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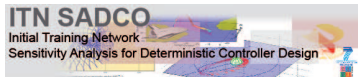
Continuous and piecewise affine Lyapunov functions using the Yoshizawa construction

Huijuan Li

Joint work with Sigurður Hafstein, Christopher M. Kellett

Institute of Mathematics, University of Bayreuth

23-27 June, 2014 NetCo conference, Tours, France



Outline

- 1 Introduction
- 2 Continuous and piecewise affine method
- 3 Yoshizawa construction of Lyapunov functions
- 4 Examples
- 5 Conclusion and open questions

Consider system described by ordinary differential equations

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n, \quad (1)$$

$f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is twice continuously differentiable.

Denote solutions to (1) with initial condition x_0 by $\phi(t, x_0) : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$.

Problem

Construction of Lyapunov functions for system (1).

Definition (Lyapunov function)

A differentiable function $V(x) : \mathbb{R}^n \rightarrow \mathbb{R}$ is a Lyapunov function for (1) if

- 1 $V(x)$ is positive definite, i.e., $V(0) = 0$, $V(x) > 0$ for $x \neq 0$.
- 2 there exists a positive definite function $\alpha : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ such that

$$\nabla V(x)^\top f(x) \leq -\alpha(|x|).$$

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Methods for the numerical construction of Lyapunov functions

- 1 such as **Zubov's method** [Zubov, 1955, F. Camilli, L. Grüne and F. Wirth, 2000], **solving a partial differential equation by collocation** [Giesl, 2007; Johansen, 2000], **graph theoretic methods** [Ban, Kalies, 2006; Kalies, Mischakow, and Vandervorst, 2005], and **semidefinite optimization for sum of squares polynomials** (SOS method) [Papachristodoulou, Prajna, 2002; Peet, Papachristodoulou, 2010].
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Definition (Suitable triangulation)

We call a finite collection $\mathcal{T} = \{\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_N\}$ of n -simplices in \mathbb{R}^n a *suitable triangulation* if

- i) $\mathcal{S}_\nu, \mathcal{S}_\mu \in \mathcal{T}$, $\nu \neq \mu$, intersect in a common face or not at all.
- ii) With $\mathcal{D}_\mathcal{T} := \cup_\nu \mathcal{S}_\nu$, $\mathcal{D}_\mathcal{T}^\circ$ is a simply connected neighbourhood of the origin. $\mathcal{D}_\mathcal{T}^\circ$ denotes the interior of $\mathcal{D}_\mathcal{T}$.
- iii) If $0 \in \mathcal{S}_\nu$, then 0 is a vertex of \mathcal{S}_ν .

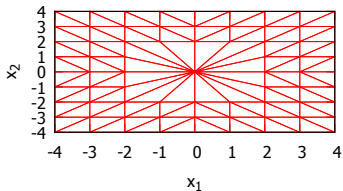


Figure: Example of \mathcal{T}

Definition (Continuous and piecewise affine function (CPA function))

For a given suitable triangulation, \mathcal{T} , and with $\mathcal{D}_{\mathcal{T}} := \cup_{S \in \mathcal{T}} S$, a function $V : \mathcal{D}_{\mathcal{T}} \rightarrow \mathbb{R}$ is called a continuous and piecewise affine function if

- 1 V is continuous, and
- 2 linearly affine on each simplex, i.e., there exist $\omega_{\nu} \in \mathbb{R}^n$, $a_{\nu} \in \mathbb{R}$ such that

$$V(x) = \omega_{\nu}^{\top} x + a_{\nu}, \quad x \in S_{\nu}.$$

Denote the set of all CPA functions on \mathcal{T} by $\text{CPA}[\mathcal{T}]$.

Properties of CPA functions

- 1 $V(x) \in \text{CPA}[\mathcal{T}]$ is determined by its values at the vertices of the simplices of \mathcal{T} , i.e.

Let $S_{\nu} = \text{co}\{x_0, x_1, \dots, x_n\} \in \mathcal{T}$.

For $x \in S_{\nu}$, $x = \sum_{i=0}^n \lambda_i^x x_i$, $\lambda_i^x \geq 0$ for all $i = 0, 1, \dots, n$, and $\sum_{i=0}^n \lambda_i^x = 1$.

The value of V at x is given by $V(x) = \sum_{i=0}^n \lambda_i^x V(x_i)$.

- 2 $V(x) \in \text{CPA}[\mathcal{T}]$ is Lipschitz continuous.

- 3 $\nabla V_{\nu} := \nabla V(x) \Big|_{x \in S_{\nu}^{\circ}} = \omega_{\nu}$.

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Definition (Upper Dini derivative)

For a function $V : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$, the upper Dini derivative at $x \in \mathbb{R}^n$ in the direction $w \in \mathbb{R}^n$ is defined by

$$D^+ V(x, w) := \limsup_{h \rightarrow 0^+} \frac{V(x + hw) - V(x)}{h}.$$

If $V : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ is differentiable then $D^+ V(x, w) = \nabla V(x)^T w$.

CPA Lyapunov function

Consider

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n. \quad (2)$$

Definition (CPA Lyapunov function)

Let \mathcal{T} be a suitable triangulation and $V \in \text{CPA}[\mathcal{T}]$ be positive definite. Assume $\varepsilon > 0$ satisfies

$$\max_{|x| \leq \varepsilon} V(x) < \min_{x \in \partial \mathcal{D}_{\mathcal{T}}} V(x).$$

If there is a constant $\alpha^* > 0$ such that

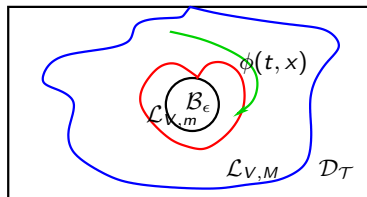
$$D^+ V(x, f(x)) \leq -\alpha^* |x|$$

for all $x \in \mathcal{D}_{\mathcal{T}} \setminus \mathcal{B}_{\varepsilon}$ we call V a CPA[\mathcal{T}] *Lyapunov function* for (2) on $\mathcal{D}_{\mathcal{T}} \setminus \mathcal{B}_{\varepsilon}$ with $\mathcal{B}_{\varepsilon} := \{x \in \mathbb{R}^n : |x| < \varepsilon\}$.

CPA Lyapunov function

Theorem

Assume that $V : \mathcal{D} \rightarrow \mathbb{R}_{\geq 0}$ is a CPA[\mathcal{T}] Lyapunov function for (2) on $\mathcal{D}_{\mathcal{T}} \setminus \mathcal{B}_{\epsilon}$. For every $c \in \mathbb{R}_{\geq 0}$ define the sublevel set $\mathcal{L}_{V,c} := \{x \in \mathcal{D}_{\mathcal{T}} : V(x) \leq c\}$ and let $m := \max_{|x| \leq \epsilon} V(x)$ and $M := \min_{x \in \partial \mathcal{D}_{\mathcal{T}}} V(x)$. Then, for every $c \in [m, M)$ we have $\mathcal{B}_{\epsilon} \subset \mathcal{L}_{V,c} \subset \mathcal{D}_{\mathcal{T}}^{\circ}$, furthermore, $\mathcal{L}_{V,m}$ is attractive and forward invariant, i.e., there exists a $T_c \geq 0$ such that $\phi(t, x) \subset \mathcal{L}_{V,m}$ for all $t \geq T_c$, $x \in \mathcal{L}_{V,c}$.



Continuous and piecewise affine method

Procedure of the CPA method for computation of Lyapunov functions

- Partition a subset $\mathcal{D} \subset \mathbb{R}^n$ into a suitable triangulation \mathcal{T} .
- Define values $V(x_i)$ at the vertices x_i of every simplex $\mathcal{S} \subset \mathcal{T}$.
- Take the convex interpolation of those vertices values, i.e., CPA function $V(x)$.
- Verify $V(x)$ satisfies a system-dependent set of linear inequalities which ensures

$$D^+ V(x, f(x)) \leq -\alpha^* |x|$$

for all $x \in \mathcal{D}_{\mathcal{T}} \setminus \mathcal{B}_{\epsilon}$.

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Theorem (A system-dependent set of linear inequalities)

Let \mathcal{T} be a suitable triangulation of an subset \mathcal{D} , $S_\nu = \text{co}\{x_0, x_1, \dots, x_n\} \in \mathcal{T}$, $\mathcal{D}_{\mathcal{T}} := \cup_{S \in \mathcal{T}} S$. Suppose $V \in \text{CPA}[\mathcal{T}]$ is positive definite and $\mu_\nu \in \mathbb{R}_{\geq 0}$ satisfies

$$\max_{\substack{i,j,k=1,2,\dots,n \\ x \in S_\nu}} \left| \frac{\partial^2 f_k}{\partial x_i \partial x_j}(x) \right| \leq \mu_\nu.$$

For each S_ν , for $i = 0, 1, \dots, n$ define the constants

$$E_{i,\nu} := \frac{n\mu_\nu}{2} |x_i - x_0| (|x_i - x_0| + \text{diam}(S_\nu)).$$

Assume there exists a constant $\epsilon > 0$ such that

$$\max_{|x| \leq \epsilon} V(x) < \min_{x \in \partial \mathcal{D}_{\mathcal{T}}} V(x).$$

Theorem (A system-dependent set of linear inequalities)

Then, for every \mathcal{S}_ν with $\mathcal{S}_\nu \cap \mathcal{B}_\epsilon = \emptyset$ such that the inequalities

$$\nabla V_\nu^T f(x_i) + |\nabla V_\nu|_1 E_{i,\nu} < 0$$

hold for all vertices $x_i \in \mathcal{S}_\nu$, $i = 0, 1, \dots, n$, we have

$$\nabla V_\nu^T f(x) < 0$$

for all $x \in \mathcal{S}_\nu$ with $\mathcal{S}_\nu \cap \mathcal{B}_\epsilon = \emptyset$. Furthermore, $V(x)$ is a CPA Lyapunov function on $\mathcal{D}_T \setminus \mathcal{B}_\epsilon$.

Continuous and piecewise affine method

Procedure of the CPA method for computation of Lyapunov functions

- Partition a subset $\mathcal{D} \subset \mathbb{R}^n$ into a suitable triangulation \mathcal{T} . ✓
- Define values $V(x_i)$ at the vertices x_i of every simplex $\mathcal{S} \subset \mathcal{T}$. ?
- Take the convex interpolation of those vertices values, i.e., CPA function $V(x)$. ✓
- Verify $V(x)$ satisfies a system-dependent set of linear inequalities for $x \in \mathcal{D}_{\mathcal{T}} \setminus \mathcal{B}_{\epsilon}$. ✓

Approaches of computation of vertices values

- Solving a linear optimization problem. See [Marinossou, 2002] [Hafstein, 2007] [Baier, Grüne, Hafstein, 2012] [Giesl, Hafstein, 2014].

Con: The size of the linear program is quite large \Rightarrow the cost of computation is very expensive

- We want to design a more efficient method, i.e., construction of CPA functions using Yoshizawa's construction for the values at the simplex vertices.

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Consider system described by

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n. \quad (3)$$

Let the open set $\mathcal{D} \subset \mathbb{R}^n$ be such that \mathcal{D} is forward invariant for (3) and the origin be contained in \mathcal{D} . Suppose (3) is \mathcal{KL} -stable on \mathcal{D} .

\mathcal{KL} -stability

(3) is \mathcal{KL} -stable on \mathcal{D} if there exists $\beta \in \mathcal{KL}$ so that

$$|\phi(t, x)| \leq \beta(|x|, t), \quad \forall x \in \mathcal{D}, \quad t \in \mathbb{R}_{\geq 0}.$$

β is called as a *stability estimate*.

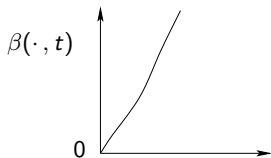


Figure: \mathcal{K}_∞ function

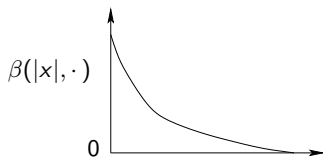


Figure: \mathcal{L} function

Lemma (Sontag, 1998)

Given $\beta \in \mathcal{KL}$ and $\lambda > 0$, there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ with α_1 smooth on $\mathbb{R}_{>0}$, so that, for all $s, t \geq 0$

$$\alpha_1(\beta(s, t)) \leq \alpha_2(s)e^{-\lambda t}.$$

Definition (Yoshizawa function, Teel and Praly, 2000, Yoshizawa, 1955)

Given a stability estimate $\beta \in \mathcal{KL}$, let $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ come from Lemma with $\lambda = 2$. We call the function $V : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ defined by

$$V(x) := \sup_{t \geq 0} \alpha_1(|\phi(t, x)|)e^t$$

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Yoshizawa function

$$V(x) := \sup_{t \geq 0} \alpha_1(|\phi(t, x)|) e^t.$$

Theorem (Properties of Yoshizawa function, Section 5.1.2, Teel and Praly, 2000)

Suppose system (3) is \mathcal{KL} -stable with stability estimate $\beta \in \mathcal{KL}$. Then the Yoshizawa function is locally Lipschitz continuous on $\mathcal{D} \setminus \{0\}$ and satisfies

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|)$$

and the decrease condition

$$V(\phi(t, x)) \leq V(x) e^{-t},$$

for all $x \in \mathcal{D}$ and all $t \in \mathbb{R}_{\geq 0}$. Thus $V(x)$ is a Lyapunov function.

Theorem (Properties of Yoshizawa function, Section 5.1.2, Teel and Praly, 2000)

Furthermore, with $T : \mathcal{D} \setminus \{0\} \rightarrow \mathbb{R}_{\geq 0}$ defined by

$$T(x) := \ln \left(\frac{\alpha_2(|x|)}{\alpha_1(|x|)} \right) + 1$$

for all $x \in \mathcal{D} \setminus \{0\}$, we have

$$V(x) = \sup_{t \geq 0} \alpha_1(|\phi(t, x)|) e^t = \max_{t \in [0, T(x)]} \alpha_1(|\phi(t, x)|) e^t.$$

Example 1-Exponentially stable estimates

Suppose the stability estimate is given by

$$|\phi(t, x)| \leq \alpha(|x|)e^{-\mu t}, \quad \mu > 0,$$

where $\alpha(s) \geq s$.

Then, for all $s \in \mathbb{R}_{\geq 0}$, we can define

$$\alpha_1(s) := s^{2/\mu}, \quad \text{and} \quad \alpha_2(s) := (\alpha(s))^{2/\mu}$$

so that, for all $x \in \mathbb{R}^n$ and $t \in \mathbb{R}_{\geq 0}$,

$$\alpha_1(\alpha(|x|)e^{-\mu t}) \leq (\alpha(|x|))^{2/\mu} e^{-2t} = \alpha_2(|x|)e^{-2t}.$$

The time horizon to optimize over is given by

$$T(x) = \frac{2}{\mu} \ln \left(\frac{\alpha(|x|)}{|x|} \right) + 1.$$

If $\alpha(s) = Ms$ for some $M > 1$, then $T(x) = T = \frac{2}{\mu} \ln M + 1$.

Suppose system

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n, \quad (4)$$

is \mathcal{KL} -stable on the open set \mathcal{D} .

Procedure of CPA method for computation of CPA Lyapunov functions using Yoshizawa function

1. Get a stability estimate $\beta(|x|, t) \in \mathcal{KL}$ such that $|\phi(t, x)| \leq \beta(|x|, t)$.
2. Choose $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that $\alpha_1(\beta(|x|, t)) \leq \alpha_2(|x|)e^{-2t}$.
3. Define Yoshizawa function by $V(x) = \max_{t \in [0, T(x)]} \alpha_1(|\phi(t, x)|)e^t$.
4. Partition the set $\mathcal{D} \subset \mathbb{R}^n$ into a suitable triangulation $\mathcal{T} = \{\mathcal{S}_1, \dots, \mathcal{S}_N\}$.
5. Compute an approximated solution to (4) on the interval $[0, T(x_i)]$ with the vertex x_i as the initial condition by numerical integration scheme (Euler scheme for numerical examples).

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$$\dot{x} = f(x), \quad x \in \mathbb{R}^n, \quad (4)$$

is \mathcal{KL} -stable on the open set \mathcal{D} .

Procedure of CPA method for computation of CPA Lyapunov functions using Yoshizawa function

1. Get a stability estimate $\beta(|x|, t) \in \mathcal{KL}$ such that $|\phi(t, x)| \leq \beta(|x|, t)$.
2. Choose $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that $\alpha_1(\beta(|x|, t)) \leq \alpha_2(|x|)e^{-2t}$.
3. Define Yoshizawa function by $V(x) = \max_{t \in [0, T(x)]} \alpha_1(|\phi(t, x)|)e^t$.
4. Partition the set $\mathcal{D} \subset \mathbb{R}^n$ into a suitable triangulation $\mathcal{T} = \{\mathcal{S}_1, \dots, \mathcal{S}_N\}$.
5. Compute an approximated solution to (4) on the interval $[0, T(x_i)]$ with the vertex x_i as the initial condition by numerical integration scheme (Euler scheme for numerical examples).

Procedure of CPA method for computation of CPA Lyapunov functions using Yoshizawa function

6. Define values $V(x_i)$ at the vertices x_i of every simplex $\mathcal{S} \in \mathcal{T}$, based on Yoshizawa function $V(x) = \max_{t \in [0, \tau(x)]} \alpha_1(|\phi(t, x)|) e^t$.
7. Take the convex interpolation of those vertices values $V(x_i)$, i.e., CPA function $V(x)$.
8. Verify if $V(x)$ satisfies the system-dependent set of linear inequalities, i.e., for every $\mathcal{S}_\nu = \text{co}\{x_0, x_1, \dots, x_n\} \in \mathcal{T} \setminus \mathcal{B}_\epsilon$ such that the inequalities

$$\nabla V_\nu^T f(x_i) + |\nabla V_\nu|_1 E_{i,\nu} < 0$$

hold for all vertices $x_i \in \mathcal{S}_\nu$, $i = 0, 1, \dots, n$.

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Outline

- 1 Introduction
- 2 Continuous and piecewise affine method
- 3 Yoshizawa construction of Lyapunov functions
- 4 Examples**
- 5 Conclusion and open questions

Example 2 -Linear system

Consider the linear system

$$\dot{x} = Ax = \begin{bmatrix} 1 & 1 \\ -5 & -3 \end{bmatrix} x. \quad (5)$$

Since the eigenvalues are at $-1 \pm i$, the origin is globally exponentially stable.

Computation of a CPA Lyapunov function:

- 1 The system satisfies the stability estimate

$$|\phi(t, x)| \leq 7|x|e^{-t}, \quad \forall x \in \mathbb{R}^2, t \in \mathbb{R}_{\geq 0}.$$

- 2 We have $\alpha_1(s) = s^2$, $\alpha_2(s) = 49s^2$ such that $\alpha_1(7|x|e^{-t}) \leq \alpha_2(|x|)e^{-2t}$ and $T(x) = \ln \frac{\alpha_2(|x|)}{\alpha_1(|x|)} = T = 4.892$. Yoshizawa function: $\max_{t \in [0, T]} \alpha_1(|\phi(t, x)|)e^t$
- 3 Partition the set $\mathcal{D} = \mathcal{B}_{0.972}$ into a suitable triangulation \mathcal{T} without the small neighbourhood of the origin $\mathcal{B}_{0.048}$.

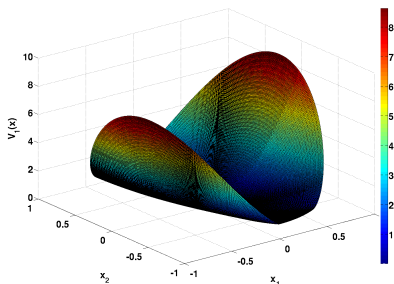


Figure: CPA Lyapunov function $V_1(x)$.

Quadratic Lyapunov function

By solving the Lyapunov equation $A^T P + PA = -\text{Id}$, a Lyapunov function is given by

$$V(x) = x^T P x = x^T \begin{bmatrix} 4.5 & 1 \\ 1 & 0.5 \end{bmatrix} x.$$

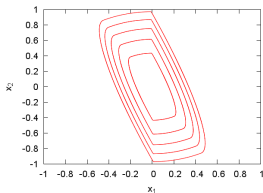


Figure: Level curves of CPA Lyapunov function $V_1(x)$ for values 0.189, 0.378, 0.567, 0.756, and 0.945.

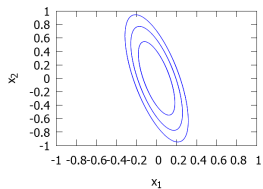


Figure: Level curves of $V(x)$ for values 0.083, 0.166, and 0.249.

Example 3 - Simple nonlinear system

Consider the system

$$\dot{x} = -x^3$$

which has solution

$$\phi(t, x) = \frac{x}{\sqrt{1 + 2x^2t}}, \quad \forall x \in \mathbb{R}, t \in \mathbb{R}_{\geq 0}.$$

$|\phi(t, x)|$ is in fact a \mathcal{KL} function.

Stability estimate: $\beta(s, t) = \frac{s}{\sqrt{1+2s^2t}}$ ($s \geq 0$).

Choose the functions

$$\alpha_1(s) = \alpha_2(s) := \begin{cases} 0 & , s = 0 \\ s \exp\left(-\frac{1}{s^2}\right) & , s > 0 \end{cases}$$

are such that

$$\alpha_1\left(\frac{s}{\sqrt{1+2s^2t}}\right) \leq \alpha_2(s)e^{-2t}.$$

Yoshizawa function: $\max_{t \in [0,1]} \alpha_1(|x|)e^t$.

The CPA Lyapunov function $V_2(x)$ is obtained on $[-4.8, 4.8] \setminus (-0.43, 0.43)$.

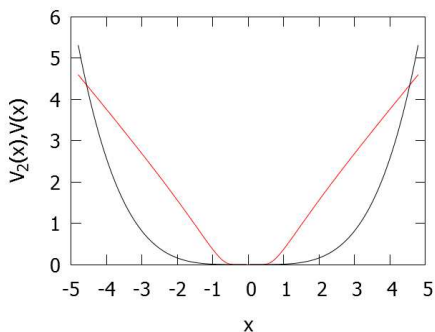


Figure: Lyapunov functions $V(x) = 0.01x^4$ (black curve) and $V_2(x)$ (red curve).

Conclusion

- 1 Given system is \mathcal{KL} -stable, a new technique for the numerical construction of Lyapunov functions is proposed.
- 2 This method delivers a true Lyapunov function.
- 3 It does not need a specific structure of dynamical systems.
- 4 The cost of computation is cheaper than the CPA method with computation vertices values by solving a linear optimization problem.

Open questions

- Find a stable estimate $\beta(|x|, t) \in \mathcal{KL}$.
- Construct $\alpha_1(\cdot), \alpha_2(\cdot) \in \mathcal{K}_\infty$ such that $\alpha_1(\beta(|x|, t)) \leq \alpha_2(|x|)e^{-2t}$.

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Thanks for your attention!