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# Indefinite Linear MPC and Approximated Economic MPC for Nonlinear Systems

Mario Zanon<sup>1</sup>, Sébastien Gros<sup>2</sup>, Moritz Diehl<sup>1</sup>

1. KU Leuven & Univ. of Freiburg, 2. Chalmers Univ.

**NetCo conference, Tours 2014**

- 1 Economic MPC and Stability Analysis
- 2 The Linear Quadratic Case
- 3 Approximated EMPC with stability guarantees

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- 2 The Linear Quadratic Case
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## MPC

Optimal Control Problem

$$\min_{x_0, u_0, \dots, x_N}$$

s.t.

## MPC

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$$\text{s.t. } x_0 - \bar{x}_i = 0,$$

Initial condition

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$$x_{k+1} - f(x_k, u_k) = 0,$$

System dynamics

## MPC

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$$\text{s.t. } x_0 - \bar{x}_i = 0,$$

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$$x_N - x_N^s = 0.$$

Initial condition

System dynamics

Terminal constraint



## MPC

### Optimal Control Problem

$$\begin{aligned} \min_{x_0, u_0, \dots, x_N} \quad & \sum_{k=0}^{N-1} l(x_k, u_k) && \text{(Quadratic) stage cost} \\ \text{s.t.} \quad & x_0 - \bar{x}_i = 0, && \text{Initial condition} \\ & x_{k+1} - f(x_k, u_k) = 0, && \text{System dynamics} \\ & x_N - x_N^s = 0. && \text{Terminal constraint} \end{aligned}$$

# MPC

## Optimal Control Problem

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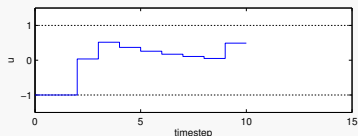
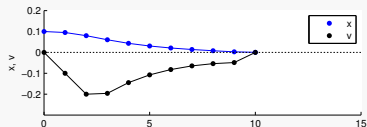
$$x_N - x_N^s = 0.$$

(Quadratic) stage cost

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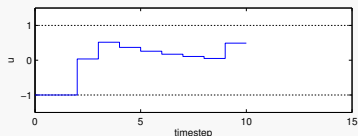
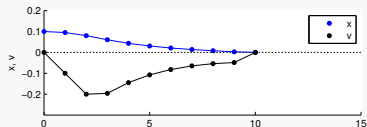
Initial condition

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Terminal constraint

At each sampling time  $i$ :

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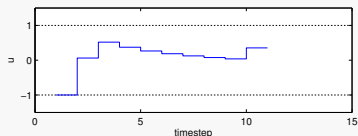
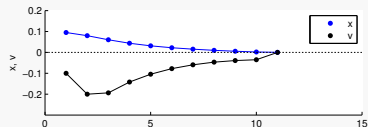
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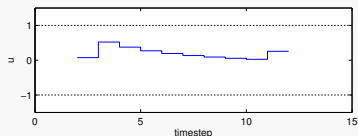
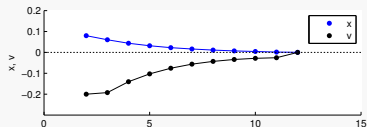
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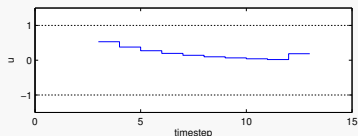
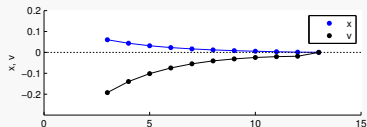
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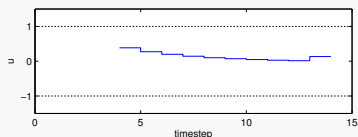
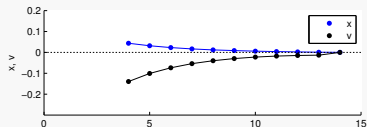
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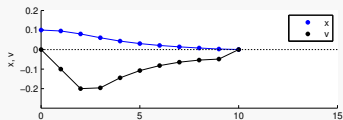
**Lyapunov stability**

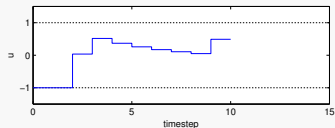
Tracking MPC:  $l(x_s, u_s) = 0$  and  $\exists \alpha \in \mathcal{K}$  s.t.  $\alpha(x - x_s) \leq l(x, u)$ ,  $\forall u \in \mathbb{U}$



## Lyapunov stability

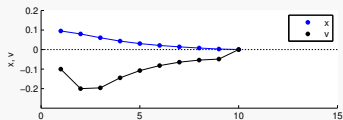
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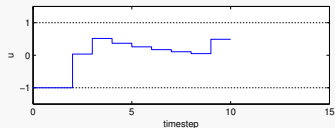
$$V_N(\bar{x}_i)$$


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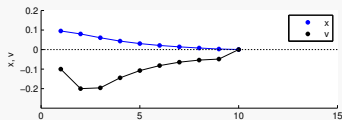


$$V_{N-1}(f(\bar{x}_i, u_0^*))$$

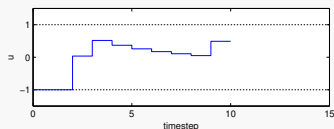


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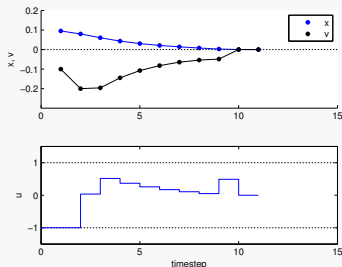
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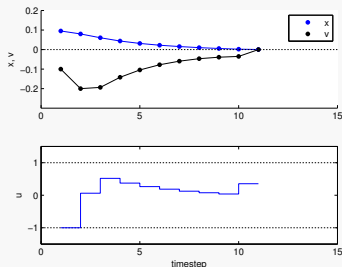


$$V_{N-1}(f(\bar{x}_i, u_0^*)) + l(x_s, u_s)$$

$$\begin{aligned}
 V_{N-1}(f(\bar{x}_i, u_0^*)) &= V_N(\bar{x}_i) - l(\bar{x}_i, u_0^*) \\
 &= V_N(\bar{x}_i) - l(\bar{x}_i, u_0^*) + \underbrace{l(x_s, u_s)}_{=0}
 \end{aligned}$$

## Lyapunov stability

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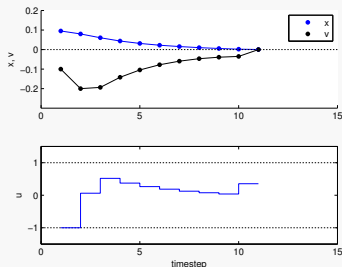
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$$V_N(f(\bar{x}_i, u_0^*)) \leq V_N(\bar{x}_i) - l(\bar{x}_i, u_0^*) + \underbrace{l(x_s, u_s)}_{=0}$$

$$V_N(f(\bar{x}_i, u_0^*)) - V_N(\bar{x}_i) \leq -l(\bar{x}_i, u_0^*)$$

**Do we always want to track?**

## Do we always want to track?





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## Do we always want to track?



**Do we always want to track?**

No!



**Do we always want to track?**

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**Then why do we track?**



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- It works



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- We have been doing it since the 80s



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- Difficult to prove stability (2008-)

## Do we always want to track?

No!

## Then why do we track?

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- We have been doing it since the 80s
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## What about Economic MPC?

- Difficult to prove stability (2008-)
- Increased “economic” gain



## **Economic vs Tracking**

## Economic vs Tracking

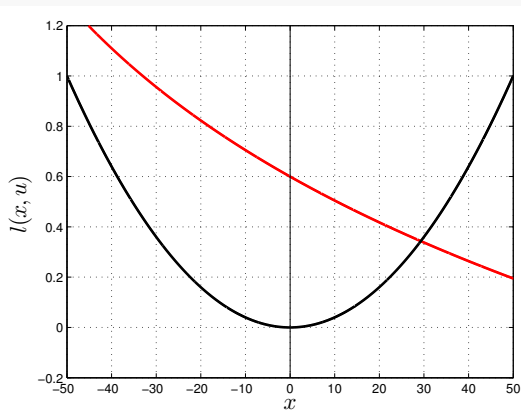
- Steady state:  $x_s = 0 = f(0, 0)$

## Economic vs Tracking

- Steady state:  $x_s = 0 = f(0, 0)$
- Stage cost: **Tracking** vs **Economic**

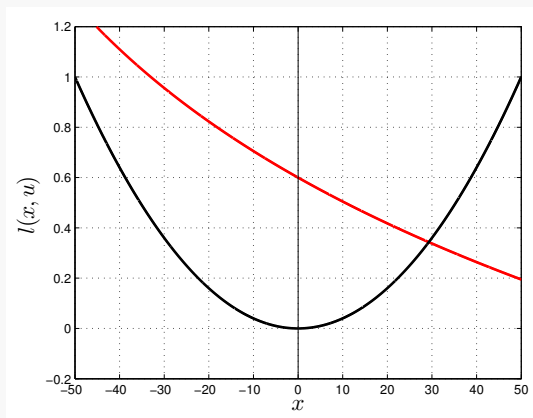
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## Economic vs Tracking

- Steady state:  $x_s = 0 = f(0, 0)$
- Stage cost: **Tracking** vs **Economic**



**The classical stability theory does not apply!**

## Economic Stage Cost

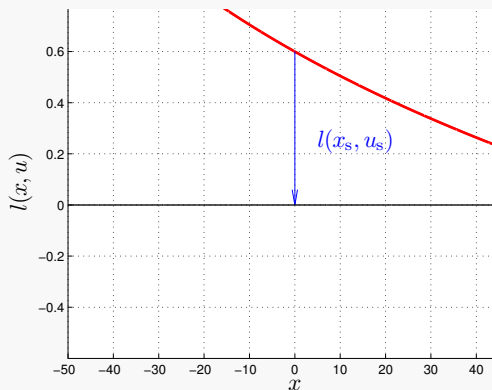
Steady state:  $(x_s, u_s) = \min_{x,u} l(x, u) \text{ s.t. } x = f(x, u)$



## Economic Stage Cost

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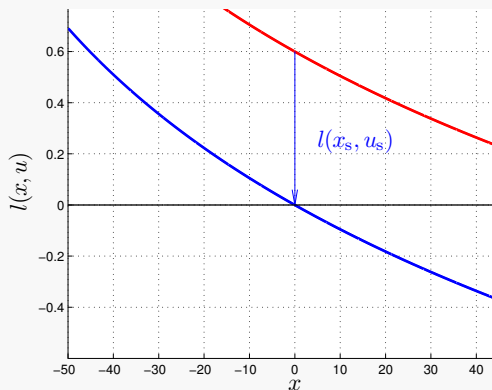
$l(x, u)$     $l(x_s, u_s)$



## Economic Stage Cost

Steady state:  $(x_s, u_s) = \min_{x,u} l(x, u)$  s.t.  $x = f(x, u)$

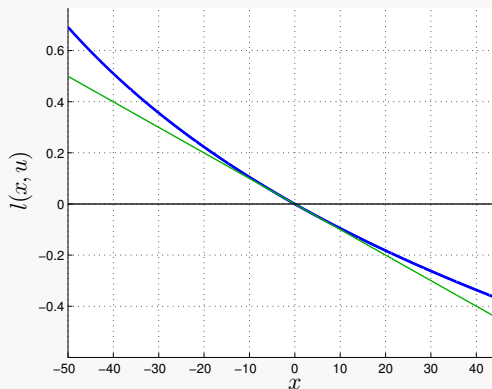
$$l(x, u) - l(x_s, u_s)$$



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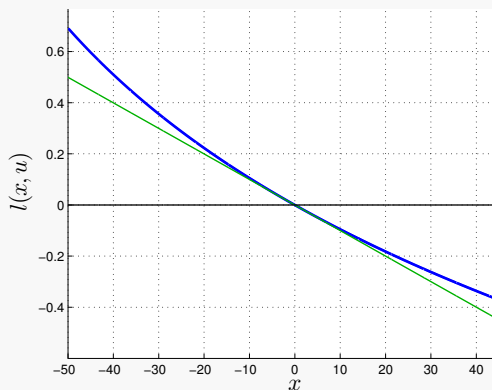
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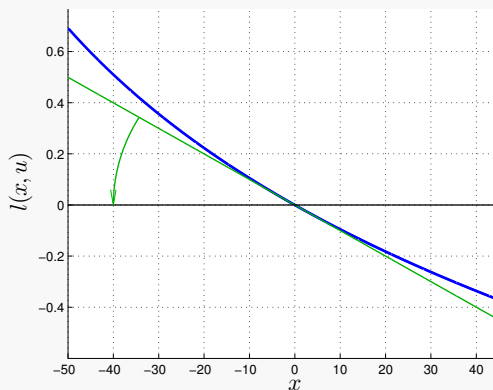
$$l(x, u) - l(x_s, u_s) \quad \underbrace{\lambda_s^T}_{\text{Lagrange multiplier}} \quad (x - f(x, u))$$



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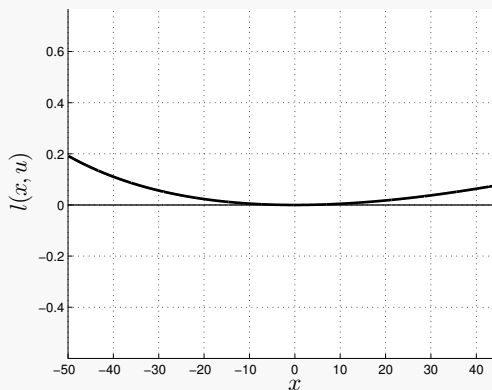
$$l(x, u) - l(x_s, u_s) + \underbrace{\lambda_s^T}_{\text{Lagrange multiplier}} (x - f(x, u))$$



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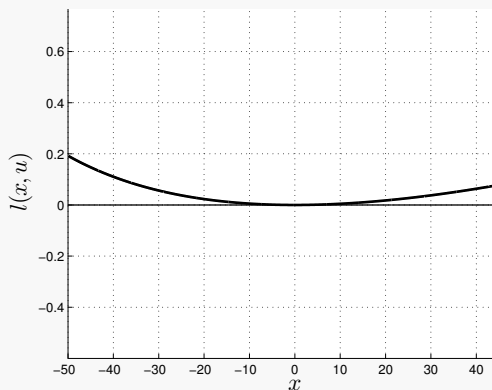


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Steady state:  $(x_s, u_s) = \min_{x,u} l(x, u) \text{ s.t. } x = f(x, u)$

**Rotated cost:**  $L(x, u) = l(x, u) - l(x_s, u_s) + \underbrace{\lambda_s^T}_{\text{Lagrange multiplier}} (x - f(x, u))$

[Diehl et al. 2011]



## Rotated MPC Problem

$$\begin{aligned} \min_{x_0, u_0, \dots, x_N} \quad & \sum_{k=0}^{N-1} L(x_k, u_k) \\ \text{s.t.} \quad & x_0 - \bar{x}_i = 0, \\ & x_{k+1} - f(x_k, u_k) = 0, \\ & x_N - x_N^s = 0. \end{aligned}$$



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Rotated Problem  $\equiv$  Original Problem

$$\sum_{k=0}^{N-1} L(x, u) = \sum_{k=0}^{N-1} l(x, u) + \underbrace{\lambda_s^T x_0 - \lambda_s^T x_N - (N-1) l(x_s, u_s)}_{\text{constant}}$$

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If

$$\alpha(x) \leq L(x, u), \quad \alpha \in \mathcal{K}$$

**the previous stability proof holds!** [Diehl et al. 2011]

## **Generalization** [Amrit et al. 2011]

Nonlinear rotating function:  $\lambda(x)$

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New rotated cost:  $L(x, u) = l(x, u) - l(x_s, u_s) + \lambda(x) - \lambda(f(x, u))$

**Generalization** [Amrit et al. 2011]

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**What conditions on  $\lambda(x)$ ?**

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**What conditions on  $\lambda(x)$ ?****Strict Dissipativity**

System strictly dissipative wrt the supply rate  $s(x, u)$  if  $\exists \lambda(x) : \mathbb{X} \rightarrow \mathbb{R}$

$$\lambda(f(x, u)) - \lambda(x) \leq -\rho(x - x_s) + s(x, u),$$

$$\forall (x, u) \in \mathbb{X} \times \mathbb{U}$$

We are interested in  $s(x, u) = l(x, u) - l(x_s, u_s)$

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We are interested in  $s(x, u) = l(x, u) - l(x_s, u_s)$

This entails

$$L(x, u) = l(x, u) - l(x_s, u_s) + \lambda(x) - \lambda(f(x, u)) \geq \rho(x - x_s)$$

- 1 Economic MPC and Stability Analysis
- 2 The Linear Quadratic Case**
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## LQ-EMPC

Consider a linear MPC problem

$$\mathcal{P}_N(A, B, Q, R, S, P_N) = \underset{x_0, u_0, \dots, x_N}{\operatorname{argmin}} \sum_{k=0}^{N-1} \begin{bmatrix} x_k \\ u_k \end{bmatrix}^T H \begin{bmatrix} x_k \\ u_k \end{bmatrix} + x_N^T P_N x_N$$

s.t.  $x_0 - \bar{x}_i = 0,$   
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## **Inifinite Horizon Case**

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$$\mathcal{D}(A, B, Q, R, S) := \{(P, K) \mid Q + A^T P A - P - (S^T + A^T P B)K = 0, \\ K = (R + B^T P B)^{-1}(S + B^T P A), \\ \rho(A - BK) < 1\}$$

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One can enforce stability by adding  $\begin{bmatrix} x \\ u \end{bmatrix}^T T \begin{bmatrix} x \\ u \end{bmatrix}$  to the cost and solving

$$\min_{P, T} \|T\|^2 \quad \text{s.t. } M + T \succeq 0,$$

**One can modify the cost-to-go without changing the problem**

$$\mathcal{P}_\infty(A, B, Q, R, S) = \mathcal{P}_\infty(A, B, Q_{\bar{P}}, R_{\bar{P}}, S_{\bar{P}}),$$

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- $K_K = 0$
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- $R$  can be replaced by any  $R_K (> 0)$



## Positive Definite LQR Formulation

### Lyapunov Stability Theorem

If  $\rho(A_K) < 1$ , then

$$\forall Q_L \succ 0 \quad \exists P_L \succ 0 \text{ s.t.} \quad Q_L + A_K^T P_L A_K - P_L = 0$$

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Pos. def. is preserved when transforming back to  $(A, B)$

## What About MPC?

In the unconstrained case, the solution is given by the Discrete Riccati Equation (DRE)

$$\begin{aligned}\mathcal{R}_N(A, B, Q, R, S, P_N) = & \{(P_0, P_1, \dots, P_N, K_0, \dots, K_{N-1}) \mid \\ & P_{k-1} = Q + A^T P_k A - (S^T + A^T P_k B) K_{k-1} \\ & K_{k-1} = (R + B^T P_k B)^{-1} (S + B^T P_k A)\}\end{aligned}$$

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### DRE equivalence

$$\mathcal{R}_N(A, B, Q, R, S, P_N) = \mathcal{R}_N(A, B, \tilde{Q}, \tilde{R}, \tilde{S}, \tilde{P}_N)$$

if

$$\begin{aligned} \tilde{P}_N - \tilde{P} &= P_N - P, \\ \tilde{R} + B^T \tilde{P} B &= R + B^T P B, \\ \tilde{S} + B^T \tilde{P} A &= S + B^T P A \end{aligned}$$

with  $P$  and  $\tilde{P}$  computed from  $\mathcal{D}(A, B, Q, R, S)$  and  $\mathcal{D}(A, B, \tilde{Q}, \tilde{R}, \tilde{S})$ .

## A Practical View on the Problem

Solve the following SDP

$$\begin{aligned}
 & \min_{\tilde{P}, \tilde{Q}, \tilde{R}, \tilde{S}, \tilde{H}} \|\tilde{P} - I\|^2 + \|\tilde{H} - I\|^2 \\
 & \text{s.t. } \tilde{H} = \begin{bmatrix} \tilde{Q} & \tilde{S}^T \\ \tilde{S} & \tilde{R} \end{bmatrix} \\
 & \tilde{H} \succeq 0 \\
 & \tilde{P} \succeq 0 \\
 & \tilde{Q} + A^T \tilde{P} A - \tilde{P} - (\tilde{S}^T + A^T \tilde{P} B) K = 0, \\
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This problem is convex!



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## Look at the Steady State Properties

Lagrangian of steady state problem

$$\mathcal{L} = l(x, u) - \lambda^T (x - f(x, u))$$

Lagrangian Hessian:

$$H = \begin{bmatrix} Q & S^T \\ S & R \end{bmatrix} = \frac{\partial^2 \mathcal{L}}{\partial (x, u)^2} \Big|_{x_S, u_S}$$

Local linearization

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- Formulate a tracking NMPC scheme using  $\tilde{H}$

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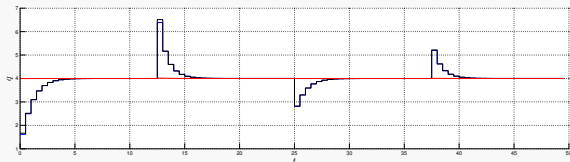
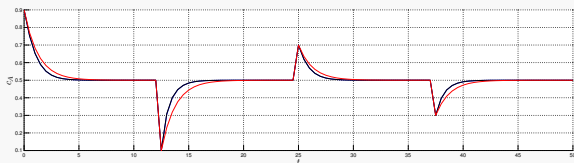
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**Thank you for your attention!**