



HAL
open science

New Contributions to Theory and Numerics for State-constrained Elliptic Optimal Control Problems

Hans-Josef Pesch

► **To cite this version:**

Hans-Josef Pesch. New Contributions to Theory and Numerics for State-constrained Elliptic Optimal Control Problems. NETCO 2014 - New Trends in Optimal Control, Jun 2014, Tours, France. hal-01028170

HAL Id: hal-01028170

<https://inria.hal.science/hal-01028170>

Submitted on 22 Jul 2014

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

New Contributions to Theory and Numerics for State-constrained Elliptic Optimal Control Problems

Hans Josef Pesch

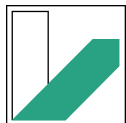
In Collaboration with

Michael Frey, Simon Bechmann, **Armin Rund**

Chair of Mathematics in Engineering Sciences
University of Bayreuth, Germany

Brose, Coburg, Germany
University of Graz, Austria

hans-josef.pesch@uni-bayreuth.de



UNIVERSITÄT
BAYREUTH

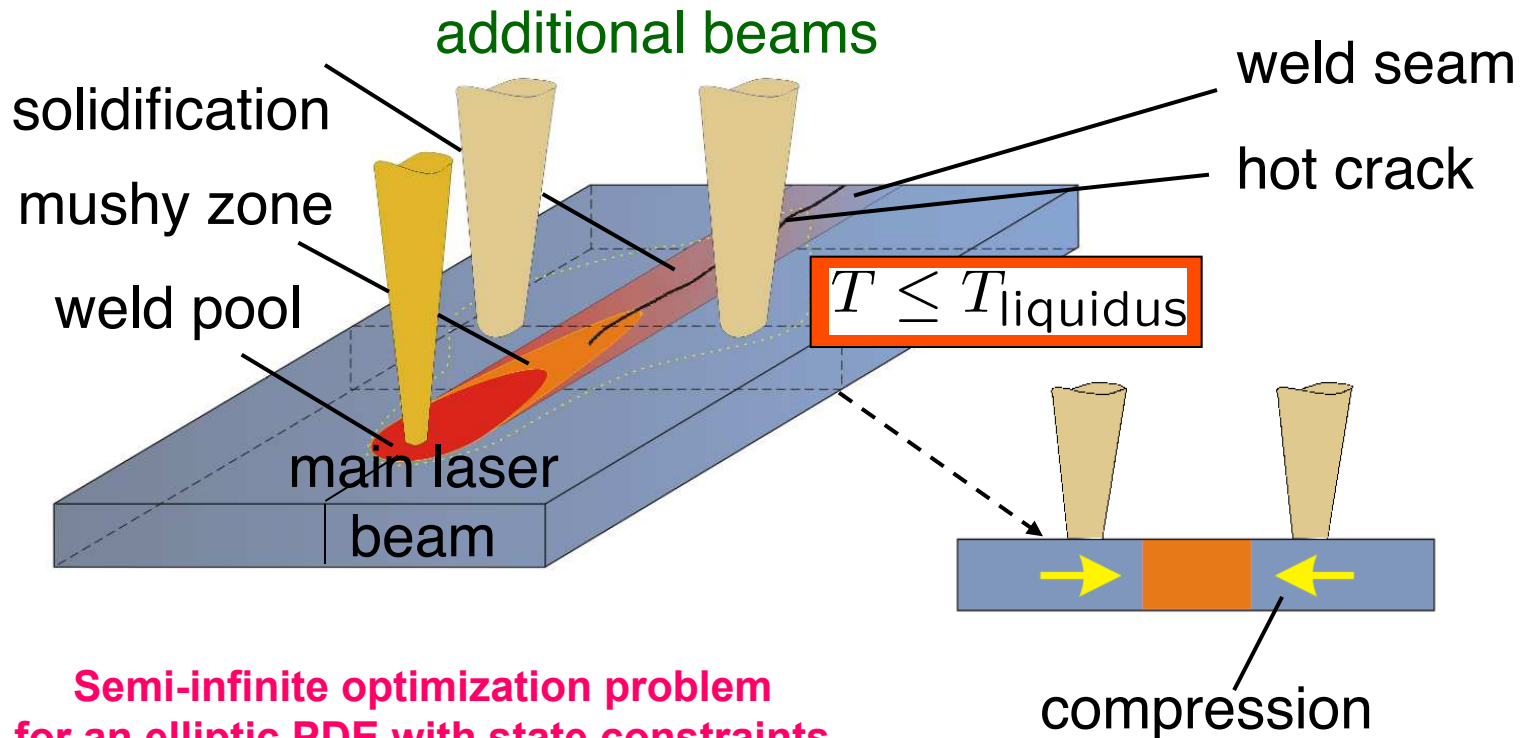
ITN SADCO NetCO 2014

Conf. on New Trends in Optimal Control, Tours, June 23-27, 14

Lehrstuhl
Ingenieur-
mathematik



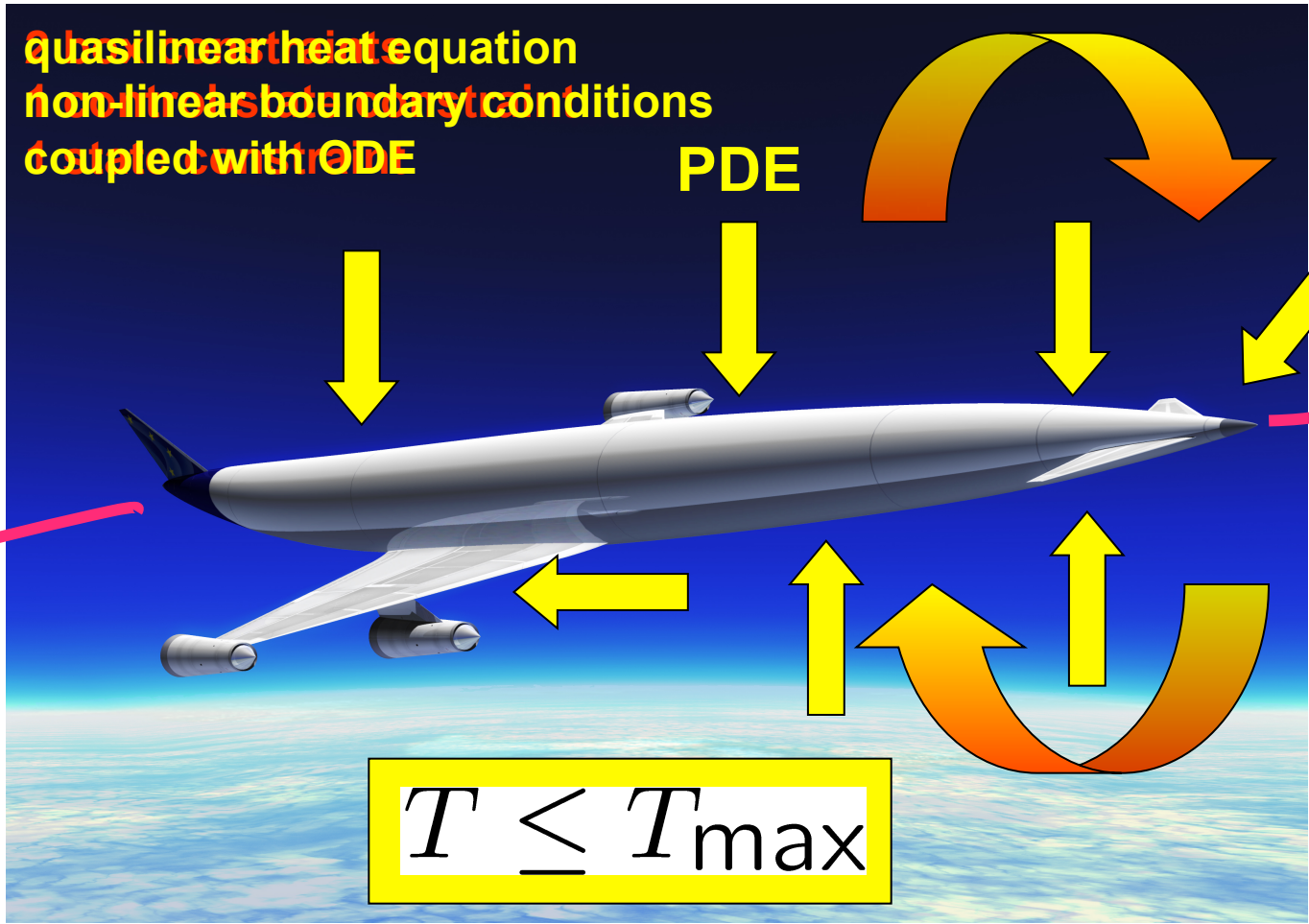
multi-beam welding



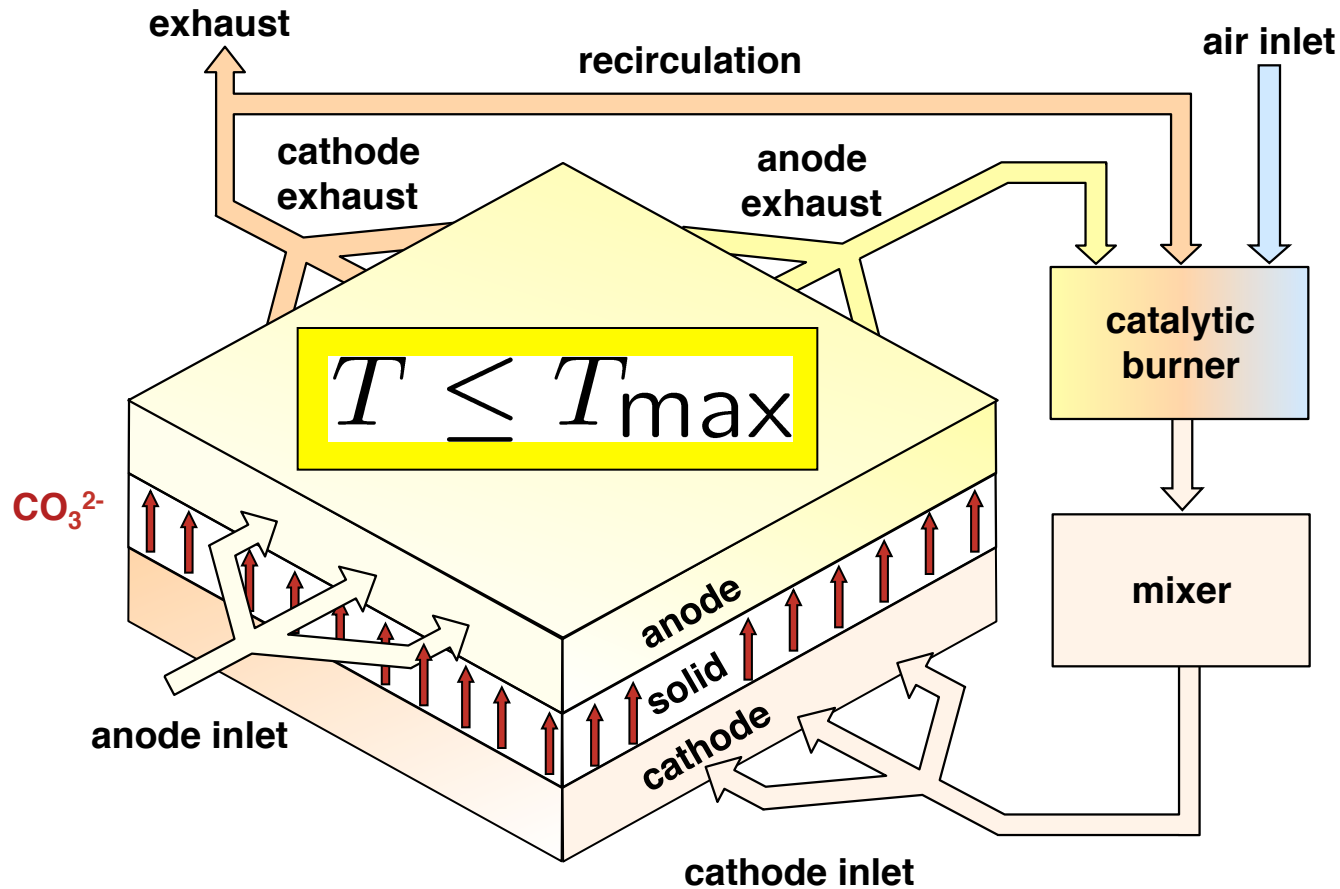
**Semi-infinite optimization problem
for an elliptic PDE with state constraints**



Motivation: Hypersonic Flights



2D cross-flow design of a high temperature fuel cell



28 quasi-linear partial integro-differential-algebraic equations with non-standard non-linear boundary conditions





- **Introduction**
- Split method
- Bryson-Denham-Dreyfus approach (BDD)
- Shape calculus and optimization on vector bundles
- Numerics



Minimize

$$J(y, u) := \frac{1}{2} \|y - y_d\|_{L^2(\Omega)}^2 + \frac{\lambda}{2} \|u\|_{L^2(\Omega)}^2$$

subject to

$$-\Delta y + y = u \quad \text{a. e. in } \Omega$$

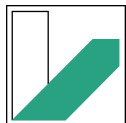
$$\partial_\nu y = 0 \quad \text{a. e. on } \Gamma$$

$$G(y) := y - y_{\max} \leq 0 \quad \text{a. e. in } \Omega$$

with

$\Omega \subset \mathbb{R}^2$ bounded domain, $\Gamma \in C^{1,1}$,

$u \in U_{\text{ad}} := L^2(\Omega)$, $y_d \in L^2(\Omega)$, $\lambda > 0$, $y_{\max} \in H^4(\Omega)$



Theorem (Casas 1986): Slater condition

$$\implies \exists p^{\text{trad}} \in W^{1,s}(\Omega) \text{ and } \mu = \mu_{\Omega} + \mu_{\Gamma} \in C(\bar{\Omega})^*$$

such that

$$-\Delta y + y = u \quad \text{a. e. in } \Omega$$

$$\partial_{\nu} y = 0 \quad \text{a. e. on } \Gamma$$

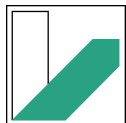
$$-\Delta p^{\text{trad}} + p^{\text{trad}} = y - y_d + \mu_{\Omega} \quad \text{a. e. in } \Omega$$

$$\partial_{\nu} p^{\text{trad}} = \mu_{\Gamma} \quad \text{a. e. on } \Gamma$$

$$\lambda u + p^{\text{trad}} = 0 \quad \text{a. e. in } \Omega$$

$$\langle y - y_{\max}, \mu \rangle_{C(\bar{\Omega}), C(\bar{\Omega})^*} = 0, \quad \mu > 0$$

low regularity causes problems in numerical treatment



Goals

- new necessary conditions with higher regularity of Lagrange multipliers
- formulate efficient numerical algorithms, which don't require any regularization technique, exploit the structure of the multiplier, and are mesh-independent

Ideas

- geometric split → set optimal control problem
- BDD approach → higher regularity
- shape calculus → necessary conditions
- optimization on vector bundles → design of algorithms



Goals

- new necessary conditions with higher regularity of Lagrange multipliers
- formulate efficient numerical algorithms, which don't require any regularization technique, exploit the structure of the multiplier, and are mesh-independent

Ideas

- geometric split → set optimal control problem
- BDD approach → higher regularity
- shape calculus → necessary conditions
- optimization on vector bundles → design of algorithms



- Introduction
- **Split method**
- Bryson-Denham-Dreyfus approach (BDD)
- Shape calculus and optimization on vector bundles
- Numerics



Definition: active / inactive set / interface

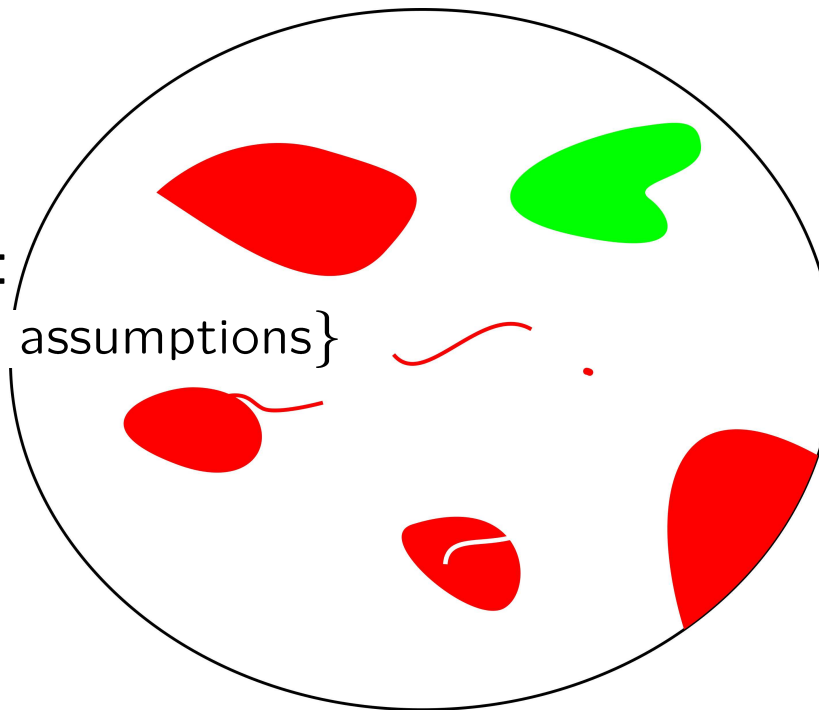
$$\mathcal{A} := \{x \in \bar{\Omega} : G(\bar{y}) = 0\}$$

$$\mathcal{I} := \Omega \setminus \mathcal{A}$$

$$\gamma := \partial\mathcal{A}$$

**Assumption
on admissible
active sets**

$$\mathcal{B}_{\text{ad}} := \{\mathcal{B} \subset \Omega : \mathcal{B} \text{ fulfills assumptions}\}$$



**No degeneracy.
No active set
of zero measure.
No common points
with boundary.**



Reformulation of the model problem

(Analog to the multipoint-boundary-value-problem formulation; Oberle)

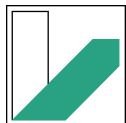
Minimize

$$J(y_{\mathcal{I}}, y_{\mathcal{A}}, u_{\mathcal{I}}, u_{\mathcal{A}}) := \frac{1}{2} \|y_{\mathcal{I}} - y_d|_{\mathcal{I}}\|_{L^2(\mathcal{I})}^2 + \frac{1}{2} \|y_{\mathcal{A}} - y_d|_{\mathcal{A}}\|_{L^2(\mathcal{A})}^2 \\ + \frac{\lambda}{2} \|u_{\mathcal{I}}\|_{L^2(\mathcal{I})}^2 + \frac{\lambda}{2} \|u_{\mathcal{A}}\|_{L^2(\mathcal{A})}^2$$

subject to

$$\begin{aligned} -\Delta y_{\mathcal{I}} + y_{\mathcal{I}} &= u_{\mathcal{I}}, & \text{a. e. in } \mathcal{I} \\ -\Delta y_{\mathcal{A}} + y_{\mathcal{A}} &= u_{\mathcal{A}}, & \text{a. e. in } \tilde{\mathcal{A}} \\ \partial_{\nu} y_{\mathcal{I}} &= 0, & \text{a. e. on } \Gamma \\ y_{\mathcal{I}} &= y_{\mathcal{A}}, & \text{a. e. on } \gamma \\ \partial_{\nu_{\mathcal{I}}} y_{\mathcal{I}} + \partial_{\nu_{\mathcal{A}}} y_{\mathcal{A}} &= 0, & \text{a. e. on } \gamma \\ y_{\mathcal{A}} &= y_{\max}, & \text{a. e. in } \tilde{\mathcal{A}} \\ y_{\mathcal{J}} &< y_{\max}, & \text{a. e. in } \tilde{\mathcal{I}} \\ y_{\mathcal{A}} &= y_{\max}, & \text{a. e. on } \gamma \end{aligned}$$

How can we determine the active set?



Reformulation as set optimal control problem

Minimize

$$J(y_{\mathcal{J}}, y_{\mathcal{B}}, u_{\mathcal{J}}, u_{\mathcal{B}}, \mathcal{B}) := \frac{1}{2} \|y_{\mathcal{J}} - y_d|_{\mathcal{J}}\|_{L^2(\mathcal{J})}^2 + \frac{1}{2} \|y_{\mathcal{B}} - y_d|_{\mathcal{B}}\|_{L^2(\mathcal{B})}^2 \\ + \frac{\lambda}{2} \|u_{\mathcal{J}}\|_{L^2(\mathcal{J})}^2 + \frac{\lambda}{2} \|u_{\mathcal{B}}\|_{L^2(\mathcal{B})}^2$$

subject to

$$\begin{aligned} -\Delta y_{\mathcal{J}} + y_{\mathcal{J}} &= u_{\mathcal{J}}, & \text{a. e. in } \mathcal{J} \\ -\Delta y_{\mathcal{B}} + y_{\mathcal{B}} &= u_{\mathcal{B}}, & \text{a. e. in } \mathring{\mathcal{B}} \\ \partial_{\nu} y_{\mathcal{J}} &= 0, & \text{a. e. on } \Gamma \\ y_{\mathcal{J}} &= y_{\mathcal{B}}, & \text{a. e. on } \beta \\ \partial_{\nu_{\mathcal{J}}} y_{\mathcal{J}} + \partial_{\nu_{\mathcal{B}}} y_{\mathcal{B}} &= 0, & \text{a. e. on } \beta \\ y_{\mathcal{B}} &= y_{\max}, & \text{a. e. in } \mathring{\mathcal{B}} \\ y_{\mathcal{J}} &< y_{\max}, & \text{a. e. in } \mathring{\mathcal{J}} \\ y_{\mathcal{B}} &= y_{\max}, & \text{a. e. on } \beta \end{aligned}$$

**How to handle
the strict inequality?**

**Theorem: The original problem and the set optimal control problem
possess the same unique solution**

Set optimal control problem as bilevel optimization problem

Minimize

$$J(y_{\mathcal{J}}, y_{\mathcal{B}}, u_{\mathcal{J}}, u_{\mathcal{B}}, \mathcal{B}) := \underbrace{\frac{1}{2} \|y_{\mathcal{J}} - y_d|_{\mathcal{J}}\|_{L^2(\mathcal{J})}^2}_{\text{inner}} + \underbrace{\frac{1}{2} \|y_{\mathcal{B}} - y_d|_{\mathcal{B}}\|_{L^2(\mathcal{B})}^2}_{\text{outer}} + \frac{\lambda}{2} \|u_{\mathcal{J}}\|_{L^2(\mathcal{J})}^2 + \frac{\lambda}{2} \|u_{\mathcal{B}}\|_{L^2(\mathcal{B})}^2$$

subject to

$$-\Delta y_{\mathcal{J}} + y_{\mathcal{J}} = u_{\mathcal{J}}, \quad \text{a. e. in } \mathcal{J}$$

$$-\Delta y_{\mathcal{B}} + y_{\mathcal{B}} = u_{\mathcal{B}}, \quad \text{a. e. in } \mathcal{B}$$

$$\partial_{\nu} y_{\mathcal{J}} = 0, \quad \text{a. e. on } \Gamma$$

$$y_{\mathcal{J}} = y_{\mathcal{B}}, \quad \text{a. e. on } \beta$$

$$\partial_{\nu_{\mathcal{J}}} y_{\mathcal{J}} + \partial_{\nu_{\mathcal{B}}} y_{\mathcal{B}} = 0, \quad \text{a. e. on } \beta$$

$$y_{\mathcal{B}} = y_{\max}, \quad \text{a. e. in } \mathring{\mathcal{B}}$$

$$y_{\mathcal{J}} < y_{\max}, \quad \text{a. e. in } \mathring{\mathcal{J}}$$

$$y_{\mathcal{B}} = y_{\max}, \quad \text{a. e. on } \beta$$

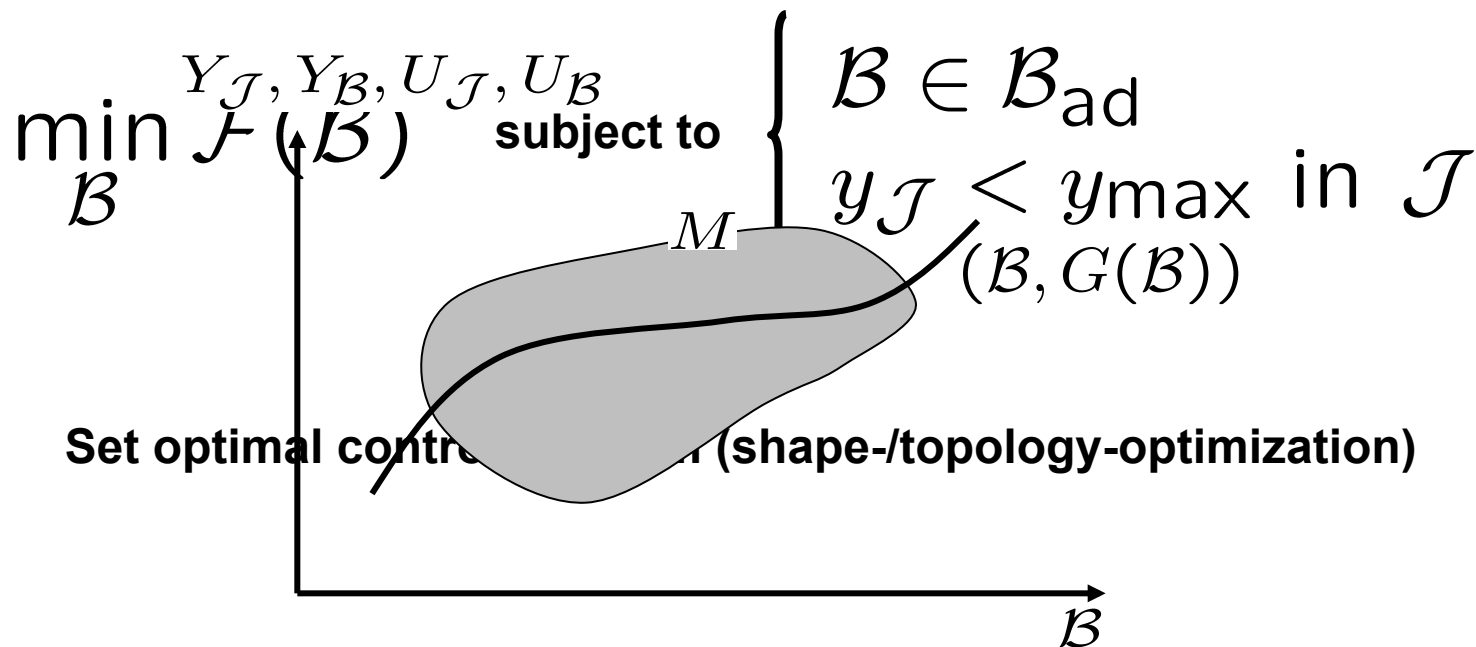
**constraint of
outer optimization**

**Theorem: The inner optimization problem
possesses a unique solution for $\mathcal{B} \in \mathcal{B}_{\text{ad}}$**

Reduced functional: $\exists G: \mathcal{B} \rightarrow (y_{\mathcal{J}}, y_{\mathcal{B}}, u_{\mathcal{J}}, u_{\mathcal{B}})$

$$\mathcal{F}(\mathcal{B}) = J(G(\mathcal{B}), \mathcal{B})$$

is well-defined on \mathcal{B}_{ad} .



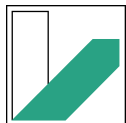
Reduced functional: $\exists G: \mathcal{B} \rightarrow (y_{\mathcal{J}}, y_{\mathcal{B}}, u_{\mathcal{J}}, u_{\mathcal{B}})$

$$\mathcal{F}(\mathcal{B}) = J(G(\mathcal{B}), \mathcal{B})$$

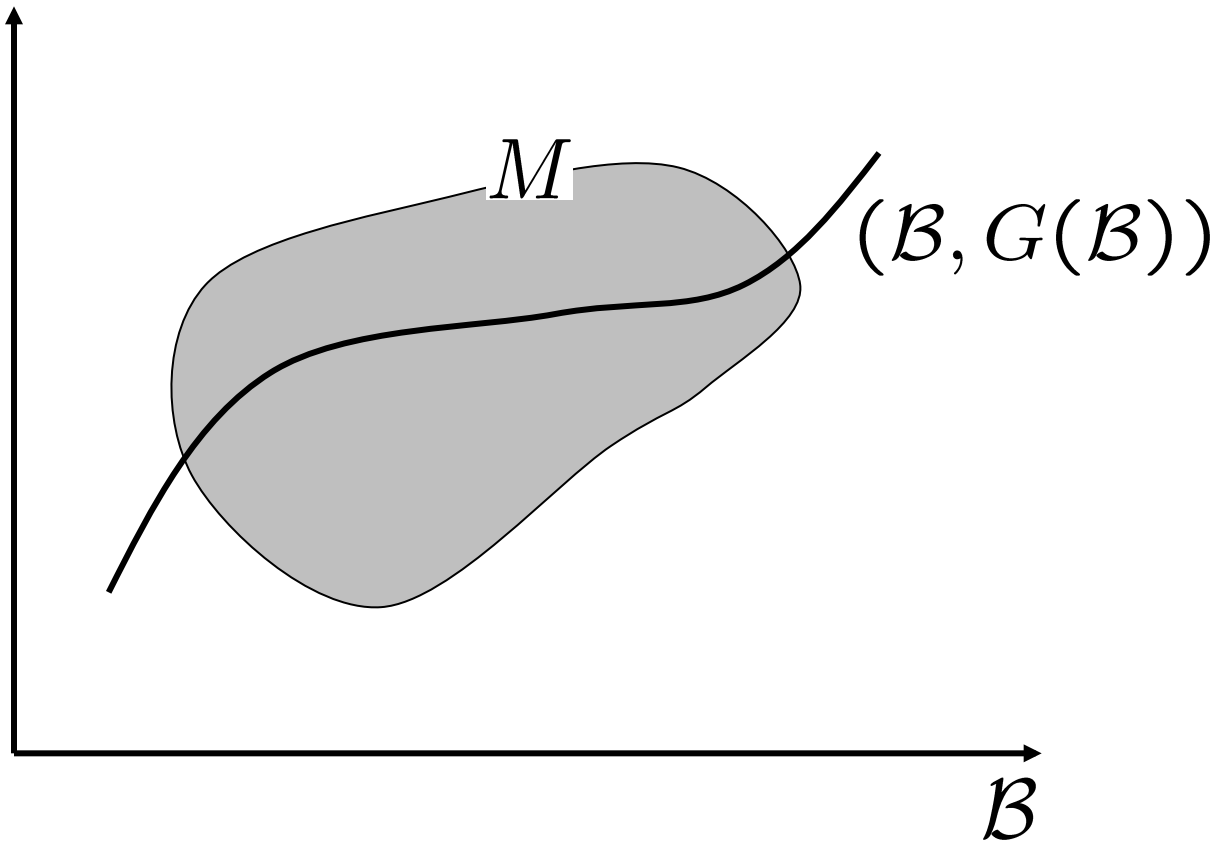
is well-defined on \mathcal{B}_{ad} :

$$\min_{\mathcal{B}} \mathcal{F}(\mathcal{B}) \quad \text{subject to} \quad \begin{cases} \mathcal{B} \in \mathcal{B}_{\text{ad}} \\ y_{\mathcal{J}} < y_{\text{max}} \text{ in } \mathcal{J} \end{cases}$$

Set optimal control problem (shape-/topology-optimization)

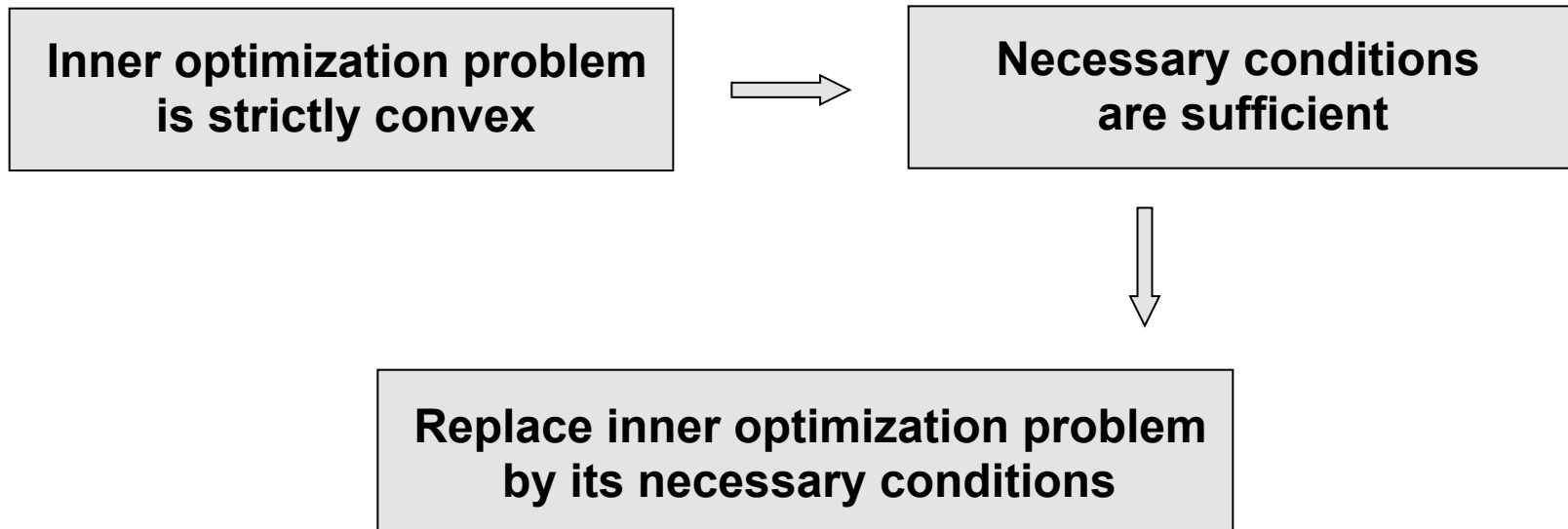


$Y_{\mathcal{J}}, Y_{\mathcal{B}}, U_{\mathcal{J}}, U_{\mathcal{B}}$



Theorem:

For any $\mathcal{B} \in \mathcal{B}_{\text{ad}}$ there exist Lagrange multipliers associated with the equality constraints of the inner optimization problem



Set optimal control problem

$$\min_{\mathcal{B} \in \mathcal{B}_{\text{ad}}} \mathcal{F}(\mathcal{B}) \quad \text{subject to} \quad y_{\mathcal{J}} < y_{\text{max}} \text{ in } \mathcal{J}$$

$$\text{and} \quad -\Delta y_{\mathcal{J}} + y_{\mathcal{J}} = -\frac{1}{\lambda} p_{\mathcal{J}}, \quad \text{a. e. in } \mathcal{J}$$

$$\partial_{\nu} y_{\mathcal{J}} = 0, \quad \text{a. e. on } \Gamma$$

$$-\Delta p_{\mathcal{J}} + p_{\mathcal{J}} = y_{\mathcal{J}} - y_d, \quad \text{a. e. in } \mathcal{J}$$

$$\partial_{\nu} p_{\mathcal{J}} = 0, \quad \text{a. e. on } \Gamma$$

$$y_{\mathcal{J}} = y_{\text{max}}, \quad \text{a. e. on } \beta$$

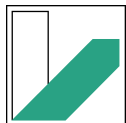
$$\partial_{\nu_{\mathcal{J}}} y_{\mathcal{J}} = \partial_{\nu_{\mathcal{J}}} y_{\text{max}}, \quad \text{a. e. on } \beta$$

no measures
involved

unusual
boundary
conditions

everything else can be computed a posteriori

$$\begin{array}{l|l} \begin{array}{l} y_{\mathcal{B}} = y_{\text{max}}, \\ u_{\mathcal{B}} = -\Delta y_{\text{max}} + y_{\text{max}}, \\ p_{\mathcal{B}} = -\lambda u_{\mathcal{B}}, \end{array} & \begin{array}{l} \text{a. e. in } \mathcal{B} \\ \text{a. e. in } \mathcal{B} \\ \text{a. e. in } \mathcal{B} \end{array} \end{array} \left| \begin{array}{l} u_{\mathcal{J}} = -\frac{1}{\lambda} p_{\mathcal{J}}, \\ \mu_{\mathcal{B}} = \partial_{\nu_{\mathcal{J}}} p_{\mathcal{J}} + \partial_{\nu_{\mathcal{B}}} p_{\mathcal{B}}, \\ \mu_{\mathcal{B}} = -\Delta p_{\mathcal{B}} + p_{\mathcal{B}} - y_{\mathcal{B}} + y_{\text{max}}, \end{array} \right. \begin{array}{l} \text{a. e. in } \mathcal{J} \\ \text{a. e. on } \beta \\ \text{a. e. in } \mathcal{B} \end{array}$$



Shape calculus for the optimal active set

$$\min_{\mathcal{B}} \underbrace{\min_{y_{\mathcal{J}}, y_{\mathcal{B}}, u_{\mathcal{J}}, u_{\mathcal{B}}} J(y_{\mathcal{J}}, y_{\mathcal{B}}, u_{\mathcal{J}}, u_{\mathcal{B}}, \mathcal{B})}_{\mathcal{F}(\mathcal{B})}$$

subject to the optimality system of the inner optimization problem

Theorem:

For each admissible \mathcal{B} the objective $\mathcal{F}(\mathcal{B})$ is shape differentiable. The semi-derivative in the direction

$$V \in \{W \in C^{1,1}(\bar{\Omega}) \mid \langle W, \nu \rangle = 0 \text{ on } \Gamma\}$$

is

$$d\mathcal{F}(\mathcal{B}; V) = \int_{\beta} -\frac{1}{2\lambda} \left[p_{\mathcal{J}} - \underbrace{\lambda (\Delta y_{\max} - y_{\max})}_{p_{\max}} \right]^2 \langle V, \nu_{\mathcal{J}} \rangle$$



determines the interface

We omit $y_{\mathcal{J}} < y_{\max}$ in the outer optimization and determine the interface β by

$$d\mathcal{F}(\mathcal{B}; V) = 0 \quad \forall \quad V \in C^{1,1}(\bar{\Omega})$$

$$\implies p_{\mathcal{J}} = p_{\max} \quad \text{on } \beta$$

$$\implies u_{\mathcal{J}} = u_{\mathcal{B}} \quad \text{on } \beta$$

Needs an a posteriori-check on feasibility: $y_{\mathcal{J}} < y_{\max}$



- Introduction
- Split method
- **Bryson-Denham-Dreyfus^{*} approach (BDD)**
- Shape calculus and optimization on vector bundles
- Numerics

* A.E. Bryson, Jr, W.F. Denham, S.E. Dreyfus: Optimal programming problems with inequality constraints I, AIAA Journal 1(11):2544-2550, 1963.

Later extended by Maurer, 1979.



Transferring the Bryson-Denham-Dreyfus approach

$$G(\bar{y}_{\mathcal{A}}) \equiv 0 \text{ a. e. in } \tilde{\mathcal{A}} \iff$$

$$-\Delta G(\bar{y}_{\mathcal{A}}) + G(\bar{y}_{\mathcal{A}}) = 0, \quad \text{a. e. in } \tilde{\mathcal{A}}$$

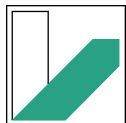
$$G(\bar{y}_{\mathcal{A}}) = 0, \quad \text{a. e. on } \gamma$$

Using the state equation

$$\Delta y_{\max} - y_{\max} + \bar{u}_{\mathcal{A}} = 0, \quad \text{a. e. in } \tilde{\mathcal{A}}$$

$$\bar{y}_{\mathcal{A}} = y_{\max}, \quad \text{a. e. on } \gamma$$

Optimal solution on $\tilde{\mathcal{A}}$ given by data, but $\tilde{\mathcal{A}}$ optimization variable



First-order necessary condition of set-OCP (split method)

(traditional adjoint state and multipliers)

a. e. in \mathcal{I} a. e. on Γ	a. e. on γ	a. e. in \mathcal{A}
$-\Delta y_I + y_I = u_I$ $\partial_\nu y_I = 0$	$y_I = y_A$ $\partial_{\nu_I} y_I + \partial_{\nu_A} y_A = 0$	$-\Delta y_A + y_A = u_A$
$y_I < y_{\max}$	$y_A = y_{\max}$	$y_A = y_{\max}$
$-\Delta p_I + p_I = y_I - y_d$ $\partial_\nu p_I = 0$	$p_I = p_A$ $\partial_{\nu_I} p_I + \partial_{\nu_A} p_A = \mu_\gamma$	$-\Delta p_A + p_A = y_A - y_d + \mu_A$
$\lambda u_I + p_I = 0$	$u_I - u_A = 0$	$\lambda u_A + p_A = 0$
	$\mu_\gamma \geq 0$	$\mu_A \geq 0$



First-order necessary condition of set-OCP (BDD method)

(new adjoint state and multipliers with higher regularity)

a. e. in \mathcal{I} a. e. on Γ	a. e. on γ	a. e. in $\tilde{\mathcal{A}}$
$-\Delta y_I + y_I = u_I$ $\partial_\nu y_I = 0$	$y_I = y_A$ $\partial_{\nu_I} y_I + \partial_{\nu_A} y_A = 0$	$-\Delta y_A + y_A = u_A$
$y_I < y_{\max}$	$y_A = y_{\max}$	$-\Delta y_{\max} + y_{\max} = u_A$
$-\Delta p_I + p_I = y_I - y_d$ $\partial_\nu p_I = 0$	$p_I = p_A \quad \downarrow$ $\partial_{\nu_I} p_I + \partial_{\nu_A} p_A = \sigma$	$-\Delta p_A + p_A = y_A - y_d$
$\lambda u_I + p_I = 0$	$u_I - u_A = 0$	$\lambda u_A + p_A + q = 0 \quad \downarrow$
	$\rightarrow \sigma \geq 0$	$\rightarrow q \geq 0$



(new adjoint state and multipliers with higher regularity)

$$-\Delta q + q = \mu_{\mathcal{A}}, \text{ a.e. in } \mathring{\mathcal{A}}$$

$$\partial_{\nu_{\mathcal{A}}} q + \sigma = \mu_{\gamma}, \text{ a.e. on } \gamma$$

**BDD approach reveals control law,
i.e. “hidden” condition (as for PDAE)**

$$u_{\mathcal{A}} = -\Delta y_{\max} + y_{\max}$$

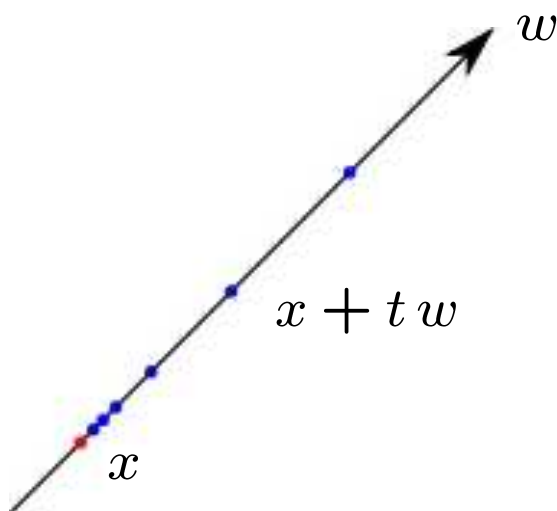


- Introduction
- Split method
- Bryson-Denham-Dreyfus approach (BDD)
- **Shape calculus and optimization on vector bundles**
- Numerics

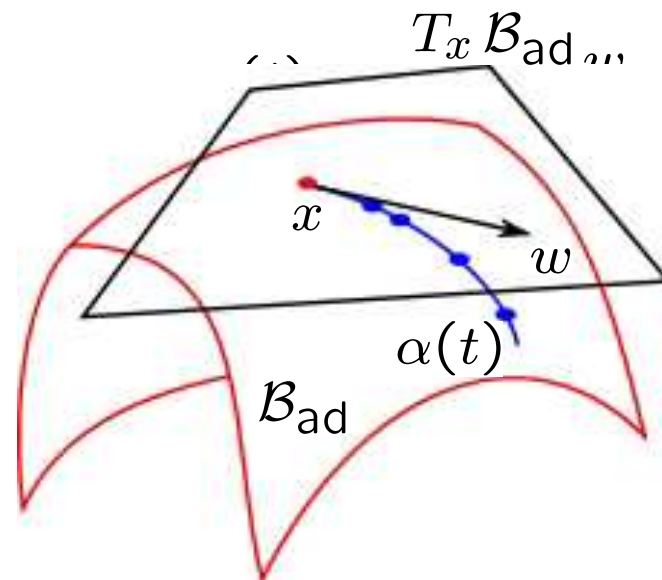


Basic considerations with respect to shape calculus

Gateaux directional derivative

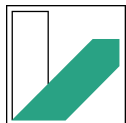


Hadamard directional derivative



$$\lim_{t \rightarrow 0} \frac{1}{t} (J(x + tw) - J(x))$$

$$\lim_{\substack{t \rightarrow 0 \\ \dot{\alpha}(0) = w}} \frac{1}{t} (J(\alpha(t)) - J(x))$$



Basic considerations with respect to shape calculus

Delfour, Zolésio, 2011

set $\mathcal{B} \in \mathcal{B}_{\text{ad}} \subset \mathcal{M}$

(no linear structure,
infinite dimensional
manifold)

vector field V

image

$$A := (\text{Id} + V)(\mathcal{B})$$

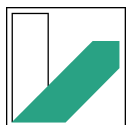
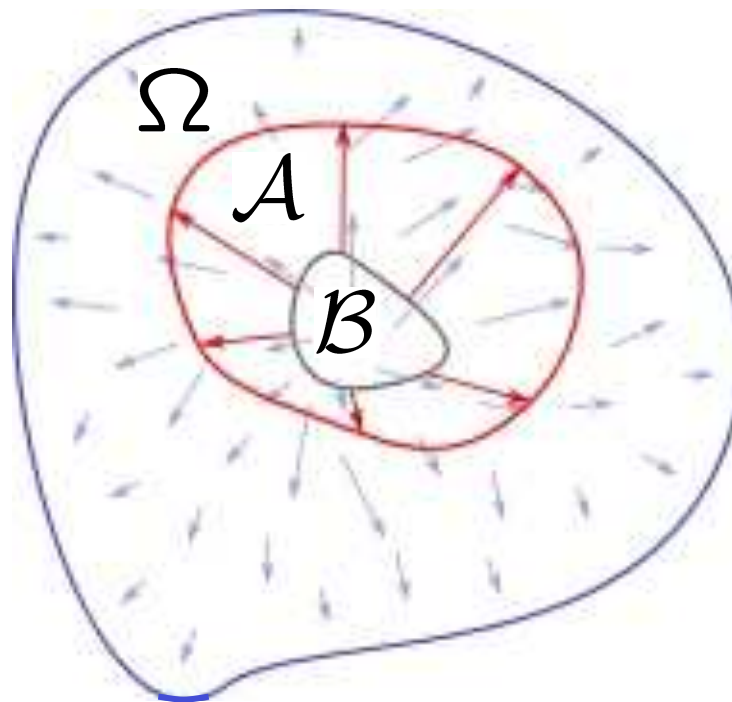
holdall Ω

perturbation of identity
defines curves \mathcal{B}_{ad}

$$\mathcal{B} \rightarrow T_t(\mathcal{B}) := (\text{Id} + tV)(\mathcal{B})$$

$$\mathcal{B}_t = T_t(\mathcal{B})$$

$$\Rightarrow dJ(\mathcal{B}; V) := \lim_{t \rightarrow 0} \frac{J(\mathcal{B}_t) - J(\mathcal{B})}{t}$$



- Hadamard directional derivative is suitable for nonlinear spaces
- Deformation of sets yields „perturbation of identity“
- Metric of function spaces induces metric in \mathcal{B}_{ad}

\mathcal{B}_{ad} has no linear structure

\mathcal{B}_{ad} is similar to an infinite dimensional manifold

- $t \mapsto \mathcal{B}_t$ defines curves in \mathcal{B}_{ad}
- suitable difference quotient

$$\lim_{t \rightarrow 0} \frac{J(\mathcal{B}_t) - J(\mathcal{B})}{t}$$



Exemplary derivation of directional shape derivative

Let

$$\mathcal{F}(\mathcal{B}) = \int_{\mathcal{B}} f(\mathcal{B}, X) \, dX$$

implicit
explicit set dependence

$$\begin{aligned} d\mathcal{F}(\mathcal{B}, V) &= \lim_{t \rightarrow 0} \frac{1}{t} \left(\int_{\mathcal{B}_t} f(\mathcal{B}_t, X) \, dX - \int_{\mathcal{B}} f(\mathcal{B}, X) \, dX \right) \\ &= \lim_{t \rightarrow 0} \frac{1}{t} \left(\int_{\mathcal{B}} f(\mathcal{B}_t, T_t(X)) \det DT_t(X) \, dX - \int_{\mathcal{B}} f(\mathcal{B}, X) \, dX \right) \\ &= \dots \\ &= \underbrace{\int_{\partial\mathcal{B}} f(\mathcal{B}, X) (V(X) \cdot \nu(X)) \, dS(X)}_{\text{explicit derivative}} + \underbrace{\int_{\mathcal{B}} \frac{\partial}{\partial t} f(\mathcal{B}_t, X) \Big|_{t=0} \, dX}_{\text{implicit derivative}} \end{aligned}$$



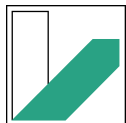
- **shape calculus is similar to calculus on manifolds**

➔ **intrinsic nonlinear behaviour**

- **shape (directional) derivative**

concentrated on boundary and

on the normal component of the vector field only



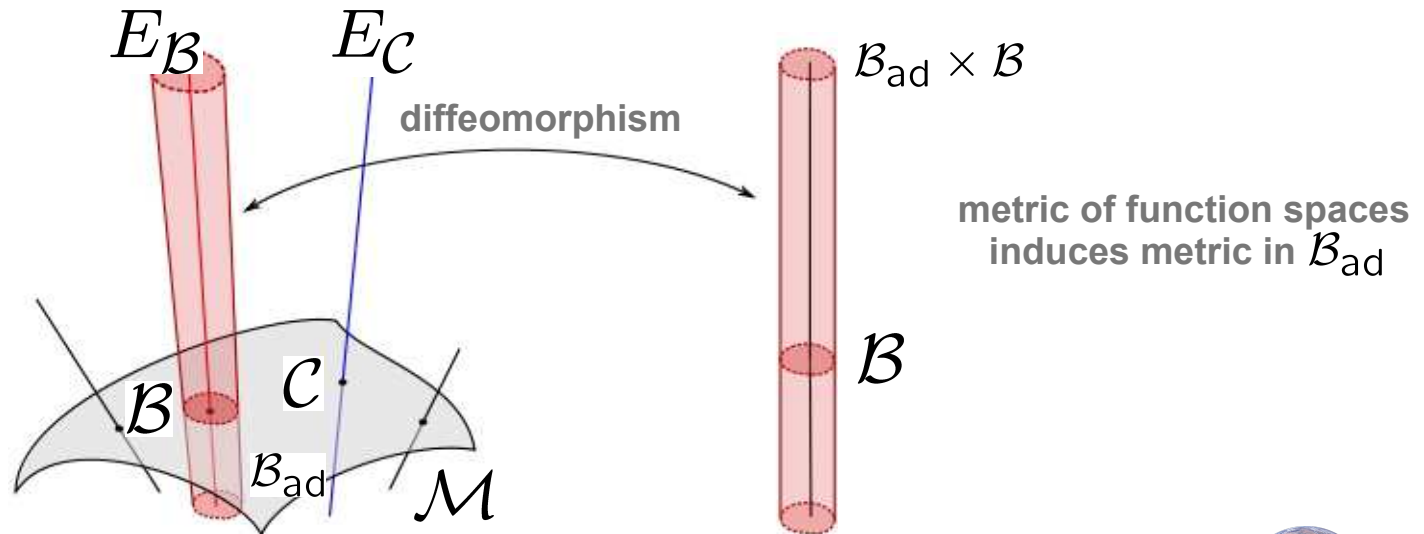
Lang, 1995

Choice of \mathcal{B} predefines the function space $E_{\mathcal{B}}$:

$$\mathcal{B} \in \mathcal{B}_{\text{ad}} \quad \longrightarrow$$

$$(y_{\mathcal{J}}, u_{\mathcal{J}}, y_{\mathcal{B}}, u_{\mathcal{B}}) \in E_{\mathcal{B}} = H^1(\mathcal{J}) \times L^2(\mathcal{J}) \times H^1(\mathcal{B}) \times L^2(\mathcal{B})$$

What is the inherent structure of $\{E_{\mathcal{B}} | \mathcal{B} \in \mathcal{B}_{\text{ad}}\}$?



Typical shape optimization problem

Minimize $J(y, \mathcal{B})$

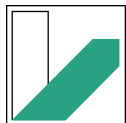
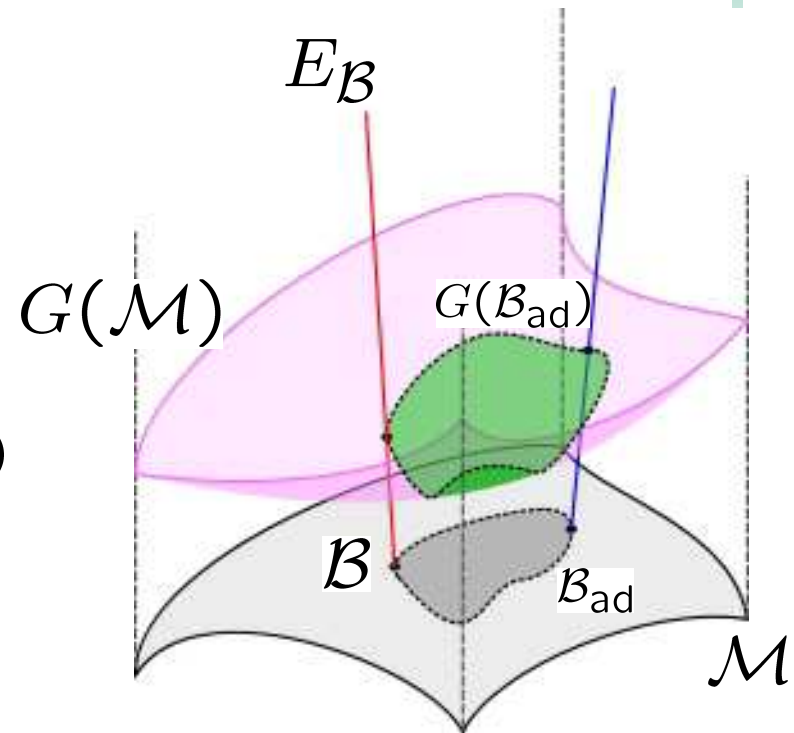
s.t. a BVP for y **on** \mathcal{B}

with $\mathcal{B} \in \mathcal{B}_{\text{ad}} \subset \mathcal{M}$, $y \in E_{\mathcal{B}}$

Unique solvability implies $y = G(\mathcal{B})$

Minimize $\mathcal{F}(\mathcal{B}) = J(G(\mathcal{B}), \mathcal{B})$

s.t. $\mathcal{B} \in \mathcal{B}_{\text{ad}} \subset \mathcal{M}$



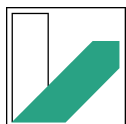
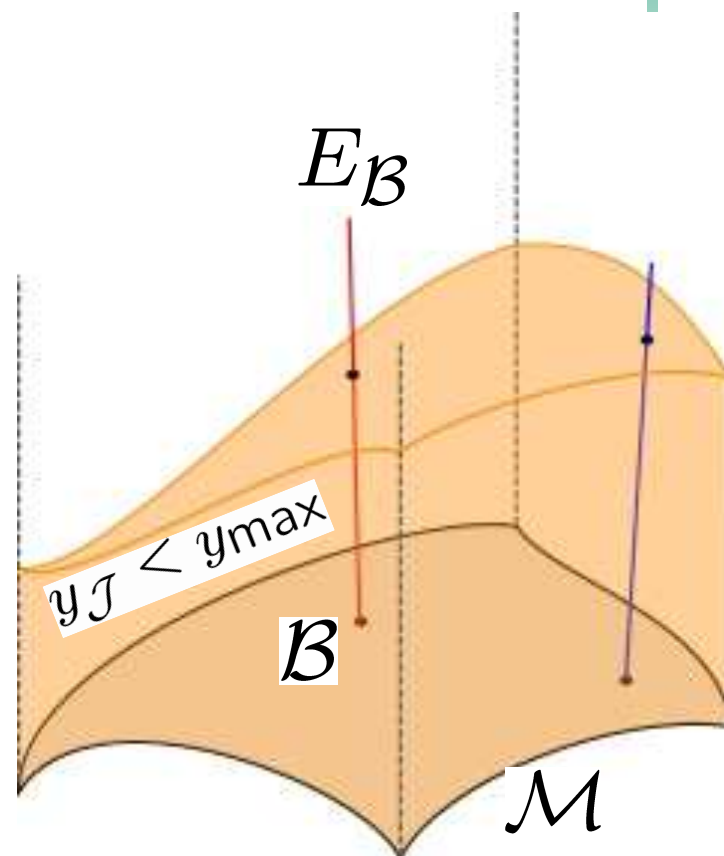
Set optimal control problem

Minimize $J(\mathcal{B}, y, u)$

s.t. a BVP for (y, u) on \mathcal{B}

with $y_{\mathcal{J}} < y_{\max}$

$$\mathcal{B} \in \mathcal{M}, (y, u) \in E_{\mathcal{B}}$$



Set optimal control problem

Minimize $J(\mathcal{B}, y, u)$

s.t. a BVP for (y, u) on \mathcal{B}

with $y_{\mathcal{J}} < y_{\max}$

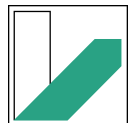
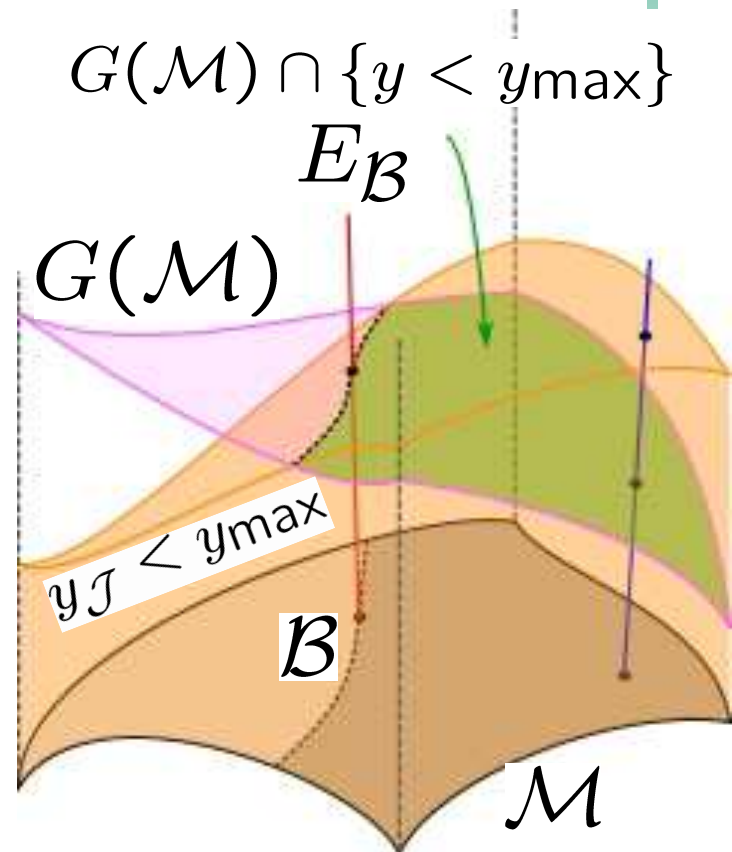
$$\mathcal{B} \in \mathcal{M}, (y, u) \in E_{\mathcal{B}}$$

Unique solvability implies $(y, u) = G(\mathcal{B})$

Minimize $\mathcal{F}(\mathcal{B}) = J(\mathcal{B}, G(\mathcal{B}))$

s.t. $\mathcal{B} \in \mathcal{B}_{\text{ad}} \subset \mathcal{M}$

$$y_{\mathcal{J}} < y_{\max}$$



- Introduction
- Split method
- Bryson-Denham-Dreyfus approach (BDD)
- Shape calculus and optimization on vector bundles
- **Numerics**



Analysis of necessary conditions

Linear PDAE

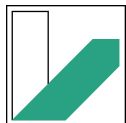
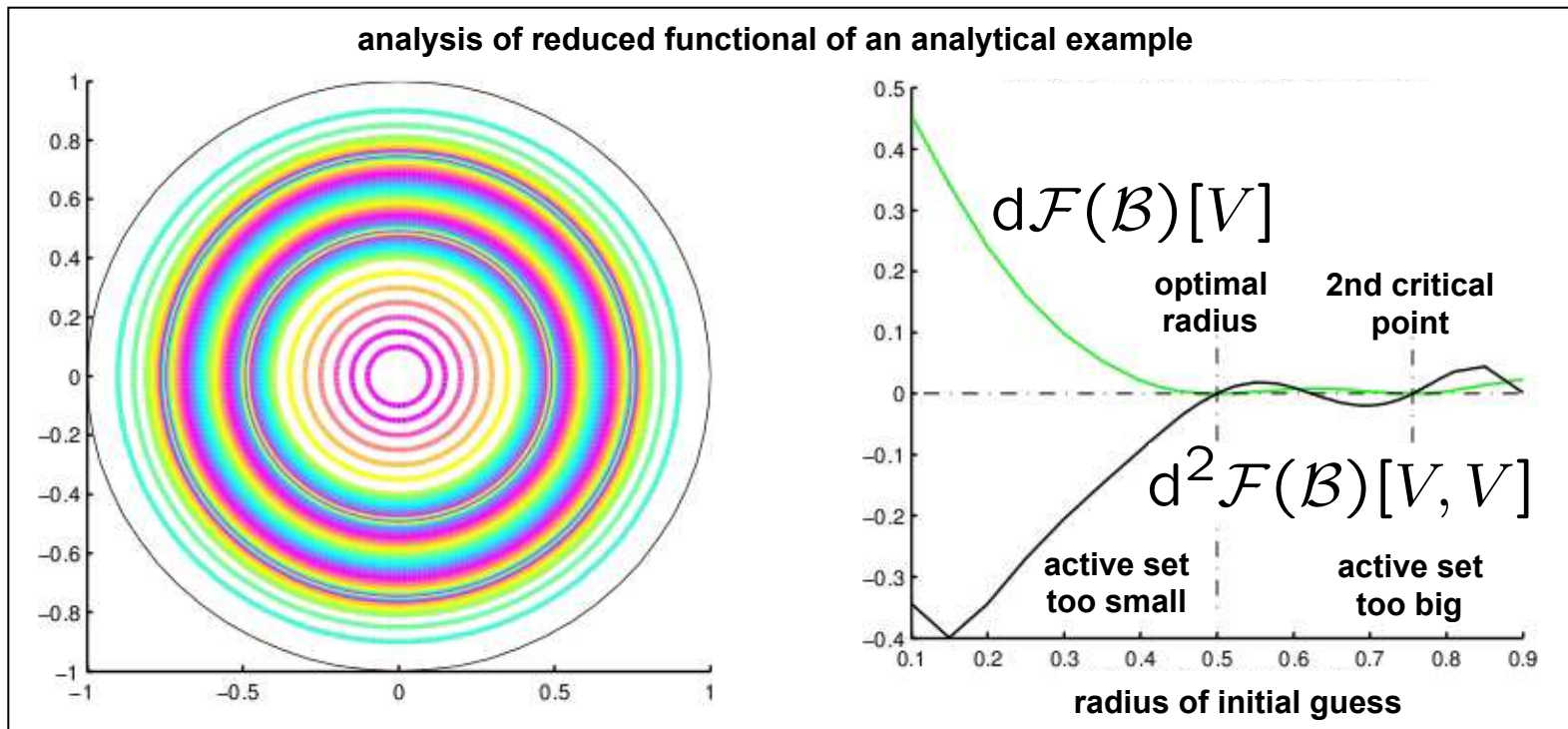
A posterior check

Interface by a free BVP
or by a nonlinear cond.

a. e. in \mathcal{I} a. e. on Γ	a. e. on γ	a. e. in \mathcal{A}
$-\Delta y_I + y_I = u_I$ $\partial_\nu y_I = 0$	$y_I = y_A$ $\partial_{\nu_I} y_I + \partial_{\nu_A} y_A = 0$	$-\Delta y_A + y_A = u_A$
$y_I < y_{\max}$	$y_A = y_{\max}$	$-\Delta y_{\max} + y_{\max} = u_A$
$-\Delta p_I + p_I = y_I - y_d$ $\partial_\nu p_I = 0$	$p_I = p_A$ $\partial_{\nu_I} p_I + \partial_{\nu_A} p_A = \sigma$	$-\Delta p_A + p_A = y_A - y_d$
$\lambda u_I + p_I = 0$	$u_I - u_A = 0$	$\lambda u_A + p_A + q = 0$
$d\mathcal{F}(\mathcal{B}; V) = 0$	$\sigma \geq 0$	$q \geq 0$

Basic considerations w.r.t. the algorithm

- $\mathcal{A} \in \mathcal{M}$ is no (local) minimum of the (unconstrained) \mathcal{F} , second semi-derivative is not definite at critical points. Hence, steepest decent algorithms not applicable, higher order methods are required.



1. Initial guess $k := 1, x_1$

2. loop on k

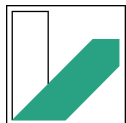
stopping criterion

Hessian and gradient require Hilbert spaces

Newton equation $\text{Hess } J(x_k)[\delta_k] = -\text{grad } J(x_k)$

update $x_{k+1} = x_k + \delta_k$

3. end of loop



1. Initial guess $k := 1, x_1$

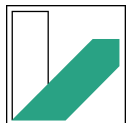
2. loop on k

stopping criterion **directional derivatives suitable for linear structure**

Newton equation $D^2 J(x_k)[\delta_k, v] = -DJ(x_k)[v] \quad \forall v \in \mathcal{V}$

update $x_{k+1} = x_k + \delta_k$

3. end of loop



Why are second derivatives more complicated?

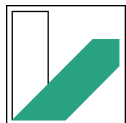
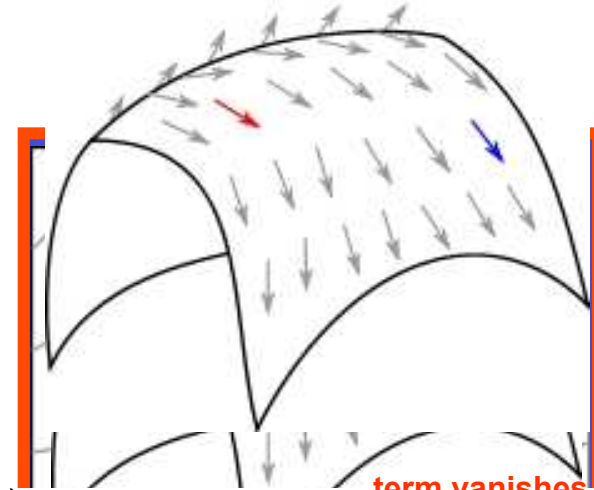
successive differentiation:

$$D^2 J[\delta_k, v] = D (DJ[v]) [\delta_k]$$

apply chain rule:

$$\begin{aligned}
 D (DJ[v]) [\delta_k] &= \left(\nabla_{\delta_k} (DJ) \right) [v] \\
 &\quad + (DJ) [\nabla_{\delta_k} v] \\
 &=: \underbrace{\nabla^2 J[\delta_k, v]}_{\text{second covariant derivative}} + \underbrace{(DJ) [\nabla_{\delta_k} v]}_{\neq 0}
 \end{aligned}$$

term vanishes
in linear spaces;
does not contain
2nd order information
on functional



1. Initial guess $k := 1, x_1$

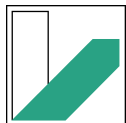
2. loop on k

stopping criterion

Newton equation $\nabla^2 J(x_k)[\delta_k, v] = -\mathbb{D}J(x_k)[v] \quad \forall v \in \mathcal{V}$

update $x_{k+1} = x_k + \delta_k$

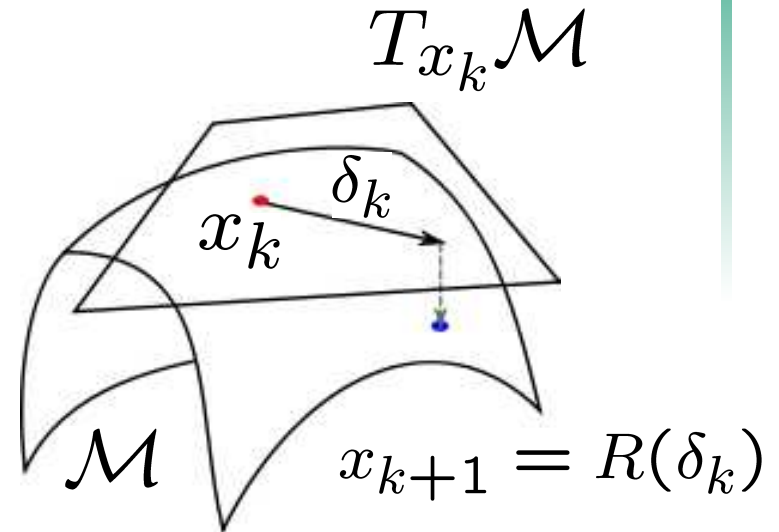
3. end of loop **sum requires linear structure**



Retraction:

$$R: T\mathcal{M} \longrightarrow \mathcal{M}$$

$$x_{k+1} = R(\delta_k)$$



$R: T\mathcal{M} \longrightarrow \mathcal{M}$ is a retraction

$:\iff$

$$R_x(0_x) = x \quad \forall x \in \mathcal{M}$$

with 0_x is zero in $T_x\mathcal{M}$ and

$$DR_x(0_x) = \text{Id}_{T_x\mathcal{M}}$$



1. Initial guess $k := 1, x_1$

2. loop on k

stopping criterion

Newton equation $\nabla^2 J(x_k)[\delta_k, v] = -DJ(x_k)[v] \quad \forall v \in \mathcal{V}$

update $x_{k+1} = R(\delta_k)$

3. end of loop



Newton's method on manifolds

1. Initial guess

$$k := 1, x_1$$

2. loop on

k

stopping criterion

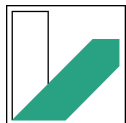
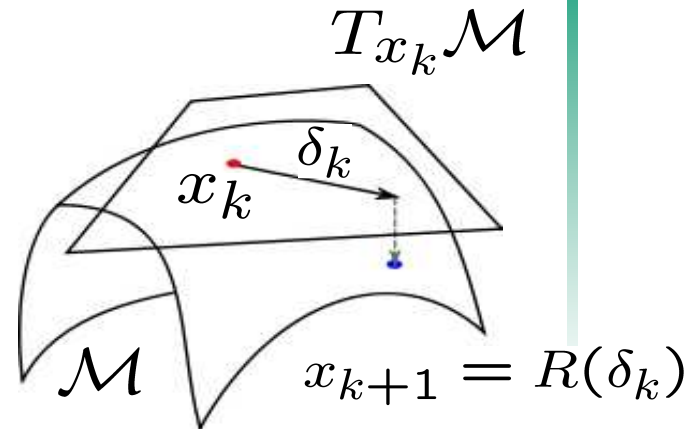
Newton equation

$$\nabla^2 J(x_k)[\delta_k, v] = -DJ(x_k)[v] \quad \forall v \in \mathcal{V}$$

update

$$x_{k+1} = R(\delta_k)$$

3. end of loop



Newton's method on manifolds

1. Initial guess

$$k := 1, x_1$$

2. loop on

$$k$$

stopping criterion

Newton equation

$$\nabla^2 J(x_k)[\delta_k, v] = -DJ(x_k)[v] \quad \forall v \in \mathcal{V}$$

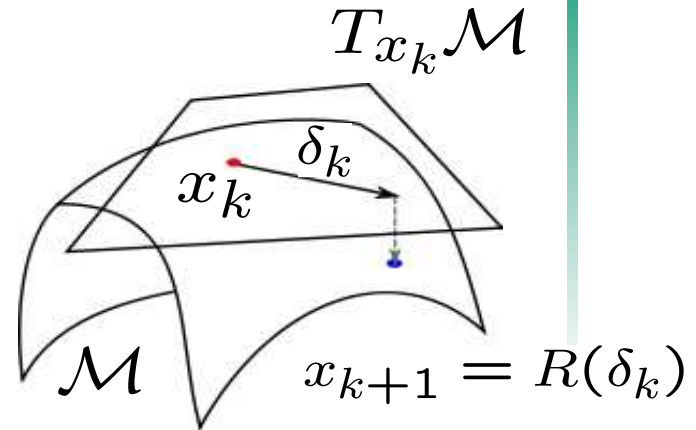
update

$$x_{k+1} = R(\delta_k) \delta_k$$

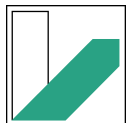
3. end of loop

second covariant derivative:
it does not contain full information
of second directional derivative,
but full second order information
on functional

directional derivative,
since Hessian and
gradient require
Hilbert spaces



retraction



1. provide initial guess \mathcal{B}_1

(by formula on data:
candidate active set)

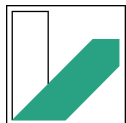
$$-\Delta y_{\mathcal{A}} + y_{\mathcal{A}} = u_{\mathcal{A}} \quad \lambda u_{\mathcal{A}} + p_{\mathcal{A}} = 0 \quad y_{\mathcal{A}} = y_{\max}$$

$$-\Delta p_{\mathcal{A}} + p_{\mathcal{A}} = y_{\mathcal{A}} - y_d + \mu$$

$$\mu(x) := -\lambda \Delta^2 y_{\max}(x) + 2\lambda \Delta y_{\max}(x) - (1 + \lambda) y_{\max}(x) + y_d(x) \geq 0$$

due to strict complementarity:

$$\mathcal{A} \subset \{x \in \Omega : \mu(x) \geq 0\}$$



Newton 's method for set optimal control problems

1. provide initial guess \mathcal{B}_1 (by formula on data:
candidate active set)

2. loop on k

stopping criterion

identify $\nabla^2 \mathcal{F}(\mathcal{B})[\cdot, \cdot]$ (complicated formula)

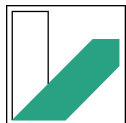
Newton equation: Find $W_k \in \mathcal{V}$

$$\nabla^2 \mathcal{F}(\mathcal{B}_k)[W_k, V] = -d\mathcal{F}(\mathcal{B}_k; V) \quad \forall V \in \mathcal{V}$$

provide retraction: deform $\mathcal{B}_k \longrightarrow \mathcal{B}_{k+1}$ (see next transparency)

3. end of loop

4. check a posteriori criteria and eventually restart with other initial guess



Newton update by second covariant derivative and retraction

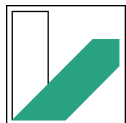
online: self-intersection

offline: violation of state constraint



Michael Frey 's dissertation, University of Bayreuth, 2012:
Shape calculus applied to state-constrained elliptic optimal control problems

<http://opus.ub.uni-bayreuth.de/opus4-ubbayreuth/frontdoor/index/index/docId/996>

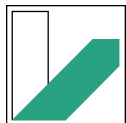


pro

- **formulation in infinite dimen. setting \Rightarrow mesh independency**
- **no regularization loops \Rightarrow better performance than PDAS**
- **feasible approximations of solutions**

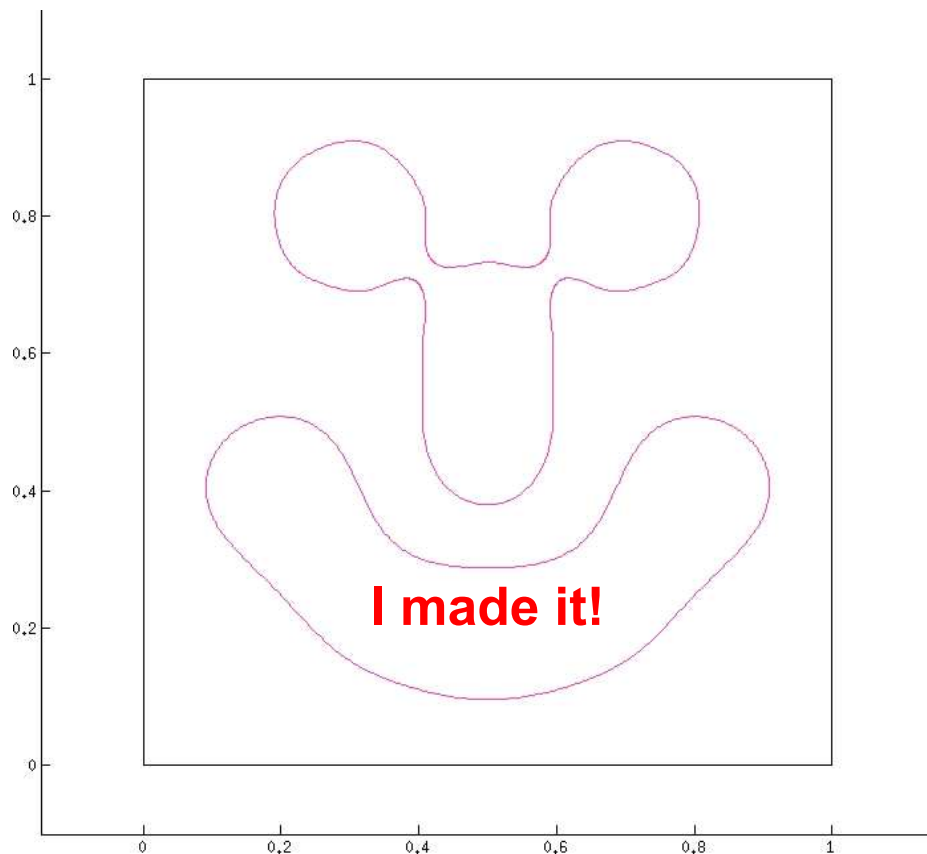
contra

- **changes of topology heuristically**
- **no convergence analysis**
- **only local convergence (with adaptive smooting for stability)**
- **assumptions on active set**



The Smiley: construction

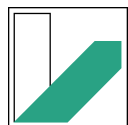
algorithm can cope with topology changes



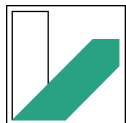
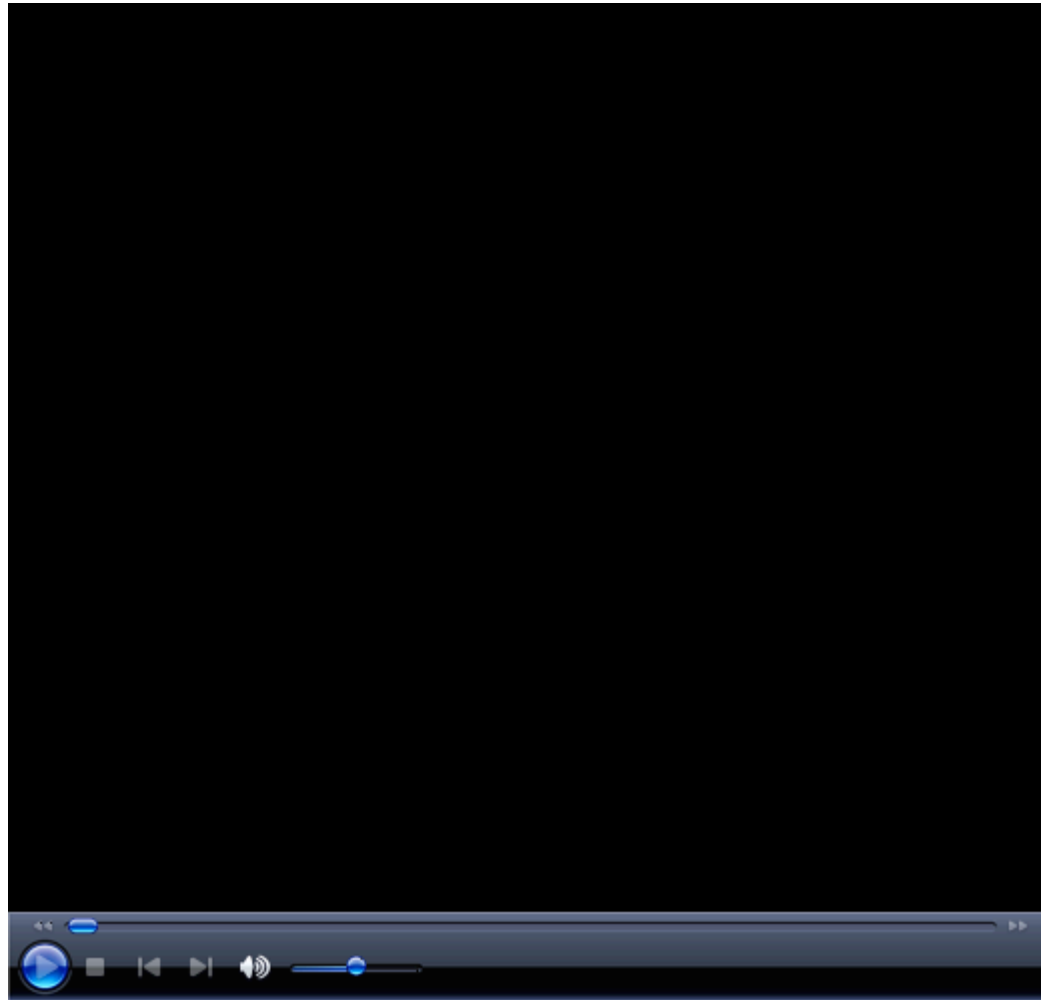
Construction:
Prescribe y_d ,
choose λ small,
press down y_{\max} .

Initial guess:
automatically from
unconstrained problem

Iter No. 



The Smiley example: rational initial guess



UNIVERSITÄT
BAYREUTH

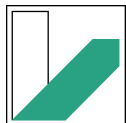
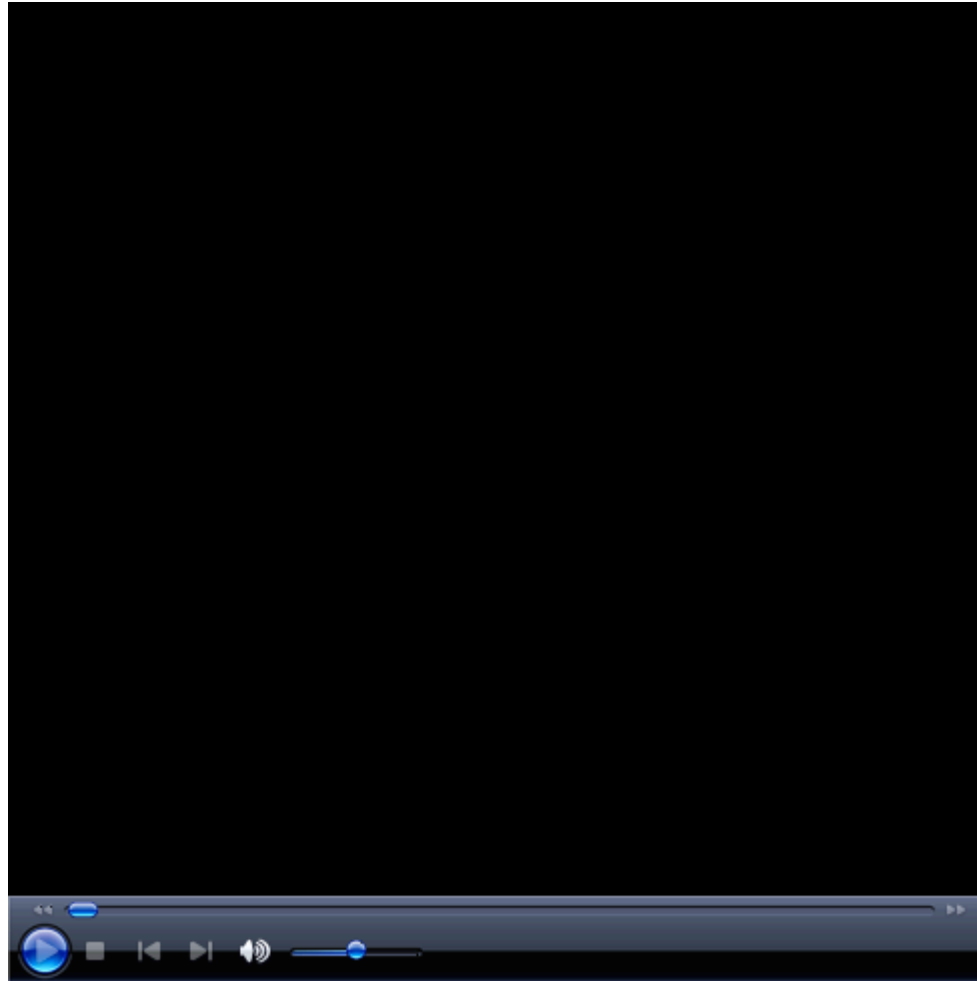
ITN SADC NetCO 2014
Conf. on New Trends in Optimal Control, Tours, June 23-27, 14

Lehrstuhl
Ingenieur-
mathematik



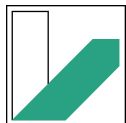
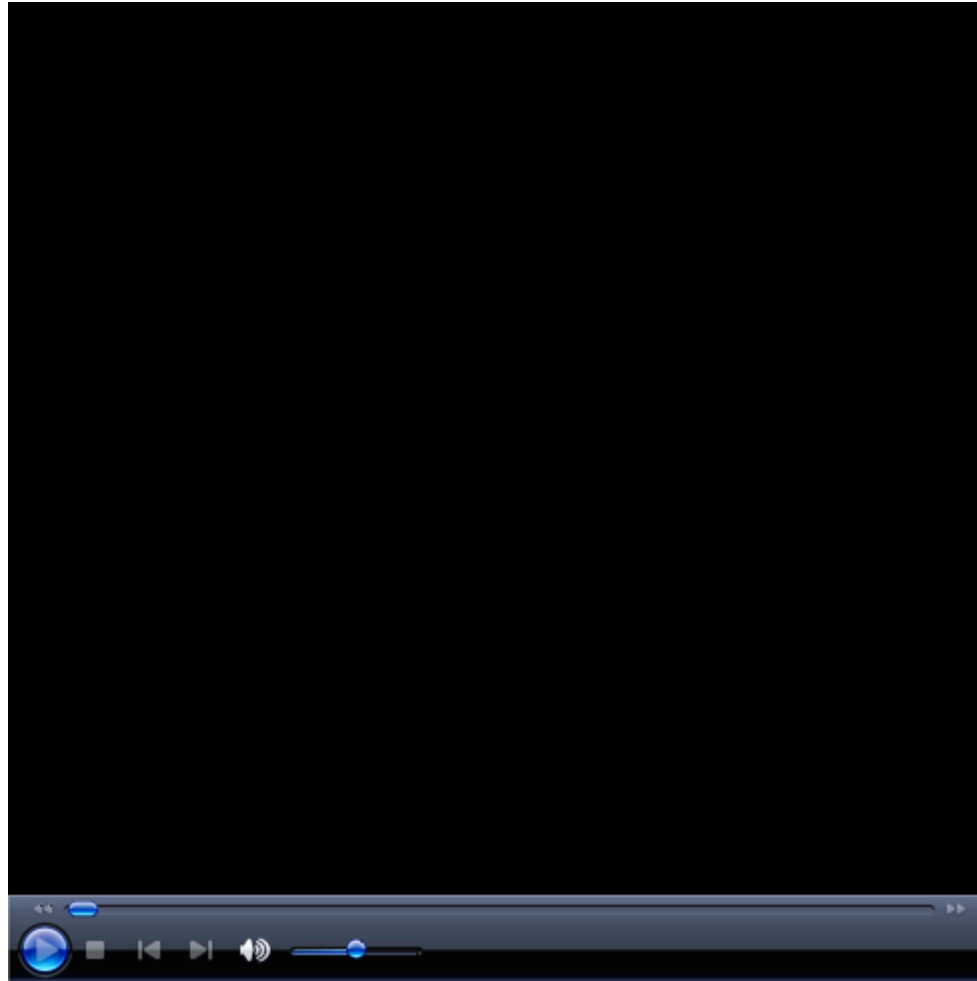
The Smiley example: bad initial guess

Algorithm can cope with topology changes to some extent



The Smiley example: bad initial guess

**Adjoint multiplier: continuous on interface,
but normal derivatives jump**

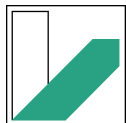


Split method

- new problem type: set optimal control problem
- bilevel formulation \Rightarrow geometry-to-solution operator
- split of constraint \Rightarrow exploitation of structure of multiplier

Extended BDD approach for PDEs

- differentiation of state constraint \Rightarrow control law
- higher regularity of multiplier
- connection with optimal control and PDAE: index reduction

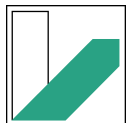


Shape calculus

- set of admissible active sets has manifold character and thus is intrinsically nonlinear
- deformation of sets: perturbation of identity
- calculation requires transformation formula

Optimization on vector bundles

- vector bundles: structure depends upon manifolds
- general basis for shape optimization / set optimal control problems
- new challenging class of optimization problems



Newton method

- several approaches towards solution of new necessary conditions
- adaption of Newton method on manifolds
- algorithm in function space without regularization
- comparable performance to sophisticated PDAS
- pays off in case of nonlinear elliptic optimal control problems

Open questions

- active sets of measure zero
- generalization to parabolic problems (first results already obtained)





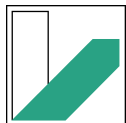
I will be glad if I have succeeded in impressing the idea that it is not only pleasant and entertaining to read at times the works of the old mathematical authors, but that this may occasionally be of use for the actual advancement of science.

Besides this there is a great lesson we can derive from the facts which I have just referred to. We have seen that even under conditions which seem most favorable very important results can be discarded for a long time and whirled away from the main stream which is carrying the vessel science. ...

If their ideas are too far in advance of their time, and if the general public is not prepared to accept them, these ideas may sleep for centuries on the shelves of our libraries ... awaiting the arrival of the prince charming who will take them home.

Constantin Carathéodory (Κωνσταντίνος Καραθεοδωρή)

* Sept. 13, 1873 in Berlin; † Feb. 2, 1950, Munich



Thank you for your attention – There 's still much to do



UNIVERSITÄT
BAYREUTH

ITN SADCO NetCO 2014

Conf. on New Trends in Optimal Control, Tours, June 23-27, 14

Lehrstuhl
Ingenieur-
mathematik

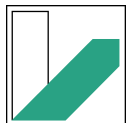


Simultaneously I want to say GOOD BYE, friends



2014-06-22

Building Bridges to Retirement



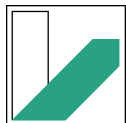
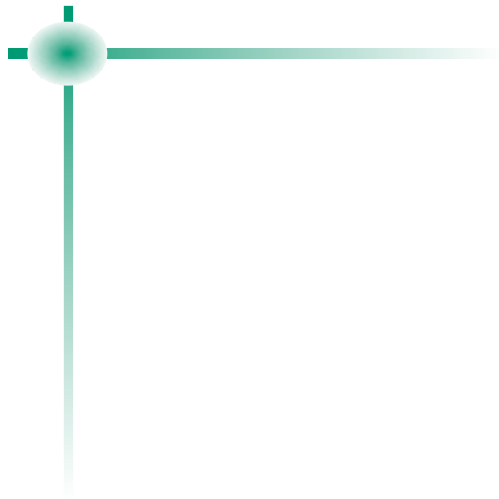
UNIVERSITÄT
BAYREUTH

ITN SADCO NetCO 2014

Conf. on New Trends in Optimal Control, Tours, June 23-27, 14

Lehrstuhl
Ingenieur-
mathematik





**UNIVERSITÄT
BAYREUTH**

ITN SADCO NetCO 2014

Conf. on New Trends in Optimal Control, Tours, June 23-27, 14

**Lehrstuhl
Ingenieur-
mathematik**

