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# New Contributions to Theory and Numerics for State-constrained Elliptic Optimal Control Problems

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# New Contributions to Theory and Numerics for State-constrained Elliptic Optimal Control Problems

Hans Josef Pesch

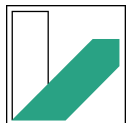
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**University of Graz, Austria**

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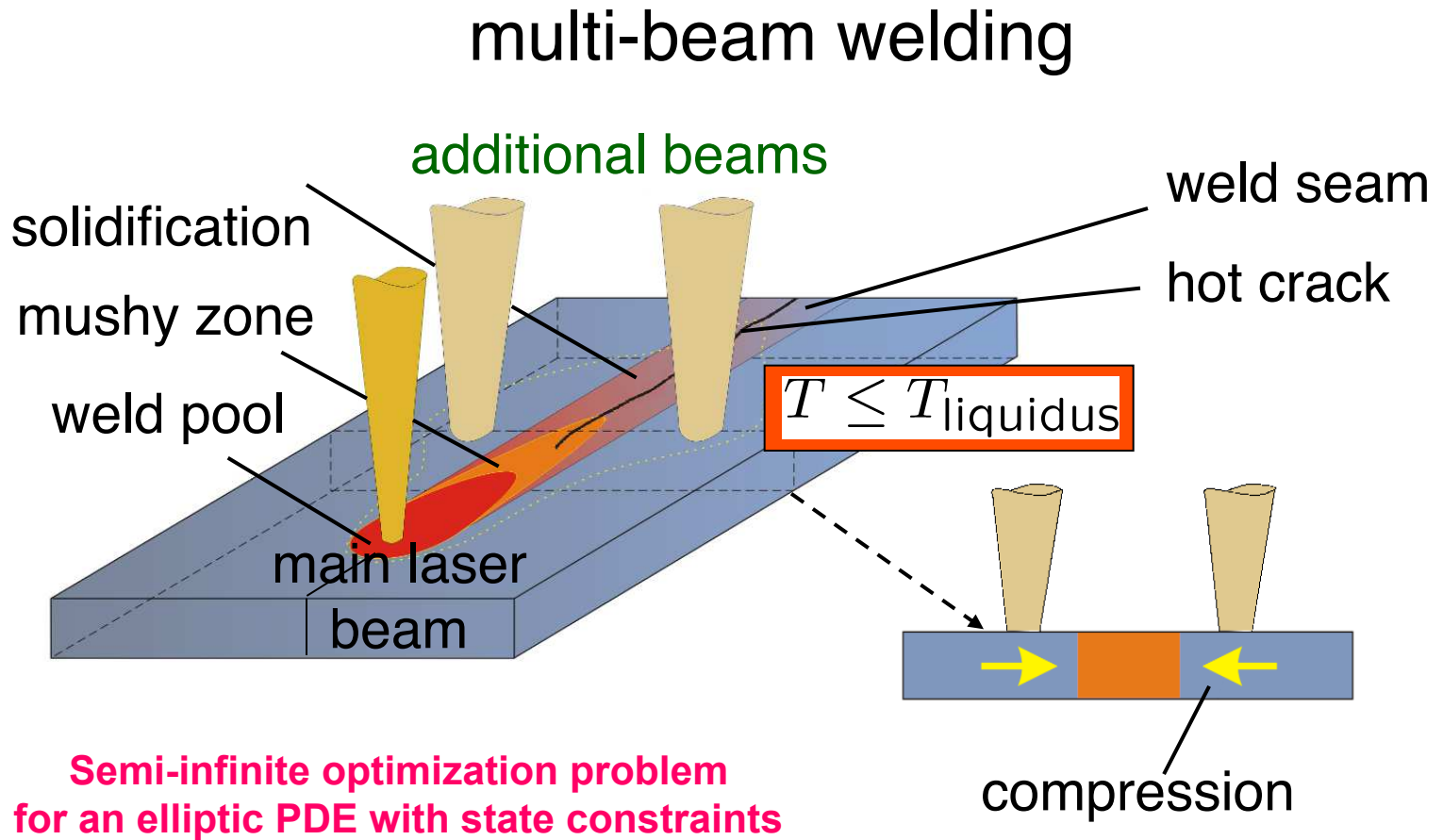
UNIVERSITÄT  
BAYREUTH

ITN SADCO NetCO 2014

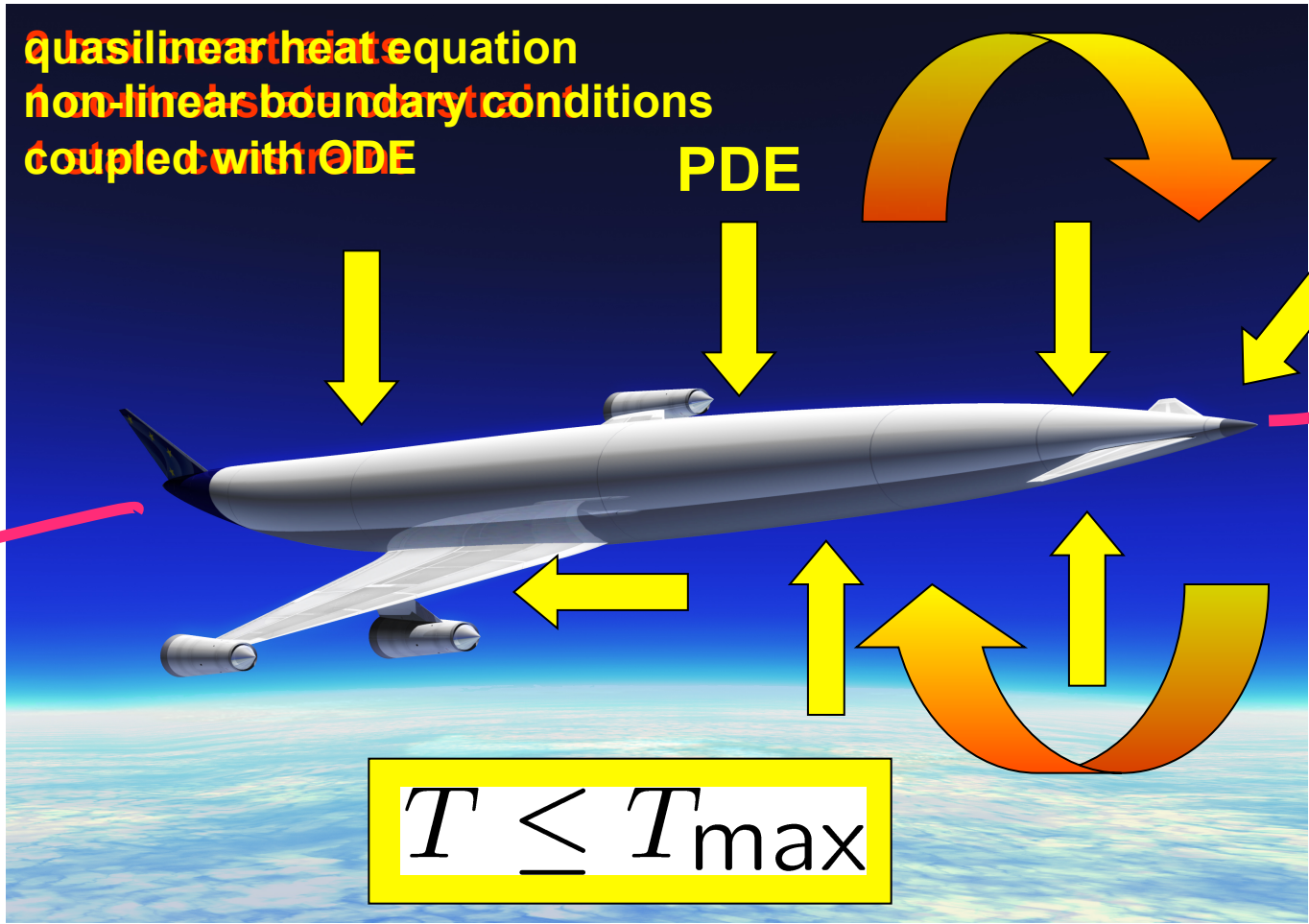
Conf. on New Trends in Optimal Control, Tours, June 23-27, 14

Lehrstuhl  
Ingenieur-  
mathematik

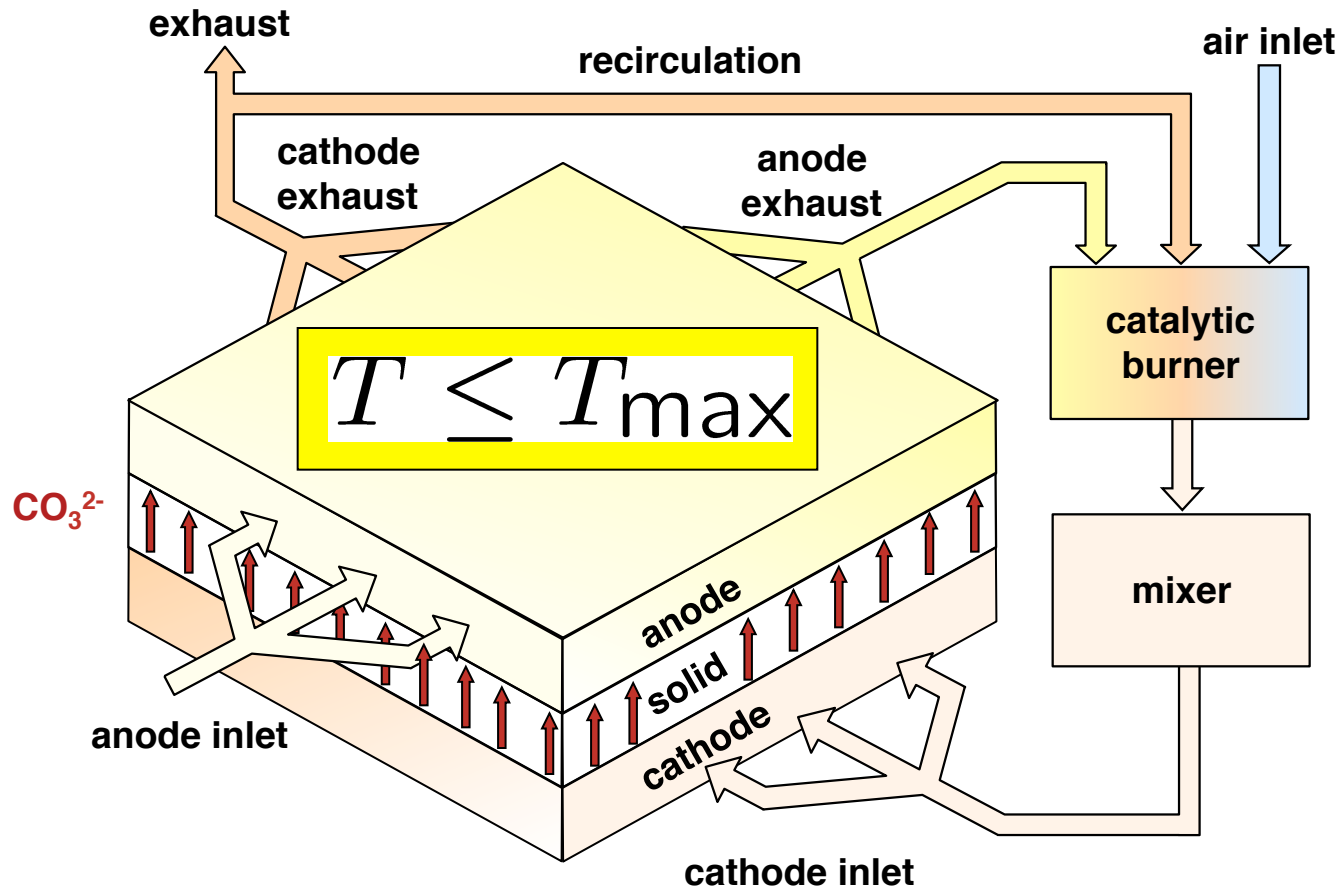




# Motivation: Hypersonic Flights



## 2D cross-flow design of a high temperature fuel cell



**28 quasi-linear partial integro-differential-algebraic equations with non-standard non-linear boundary conditions**





- **Introduction**
- Split method
- Bryson-Denham-Dreyfus approach (BDD)
- Shape calculus and optimization on vector bundles
- Numerics



**Minimize**

$$J(y, u) := \frac{1}{2} \|y - y_d\|_{L^2(\Omega)}^2 + \frac{\lambda}{2} \|u\|_{L^2(\Omega)}^2$$

**subject to**

$$-\Delta y + y = u \quad \text{a. e. in } \Omega$$

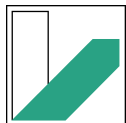
$$\partial_\nu y = 0 \quad \text{a. e. on } \Gamma$$

$$G(y) := y - y_{\max} \leq 0 \quad \text{a. e. in } \Omega$$

**with**

$\Omega \subset \mathbb{R}^2$  bounded domain,  $\Gamma \in C^{1,1}$ ,

$u \in U_{\text{ad}} := L^2(\Omega)$ ,  $y_d \in L^2(\Omega)$ ,  $\lambda > 0$ ,  $y_{\max} \in H^4(\Omega)$





## Theorem (Casas 1986): Slater condition

$$\implies \exists p^{\text{trad}} \in W^{1,s}(\Omega) \text{ and } \mu = \mu_{\Omega} + \mu_{\Gamma} \in C(\bar{\Omega})^*$$

such that

$$-\Delta y + y = u \quad \text{a. e. in } \Omega$$

$$\partial_{\nu} y = 0 \quad \text{a. e. on } \Gamma$$

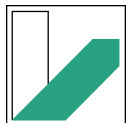
$$-\Delta p^{\text{trad}} + p^{\text{trad}} = y - y_d + \mu_{\Omega} \quad \text{a. e. in } \Omega$$

$$\partial_{\nu} p^{\text{trad}} = \mu_{\Gamma} \quad \text{a. e. on } \Gamma$$

$$\lambda u + p^{\text{trad}} = 0 \quad \text{a. e. in } \Omega$$

$$\langle y - y_{\max}, \mu \rangle_{C(\bar{\Omega}), C(\bar{\Omega})^*} = 0, \quad \mu > 0$$

**low regularity causes problems in numerical treatment**



## Goals

- new necessary conditions with higher regularity of Lagrange multipliers
- formulate efficient numerical algorithms, which don't require any regularization technique, exploit the structure of the multiplier, and are mesh-independent

## Ideas

- geometric split → set optimal control problem
- BDD approach → higher regularity
- shape calculus → necessary conditions
- optimization on vector bundles → design of algorithms



## Goals

- new necessary conditions with higher regularity of Lagrange multipliers
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- BDD approach → higher regularity
- shape calculus → necessary conditions
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- Introduction
- **Split method**
- Bryson-Denham-Dreyfus approach (BDD)
- Shape calculus and optimization on vector bundles
- Numerics



**Definition: active / inactive set / interface**

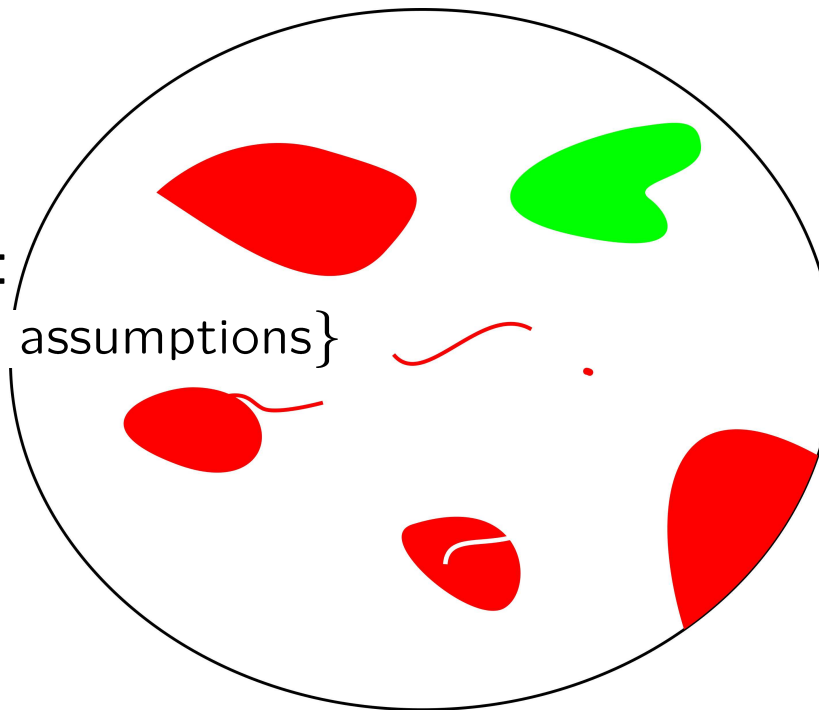
$$\mathcal{A} := \{x \in \bar{\Omega} : G(\bar{y}) = 0\}$$

$$\mathcal{I} := \Omega \setminus \mathcal{A}$$

$$\gamma := \partial\mathcal{A}$$

**Assumption  
on admissible  
active sets**

$$\mathcal{B}_{\text{ad}} := \{\mathcal{B} \subset \Omega : \mathcal{B} \text{ fulfills assumptions}\}$$



**No degeneracy.  
No active set  
of zero measure.  
No common points  
with boundary.**



# Reformulation of the model problem

(Analog to the multipoint-boundary-value-problem formulation; Oberle)

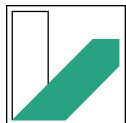
**Minimize**

$$J(y_{\mathcal{I}}, y_{\mathcal{A}}, u_{\mathcal{I}}, u_{\mathcal{A}}) := \frac{1}{2} \|y_{\mathcal{I}} - y_d|_{\mathcal{I}}\|_{L^2(\mathcal{I})}^2 + \frac{1}{2} \|y_{\mathcal{A}} - y_d|_{\mathcal{A}}\|_{L^2(\mathcal{A})}^2 \\ + \frac{\lambda}{2} \|u_{\mathcal{I}}\|_{L^2(\mathcal{I})}^2 + \frac{\lambda}{2} \|u_{\mathcal{A}}\|_{L^2(\mathcal{A})}^2$$

**subject to**

$$\begin{aligned} -\Delta y_{\mathcal{I}} + y_{\mathcal{I}} &= u_{\mathcal{I}}, & \text{a. e. in } \mathcal{I} \\ -\Delta y_{\mathcal{A}} + y_{\mathcal{A}} &= u_{\mathcal{A}}, & \text{a. e. in } \tilde{\mathcal{A}} \\ \partial_{\nu} y_{\mathcal{I}} &= 0, & \text{a. e. on } \Gamma \\ y_{\mathcal{I}} &= y_{\mathcal{A}}, & \text{a. e. on } \gamma \\ \partial_{\nu_{\mathcal{I}}} y_{\mathcal{I}} + \partial_{\nu_{\mathcal{A}}} y_{\mathcal{A}} &= 0, & \text{a. e. on } \gamma \\ y_{\mathcal{A}} &= y_{\max}, & \text{a. e. in } \tilde{\mathcal{A}} \\ y_{\mathcal{J}} &< y_{\max}, & \text{a. e. in } \tilde{\mathcal{I}} \\ y_{\mathcal{A}} &= y_{\max}, & \text{a. e. on } \gamma \end{aligned}$$

**How can we determine the active set?**



# Reformulation as set optimal control problem

**Minimize**

$$J(y_{\mathcal{J}}, y_{\mathcal{B}}, u_{\mathcal{J}}, u_{\mathcal{B}}, \mathcal{B}) := \frac{1}{2} \|y_{\mathcal{J}} - y_d|_{\mathcal{J}}\|_{L^2(\mathcal{J})}^2 + \frac{1}{2} \|y_{\mathcal{B}} - y_d|_{\mathcal{B}}\|_{L^2(\mathcal{B})}^2 \\ + \frac{\lambda}{2} \|u_{\mathcal{J}}\|_{L^2(\mathcal{J})}^2 + \frac{\lambda}{2} \|u_{\mathcal{B}}\|_{L^2(\mathcal{B})}^2$$

**subject to**

$$\begin{aligned} -\Delta y_{\mathcal{J}} + y_{\mathcal{J}} &= u_{\mathcal{J}}, & \text{a. e. in } \mathcal{J} \\ -\Delta y_{\mathcal{B}} + y_{\mathcal{B}} &= u_{\mathcal{B}}, & \text{a. e. in } \mathring{\mathcal{B}} \\ \partial_{\nu} y_{\mathcal{J}} &= 0, & \text{a. e. on } \Gamma \\ y_{\mathcal{J}} &= y_{\mathcal{B}}, & \text{a. e. on } \beta \\ \partial_{\nu_{\mathcal{J}}} y_{\mathcal{J}} + \partial_{\nu_{\mathcal{B}}} y_{\mathcal{B}} &= 0, & \text{a. e. on } \beta \\ y_{\mathcal{B}} &= y_{\max}, & \text{a. e. in } \mathring{\mathcal{B}} \\ y_{\mathcal{J}} &< y_{\max}, & \text{a. e. in } \mathring{\mathcal{J}} \\ y_{\mathcal{B}} &= y_{\max}, & \text{a. e. on } \beta \end{aligned}$$

**How to handle  
the strict inequality?**

**Theorem: The original problem and the set optimal control problem  
possess the same unique solution**

# Set optimal control problem as bilevel optimization problem

**Minimize**

$$J(y_{\mathcal{J}}, y_{\mathcal{B}}, u_{\mathcal{J}}, u_{\mathcal{B}}, \mathcal{B}) := \underbrace{\frac{1}{2} \|y_{\mathcal{J}} - y_d|_{\mathcal{J}}\|_{L^2(\mathcal{J})}^2}_{\text{inner}} + \underbrace{\frac{1}{2} \|y_{\mathcal{B}} - y_d|_{\mathcal{B}}\|_{L^2(\mathcal{B})}^2}_{\text{outer}} + \frac{\lambda}{2} \|u_{\mathcal{J}}\|_{L^2(\mathcal{J})}^2 + \frac{\lambda}{2} \|u_{\mathcal{B}}\|_{L^2(\mathcal{B})}^2$$

**subject to**

$$-\Delta y_{\mathcal{J}} + y_{\mathcal{J}} = u_{\mathcal{J}}, \quad \text{a. e. in } \mathcal{J}$$

$$-\Delta y_{\mathcal{B}} + y_{\mathcal{B}} = u_{\mathcal{B}}, \quad \text{a. e. in } \mathring{\mathcal{B}}$$

$$\partial_{\nu} y_{\mathcal{J}} = 0, \quad \text{a. e. on } \Gamma$$

$$y_{\mathcal{J}} = y_{\mathcal{B}}, \quad \text{a. e. on } \beta$$

$$\partial_{\nu_{\mathcal{J}}} y_{\mathcal{J}} + \partial_{\nu_{\mathcal{B}}} y_{\mathcal{B}} = 0, \quad \text{a. e. on } \beta$$

$$y_{\mathcal{B}} = y_{\max}, \quad \text{a. e. in } \mathring{\mathcal{B}}$$

$$y_{\mathcal{J}} < y_{\max}, \quad \text{a. e. in } \mathring{\mathcal{J}}$$

$$y_{\mathcal{B}} = y_{\max}, \quad \text{a. e. on } \beta$$

**constraint of  
outer optimization**

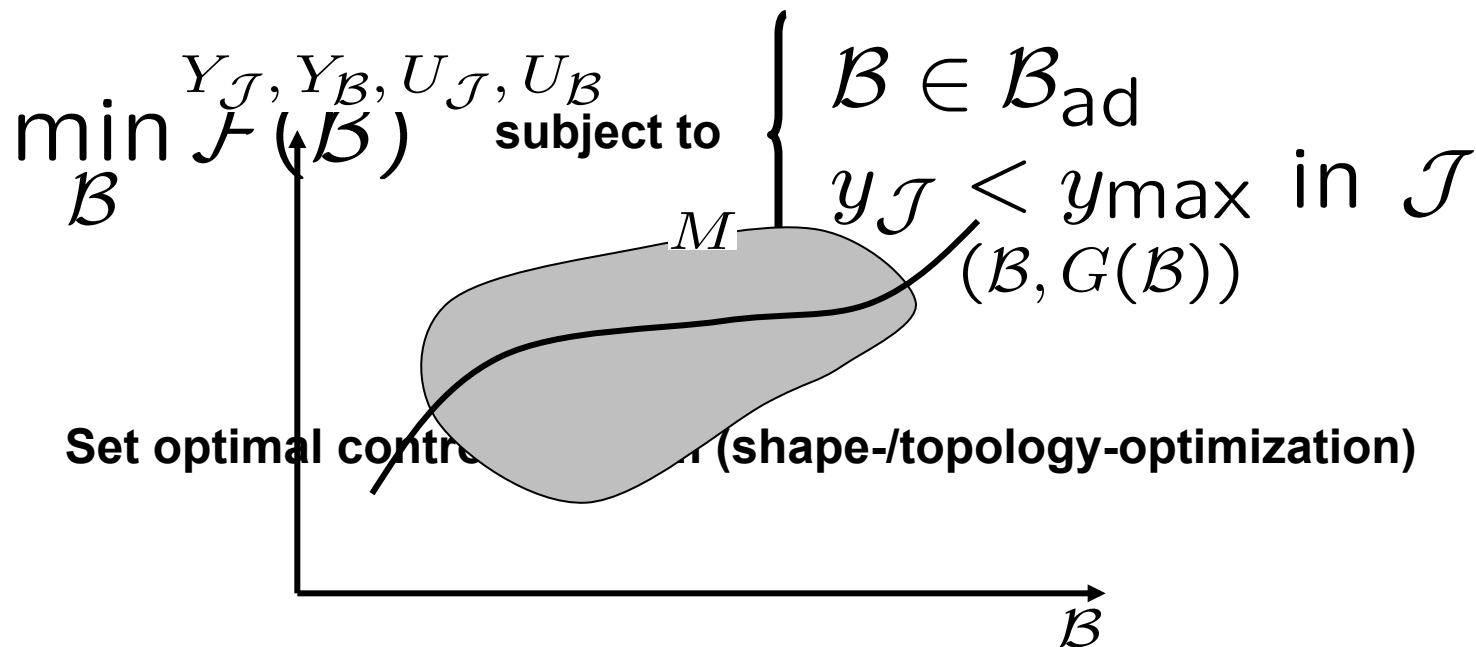
**Theorem: The inner optimization problem  
possesses a unique solution for  $\mathcal{B} \in \mathcal{B}_{\text{ad}}$**



Reduced functional:  $\exists G: \mathcal{B} \rightarrow (y_{\mathcal{J}}, y_{\mathcal{B}}, u_{\mathcal{J}}, u_{\mathcal{B}})$

$$\mathcal{F}(\mathcal{B}) = J(G(\mathcal{B}), \mathcal{B})$$

is well-defined on  $\mathcal{B}_{\text{ad}}$ .



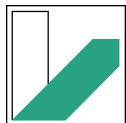
Reduced functional:  $\exists G: \mathcal{B} \rightarrow (y_{\mathcal{J}}, y_{\mathcal{B}}, u_{\mathcal{J}}, u_{\mathcal{B}})$

$$\mathcal{F}(\mathcal{B}) = J(G(\mathcal{B}), \mathcal{B})$$

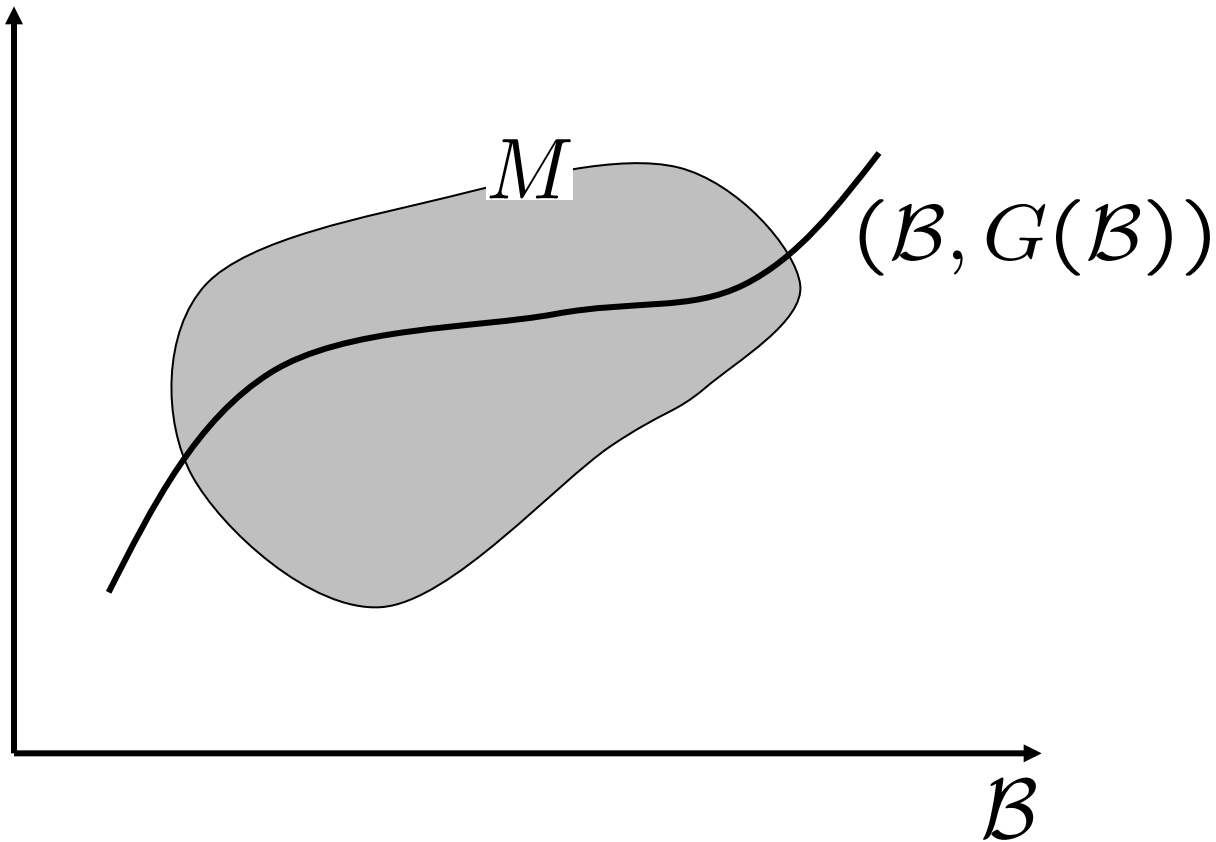
is well-defined on  $\mathcal{B}_{\text{ad}}$  :

$$\min_{\mathcal{B}} \mathcal{F}(\mathcal{B}) \quad \text{subject to} \quad \begin{cases} \mathcal{B} \in \mathcal{B}_{\text{ad}} \\ y_{\mathcal{J}} < y_{\text{max}} \text{ in } \mathcal{J} \end{cases}$$

**Set optimal control problem (shape-/topology-optimization)**

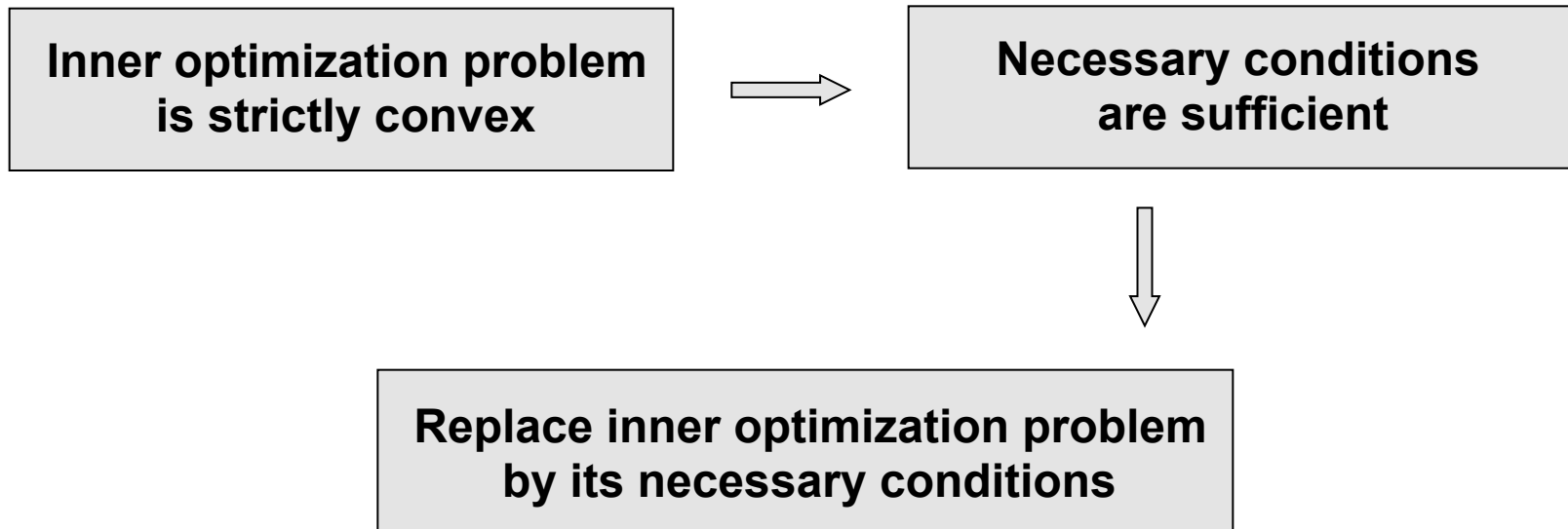


$Y_{\mathcal{J}}, Y_{\mathcal{B}}, U_{\mathcal{J}}, U_{\mathcal{B}}$



## Theorem:

For any  $\mathcal{B} \in \mathcal{B}_{\text{ad}}$  there exist Lagrange multipliers associated with the equality constraints of the inner optimization problem



# Set optimal control problem

$$\min_{\mathcal{B} \in \mathcal{B}_{\text{ad}}} \mathcal{F}(\mathcal{B}) \quad \text{subject to} \quad y_{\mathcal{J}} < y_{\text{max}} \text{ in } \mathcal{J}$$

$$\text{and} \quad -\Delta y_{\mathcal{J}} + y_{\mathcal{J}} = -\frac{1}{\lambda} p_{\mathcal{J}}, \quad \text{a. e. in } \mathcal{J}$$

$$\partial_{\nu} y_{\mathcal{J}} = 0, \quad \text{a. e. on } \Gamma$$

$$-\Delta p_{\mathcal{J}} + p_{\mathcal{J}} = y_{\mathcal{J}} - y_d, \quad \text{a. e. in } \mathcal{J}$$

$$\partial_{\nu} p_{\mathcal{J}} = 0, \quad \text{a. e. on } \Gamma$$

$$y_{\mathcal{J}} = y_{\text{max}}, \quad \text{a. e. on } \beta$$

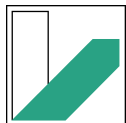
$$\partial_{\nu_{\mathcal{J}}} y_{\mathcal{J}} = \partial_{\nu_{\mathcal{J}}} y_{\text{max}}, \quad \text{a. e. on } \beta$$

no measures  
involved

unusual  
boundary  
conditions

everything else can be computed a posteriori

$$\begin{array}{l|l} \begin{array}{l} y_{\mathcal{B}} = y_{\text{max}}, \\ u_{\mathcal{B}} = -\Delta y_{\text{max}} + y_{\text{max}}, \\ p_{\mathcal{B}} = -\lambda u_{\mathcal{B}}, \end{array} & \begin{array}{l} \text{a. e. in } \mathcal{B} \\ \text{a. e. in } \mathcal{B} \\ \text{a. e. in } \mathcal{B} \end{array} \end{array} \quad \left| \quad \begin{array}{l} u_{\mathcal{J}} = -\frac{1}{\lambda} p_{\mathcal{J}}, \\ \mu_{\mathcal{B}} = \partial_{\nu_{\mathcal{J}}} p_{\mathcal{J}} + \partial_{\nu_{\mathcal{B}}} p_{\mathcal{B}}, \\ \mu_{\mathcal{B}} = -\Delta p_{\mathcal{B}} + p_{\mathcal{B}} - y_{\mathcal{B}} + y_{\text{max}}, \end{array} \quad \begin{array}{l} \text{a. e. in } \mathcal{J} \\ \text{a. e. on } \beta \\ \text{a. e. in } \mathcal{B} \end{array}$$



# Shape calculus for the optimal active set

$$\min_{\mathcal{B}} \underbrace{\min_{y_{\mathcal{J}}, y_{\mathcal{B}}, u_{\mathcal{J}}, u_{\mathcal{B}}} J(y_{\mathcal{J}}, y_{\mathcal{B}}, u_{\mathcal{J}}, u_{\mathcal{B}}, \mathcal{B})}_{\mathcal{F}(\mathcal{B})}$$

subject to the optimality system of the inner optimization problem

**Theorem:**

For each admissible  $\mathcal{B}$  the objective  $\mathcal{F}(\mathcal{B})$  is shape differentiable. The semi-derivative in the direction

$$V \in \{W \in C^{1,1}(\bar{\Omega}) \mid \langle W, \nu \rangle = 0 \text{ on } \Gamma\}$$

is

$$d\mathcal{F}(\mathcal{B}; V) = \int_{\beta} -\frac{1}{2\lambda} \left[ p_{\mathcal{J}} - \underbrace{\lambda (\Delta y_{\max} - y_{\max})}_{p_{\max}} \right]^2 \langle V, \nu_{\mathcal{J}} \rangle$$



determines the interface

We omit  $y_{\mathcal{J}} < y_{\max}$  in the outer optimization and determine the interface  $\beta$  by

$$d\mathcal{F}(\mathcal{B}; V) = 0 \quad \forall \quad V \in C^{1,1}(\bar{\Omega})$$

$$\implies p_{\mathcal{J}} = p_{\max} \quad \text{on } \beta$$

$$\implies u_{\mathcal{J}} = u_{\mathcal{B}} \quad \text{on } \beta$$

Needs an a posteriori-check on feasibility:  $y_{\mathcal{J}} < y_{\max}$



- Introduction
- Split method
- **Bryson-Denham-Dreyfus<sup>\*</sup> approach (BDD)**
- Shape calculus and optimization on vector bundles
- Numerics

\* A.E. Bryson, Jr, W.F. Denham, S.E. Dreyfus: Optimal programming problems with inequality constraints I, AIAA Journal 1(11):2544-2550, 1963.

Later extended by Maurer, 1979.





## Transferring the Bryson-Denham-Dreyfus approach

$$G(\bar{y}_{\mathcal{A}}) \equiv 0 \text{ a. e. in } \tilde{\mathcal{A}} \iff$$

$$-\Delta G(\bar{y}_{\mathcal{A}}) + G(\bar{y}_{\mathcal{A}}) = 0, \quad \text{a. e. in } \tilde{\mathcal{A}}$$

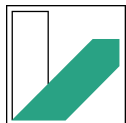
$$G(\bar{y}_{\mathcal{A}}) = 0, \quad \text{a. e. on } \gamma$$

## Using the state equation

$$\Delta y_{\max} - y_{\max} + \bar{u}_{\mathcal{A}} = 0, \quad \text{a. e. in } \tilde{\mathcal{A}}$$

$$\bar{y}_{\mathcal{A}} = y_{\max}, \quad \text{a. e. on } \gamma$$

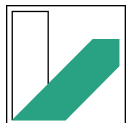
Optimal solution on  $\tilde{\mathcal{A}}$  given by data, but  $\tilde{\mathcal{A}}$  optimization variable



# First-order necessary condition of set-OCP (split method)

(traditional adjoint state and multipliers)

a. e. in $\mathcal{I}$ a. e. on $\Gamma$	a. e. on $\gamma$	a. e. in $\mathcal{A}$
$-\Delta y_I + y_I = u_I$ $\partial_\nu y_I = 0$	$y_I = y_A$ $\partial_{\nu_I} y_I + \partial_{\nu_A} y_A = 0$	$-\Delta y_A + y_A = u_A$
$y_I < y_{\max}$	$y_A = y_{\max}$	$y_A = y_{\max}$
$-\Delta p_I + p_I = y_I - y_d$ $\partial_\nu p_I = 0$	$p_I = p_A$ $\partial_{\nu_I} p_I + \partial_{\nu_A} p_A = \mu_\gamma$	$-\Delta p_A + p_A = y_A - y_d + \mu_A$
$\lambda u_I + p_I = 0$	$u_I - u_A = 0$	$\lambda u_A + p_A = 0$
	$\mu_\gamma \geq 0$	$\mu_A \geq 0$



# First-order necessary condition of set-OCP (BDD method)

(new adjoint state and multipliers with higher regularity)

a. e. in $\mathcal{I}$ a. e. on $\Gamma$	a. e. on $\gamma$	a. e. in $\tilde{\mathcal{A}}$
$-\Delta y_I + y_I = u_I$ $\partial_\nu y_I = 0$	$y_I = y_A$ $\partial_{\nu_I} y_I + \partial_{\nu_A} y_A = 0$	$-\Delta y_A + y_A = u_A$
$y_I < y_{\max}$	$y_A = y_{\max}$	$-\Delta y_{\max} + y_{\max} = u_A$
$-\Delta p_I + p_I = y_I - y_d$ $\partial_\nu p_I = 0$	$p_I = p_A \quad \downarrow$ $\partial_{\nu_I} p_I + \partial_{\nu_A} p_A = \sigma$	$-\Delta p_A + p_A = y_A - y_d$
$\lambda u_I + p_I = 0$	$u_I - u_A = 0$	$\lambda u_A + p_A + q = 0 \quad \downarrow$
	$\rightarrow \sigma \geq 0$	$\rightarrow q \geq 0$



(new adjoint state and multipliers with higher regularity)

$$-\Delta q + q = \mu_{\mathcal{A}}, \text{ a.e. in } \mathring{\mathcal{A}}$$

$$\partial_{\nu_{\mathcal{A}}} q + \sigma = \mu_{\gamma}, \text{ a.e. on } \gamma$$

**BDD approach reveals control law,  
i.e. “hidden” condition (as for PDAE)**

$$u_{\mathcal{A}} = -\Delta y_{\max} + y_{\max}$$

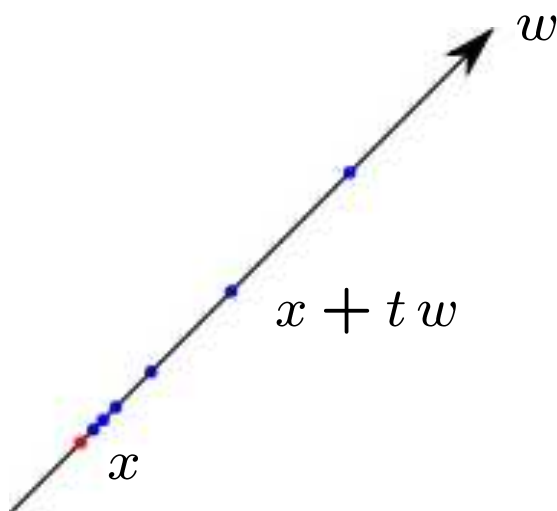


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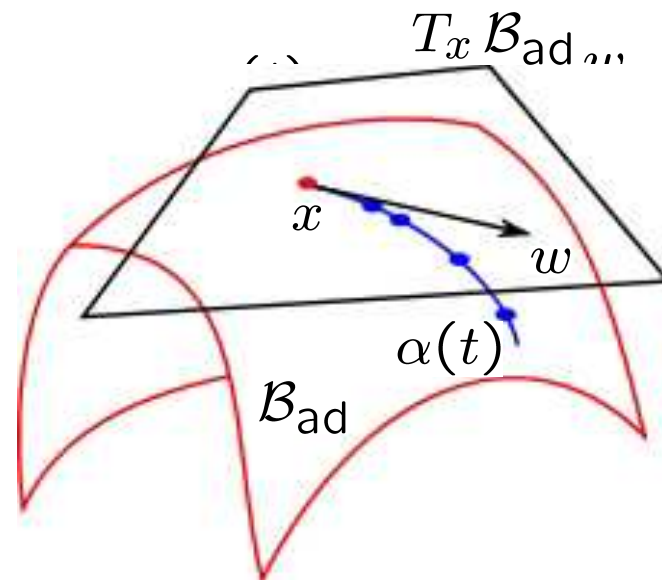


# Basic considerations with respect to shape calculus

## Gateaux directional derivative

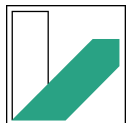


## Hadamard directional derivative



$$\lim_{t \rightarrow 0} \frac{1}{t} (J(x + tw) - J(x))$$

$$\lim_{\substack{t \rightarrow 0 \\ \dot{\alpha}(0) = w}} \frac{1}{t} (J(\alpha(t)) - J(x))$$



# Basic considerations with respect to shape calculus

Delfour, Zolésio, 2011

set  $\mathcal{B} \in \mathcal{B}_{\text{ad}} \subset \mathcal{M}$

(no linear structure,  
infinite dimensional  
manifold)

vector field  $V$

image

$$A := (\text{Id} + V)(\mathcal{B})$$

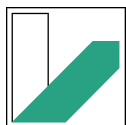
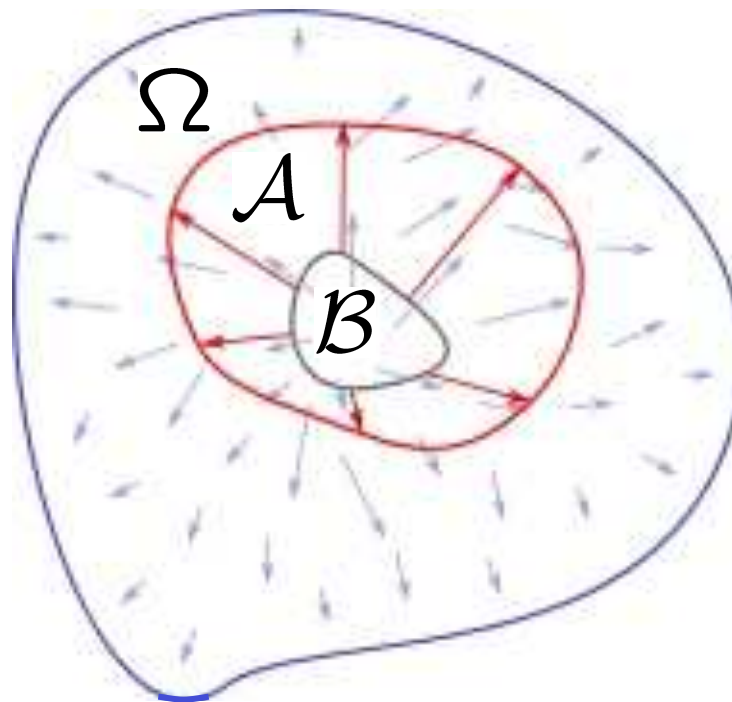
holdall  $\Omega$

perturbation of identity  
defines curves  $\mathcal{B}_{\text{ad}}$

$$\mathcal{B} \rightarrow T_t(\mathcal{B}) := (\text{Id} + tV)(\mathcal{B})$$

$$\mathcal{B}_t = T_t(\mathcal{B})$$

$$\Rightarrow dJ(\mathcal{B}; V) := \lim_{t \rightarrow 0} \frac{J(\mathcal{B}_t) - J(\mathcal{B})}{t}$$



- Hadamard directional derivative is suitable for nonlinear spaces
- Deformation of sets yields „perturbation of identity“
- Metric of function spaces induces metric in  $\mathcal{B}_{\text{ad}}$

$\mathcal{B}_{\text{ad}}$  has no linear structure

$\mathcal{B}_{\text{ad}}$  is similar to an infinite dimensional manifold

- $t \mapsto \mathcal{B}_t$  defines curves in  $\mathcal{B}_{\text{ad}}$
- suitable difference quotient

$$\lim_{t \rightarrow 0} \frac{J(\mathcal{B}_t) - J(\mathcal{B})}{t}$$





# Exemplary derivation of directional shape derivative

Let

$$\mathcal{F}(\mathcal{B}) = \int_{\mathcal{B}} f(\mathcal{B}, X) \, dX$$

implicit  
explicit set dependence

$$\begin{aligned} d\mathcal{F}(\mathcal{B}, V) &= \lim_{t \rightarrow 0} \frac{1}{t} \left( \int_{\mathcal{B}_t} f(\mathcal{B}_t, X) \, dX - \int_{\mathcal{B}} f(\mathcal{B}, X) \, dX \right) \\ &= \lim_{t \rightarrow 0} \frac{1}{t} \left( \int_{\mathcal{B}} f(\mathcal{B}_t, T_t(X)) \det DT_t(X) \, dX - \int_{\mathcal{B}} f(\mathcal{B}, X) \, dX \right) \\ &= \dots \\ &= \underbrace{\int_{\partial\mathcal{B}} f(\mathcal{B}, X) (V(X) \cdot \nu(X)) \, dS(X)}_{\text{explicit derivative}} + \underbrace{\int_{\mathcal{B}} \frac{\partial}{\partial t} f(\mathcal{B}_t, X) \Big|_{t=0} \, dX}_{\text{implicit derivative}} \end{aligned}$$



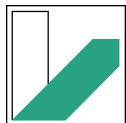
- **shape calculus is similar to calculus on manifolds**

➔ **intrinsic nonlinear behaviour**

- **shape (directional) derivative**

**concentrated on boundary and**

**on the normal component of the vector field only**



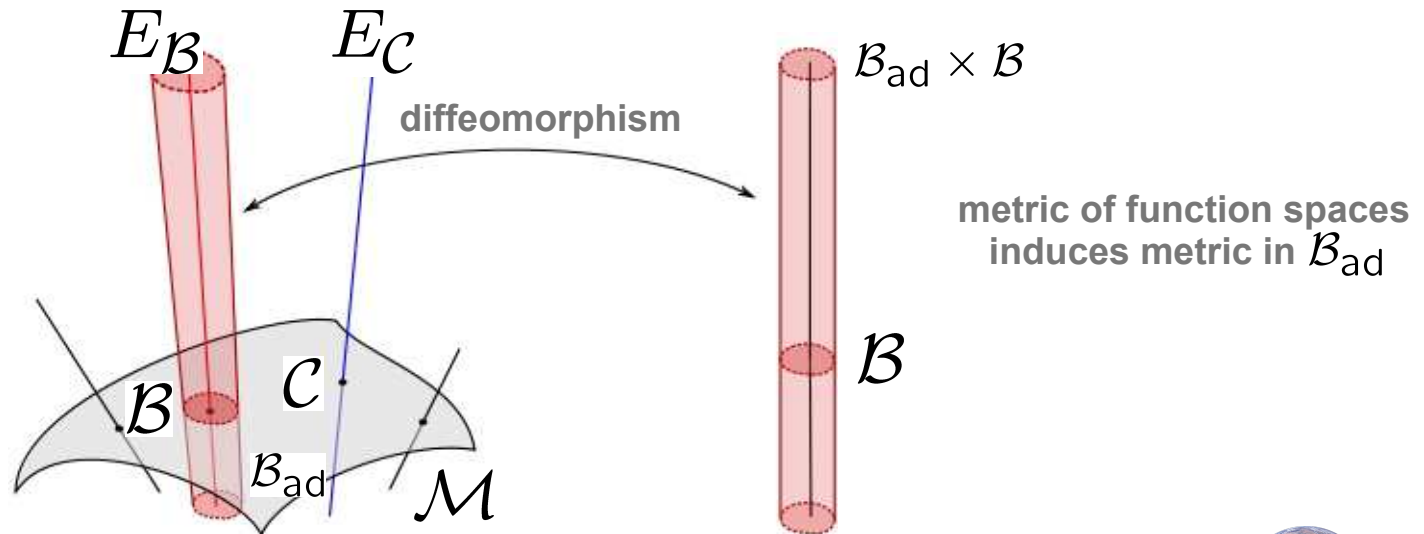
Lang, 1995

Choice of  $\mathcal{B}$  predefines the function space  $E_{\mathcal{B}}$  :

$$\mathcal{B} \in \mathcal{B}_{\text{ad}} \quad \longrightarrow$$

$$(y_{\mathcal{J}}, u_{\mathcal{J}}, y_{\mathcal{B}}, u_{\mathcal{B}}) \in E_{\mathcal{B}} = H^1(\mathcal{J}) \times L^2(\mathcal{J}) \times H^1(\mathcal{B}) \times L^2(\mathcal{B})$$

What is the inherent structure of  $\{E_{\mathcal{B}} | \mathcal{B} \in \mathcal{B}_{\text{ad}}\}$  ?



## Typical shape optimization problem

**Minimize**  $J(y, \mathcal{B})$

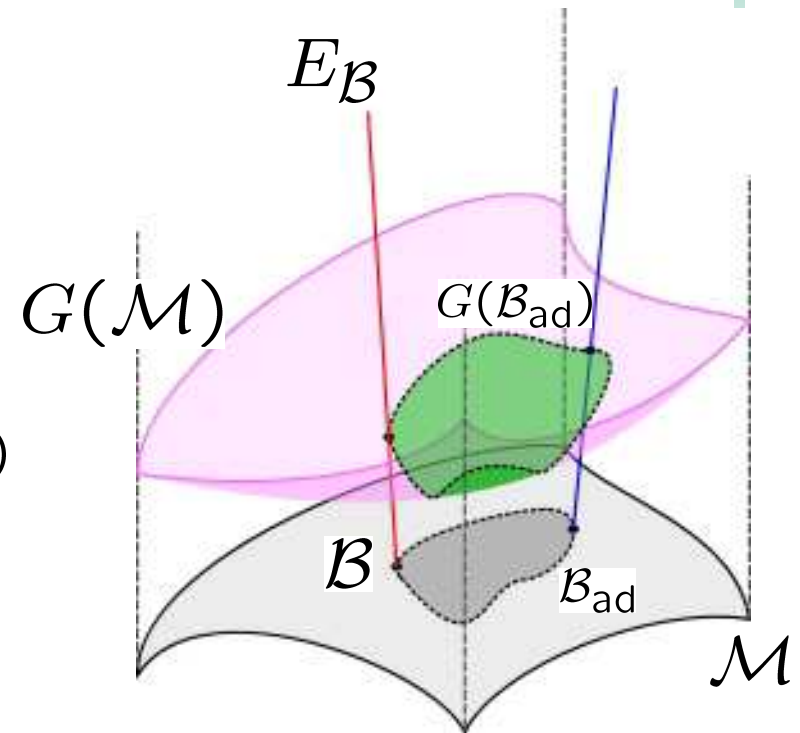
**s.t. a BVP for**  $y$  **on**  $\mathcal{B}$

**with**  $\mathcal{B} \in \mathcal{B}_{\text{ad}} \subset \mathcal{M}$ ,  $y \in E_{\mathcal{B}}$

**Unique solvability implies**  $y = G(\mathcal{B})$

**Minimize**  $\mathcal{F}(\mathcal{B}) = J(G(\mathcal{B}), \mathcal{B})$

**s.t.**  $\mathcal{B} \in \mathcal{B}_{\text{ad}} \subset \mathcal{M}$



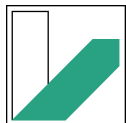
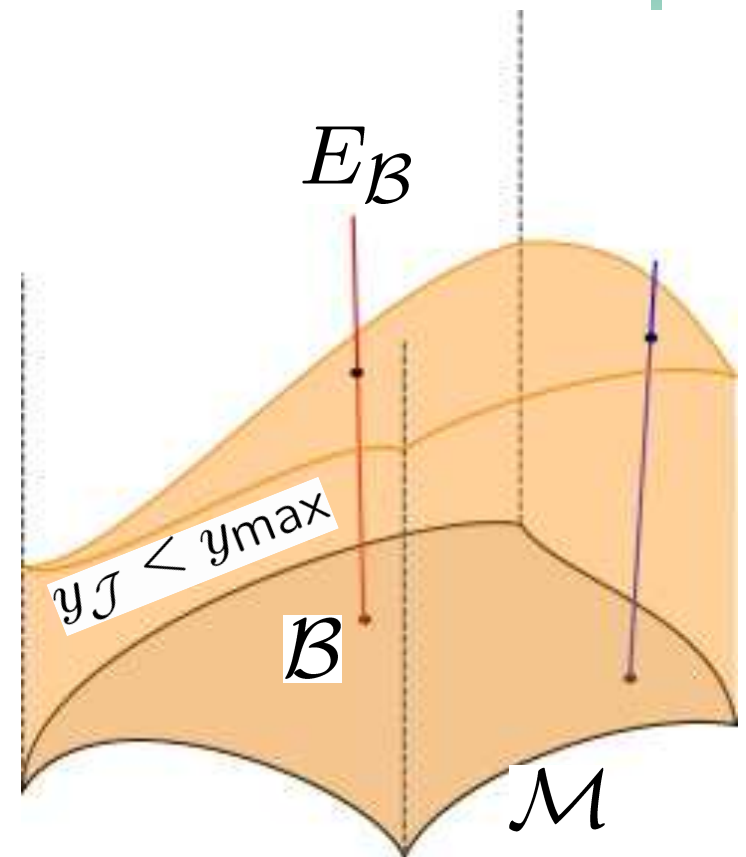
## Set optimal control problem

Minimize  $J(\mathcal{B}, y, u)$

s.t. a BVP for  $(y, u)$  on  $\mathcal{B}$

with  $y_{\mathcal{J}} < y_{\max}$

$$\mathcal{B} \in \mathcal{M}, (y, u) \in E_{\mathcal{B}}$$



## Set optimal control problem

Minimize  $J(\mathcal{B}, y, u)$

s.t. a BVP for  $(y, u)$  on  $\mathcal{B}$

with  $y_{\mathcal{J}} < y_{\max}$

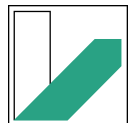
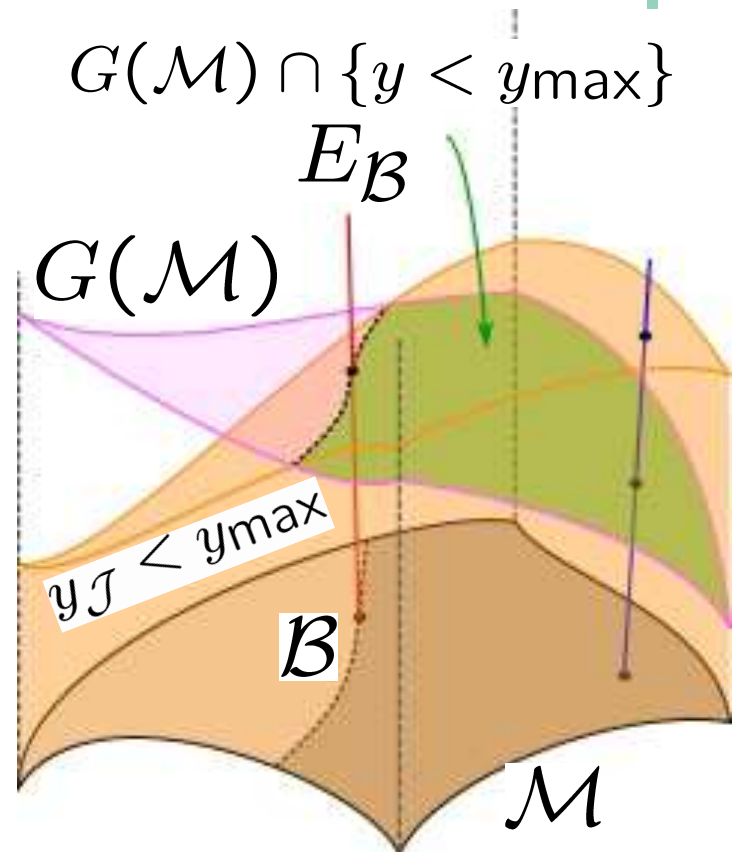
$$\mathcal{B} \in \mathcal{M}, (y, u) \in E_{\mathcal{B}}$$

Unique solvability implies  $(y, u) = G(\mathcal{B})$

Minimize  $\mathcal{F}(\mathcal{B}) = J(\mathcal{B}, G(\mathcal{B}))$

s.t.  $\mathcal{B} \in \mathcal{B}_{\text{ad}} \subset \mathcal{M}$

$$y_{\mathcal{J}} < y_{\max}$$



- Introduction
- Split method
- Bryson-Denham-Dreyfus approach (BDD)
- Shape calculus and optimization on vector bundles
- **Numerics**



# Analysis of necessary conditions

Linear PDAE

A posterior check

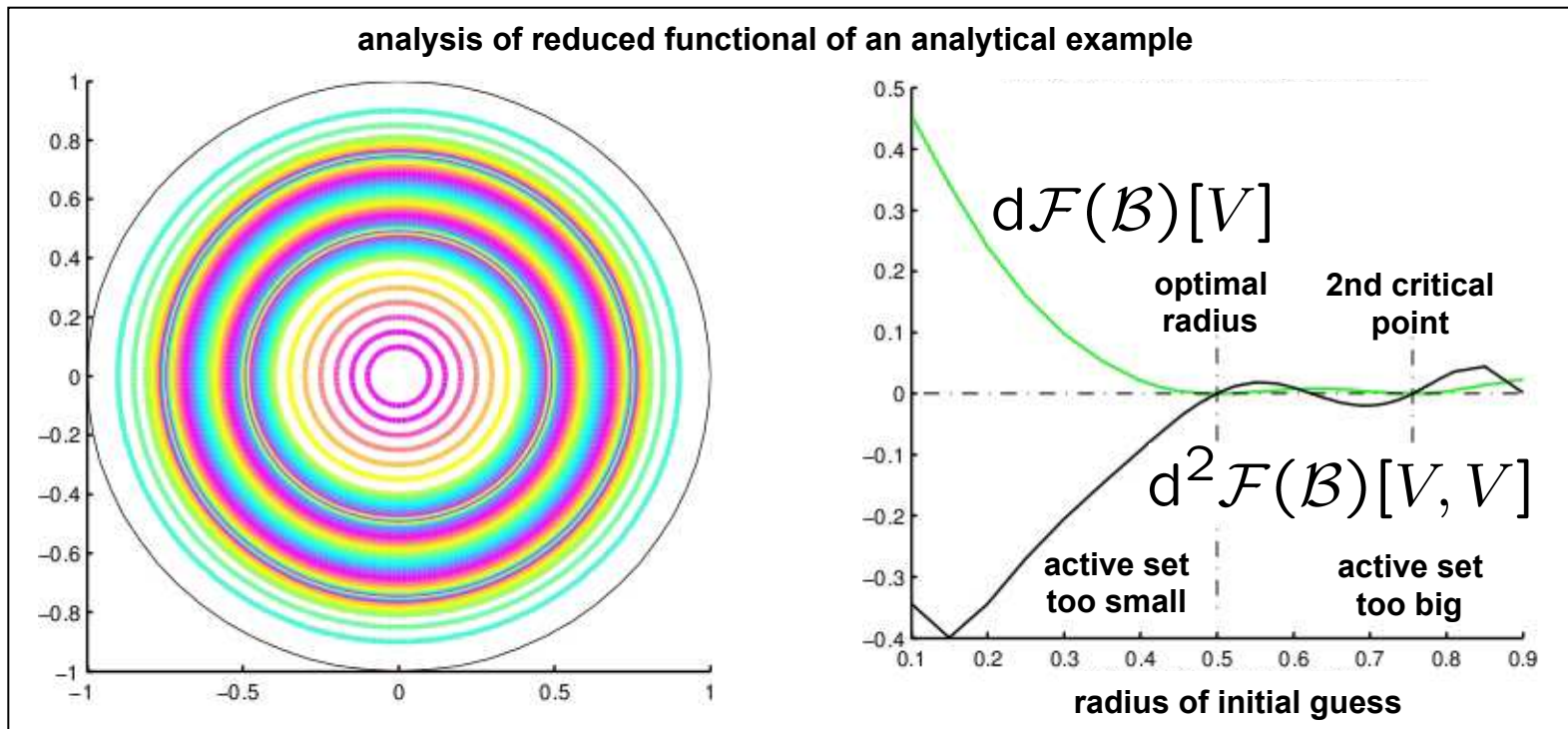
Interface by a free BVP  
or by a nonlinear cond.

a. e. in $\mathcal{I}$ a. e. on $\Gamma$	a. e. on $\gamma$	a. e. in $\mathcal{A}$
$-\Delta y_I + y_I = u_I$ $\partial_\nu y_I = 0$	$y_I = y_A$ $\partial_{\nu_I} y_I + \partial_{\nu_A} y_A = 0$	$-\Delta y_A + y_A = u_A$
$y_I < y_{\max}$	$y_A = y_{\max}$	$-\Delta y_{\max} + y_{\max} = u_A$
$-\Delta p_I + p_I = y_I - y_d$ $\partial_\nu p_I = 0$	$p_I = p_A$ $\partial_{\nu_I} p_I + \partial_{\nu_A} p_A = \sigma$	$-\Delta p_A + p_A = y_A - y_d$
$\lambda u_I + p_I = 0$	$u_I - u_A = 0$	$\lambda u_A + p_A + q = 0$
$d\mathcal{F}(\mathcal{B}; V) = 0$	$\sigma \geq 0$	$q \geq 0$



# Basic considerations w.r.t. the algorithm

- $\mathcal{A} \in \mathcal{M}$  is no (local) minimum of the (unconstrained)  $\mathcal{F}$ , second semi-derivative is not definite at critical points. Hence, steepest decent algorithms not applicable, higher order methods are required.



1. Initial guess  $k := 1, x_1$

2. loop on  $k$

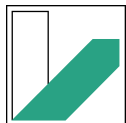
stopping criterion

Hessian and gradient require Hilbert spaces

Newton equation  $\text{Hess } J(x_k)[\delta_k] = -\text{grad } J(x_k)$

update  $x_{k+1} = x_k + \delta_k$

3. end of loop



1. Initial guess  $k := 1, x_1$

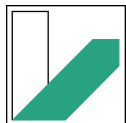
2. loop on  $k$

stopping criterion **directional derivatives suitable for linear structure**

Newton equation  $D^2 J(x_k)[\delta_k, v] = -DJ(x_k)[v] \quad \forall v \in \mathcal{V}$

update  $x_{k+1} = x_k + \delta_k$

3. end of loop



Why are second derivatives more complicated?

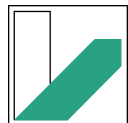
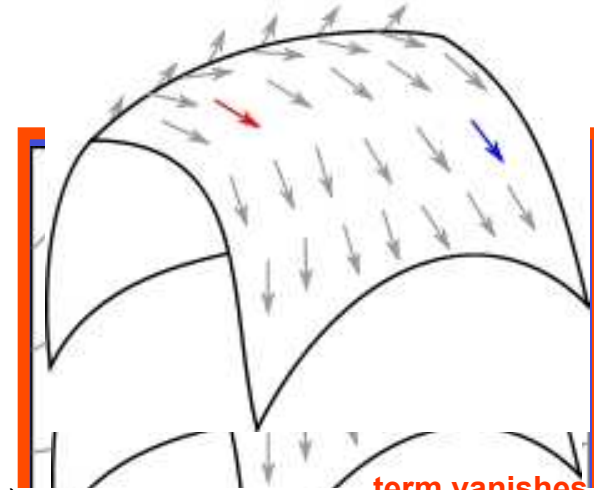
successive differentiation:

$$D^2 J[\delta_k, v] = D (D J[v]) [\delta_k]$$

apply chain rule:

$$\begin{aligned}
 D (D J[v]) [\delta_k] &= \left( \nabla_{\delta_k} (D J) \right) [v] \\
 &\quad + (D J) [\nabla_{\delta_k} v] \\
 &=: \underbrace{\nabla^2 J[\delta_k, v]}_{\text{second covariant derivative}} + \underbrace{(D J) [\nabla_{\delta_k} v]}_{\neq 0}
 \end{aligned}$$

term vanishes  
 in linear spaces;  
 does not contain  
 2nd order information  
 on functional



1. Initial guess  $k := 1, x_1$

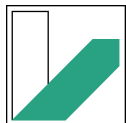
2. loop on  $k$

stopping criterion

Newton equation  $\nabla^2 J(x_k)[\delta_k, v] = -\mathbb{D}J(x_k)[v] \quad \forall v \in \mathcal{V}$

update  $x_{k+1} = x_k + \delta_k$

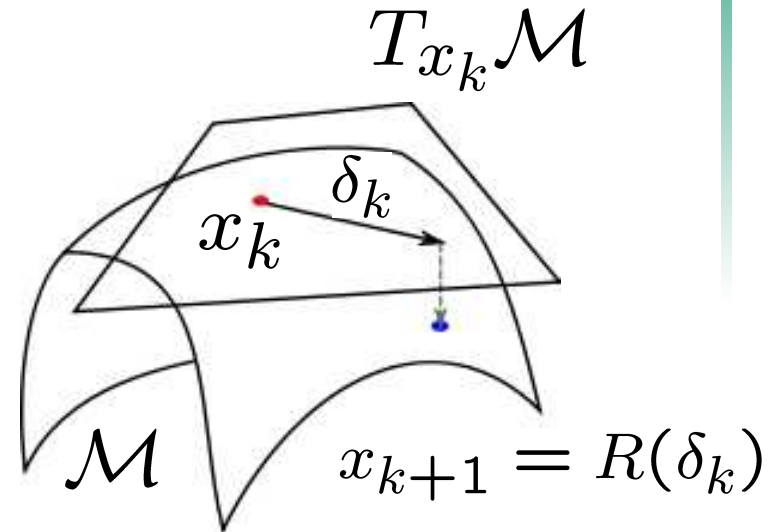
3. end of loop **sum requires linear structure**



**Retraction:**

$$R: T\mathcal{M} \longrightarrow \mathcal{M}$$

$$x_{k+1} = R(\delta_k)$$



$R: T\mathcal{M} \longrightarrow \mathcal{M}$  is a retraction

$:\iff$

$$R_x(0_x) = x \quad \forall x \in \mathcal{M}$$

with  $0_x$  is zero in  $T_x\mathcal{M}$  and

$$DR_x(0_x) = \text{Id}_{T_x\mathcal{M}}$$



1. Initial guess  $k := 1, x_1$

2. loop on  $k$

stopping criterion

Newton equation  $\nabla^2 J(x_k)[\delta_k, v] = -DJ(x_k)[v] \quad \forall v \in \mathcal{V}$

update  $x_{k+1} = R(\delta_k)$

3. end of loop



# Newton's method on manifolds

1. Initial guess

$$k := 1, x_1$$

2. loop on

$k$

stopping criterion

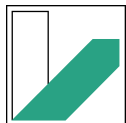
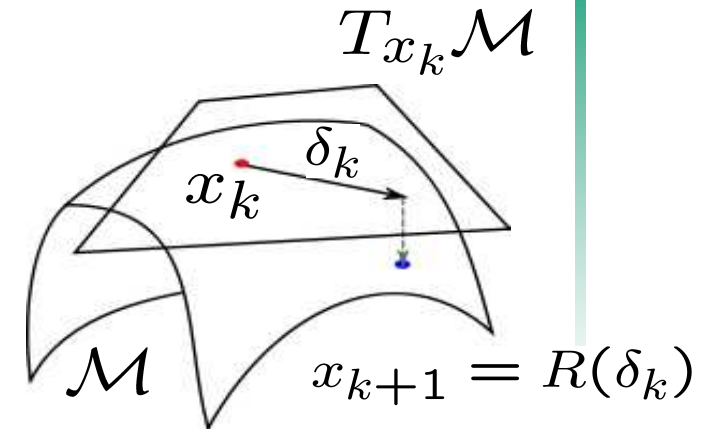
Newton equation

$$\nabla^2 J(x_k)[\delta_k, v] = -DJ(x_k)[v] \quad \forall v \in \mathcal{V}$$

update

$$x_{k+1} = R(\delta_k)$$

3. end of loop





# Newton's method on manifolds

1. Initial guess

$$k := 1, x_1$$

2. loop on

$$k$$

stopping criterion

Newton equation

$$\nabla^2 J(x_k)[\delta_k, v] = -DJ(x_k)[v] \quad \forall v \in \mathcal{V}$$

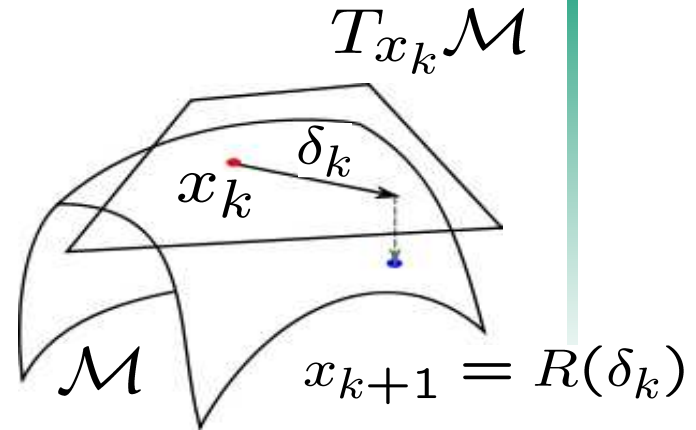
update

$$x_{k+1} = R(\delta_k) \delta_k$$

3. end of loop

**second covariant derivative:**  
it does not contain full information  
of second directional derivative,  
but full second order information  
on functional

**directional derivative,**  
since Hessian and  
gradient require  
Hilbert spaces



**retraction**



1. provide initial guess  $\mathcal{B}_1$

(by formula on data:  
candidate active set)

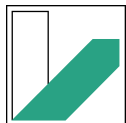
$$-\Delta y_{\mathcal{A}} + y_{\mathcal{A}} = u_{\mathcal{A}} \quad \lambda u_{\mathcal{A}} + p_{\mathcal{A}} = 0 \quad y_{\mathcal{A}} = y_{\max}$$

$$-\Delta p_{\mathcal{A}} + p_{\mathcal{A}} = y_{\mathcal{A}} - y_d + \mu$$

$$\mu(x) := -\lambda \Delta^2 y_{\max}(x) + 2\lambda \Delta y_{\max}(x) - (1 + \lambda) y_{\max}(x) + y_d(x) \geq 0$$

due to strict complementarity:

$$\mathcal{A} \subset \{x \in \Omega : \mu(x) \geq 0\}$$



# Newton's method for set optimal control problems

1. provide initial guess  $\mathcal{B}_1$  (by formula on data:  
candidate active set)

2. loop on  $k$

stopping criterion

identify  $\nabla^2 \mathcal{F}(\mathcal{B})[\cdot, \cdot]$  (complicated formula)

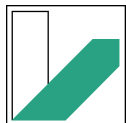
Newton equation: Find  $W_k \in \mathcal{V}$

$$\nabla^2 \mathcal{F}(\mathcal{B}_k)[W_k, V] = -d\mathcal{F}(\mathcal{B}_k; V) \quad \forall V \in \mathcal{V}$$

provide retraction: deform  $\mathcal{B}_k \longrightarrow \mathcal{B}_{k+1}$  (see next transparency)

3. end of loop

4. check a posteriori criteria and eventually restart with other initial guess



Newton update by second covariant derivative and retraction

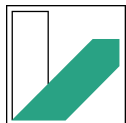
online: self-intersection

offline: violation of state constraint



Michael Frey 's dissertation, University of Bayreuth, 2012:  
Shape calculus applied to state-constrained elliptic optimal control problems

<http://opus.ub.uni-bayreuth.de/opus4-ubbayreuth/frontdoor/index/index/docId/996>

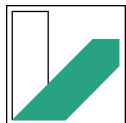


## pro

- **formulation in infinite dimen. setting  $\Rightarrow$  mesh independency**
- **no regularization loops  $\Rightarrow$  better performance than PDAS**
- **feasible approximations of solutions**

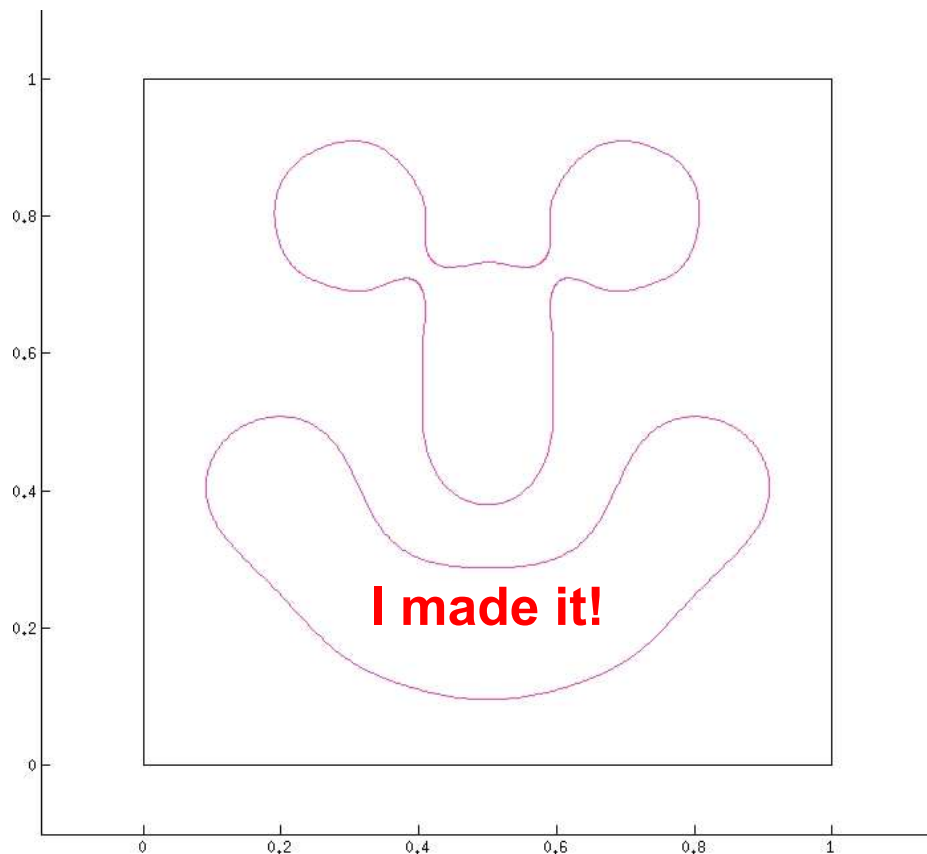
## contra

- **changes of topology heuristically**
- **no convergence analysis**
- **only local convergence (with adaptive smooting for stability)**
- **assumptions on active set**



# The Smiley: construction

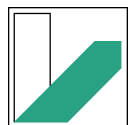
algorithm can cope with topology changes



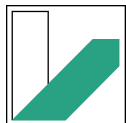
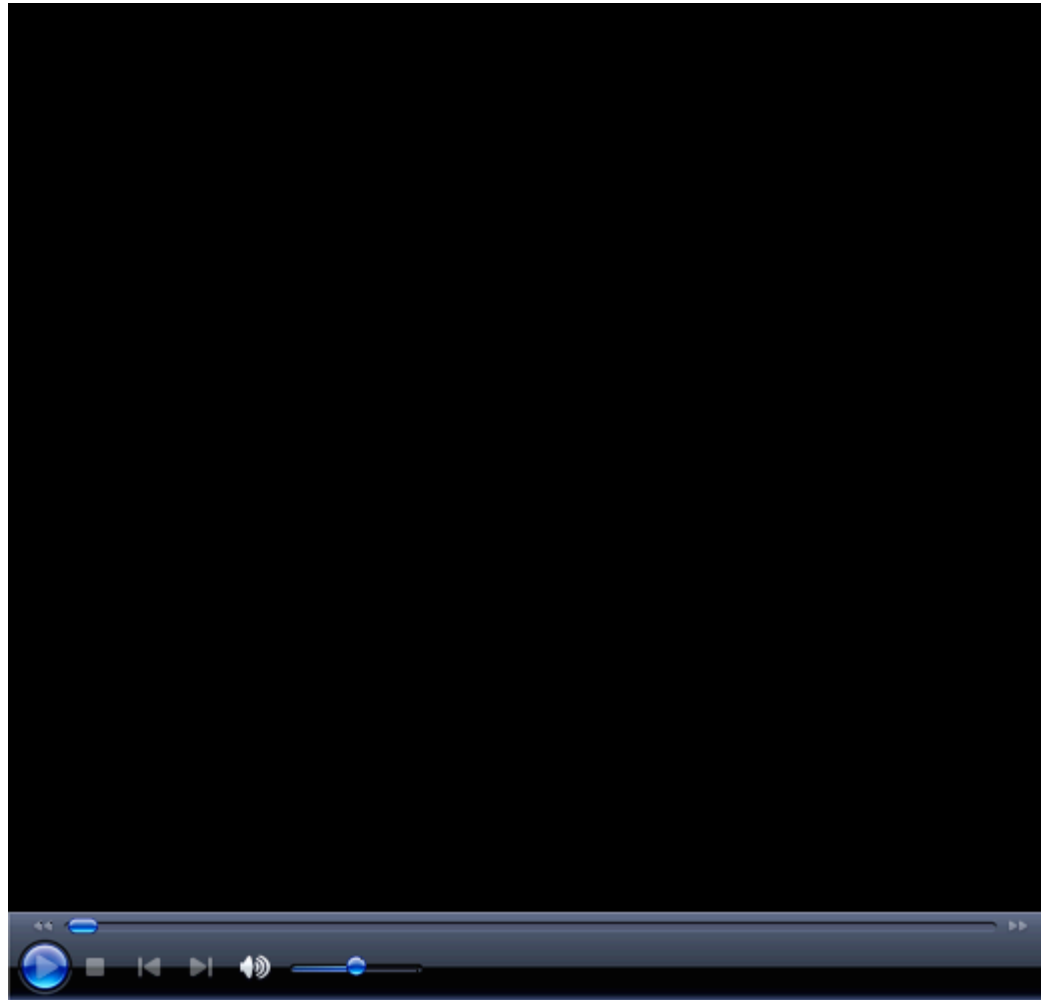
**Construction:**  
Prescribe  $y_d$ ,  
choose  $\lambda$  small,  
press down  $y_{\max}$ .

**Initial guess:**  
automatically from  
unconstrained problem

Iter No. 



# The Smiley example: rational initial guess



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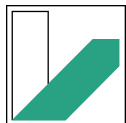
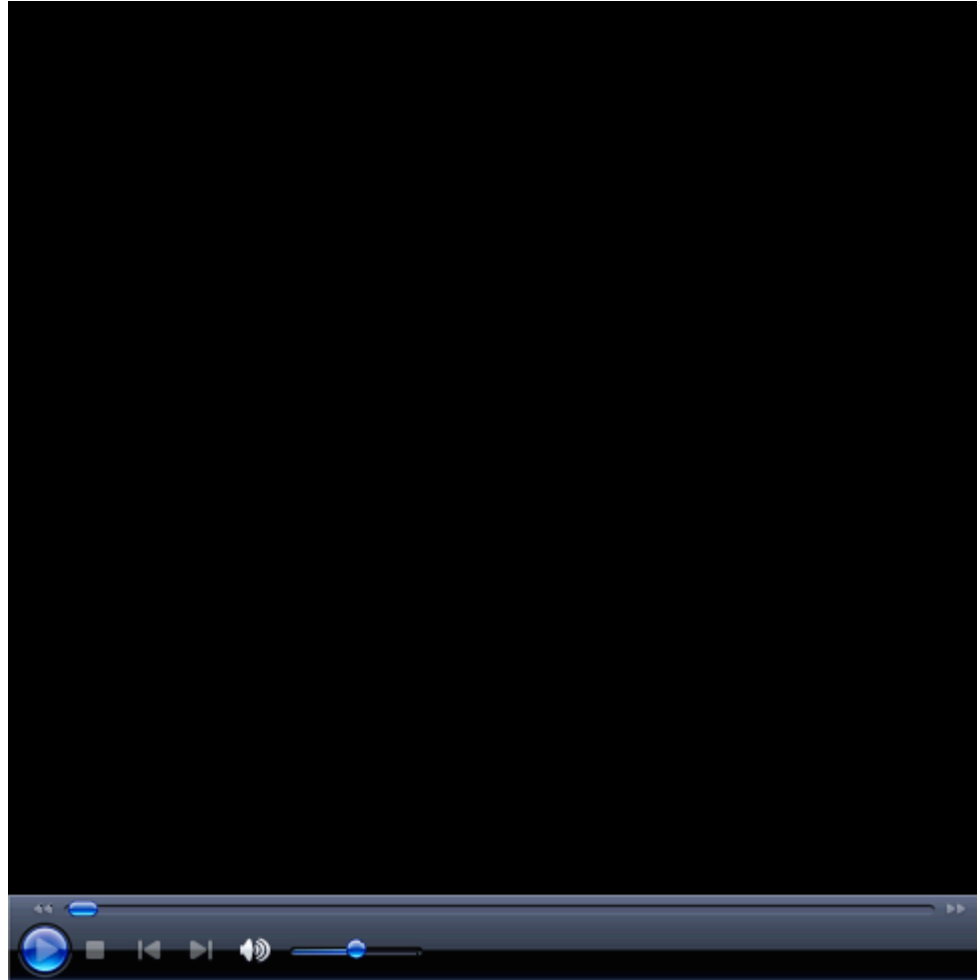
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# The Smiley example: bad initial guess

Algorithm can cope with topology changes to some extent



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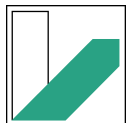
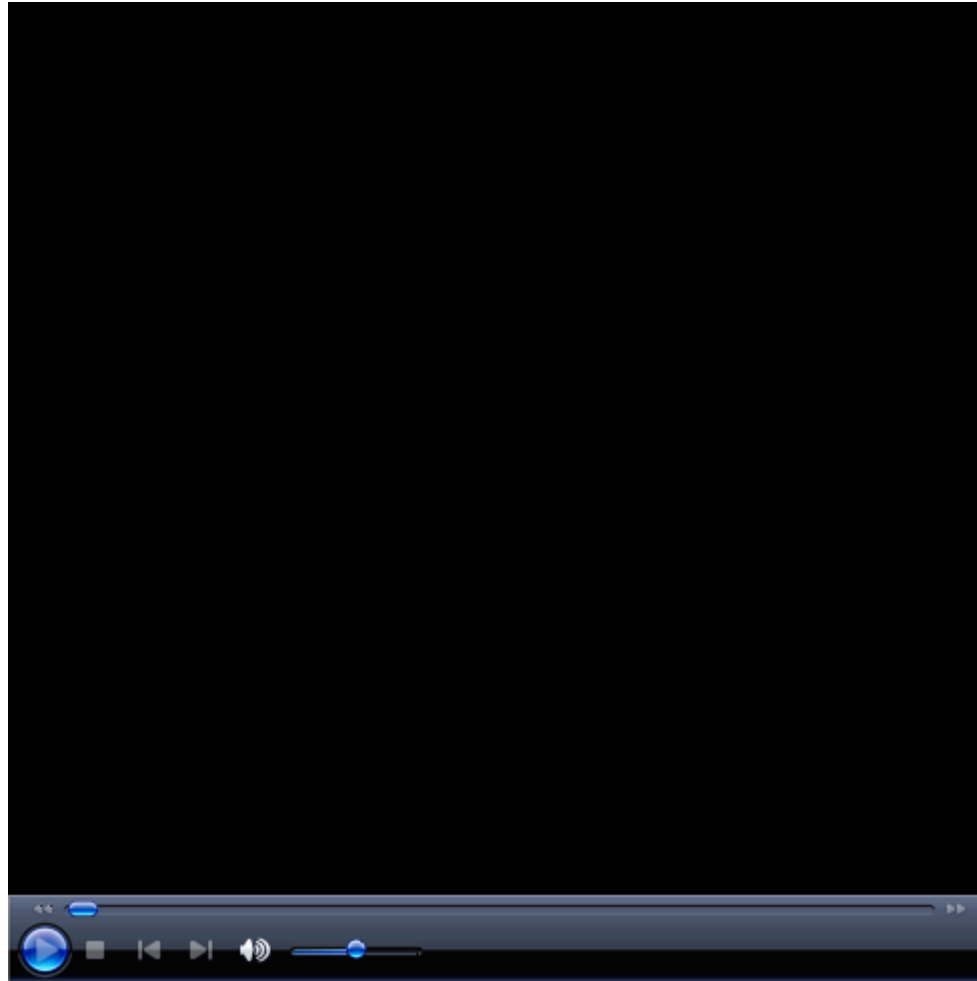
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# The Smiley example: bad initial guess

**Adjoint multiplier: continuous on interface,  
but normal derivatives jump**

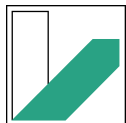


## Split method

- new problem type: set optimal control problem
- bilevel formulation  $\Rightarrow$  geometry-to-solution operator
- split of constraint  $\Rightarrow$  exploitation of structure of multiplier

## Extended BDD approach for PDEs

- differentiation of state constraint  $\Rightarrow$  control law
- higher regularity of multiplier
- connection with optimal control and PDAE: index reduction

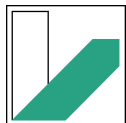


## Shape calculus

- set of admissible active sets has manifold character and thus is intrinsically nonlinear
- deformation of sets: perturbation of identity
- calculation requires transformation formula

## Optimization on vector bundles

- vector bundles: structure depends upon manifolds
- general basis for shape optimization / set optimal control problems
- new challenging class of optimization problems



## Newton method

- several approaches towards solution of new necessary conditions
- adaption of Newton method on manifolds
- algorithm in function space without regularization
- comparable performance to sophisticated PDAS
- pays off in case of nonlinear elliptic optimal control problems

## Open questions

- active sets of measure zero
- generalization to parabolic problems (first results already obtained)





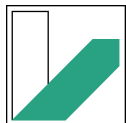
I will be glad if I have succeeded in impressing the idea that it is not only pleasant and entertaining to read at times the works of the old mathematical authors, but that this may occasionally be of use for the actual advancement of science.

Besides this there is a great lesson we can derive from the facts which I have just referred to. We have seen that even under conditions which seem most favorable very important results can be discarded for a long time and whirled away from the main stream which is carrying the vessel science. ...

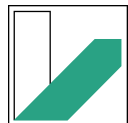
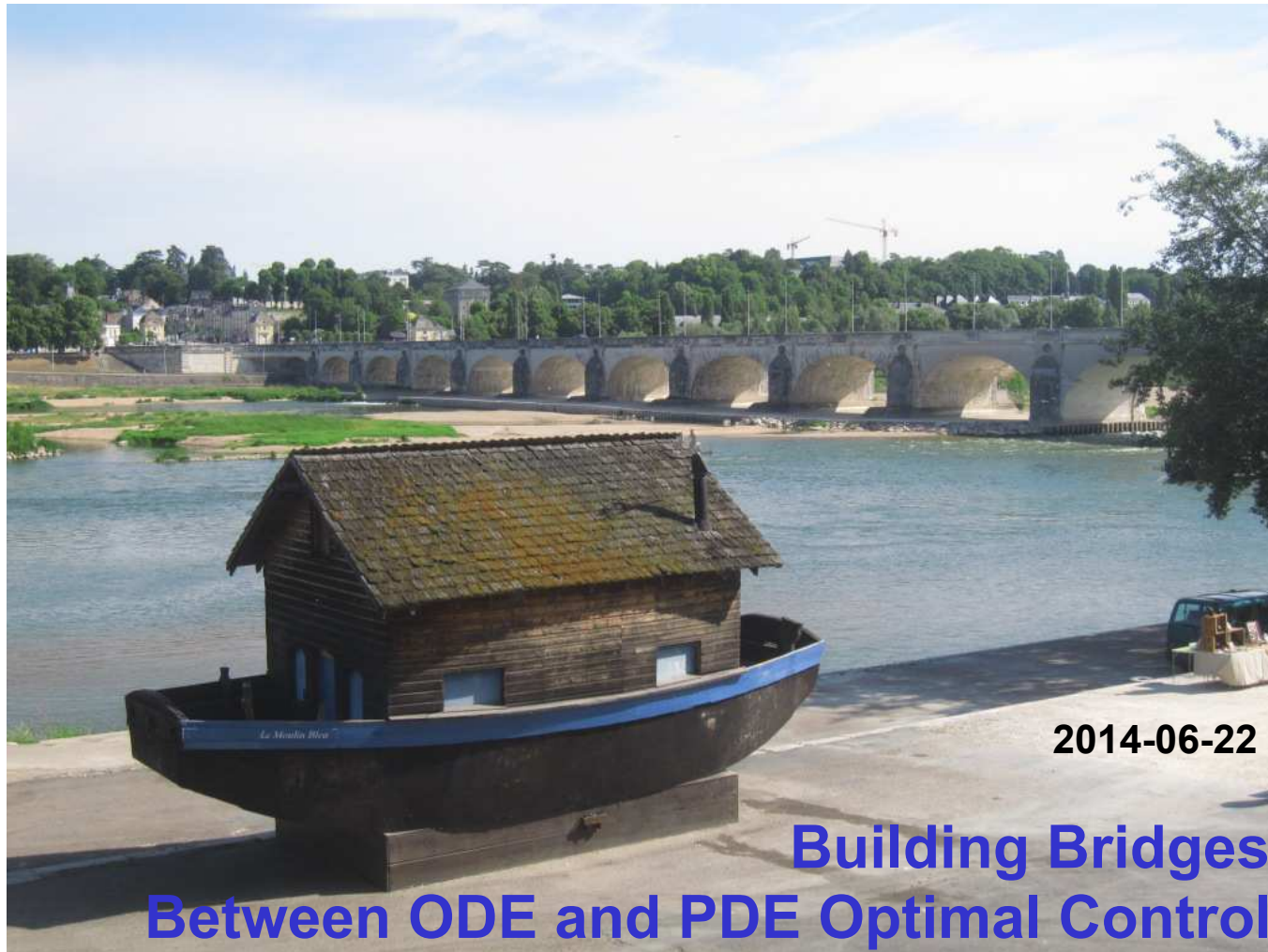
If their ideas are too far in advance of their time, and if the general public is not prepared to accept them, these ideas may sleep for centuries on the shelves of our libraries ... awaiting the arrival of the prince charming who will take them home.

**Constantin Carathéodory** (Κωνσταντίνος Καραθεοδωρή)

\* Sept. 13, 1873 in Berlin; † Feb. 2, 1950, Munich



Thank you for your attention – There 's still much to do



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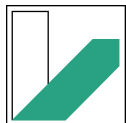


# Simultaneously I want to say GOOD BYE, friends



2014-06-22

**Building Bridges to Retirement**



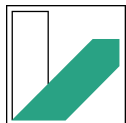
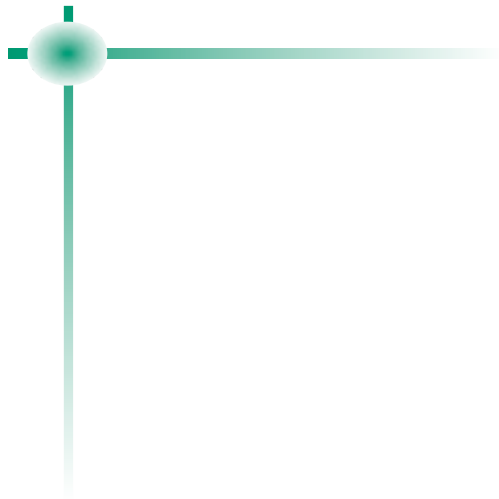
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