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Indefinite Linear MPC and Approximated Economic MPC for Nonlinear Systems

Mario Zanon¹, Sébastien Gros², Moritz Diehl¹

1. KU Leuven & Univ. of Freiburg, 2. Chalmers Univ.

NetCo conference, Tours 2014

- 1 Economic MPC and Stability Analysis
- 2 The Linear Quadratic Case
- 3 Approximated EMPC with stability guarantees

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MPC

Optimal Control Problem

$$\min_{x_0, u_0, \dots, x_N}$$

s.t.

MPC

Optimal Control Problem

$$\min_{x_0, u_0, \dots, x_N}$$

$$\text{s.t. } x_0 - \bar{x}_i = 0,$$

Initial condition

MPC

Optimal Control Problem

$$\min_{x_0, u_0, \dots, x_N}$$

$$\text{s.t. } x_0 - \bar{x}_i = 0,$$

Initial condition

$$x_{k+1} - f(x_k, u_k) = 0,$$

System dynamics

MPC

Optimal Control Problem

$$\min_{x_0, u_0, \dots, x_N}$$

$$\text{s.t. } x_0 - \bar{x}_i = 0,$$

$$x_{k+1} - f(x_k, u_k) = 0,$$

$$x_N - x_N^s = 0.$$

Initial condition

System dynamics

Terminal constraint

MPC

Optimal Control Problem

$$\begin{aligned} \min_{x_0, u_0, \dots, x_N} \quad & \sum_{k=0}^{N-1} l(x_k, u_k) && \text{(Quadratic) stage cost} \\ \text{s.t.} \quad & x_0 - \bar{x}_i = 0, && \text{Initial condition} \\ & x_{k+1} - f(x_k, u_k) = 0, && \text{System dynamics} \\ & x_N - x_N^s = 0. && \text{Terminal constraint} \end{aligned}$$

MPC

Optimal Control Problem

$$\min_{x_0, u_0, \dots, x_N} \sum_{k=0}^{N-1} l(x_k, u_k)$$

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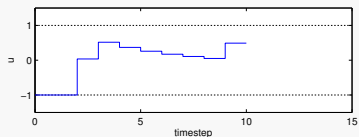
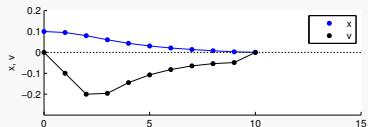
$$x_N - x_N^s = 0.$$

(Quadratic) stage cost

Initial condition

System dynamics

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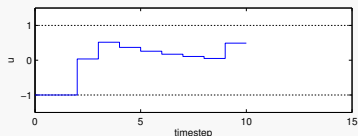
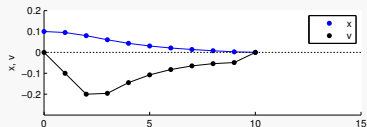
Initial condition

System dynamics

Terminal constraint

At each sampling time i :

- get the initial state \bar{x}_i
- solve the MPC OCP
- apply the first control u_0^*



MPC

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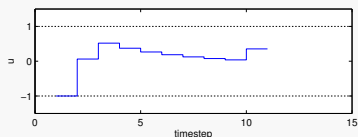
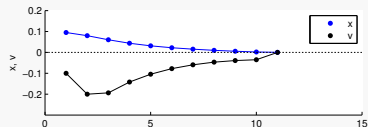
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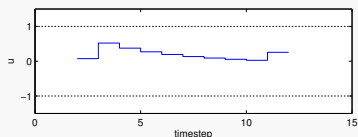
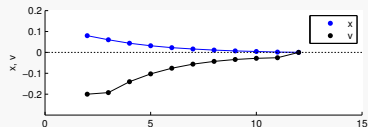
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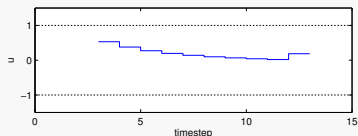
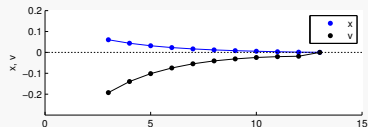
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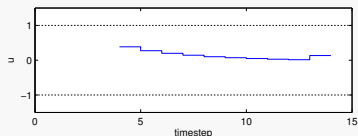
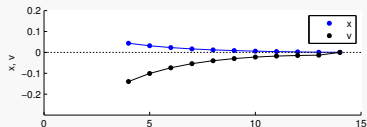
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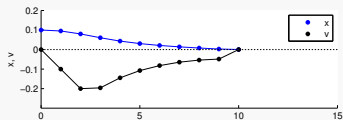


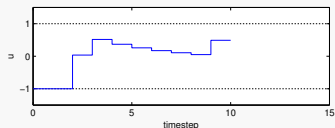
Lyapunov stability

Tracking MPC: $l(x_s, u_s) = 0$ and $\exists \alpha \in \mathcal{K}$ s.t. $\alpha(x - x_s) \leq l(x, u)$, $\forall u \in \mathbb{U}$

Lyapunov stability

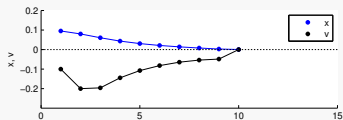
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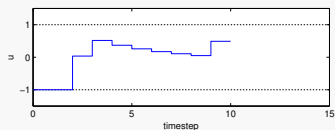
$$V_N(\bar{x}_i)$$


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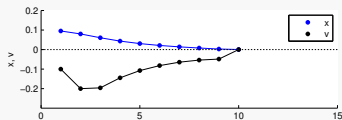


$$V_{N-1}(f(\bar{x}_i, u_0^*))$$

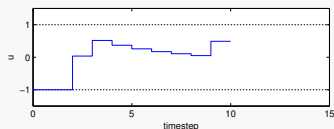


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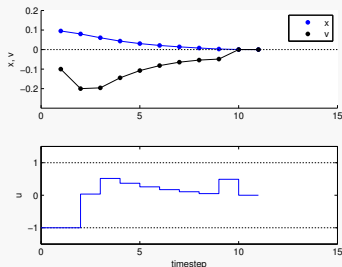
$$V_{N-1}(f(\bar{x}_i, u_0^*))$$



$$V_{N-1}(f(\bar{x}_i, u_0^*)) = V_N(\bar{x}_i) - l(\bar{x}_i, u_0^*)$$

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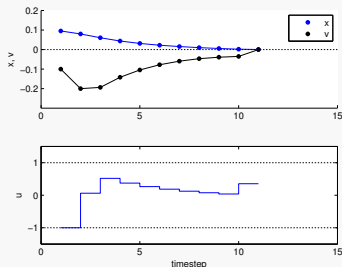


$$V_{N-1}(f(\bar{x}_i, u_0^*)) + l(x_s, u_s)$$

$$\begin{aligned}
 V_{N-1}(f(\bar{x}_i, u_0^*)) &= V_N(\bar{x}_i) - l(\bar{x}_i, u_0^*) \\
 &= V_N(\bar{x}_i) - l(\bar{x}_i, u_0^*) + \underbrace{l(x_s, u_s)}_{=0}
 \end{aligned}$$

Lyapunov stability

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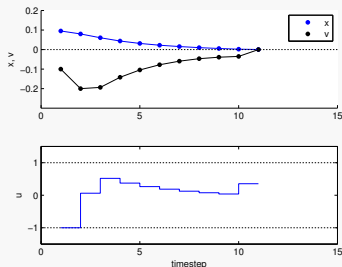
$$V_N(f(\bar{x}_i, u_0^*))$$

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$$V_N(f(\bar{x}_i, u_0^*)) \leq V_N(\bar{x}_i) - l(\bar{x}_i, u_0^*) + \underbrace{l(x_s, u_s)}_{=0}$$

$$V_N(f(\bar{x}_i, u_0^*)) - V_N(\bar{x}_i) \leq -l(\bar{x}_i, u_0^*)$$

Do we always want to track?

Do we always want to track?



Do we always want to track?



Do we always want to track?



Do we always want to track?

No!



Do we always want to track?

No!

Then why do we track?



Do we always want to track?

No!

Then why do we track?

- It works



Do we always want to track?

No!

Then why do we track?

- It works
- We have been doing it since the 80s



Do we always want to track?

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Then why do we track?

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What about Economic MPC?



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What about Economic MPC?

- Difficult to prove stability (2008-)



Do we always want to track?

No!

Then why do we track?

- It works
- We have been doing it since the 80s
- We have stability guarantees

What about Economic MPC?

- Difficult to prove stability (2008-)
- Increased “economic” gain



Economic vs Tracking

Economic vs Tracking

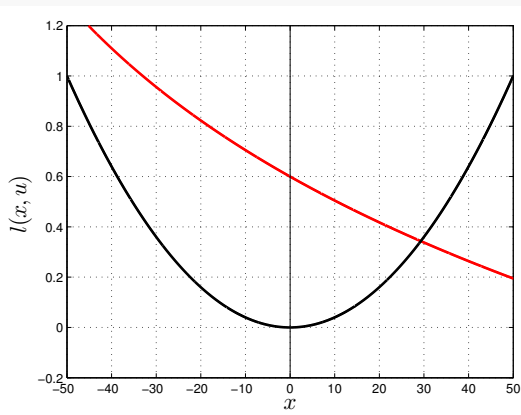
- Steady state: $x_s = 0 = f(0, 0)$

Economic vs Tracking

- Steady state: $x_s = 0 = f(0, 0)$
- Stage cost: **Tracking** vs **Economic**

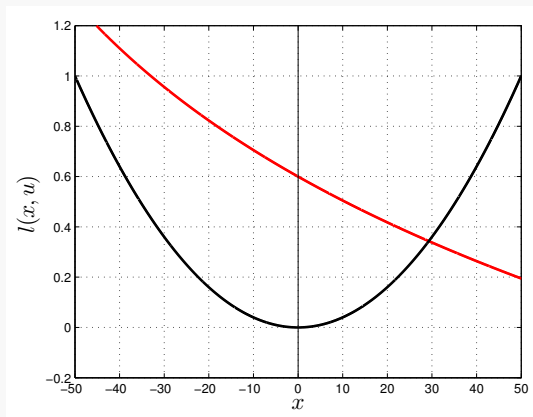
Economic vs Tracking

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Economic vs Tracking

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The classical stability theory does not apply!

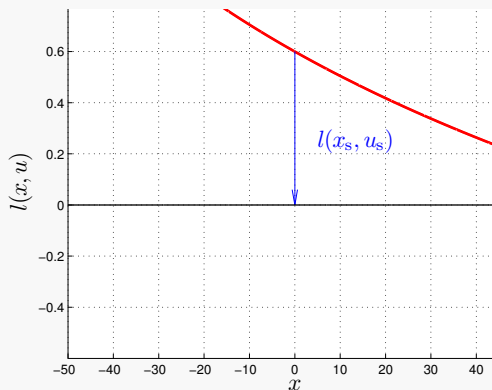
Economic Stage Cost

Steady state: $(x_s, u_s) = \min_{x,u} l(x, u)$ s.t. $x = f(x, u)$

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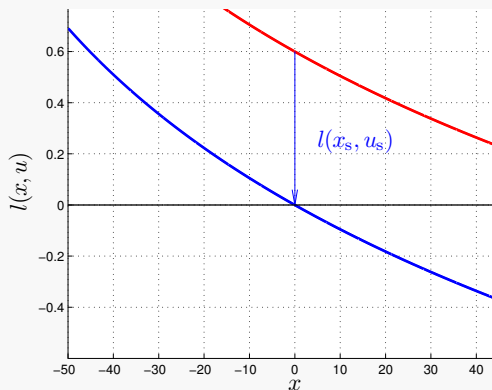
$l(x, u)$ $l(x_s, u_s)$



Economic Stage Cost

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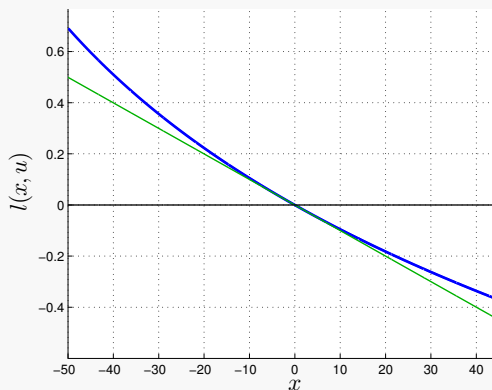
$$l(x, u) - l(x_s, u_s)$$



Economic Stage Cost

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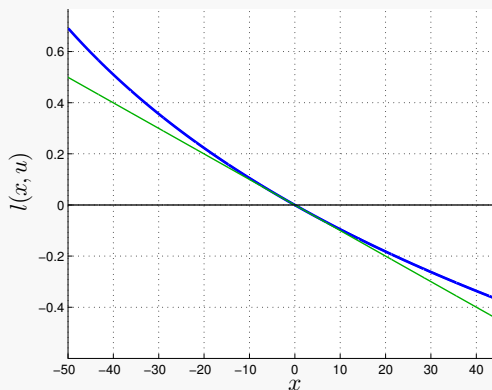
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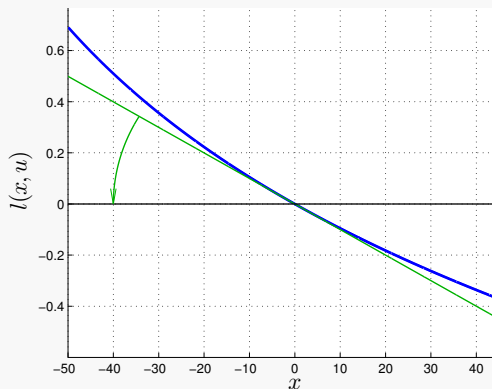
$$l(x, u) - l(x_s, u_s) \quad \underbrace{\lambda_s^T}_{\text{Lagrange multiplier}} \quad (x - f(x, u))$$



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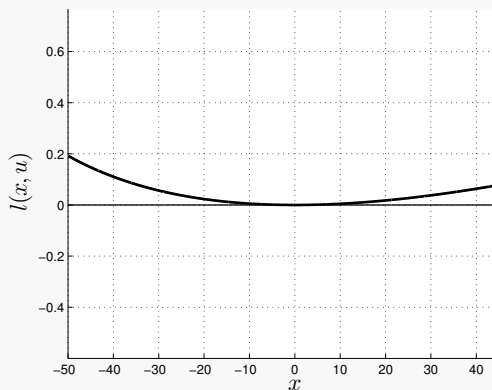
$$l(x, u) - l(x_s, u_s) + \underbrace{\lambda_s^T}_{\text{Lagrange multiplier}} (x - f(x, u))$$



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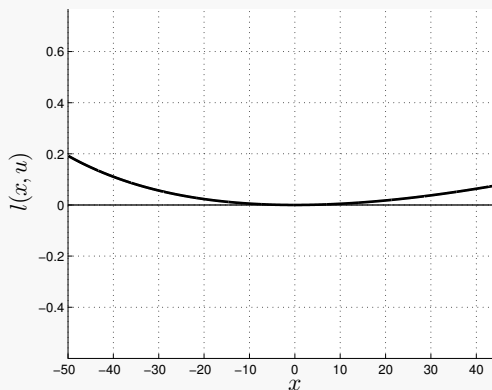
$$L(x, u) = l(x, u) - l(x_s, u_s) + \underbrace{\lambda_s^T}_{\text{Lagrange multiplier}} (x - f(x, u))$$



Economic Stage Cost

$$\text{Steady state: } (x_s, u_s) = \min_{x, u} l(x, u) \text{ s.t. } x = f(x, u)$$

$$\underbrace{\text{Rotated cost:}}_{\text{[Diehl et al. 2011]}} L(x, u) = l(x, u) - l(x_s, u_s) + \underbrace{\lambda_s^T}_{\text{Lagrange multiplier}} (x - f(x, u))$$



Rotated MPC Problem

$$\begin{aligned} \min_{x_0, u_0, \dots, x_N} \quad & \sum_{k=0}^{N-1} L(x_k, u_k) \\ \text{s.t.} \quad & x_0 - \bar{x}_i = 0, \\ & x_{k+1} - f(x_k, u_k) = 0, \\ & x_N - x_N^s = 0. \end{aligned}$$

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Rotated Problem \equiv Original Problem

$$\sum_{k=0}^{N-1} L(x, u) = \sum_{k=0}^{N-1} l(x, u) + \underbrace{\lambda_s^T x_0 - \lambda_s^T x_N - (N-1) l(x_s, u_s)}_{\text{constant}}$$

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Rotated Problem \equiv Original Problem

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If

$$\alpha(x) \leq L(x, u), \quad \alpha \in \mathcal{K}$$

the previous stability proof holds! [Diehl et al. 2011]

Generalization [Amrit et al. 2011]

Nonlinear rotating function: $\lambda(x)$

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Nonlinear rotating function: $\lambda(x)$

New rotated cost: $L(x, u) = l(x, u) - l(x_s, u_s) + \lambda(x) - \lambda(f(x, u))$

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What conditions on $\lambda(x)$?

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What conditions on $\lambda(x)$?**Strict Dissipativity**

System strictly dissipative wrt the supply rate $s(x, u)$ if $\exists \lambda(x) : \mathbb{X} \rightarrow \mathbb{R}$

$$\lambda(f(x, u)) - \lambda(x) \leq -\rho(x - x_s) + s(x, u),$$

$$\forall (x, u) \in \mathbb{X} \times \mathbb{U}$$

We are interested in $s(x, u) = l(x, u) - l(x_s, u_s)$

Generalization [Amrit et al. 2011]

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$$\forall (x, u) \in \mathbb{X} \times \mathbb{U}$$

We are interested in $s(x, u) = l(x, u) - l(x_s, u_s)$

This entails

$$L(x, u) = l(x, u) - l(x_s, u_s) + \lambda(x) - \lambda(f(x, u)) \geq \rho(x - x_s)$$

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LQ-EMPC

Consider a linear MPC problem

$$\mathcal{P}_N(A, B, Q, R, S, P_N) = \underset{x_0, u_0, \dots, x_N}{\operatorname{argmin}} \sum_{k=0}^{N-1} \begin{bmatrix} x_k \\ u_k \end{bmatrix}^T H \begin{bmatrix} x_k \\ u_k \end{bmatrix} + x_N^T P_N x_N$$

s.t. $x_0 - \bar{x}_i = 0,$
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- If $H \neq 0$ this is an **Economic MPC** problem
- When is it **stabilizing**?

Inifinite Horizon Case

Infinite Horizon Case

This is an (indefinite) LQR

$$\mathcal{D}(A, B, Q, R, S) := \{(P, K) \mid Q + A^T P A - P - (S^T + A^T P B)K = 0, \\ K = (R + B^T P B)^{-1}(S + B^T P A), \\ \rho(A - BK) < 1\}$$

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One can enforce stability by adding $\begin{bmatrix} x \\ u \end{bmatrix}^T T \begin{bmatrix} x \\ u \end{bmatrix}$ to the cost and solving

$$\min_{P, T} \|T\|^2 \quad \text{s.t. } M + T \succeq 0,$$

One can modify the cost-to-go without changing the problem

$$\mathcal{P}_\infty(A, B, Q, R, S) = \mathcal{P}_\infty(A, B, Q_{\bar{P}}, R_{\bar{P}}, S_{\bar{P}}),$$

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- R can be replaced by any $R_K (> 0)$

Positive Definite LQR Formulation

Lyapunov Stability Theorem

If $\rho(A_K) < 1$, then

$$\forall Q_L \succ 0 \quad \exists P_L \succ 0 \text{ s.t.} \quad Q_L + A_K^T P_L A_K - P_L = 0$$

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Pos. def. is preserved when transforming back to (A, B)

What About MPC?

In the unconstrained case, the solution is given by the Discrete Riccati Equation (DRE)

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DRE equivalence

$$\mathcal{R}_N(A, B, Q, R, S, P_N) = \mathcal{R}_N(A, B, \tilde{Q}, \tilde{R}, \tilde{S}, \tilde{P}_N)$$

if

$$\begin{aligned} \tilde{P}_N - \tilde{P} &= P_N - P, \\ \tilde{R} + B^T \tilde{P} B &= R + B^T P B, \\ \tilde{S} + B^T \tilde{P} A &= S + B^T P A \end{aligned}$$

with P and \tilde{P} computed from $\mathcal{D}(A, B, Q, R, S)$ and $\mathcal{D}(A, B, \tilde{Q}, \tilde{R}, \tilde{S})$.

A Practical View on the Problem

Solve the following SDP

$$\begin{aligned}
 & \min_{\tilde{P}, \tilde{Q}, \tilde{R}, \tilde{S}, \tilde{H}} \|\tilde{P} - I\|^2 + \|\tilde{H} - I\|^2 \\
 & \text{s.t. } \tilde{H} = \begin{bmatrix} \tilde{Q} & \tilde{S}^T \\ \tilde{S} & \tilde{R} \end{bmatrix} \\
 & \tilde{H} \succeq 0 \\
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This problem is convex!

- 1 Economic MPC and Stability Analysis
- 2 The Linear Quadratic Case
- 3 **Approximated EMPC with stability guarantees**

Look at the Steady State Properties

Lagrangian of steady state problem

$$\mathcal{L} = l(x, u) - \lambda^T (x - f(x, u))$$

Lagrangian Hessian:

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$$A = \frac{\partial f(x, u)}{\partial x} \Big|_{x_S, u_S} \qquad B = \frac{\partial f(x, u)}{\partial u} \Big|_{x_S, u_S}$$

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- Formulate a tracking NMPC scheme using \tilde{H}

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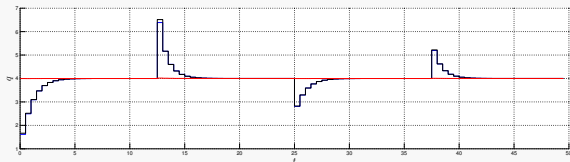
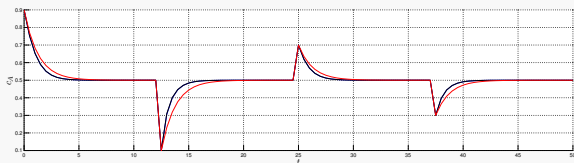
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Thank you for your attention!