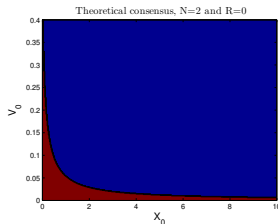
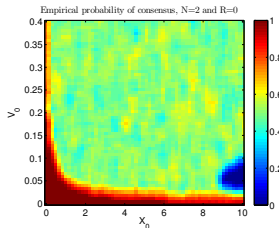
via the Ha-Ha-Kim Theorem. □

- Result generalizable to different forms of η_R and $\chi_{[0,R]}$.

Variation of the empirical and theoretical consensus region in function of R



Sharpness of the result for $N = 2$



Efficiency of stabilization

- We have shown that an unconditional (w.r.t. the initial data) convergence to consensus is guaranteed only in the case $R = \infty$, i.e., the agents are completely “open-minded”.
In this case, the Lyapunov functional V decreases at least as

$$V(t) \leq V(0)e^{\gamma t}.$$

Efficiency of stabilization

- We have shown that an unconditional (w.r.t. the initial data) convergence to consensus is guaranteed only in the case $R = \infty$, i.e., the agents are completely “open-minded”.
In this case, the Lyapunov functional V decreases at least as

$$V(t) \leq V(0)e^{\gamma t}.$$

- Question: are there more economical and efficient strategies which guarantees unconditional stabilization?

Efficiency of stabilization

- We have shown that an unconditional (w.r.t. the initial data) convergence to consensus is guaranteed only in the case $R = \infty$, i.e., the agents are completely “open-minded”.
In this case, the Lyapunov functional V decreases at least as

$$V(t) \leq V(0)e^{\gamma t}.$$

- **Question: are there more economical and efficient strategies which guarantees unconditional stabilization?**
- More precisely, we wish to make the quantity $\frac{d}{dt}V(t)$ the smallest possible and use the minimal amount of intervention (e.g., by imposing sparsifying constraints).

Self-organization vs organization by intervention

We introduce a control term in the Cucker-Smale model:

$$\begin{cases} \dot{x}_i = v_i \\ \dot{v}_i = \frac{1}{N} \sum_{j=1}^N a(\|x_j - x_i\|)(v_j - v_i) + u_i, \text{ for } i = 1, \dots, N. \end{cases}$$

The admissible controls will be measurable functions $u = (u_1, \dots, u_N) : [0, +\infty) \rightarrow \mathbb{R}^N$ such that $\sum_{i=1}^N \|u_i(t)\| \leq M$ for every $t > 0$, for a given constant M .

Self-organization vs organization by intervention

We introduce a control term in the Cucker-Smale model:

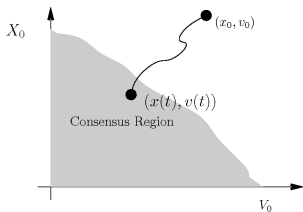
$$\begin{cases} \dot{x}_i = v_i \\ \dot{v}_i = \frac{1}{N} \sum_{j=1}^N a(\|x_j - x_i\|)(v_j - v_i) + u_i, \text{ for } i = 1, \dots, N. \end{cases}$$

The admissible controls will be measurable functions $u = (u_1, \dots, u_N) : [0, +\infty) \rightarrow \mathbb{R}^N$ such that $\sum_{i=1}^N \|u_i(t)\| \leq M$ for every $t > 0$, for a given constant M .

Our aim is then to find admissible controls steering the system to the consensus region and minimizing

$$\mathcal{R}(t, u) := \mathcal{R}(t) = \frac{d}{dt} V(t),$$

in a large class of controls.



Greedy sparse control

Theorem (Caponigro-Fornasier-Piccoli-Trélat)

For every initial condition $(x_0, v_0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$ and $M > 0$ there exist $T > 0$ and a *sparse* control $u : [0, T] \rightarrow (\mathbb{R}^d)^N$, with $\sum_{i=1}^N \|u_i(t)\| \leq M$ for every $t \in [0, T]$ such that the associated solution tends to consensus.

Greedy sparse control

Theorem (Caponigro-Fornasier-Piccoli-Trélat)

For every initial condition $(x_0, v_0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$ and $M > 0$ there exist $T > 0$ and a *sparse* control $u : [0, T] \rightarrow (\mathbb{R}^d)^N$, with $\sum_{i=1}^N \|u_i(t)\| \leq M$ for every $t \in [0, T]$ such that the associated solution tends to consensus.

We can choose adaptively the control law explicitly as *one of the solutions* of the variational problem

$$\min B(v, u) + \frac{\mathcal{T}(x)}{N} \sum_{i=1}^N \|u_i\| \quad \text{subject to} \quad \sum_{i=1}^N \|u_i\| \leq M,$$

where

$$\mathcal{T}(x) = \int_{\sqrt{B(x,x)}}^{\infty} a(\sqrt{2Nr}) dr \quad (\text{threshold function by Ha-Ha-Kim})$$

The 1-sparse feedback strategy

Definition

For every $t \geq 0$ we define the **1-sparse feedback control** as

$$u_i(t) = \begin{cases} -M \frac{v_i^\perp(t)}{\|v_i^\perp(t)\|} & \text{if } i = \hat{i}(t) \\ 0 & \text{if } i \neq \hat{i}(t) \end{cases}$$

where $\hat{i}(t) \in \{1, \dots, N\}$ is the minimum index such that

$$\|v_{\hat{i}(t)}^\perp(t)\| = \max_{j=1, \dots, N} \|v_j^\perp(t)\|.$$



The 1-sparse feedback strategy

Definition

For every $t \geq 0$ we define the **1-sparse feedback control** as

$$u_i(t) = \begin{cases} -M \frac{v_i^\perp(t)}{\|v_i^\perp(t)\|} & \text{if } i = \hat{i}(t) \\ 0 & \text{if } i \neq \hat{i}(t) \end{cases}$$

where $\hat{i}(t) \in \{1, \dots, N\}$ is the minimum index such that

$$\|v_{\hat{i}(t)}^\perp(t)\| = \max_{j=1, \dots, N} \|v_j^\perp(t)\|.$$

Theorem

The 1-sparse control is the minimizer of

$$\mathcal{R}(t, u) := \mathcal{R}(t) = \frac{d}{dt} V(t),$$

among all the solution of the variational problem.

The 1-sparse feedback strategy

Definition

For every $t \geq 0$ we define the **1-sparse feedback control** as

$$u_i(t) = \begin{cases} -M \frac{v_i^\perp(t)}{\|v_i^\perp(t)\|} & \text{if } i = \hat{i}(t) \\ 0 & \text{if } i \neq \hat{i}(t) \end{cases}$$

where $\hat{i}(t) \in \{1, \dots, N\}$ is the minimum index such that

$$\|v_{\hat{i}(t)}^\perp(t)\| = \max_{j=1, \dots, N} \|v_j^\perp(t)\|.$$

Theorem

The 1-sparse control is the minimizer of

$$\mathcal{R}(t, u) := \mathcal{R}(t) = \frac{d}{dt} V(t),$$

among all the solution of the variational problem.



The control acts on the most “stubborn” agent. We may call it the “shepherd dog strategy”

Dynamical systems driven by attraction and repulsion forces

The **Cucker-Dong model**: for every $1 \leq i \leq N$

$$\begin{cases} \dot{x}_i = v_i \in \mathbb{R}^d \\ \dot{v}_i = -b_i v_i + \sum_{j=1}^N a(\|x_i - x_j\|^2) (x_j - x_i) + \sum_{\substack{j=1 \\ j \neq i}}^N f(\|x_i - x_j\|^2) (x_i - x_j) \in \mathbb{R}^d \end{cases}$$

Dynamical systems driven by attraction and repulsion forces

The **Cucker-Dong model**: for every $1 \leq i \leq N$

$$\begin{cases} \dot{x}_i = v_i \in \mathbb{R}^d \\ \dot{v}_i = -b_i v_i + \sum_{j=1}^N a(\|x_i - x_j\|^2) (x_j - x_i) + \sum_{\substack{j=1 \\ j \neq i}}^N f(\|x_i - x_j\|^2) (x_i - x_j) \in \mathbb{R}^d \end{cases}$$

where

- $b_i : [0, +\infty) \rightarrow [0, \Lambda]$ is the friction acting on the system,

Dynamical systems driven by attraction and repulsion forces

The **Cucker-Dong model**: for every $1 \leq i \leq N$

$$\begin{cases} \dot{x}_i = v_i \in \mathbb{R}^d \\ \dot{v}_i = -b_i v_i + \sum_{j=1}^N a(\|x_i - x_j\|^2)(x_j - x_i) + \sum_{\substack{j=1 \\ j \neq i}}^N f(\|x_i - x_j\|^2)(x_i - x_j) \in \mathbb{R}^d \end{cases}$$

where

- $b_i : [0, +\infty) \rightarrow [0, \Lambda]$ is the friction acting on the system,
- $a : [0, +\infty) \rightarrow [0, +\infty)$ is the rate of communication,

Dynamical systems driven by attraction and repulsion forces

The **Cucker-Dong model**: for every $1 \leq i \leq N$

$$\begin{cases} \dot{x}_i = v_i \in \mathbb{R}^d \\ \dot{v}_i = -b_i v_i + \sum_{j=1}^N a(\|x_i - x_j\|^2) (x_j - x_i) + \sum_{\substack{j=1 \\ j \neq i}}^N f(\|x_i - x_j\|^2) (x_i - x_j) \in \mathbb{R}^d \end{cases}$$

where

- $b_i : [0, +\infty) \rightarrow [0, \Lambda]$ is the friction acting on the system,
- $a : [0, +\infty) \rightarrow [0, +\infty)$ is the rate of communication,
- $f : (0, +\infty) \rightarrow (0, +\infty)$ such that

$$\int_{\delta}^{+\infty} f(r) dr < \infty \text{ for every } \delta > 0, \quad \int_0^{+\infty} f(r) dr = +\infty$$

models the repulsion between agents.

Example: Lennard-Jones potential

- It is the potential of the Van der Waals force.

Example: Lennard-Jones potential

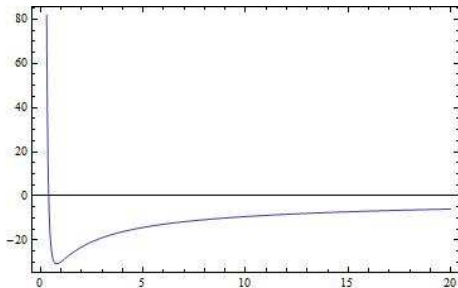
- It is the potential of the Van der Waals force.
- It can be seen as a Cucker-Dong system with

$$a(r) = \frac{\sigma_a}{r^7} \quad \text{and} \quad f(r) = \frac{\sigma_f}{r^{13}}.$$

Example: Lennard-Jones potential

- It is the potential of the Van der Waals force.
- It can be seen as a Cucker-Dong system with

$$a(r) = \frac{\sigma_a}{r^7} \quad \text{and} \quad f(r) = \frac{\sigma_f}{r^{13}}.$$



Difference $f(r) - a(r)$ for Lennard-Jones potentials.

Total Energy of Cucker-Dong Systems

We introduce

- the **kinetic energy** $K(t) := \frac{1}{2} \sum_{i=1}^N \|v_i(t)\|^2$,

Total Energy of Cucker-Dong Systems

We introduce

- the **kinetic energy** $K(t) := \frac{1}{2} \sum_{i=1}^N \|v_i(t)\|^2$,
- the **potential energy**

$$P(t) := \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^N \int_0^{\|x_i(t) - x_j(t)\|^2} a(r) dr + \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^N \int_{\|x_i(t) - x_j(t)\|^2}^{\infty} f(r) dr,$$

Total Energy of Cucker-Dong Systems

We introduce

- the **kinetic energy** $K(t) := \frac{1}{2} \sum_{i=1}^N \|v_i(t)\|^2$,
- the **potential energy**

$$P(t) := \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^N \int_0^{\|x_i(t) - x_j(t)\|^2} a(r) dr + \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^N \int_{\|x_i(t) - x_j(t)\|^2}^{\infty} f(r) dr,$$

- the **total energy** $E(t) := K(t) + P(t)$.



Conditional consensus emergence

Theorem (Cucker - Dong)

Consider a population of N agents modeled by a Cucker-Dong system with $a(t) := a_\beta(t) = \frac{1}{(1+t^2)^\beta}$, $\beta > 0$

$$\|x_i(0) - x_j(0)\| > 0 \text{ for all } i \neq j.$$

Then there exists a unique solution $(x(t), v(t))$ of the system with initial state $(x(0), v(0))$. Moreover if one of the two following hypotheses holds:



Conditional consensus emergence

Theorem (Cucker - Dong)

Consider a population of N agents modeled by a Cucker-Dong system with $a(t) := a_\beta(t) = \frac{1}{(1+t^2)^\beta}$, $\beta > 0$

$$\|x_i(0) - x_j(0)\| > 0 \text{ for all } i \neq j.$$

Then there exists a unique solution $(x(t), v(t))$ of the system with initial state $(x(0), v(0))$. Moreover if one of the two following hypotheses holds:

1. $\beta \leq 1$



Conditional consensus emergence

Theorem (Cucker - Dong)

Consider a population of N agents modeled by a Cucker-Dong system with $a(t) := a_\beta(t) = \frac{1}{(1+t^2)^\beta}$, $\beta > 0$

$$\|x_i(0) - x_j(0)\| > 0 \text{ for all } i \neq j.$$

Then there exists a unique solution $(x(t), v(t))$ of the system with initial state $(x(0), v(0))$. Moreover if one of the two following hypotheses holds:

1. $\beta \leq 1$
2. $\beta > 1$ and $E(0) < \vartheta := (N - 1) \int_0^\infty a(r) dr$



Conditional consensus emergence

Theorem (Cucker - Dong)

Consider a population of N agents modeled by a Cucker-Dong system with $a(t) := a_\beta(t) = \frac{1}{(1+t^2)^\beta}$, $\beta > 0$

$$\|x_i(0) - x_j(0)\| > 0 \text{ for all } i \neq j.$$

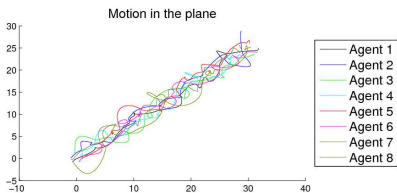
Then there exists a unique solution $(x(t), v(t))$ of the system with initial state $(x(0), v(0))$. Moreover if one of the two following hypotheses holds:

1. $\beta \leq 1$
2. $\beta > 1$ and $E(0) < \vartheta := (N - 1) \int_0^\infty a(r) dr$

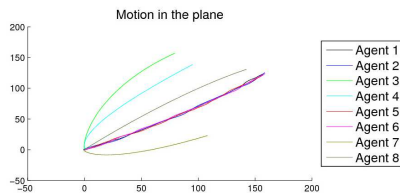
then the population is *cohesive and collision-avoiding*, i.e., there exist two constants B_0 and $b_0 > 0$ such that, for every $t \geq 0$

$$b_0 \leq \|x_i(t) - x_j(t)\| \leq B_0 \text{ for all } 1 \leq i \neq j \leq N.$$

Non-consensus events need intervention

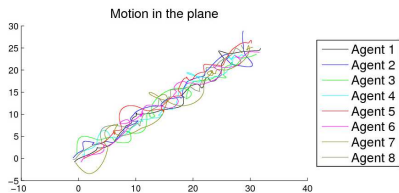


$$\beta \leq 1$$

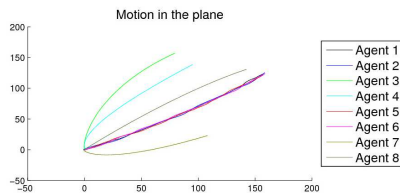


$$\beta > 1 \text{ and } E(0) \geq \vartheta$$

Non-consensus events need intervention



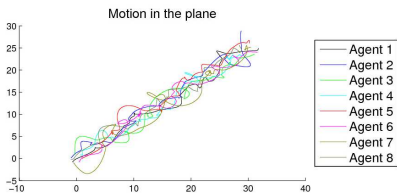
$$\beta \leq 1$$



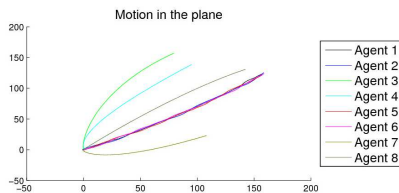
$$\beta > 1 \text{ and } E(0) \geq \vartheta$$

- Assume we are in the case $\beta > 1$ and $E(0) \geq \vartheta$. Can we again stabilize the society by external parsimonious intervention?

Non-consensus events need intervention



$$\beta \leq 1$$



$$\beta > 1 \text{ and } E(0) \geq \vartheta$$

- Assume we are in the case $\beta > 1$ and $E(0) \geq \vartheta$. Can we again stabilize the society by external parsimonious intervention?
- We again introduce a control term inside the model

$$\begin{cases} \dot{x}_i = v_i \\ \dot{v}_i = -b_i v_i + \sum_{j=1}^N a(\|x_i - x_j\|^2) (x_j - x_i) + \sum_{\substack{j=1 \\ j \neq i}}^N f(\|x_i - x_j\|^2) (x_i - x_j) + u_i \end{cases}$$

under the sparsifying constraint $\sum_{i=1}^N \|u_i(t)\| \leq M$ for $M > 0$.



Sparse control strategy for the Cucker-Dong model

Definition

For every $t \geq 0$ we define the **1-sparse feedback control** as

$$u_i(t) = \begin{cases} -\frac{M}{E(0)} E(t) \frac{v_i(t)}{\|v_i(t)\|} & \text{if } i = \hat{i}(t) \\ 0 & \text{if } i \neq \hat{i}(t) \end{cases}$$

where $\hat{i}(t) \in \{1, \dots, N\}$ is the minimum index such that

$$\|v_{\hat{i}(t)}(t)\| = \max_{j=1, \dots, N} \|v_j(t)\|.$$

Sparse control strategy for the Cucker-Dong model

Definition

For every $t \geq 0$ we define the **1-sparse feedback control** as

$$u_i(t) = \begin{cases} -\frac{M}{E(0)} E(t) \frac{v_i(t)}{\|v_i(t)\|} & \text{if } i = \widehat{i}(t) \\ 0 & \text{if } i \neq \widehat{i}(t) \end{cases}$$

where $\widehat{i}(t) \in \{1, \dots, N\}$ is the minimum index such that

$$\|v_{\widehat{i}(t)}(t)\| = \max_{j=1, \dots, N} \|v_j(t)\|.$$

Define for every $t > 0$ the set

$$K(t) := \left\{ u \in (\mathbb{R}^d)^N \mid \sum_{i=1}^N \|u_i\| \leq M \cdot \frac{E(t)}{E(0)} \right\},$$

and for every $t > 0$ and $q > 0$ the functional $\mathcal{J}_{t,q} : (\mathbb{R}^d)^N \rightarrow \mathbb{R}$

$$\mathcal{J}_{t,q}(u) = \langle v(t), u \rangle + \frac{\left\| \frac{1}{N} \sum_{i=1}^N v_i(0) \right\|}{q} \sum_{i=1}^N \|u_i\|.$$

Under the following hypotheses...

Theorem

Fix $M > 0$. Let $(x_0, v_0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$ be such that the following hold:

1. $\|x_{0i} - x_{0j}\| > 0$ for all $i \neq j$,

Under the following hypotheses...

Theorem

Fix $M > 0$. Let $(x_0, v_0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$ be such that the following hold:

1. $\|x_{0i} - x_{0j}\| > 0$ for all $i \neq j$,
2. $\|\frac{1}{N} \sum_{i=1}^N v_i(0)\| > 0$,



Under the following hypotheses...

Theorem

Fix $M > 0$. Let $(x_0, v_0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$ be such that the following hold:

1. $\|x_{0i} - x_{0j}\| > 0$ for all $i \neq j$,
2. $\|\frac{1}{N} \sum_{i=1}^N v_i(0)\| > 0$,
3. $E(0) \geq \vartheta > E(0) \exp\left(-\frac{2\sqrt{3}}{9} \frac{M \|\frac{1}{N} \sum_{i=1}^N v_i(0)\|^3}{E(0)\sqrt{E(0)}\left(\Lambda\sqrt{E(0)} + \frac{M}{N}\right)}\right)$.

Then...

... we can prove the following.

Theorem (continues...)

... then there exist $T > 0$ and $q > 0$ such that

- the 1-sparse feedback control belongs to the set $\operatorname{argmin}_{u \in K(t)} \mathcal{J}_{t,q}(u)$ for every $t \leq T$;

... we can prove the following.

Theorem (continues...)

... then there exist $T > 0$ and $q > 0$ such that

- the 1-sparse feedback control belongs to the set $\operatorname{argmin}_{u \in K(t)} \mathcal{J}_{t,q}(u)$ for every $t \leq T$;
- the 1-sparse feedback control is an instantaneous minimizer of the functional

$$\mathcal{D}(t, u) = \frac{d}{dt} E(t)$$

over all possible feedback controls in $\operatorname{argmin}_{u \in K(t)} \mathcal{J}_{t,q}(u)$;

... we can prove the following.

Theorem (continues...)

... then there exist $T > 0$ and $q > 0$ such that

- the 1-sparse feedback control belongs to the set $\operatorname{argmin}_{u \in K(t)} \mathcal{J}_{t,q}(u)$ for every $t \leq T$;
- the 1-sparse feedback control is an instantaneous minimizer of the functional

$$\mathcal{D}(t, u) = \frac{d}{dt} E(t)$$

over all possible feedback controls in $\operatorname{argmin}_{u \in K(t)} \mathcal{J}_{t,q}(u)$;

- there exists a solution of the controlled Cucker-Dong system associated to a control solving the variational problem

$$\tilde{u} \in \operatorname{argmin}_{u \in K(t)} \mathcal{J}_{t,q}(u) \text{ for every } t \leq T.$$

Summing up our results

- We have shown a quantitative link between a degree of “narrow-mindedness” and convergence to global patterns;

Summing up our results

- We have shown a quantitative link between a degree of “narrow-mindedness” and convergence to global patterns;
- a sparse control strategy is more efficient than local decentralized control in order to enforce convergence: if we pay our attention solely to the “most stubborn” agent while leaving the other free to adjust themselves, we actually have the “best greedy” strategy;

Summing up our results

- We have shown a quantitative link between a degree of “narrow-mindedness” and convergence to global patterns;
- a sparse control strategy is more efficient than local decentralized control in order to enforce convergence: if we pay our attention solely to the “most stubborn” agent while leaving the other free to adjust themselves, we actually have the “best greedy” strategy;
- in contrast to what happen with the Cucker-Smale model, the result for the 1-sparse control feedback in the Cucker-Dong model is **conditional** (it depends on the initial conditions of the system)

Summing up our results

- We have shown a quantitative link between a degree of “narrow-mindedness” and convergence to global patterns;
- a sparse control strategy is more efficient than local decentralized control in order to enforce convergence: if we pay our attention solely to the “most stubborn” agent while leaving the other free to adjust themselves, we actually have the “best greedy” strategy;
- in contrast to what happen with the Cucker-Smale model, the result for the 1-sparse control feedback in the Cucker-Dong model is **conditional** (it depends on the initial conditions of the system)
 ⇒ we don't know if the conditions are necessary.

A few info

- **WWW:** <http://www-m15.ma.tum.de/>
- **References:**
 - M. Bongini, D. Kalise, and M. Fornasier, *(Un)conditional consensus emergence under perturbed and decentralized feedback controls*, submitted to Discrete and Continuous Dynamical Systems, pp. 27
 - M. Bongini and M. Fornasier, *Sparse stabilization of dynamical systems driven by attraction and avoidance forces*, Networks and Heterogeneous Media, Pages 1 - 31, Volume 9, Issue 1, March 2014
 - M. Caponigro, M. Fornasier, B. Piccoli, and E. Trélat, *Sparse stabilization and control of alignment models*, to appear in Mathematical Models and Methods in Applied Sciences, pp. 33
 - F. Cucker and J. Dong, *A conditional, collision-avoiding, model for swarming*, IEEE Trans. Automat. Control, 56(5):1124–1129, 2011
 - S.-Y. Ha, T. Ha, and J.-H. Kim, *Emergent behavior of a Cucker-Smale type particle model with nonlinear velocity couplings*, IEEE Trans. Aut. Contr. 55.7 (2010): 1679-1683