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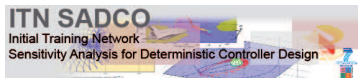
# Continuous and piecewise affine Lyapunov functions using the Yoshizawa construction

Huijuan Li

Joint work with Sigurður Hafstein, Christopher M. Kellett

Institute of Mathematics, University of Bayreuth

23-27 June, 2014 NetCo conference, Tours, France



# Outline

- 1 Introduction
- 2 Continuous and piecewise affine method
- 3 Yoshizawa construction of Lyapunov functions
- 4 Examples
- 5 Conclusion and open questions

Consider system described by ordinary differential equations

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n, \quad (1)$$

$f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is twice continuously differentiable.

Denote solutions to (1) with initial condition  $x_0$  by  $\phi(t, x_0) : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ .

### Problem

Construction of Lyapunov functions for system (1).

### Definition (Lyapunov function)

A differentiable function  $V(x) : \mathbb{R}^n \rightarrow \mathbb{R}$  is a Lyapunov function for (1) if

- 1  $V(x)$  is positive definite, i.e.,  $V(0) = 0$ ,  $V(x) > 0$  for  $x \neq 0$ .
- 2 there exists a positive definite function  $\alpha : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  such that

$$\nabla V(x)^\top f(x) \leq -\alpha(|x|).$$

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## Methods for the numerical construction of Lyapunov functions

- 1 such as **Zubov's method** [Zubov, 1955, F. Camilli, L. Grüne and F. Wirth, 2000], **solving a partial differential equation by collocation** [Giesl, 2007; Johansen, 2000], **graph theoretic methods** [Ban, Kalies, 2006; Kalies, Mischakow, and Vandervorst, 2005], and **semidefinite optimization for sum of squares polynomials** (SOS method) [Papachristodoulou, Prajna, 2002; Peet, Papachristodoulou, 2010].
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## Definition (Suitable triangulation)

We call a finite collection  $\mathcal{T} = \{\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_N\}$  of  $n$ -simplices in  $\mathbb{R}^n$  a *suitable triangulation* if

- i)  $\mathcal{S}_\nu, \mathcal{S}_\mu \in \mathcal{T}$ ,  $\nu \neq \mu$ , intersect in a common face or not at all.
- ii) With  $\mathcal{D}_\mathcal{T} := \cup_\nu \mathcal{S}_\nu$ ,  $\mathcal{D}_\mathcal{T}^\circ$  is a simply connected neighbourhood of the origin.  $\mathcal{D}_\mathcal{T}^\circ$  denotes the interior of  $\mathcal{D}_\mathcal{T}$ .
- iii) If  $0 \in \mathcal{S}_\nu$ , then  $0$  is a vertex of  $\mathcal{S}_\nu$ .

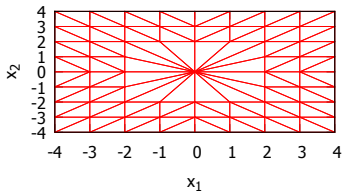


Figure: Example of  $\mathcal{T}$

## Definition (Continuous and piecewise affine function (CPA function))

For a given suitable triangulation,  $\mathcal{T}$ , and with  $\mathcal{D}_{\mathcal{T}} := \cup_{S \in \mathcal{T}} S$ , a function  $V : \mathcal{D}_{\mathcal{T}} \rightarrow \mathbb{R}$  is called a continuous and piecewise affine function if

- 1  $V$  is continuous, and
- 2 linearly affine on each simplex, i.e., there exist  $\omega_{\nu} \in \mathbb{R}^n$ ,  $a_{\nu} \in \mathbb{R}$  such that

$$V(x) = \omega_{\nu}^{\top} x + a_{\nu}, \quad x \in S_{\nu}.$$

Denote the set of all CPA functions on  $\mathcal{T}$  by  $\text{CPA}[\mathcal{T}]$ .

## Properties of CPA functions

- 1  $V(x) \in \text{CPA}[\mathcal{T}]$  is determined by its values at the vertices of the simplices of  $\mathcal{T}$ , i.e.

Let  $S_{\nu} = \text{co}\{x_0, x_1, \dots, x_n\} \in \mathcal{T}$ .

For  $x \in S_{\nu}$ ,  $x = \sum_{i=0}^n \lambda_i^x x_i$ ,  $\lambda_i^x \geq 0$  for all  $i = 0, 1, \dots, n$ , and  $\sum_{i=0}^n \lambda_i^x = 1$ .

The value of  $V$  at  $x$  is given by  $V(x) = \sum_{i=0}^n \lambda_i^x V(x_i)$ .

- 2  $V(x) \in \text{CPA}[\mathcal{T}]$  is Lipschitz continuous.

- 3  $\nabla V_{\nu} := \nabla V(x) \Big|_{x \in S_{\nu}^{\circ}} = \omega_{\nu}$ .

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### Definition (Upper Dini derivative)

For a function  $V : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ , the upper Dini derivative at  $x \in \mathbb{R}^n$  in the direction  $w \in \mathbb{R}^n$  is defined by

$$D^+ V(x, w) := \limsup_{h \rightarrow 0^+} \frac{V(x + hw) - V(x)}{h}.$$

If  $V : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$  is differentiable then  $D^+ V(x, w) = \nabla V(x)^T w$ .

# CPA Lyapunov function

Consider

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n. \quad (2)$$

Definition (CPA Lyapunov function)

Let  $\mathcal{T}$  be a suitable triangulation and  $V \in \text{CPA}[\mathcal{T}]$  be positive definite. Assume  $\varepsilon > 0$  satisfies

$$\max_{|x| \leq \varepsilon} V(x) < \min_{x \in \partial \mathcal{D}_{\mathcal{T}}} V(x).$$

If there is a constant  $\alpha^* > 0$  such that

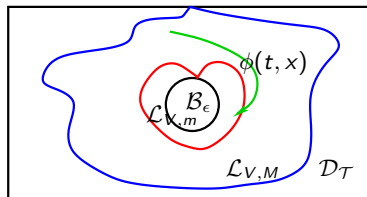
$$D^+ V(x, f(x)) \leq -\alpha^* |x|$$

for all  $x \in \mathcal{D}_{\mathcal{T}} \setminus \mathcal{B}_{\varepsilon}$  we call  $V$  a CPA[ $\mathcal{T}$ ] *Lyapunov function* for (2) on  $\mathcal{D}_{\mathcal{T}} \setminus \mathcal{B}_{\varepsilon}$  with  $\mathcal{B}_{\varepsilon} := \{x \in \mathbb{R}^n : |x| < \varepsilon\}$ .

# CPA Lyapunov function

## Theorem

Assume that  $V : \mathcal{D} \rightarrow \mathbb{R}_{\geq 0}$  is a CPA[ $\mathcal{T}$ ] Lyapunov function for (2) on  $\mathcal{D}_{\mathcal{T}} \setminus \mathcal{B}_{\epsilon}$ . For every  $c \in \mathbb{R}_{\geq 0}$  define the sublevel set  $\mathcal{L}_{V,c} := \{x \in \mathcal{D}_{\mathcal{T}} : V(x) \leq c\}$  and let  $m := \max_{|x| \leq \epsilon} V(x)$  and  $M := \min_{x \in \partial \mathcal{D}_{\mathcal{T}}} V(x)$ . Then, for every  $c \in [m, M)$  we have  $\mathcal{B}_{\epsilon} \subset \mathcal{L}_{V,c} \subset \mathcal{D}_{\mathcal{T}}^{\circ}$ , furthermore,  $\mathcal{L}_{V,m}$  is attractive and forward invariant, i.e., there exists a  $T_c \geq 0$  such that  $\phi(t, x) \subset \mathcal{L}_{V,m}$  for all  $t \geq T_c$ ,  $x \in \mathcal{L}_{V,c}$ .



# Continuous and piecewise affine method

## Procedure of the CPA method for computation of Lyapunov functions

- Partition a subset  $\mathcal{D} \subset \mathbb{R}^n$  into a suitable triangulation  $\mathcal{T}$ .
- Define values  $V(x_i)$  at the vertices  $x_i$  of every simplex  $\mathcal{S} \in \mathcal{T}$ .
- Take the convex interpolation of those vertices values, i.e., CPA function  $V(x)$ .
- Verify  $V(x)$  satisfies a system-dependent set of linear inequalities which ensures

$$D^+ V(x, f(x)) \leq -\alpha^* |x|$$

for all  $x \in \mathcal{D}_{\mathcal{T}} \setminus \mathcal{B}_{\epsilon}$ .



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## Theorem (A system-dependent set of linear inequalities)

Let  $\mathcal{T}$  be a suitable triangulation of an subset  $\mathcal{D}$ ,  $S_\nu = \text{co}\{x_0, x_1, \dots, x_n\} \in \mathcal{T}$ ,  $\mathcal{D}_{\mathcal{T}} := \cup_{S \in \mathcal{T}} S$ . Suppose  $V \in \text{CPA}[\mathcal{T}]$  is positive definite and  $\mu_\nu \in \mathbb{R}_{\geq 0}$  satisfies

$$\max_{\substack{i,j,k=1,2,\dots,n \\ x \in S_\nu}} \left| \frac{\partial^2 f_k}{\partial x_i \partial x_j}(x) \right| \leq \mu_\nu.$$

For each  $S_\nu$ , for  $i = 0, 1, \dots, n$  define the constants

$$E_{i,\nu} := \frac{n\mu_\nu}{2} |x_i - x_0| (|x_i - x_0| + \text{diam}(S_\nu)).$$

Assume there exists a constant  $\epsilon > 0$  such that

$$\max_{|x| \leq \epsilon} V(x) < \min_{x \in \partial \mathcal{D}_{\mathcal{T}}} V(x).$$

### Theorem (A system-dependent set of linear inequalities)

Then, for every  $S_\nu$  with  $S_\nu \cap \mathcal{B}_\epsilon = \emptyset$  such that the inequalities

$$\nabla V_\nu^T f(x_i) + |\nabla V_\nu|_1 E_{i,\nu} < 0$$

hold for all vertices  $x_i \in S_\nu$ ,  $i = 0, 1, \dots, n$ , we have

$$\nabla V_\nu^T f(x) < 0$$

for all  $x \in S_\nu$  with  $S_\nu \cap \mathcal{B}_\epsilon = \emptyset$ . Furthermore,  $V(x)$  is a CPA Lyapunov function on  $\mathcal{D}_T \setminus \mathcal{B}_\epsilon$ .

# Continuous and piecewise affine method

## Procedure of the CPA method for computation of Lyapunov functions

- Partition a subset  $\mathcal{D} \subset \mathbb{R}^n$  into a suitable triangulation  $\mathcal{T}$ . ✓
- Define values  $V(x_i)$  at the vertices  $x_i$  of every simplex  $\mathcal{S} \subset \mathcal{T}$ . ?
- Take the convex interpolation of those vertices values, i.e., CPA function  $V(x)$ . ✓
- Verify  $V(x)$  satisfies a system-dependent set of linear inequalities for  $x \in \mathcal{D}_{\mathcal{T}} \setminus \mathcal{B}_{\epsilon}$ . ✓

## Approaches of computation of vertices values

- Solving a linear optimization problem. See [Marinossou, 2002] [Hafstein, 2007] [Baier, Grüne, Hafstein, 2012] [Giesl, Hafstein, 2014].

**Con:** The size of the linear program is quite large  $\Rightarrow$  the cost of computation is very expensive

- We want to design a more efficient method, i.e., construction of CPA functions using Yoshizawa's construction for the values at the simplex vertices.

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Consider system described by

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n. \quad (3)$$

Let the open set  $\mathcal{D} \subset \mathbb{R}^n$  be such that  $\mathcal{D}$  is forward invariant for (3) and the origin be contained in  $\mathcal{D}$ . Suppose (3) is  $\mathcal{KL}$ -stable on  $\mathcal{D}$ .

### $\mathcal{KL}$ -stability

(3) is  $\mathcal{KL}$ -stable on  $\mathcal{D}$  if there exists  $\beta \in \mathcal{KL}$  so that

$$|\phi(t, x)| \leq \beta(|x|, t), \quad \forall x \in \mathcal{D}, \quad t \in \mathbb{R}_{\geq 0}.$$

$\beta$  is called as a *stability estimate*.

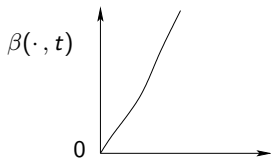


Figure:  $\mathcal{K}_\infty$  function

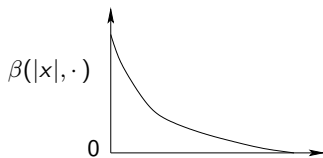


Figure:  $\mathcal{L}$  function

## Lemma (Sontag, 1998)

Given  $\beta \in \mathcal{KL}$  and  $\lambda > 0$ , there exist  $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$  with  $\alpha_1$  smooth on  $\mathbb{R}_{>0}$ , so that, for all  $s, t \geq 0$

$$\alpha_1(\beta(s, t)) \leq \alpha_2(s)e^{-\lambda t}.$$

Definition (Yoshizawa function, Teel and Praly, 2000, Yoshizawa, 1955)

Given a stability estimate  $\beta \in \mathcal{KL}$ , let  $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$  come from Lemma with  $\lambda = 2$ . We call the function  $V : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$  defined by

$$V(x) := \sup_{t \geq 0} \alpha_1(|\phi(t, x)|)e^t$$

a *Yoshizawa function* for (3).

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## Yoshizawa function

$$V(x) := \sup_{t \geq 0} \alpha_1(|\phi(t, x)|) e^t.$$

Theorem (Properties of Yoshizawa function, Section 5.1.2, Teel and Praly, 2000)

*Suppose system (3) is  $\mathcal{KL}$ -stable with stability estimate  $\beta \in \mathcal{KL}$ . Then the Yoshizawa function is locally Lipschitz continuous on  $\mathcal{D} \setminus \{0\}$  and satisfies*

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|)$$

*and the decrease condition*

$$V(\phi(t, x)) \leq V(x) e^{-t},$$

*for all  $x \in \mathcal{D}$  and all  $t \in \mathbb{R}_{\geq 0}$ . Thus  $V(x)$  is a Lyapunov function.*

Theorem (Properties of Yoshizawa function, Section 5.1.2, Teel and Praly, 2000)

Furthermore, with  $T : \mathcal{D} \setminus \{0\} \rightarrow \mathbb{R}_{\geq 0}$  defined by

$$T(x) := \ln \left( \frac{\alpha_2(|x|)}{\alpha_1(|x|)} \right) + 1$$

for all  $x \in \mathcal{D} \setminus \{0\}$ , we have

$$V(x) = \sup_{t \geq 0} \alpha_1(|\phi(t, x)|) e^t = \max_{t \in [0, T(x)]} \alpha_1(|\phi(t, x)|) e^t.$$

## Example 1-Exponentially stable estimates

Suppose the stability estimate is given by

$$|\phi(t, x)| \leq \alpha(|x|)e^{-\mu t}, \quad \mu > 0,$$

where  $\alpha(s) \geq s$ .

Then, for all  $s \in \mathbb{R}_{\geq 0}$ , we can define

$$\alpha_1(s) := s^{2/\mu}, \quad \text{and} \quad \alpha_2(s) := (\alpha(s))^{2/\mu}$$

so that, for all  $x \in \mathbb{R}^n$  and  $t \in \mathbb{R}_{\geq 0}$ ,

$$\alpha_1(\alpha(|x|)e^{-\mu t}) \leq (\alpha(|x|))^{2/\mu} e^{-2t} = \alpha_2(|x|)e^{-2t}.$$

The time horizon to optimize over is given by

$$T(x) = \frac{2}{\mu} \ln \left( \frac{\alpha(|x|)}{|x|} \right) + 1.$$

If  $\alpha(s) = Ms$  for some  $M > 1$ , then  $T(x) = T = \frac{2}{\mu} \ln M + 1$ .

Suppose system

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n, \quad (4)$$

is  $\mathcal{KL}$ -stable on the open set  $\mathcal{D}$ .

Procedure of CPA method for computation of CPA Lyapunov functions using Yoshizawa function

1. Get a stability estimate  $\beta(|x|, t) \in \mathcal{KL}$  such that  $|\phi(t, x)| \leq \beta(|x|, t)$ .
2. Choose  $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$  such that  $\alpha_1(\beta(|x|, t)) \leq \alpha_2(|x|)e^{-2t}$ .
3. Define Yoshizawa function by  $V(x) = \max_{t \in [0, T(x)]} \alpha_1(|\phi(t, x)|)e^t$ .
4. Partition the set  $\mathcal{D} \subset \mathbb{R}^n$  into a suitable triangulation  $\mathcal{T} = \{\mathcal{S}_1, \dots, \mathcal{S}_N\}$ .
5. Compute an approximated solution to (4) on the interval  $[0, T(x_i)]$  with the vertex  $x_i$  as the initial condition by numerical integration scheme (Euler scheme for numerical examples).



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## Procedure of CPA method for computation of CPA Lyapunov functions using Yoshizawa function

- Define values  $V(x_i)$  at the vertices  $x_i$  of every simplex  $\mathcal{S} \in \mathcal{T}$ , based on Yoshizawa function  $V(x) = \max_{t \in [0, \tau(x)]} \alpha_1(|\phi(t, x)|) e^t$ .
- Take the convex interpolation of those vertices values  $V(x_i)$ , i.e., CPA function  $V(x)$ .
- Verify if  $V(x)$  satisfies the system-dependent set of linear inequalities, i.e., for every  $\mathcal{S}_\nu = \text{co}\{x_0, x_1, \dots, x_n\} \in \mathcal{T} \setminus \mathcal{B}_\epsilon$  such that the inequalities

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hold for all vertices  $x_i \in \mathcal{S}_\nu$ ,  $i = 0, 1, \dots, n$ .

## Procedure of CPA method for computation of CPA Lyapunov functions using Yoshizawa function

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# Outline

- 1 Introduction
- 2 Continuous and piecewise affine method
- 3 Yoshizawa construction of Lyapunov functions
- 4 Examples**
- 5 Conclusion and open questions



## Example 2 -Linear system

Consider the linear system

$$\dot{x} = Ax = \begin{bmatrix} 1 & 1 \\ -5 & -3 \end{bmatrix} x. \quad (5)$$

Since the eigenvalues are at  $-1 \pm i$ , the origin is globally exponentially stable.

### Computation of a CPA Lyapunov function:

- 1 The system satisfies the stability estimate

$$|\phi(t, x)| \leq 7|x|e^{-t}, \quad \forall x \in \mathbb{R}^2, t \in \mathbb{R}_{\geq 0}.$$

- 2 We have  $\alpha_1(s) = s^2$ ,  $\alpha_2(s) = 49s^2$  such that  $\alpha_1(7|x|e^{-t}) \leq \alpha_2(|x|)e^{-2t}$  and  $T(x) = \ln \frac{\alpha_2(|x|)}{\alpha_1(|x|)} = T = 4.892$ . Yoshizawa function:  $\max_{t \in [0, T]} \alpha_1(|\phi(t, x)|)e^t$
- 3 Partition the set  $\mathcal{D} = \mathcal{B}_{0.972}$  into a suitable triangulation  $\mathcal{T}$  without the small neighbourhood of the origin  $\mathcal{B}_{0.048}$ .

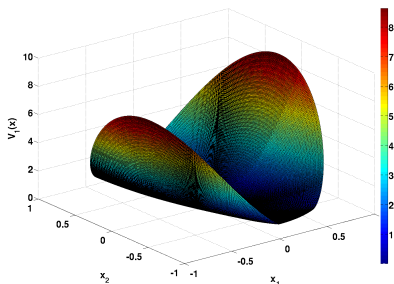


Figure: CPA Lyapunov function  $V_1(x)$ .

### Quadratic Lyapunov function

By solving the Lyapunov equation  $A^T P + PA = -\text{Id}$ , a Lyapunov function is given by

$$V(x) = x^T P x = x^T \begin{bmatrix} 4.5 & 1 \\ 1 & 0.5 \end{bmatrix} x.$$

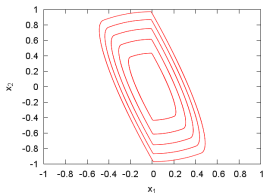


Figure: Level curves of CPA Lyapunov function  $V_1(x)$  for values 0.189, 0.378, 0.567, 0.756, and 0.945.

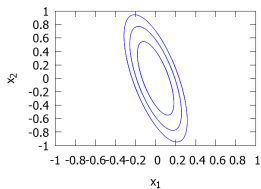


Figure: Level curves of  $V(x)$  for values 0.083, 0.166, and 0.249.

### Example 3 - Simple nonlinear system

Consider the system

$$\dot{x} = -x^3$$

which has solution

$$\phi(t, x) = \frac{x}{\sqrt{1 + 2x^2t}}, \quad \forall x \in \mathbb{R}, t \in \mathbb{R}_{\geq 0}.$$

$|\phi(t, x)|$  is in fact a  $\mathcal{KL}$  function.

Stability estimate:  $\beta(s, t) = \frac{s}{\sqrt{1+2s^2t}}$  ( $s \geq 0$ ).

Choose the functions

$$\alpha_1(s) = \alpha_2(s) := \begin{cases} 0 & , s = 0 \\ s \exp\left(-\frac{1}{s^2}\right) & , s > 0 \end{cases}$$

are such that

$$\alpha_1\left(\frac{s}{\sqrt{1+2s^2t}}\right) \leq \alpha_2(s)e^{-2t}.$$

Yoshizawa function:  $\max_{t \in [0,1]} \alpha_1(|x|)e^t$ .

The CPA Lyapunov function  $V_2(x)$  is obtained on  $[-4.8, 4.8] \setminus (-0.43, 0.43)$ .

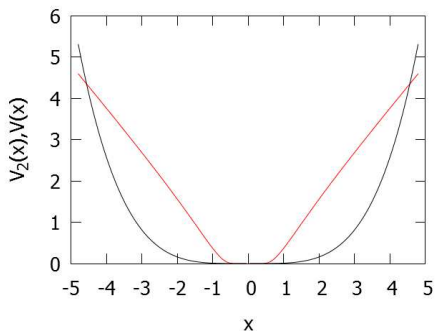


Figure: Lyapunov functions  $V(x) = 0.01x^4$  (black curve) and  $V_2(x)$  (red curve).

## Conclusion

- 1 Given system is  $\mathcal{KL}$ -stable, a new technique for the numerical construction of Lyapunov functions is proposed.
- 2 This method delivers a true Lyapunov function.
- 3 It does not need a specific structure of dynamical systems.
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## Open questions

- Find a stable estimate  $\beta(|x|, t) \in \mathcal{KL}$ .
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Thanks for your attention!