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From discrete microscopic models to macroscopic models and applications to traffic flow

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From discrete microscopic models to macroscopic models and applications to traffic flow.

Nicolas Forcadel

Laboratoire de Mathématique de L'INSA de Rouen

Joint work with W. Salazar

Conference on "New Trends in Optimal Control"

23-27 June 2014, Tours

Plan

- 1 Motivations
- 2 Homogenization of traffic flows models

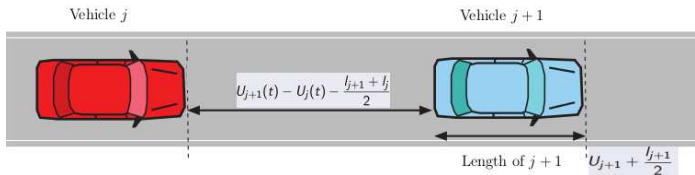
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Bando model

- Second order discrete model with $n_0 \in \mathbb{N}$ types of vehicles :

$$\ddot{U}_j(t) = a_j \left(V_j \left(U_{j+1}(t) - U_j(t) - \frac{l_{j+1} + l_j}{2} \right) - \dot{U}_j(t) \right). \quad (1)$$



- U_j : position of the vehicle j .
- a_j : sensibility of the driver j ($a_{j+n_0} = a_j$).
- V_j : Optimal velocity function (OVF) of the driver j ($V_{j+n_0} = V_j$).

Passing from micro to macro

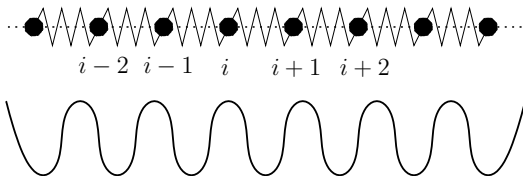
- Goal : Describe the traffic in term of density of vehicles, i.e. passing from the microscopic model to a macroscopic one.
- LWR macroscopic model:

$$\rho_t + (\rho v(\rho))_x = 0$$

where v is the average speed of vehicles.

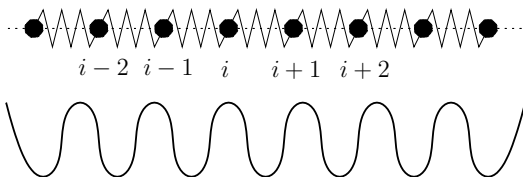
Existing results for the Frenkel-Kontorova model

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$$m \frac{d^2 U_j}{dt^2} + \gamma \frac{dU_j}{dt} = (U_{j+1} - U_j) + (U_{j-1} - U_j) + \sin(2\pi U_j) + f$$

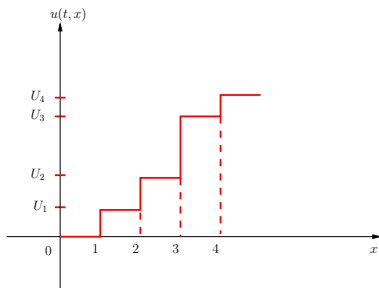
Existing results for the Frenkel-Kontorova model



$$m \frac{d^2 U_j}{dt^2} + \gamma \frac{dU_j}{dt} = F_j(U_{j-m}, \dots, U_{j+m})$$

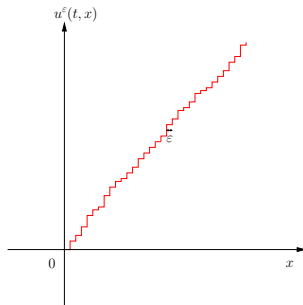
Homogenization

$$u(t, x) = U_{[x]}(t)$$



Rescaling

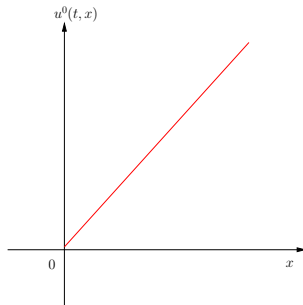
$$u^\varepsilon(t, x) = \varepsilon u\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right)$$



$$u^\varepsilon \rightarrow ?$$

Rescaling

$$u^\varepsilon(t, x) = \varepsilon u\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right)$$



$$u^\varepsilon \rightarrow u^0$$

Homogenization results

For $\varepsilon > 0$, we define

$$\begin{cases} u^\varepsilon(t, x) = \varepsilon u\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right), \\ u^\varepsilon(0, x) = u_0(x) \end{cases}$$

Theorem (F., Imbert, Monneau)

Under certain assumptions on u_0 , we have $u^\varepsilon \rightarrow u^0$ with

$$\begin{cases} u_t^0 = \bar{F}(u_x^0), \\ u^0(0, x) = u_0(x) \end{cases}$$

- The density of particles $\rho = \frac{1}{u_x^0}$ satisfies formally a conservation law

$$\rho_t = (\bar{H}(\rho))_x \quad \text{with} \quad \bar{H}(\rho) = -\rho \bar{F}(1/\rho)$$

Ingredients of the proof

- Main idea: inject the system of particles into a (system of) PDE.
- If m is small, the system of PDE satisfies a comparison principle
- Order of the particles (link between the F_j).
- Notion of hull functions

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Change of variables

$$\ddot{U}_j(t) = a_j \left(V_j (U_{j+1}(t) - U_j(t)) - \dot{U}_j(t) \right).$$

- We set

$$\Xi_j(t) = U_j(t) + \frac{1}{\alpha} \dot{U}_j(t) \quad \text{with} \quad \alpha = \frac{1}{2} \min_{j \in \{1, \dots, n_0\}} (a_j).$$

Then

$$\begin{cases} \dot{U}_j(t) = \alpha (\Xi_j(t) - U_j(t)) \\ \dot{\Xi}_j(t) = (a_j - \alpha)(U_j(t) - \Xi_j(t)) + \frac{a_j}{\alpha} V_j (U_{j+1}(t) - U_j(t)). \end{cases}$$

Change of variables

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- This system is monotone if $a_j \geq 4\|V'\|_\infty$

Injection in a system of PDE

For $j \in \mathbb{Z}$, pour tout $(t, x) \in (0, +\infty) \times \mathbb{R}$,

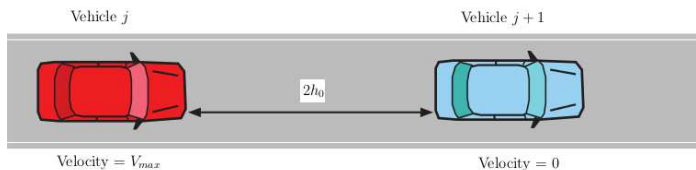
$$u_j(t, x) = U_{j+n_0[x]}(t) \quad \text{and} \quad \xi_j(t, x) = \Xi_{j+n_0[x]}(t).$$

Then

$$\left\{ \begin{array}{l} \frac{\partial u_j}{\partial t} = \alpha(\xi_j(t, x) - u_j(t, x)) \\ \frac{\partial \xi_j}{\partial t} = (a_j - \alpha)(u_j - \xi_j) + \frac{a_j}{\alpha} V_j(u_{j+1} - u_j) \\ u_{j+n_0}(t, x) = u_j(t, x + 1) \\ \xi_{j+n_0}(t, x) = \xi_j(t, x + 1). \end{array} \right. \quad (2)$$

Ordering the vehicles

- We assume that $V_j(h) = 0$ for $h \leq 2h_0$ and we consider the worst case



- Using the solution of this ode as barrier, we get $U_{j+1} \geq U_j + h_0$ for a good choice of h_0
- This implies that $\Xi_{j+1} \geq \Xi_j$

Notion of hull function

Hull function h

$$\begin{cases} \frac{\partial u_j}{\partial t} = \alpha(\xi_j(t, x) - u_j(t, x)) \\ \frac{\partial \xi_j}{\partial t} = (a_j - \alpha)(u_j - \xi_j) + \frac{a_j}{\alpha} V_j (u_{j+1} - u_j) \end{cases}$$

- We search particular solutions of the form

$$u_j(t, x) = h_j(\lambda t + px), \quad \xi_j(t, x) = g_j(\lambda t + px)$$

(h_j, g_j) = hull functions

λ = mean velocity

$\frac{n_0}{p}$ = mean density

$$\frac{U_{i+n_0 l} - U_i}{l} \rightarrow p \quad \text{as } l \rightarrow +\infty$$

Equation of the hull functions

$$\begin{cases} \lambda = \alpha(g_j(z) - h_j(z)) \\ \lambda = (a_j - \alpha)(h_j(z) - g_j(z)) + \frac{a_j}{\alpha} V_j (h_{j+1}(z) - h_j(z)) \end{cases}$$

Existence of hull functions

Theorem (F., Salazar)

For every $p \in (0, +\infty)$, there exists a unique $\lambda := \overline{F}(p)$ for which there exists hull functions (h_j, g_j) . Moreover, (h_j, g_j) can be constructed such that

$$h_j(z) = z + h_j(0), \quad g_j(z) = z + g_j(0).$$

Properties of λ

Theorem (F., Salazar)

- *Monotonicity* : λ is non-decreasing
- *Upper boundary* : $\lim_{p \rightarrow +\infty} \lambda(p) = \min_{j \in \{1, \dots, n_0\}} \|V_j\|_\infty$
- *zero velocity* : If $p \leq 2h_0n_0$, then $\lambda = 0$

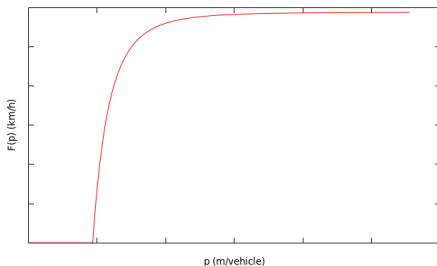


Figure : Schematic representation of the effective Hamiltonian.

Construction of the hull functions

Existence of hull functions

Theorem (F., Salazar)

For every $p > 0$, there exists some $(\bar{u}_j^\infty, \bar{\xi}_j^\infty)$ such that there exists a unique $\lambda =: \bar{F}(q)$ such that

$$\left((u_j^\infty(t, x))_j, (\xi_j^\infty(t, x))_j \right) = \left((px + \lambda t + \bar{u}_j^\infty)_j, (px + \lambda t + \bar{\xi}_j^\infty)_j \right),$$

is a solution of (2). Moreover, if we define (h_j, g_j) such that

$$u_j^\infty(t, x) = h_j(\lambda t + qx), \quad \text{and} \quad \xi_j^\infty(t, x) = g_j(\lambda t + qx)$$

then (h_j, g_j) is a hull function and satisfies

$$h_j(z) = z + h_j(0) \quad \text{and} \quad g_j(z) = z + g_j(0).$$

Ideas to construct h

① **(Initial data)** $u(0, x) = \xi(0, x) = px$

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$$|u_j(t, 0) - \lambda t| \leq C \quad \text{and} \quad |\xi_j(t, 0) - \lambda t| \leq C$$

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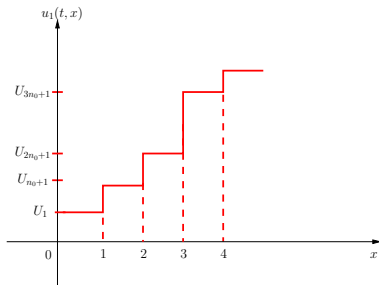
⑤ **(Translation at infinity)** By considering

$(u_j(t + n, y) - \lambda n, \xi_j(t + n, y) - \lambda n)$ as $n \rightarrow +\infty$, one can construct global in time solution $(u_j^\infty, \xi_j^\infty)$

Homogenization results

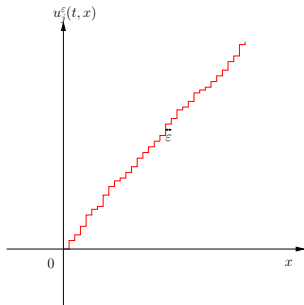
Homogenization

$$u_j(t, x) = U_{j+n_0 \lfloor x \rfloor}(t)$$



Rescaling

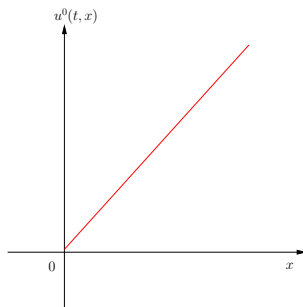
$$u_j^\varepsilon(t, x) = \varepsilon u_j \left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon} \right)$$



$$u_j^\varepsilon \rightarrow ?$$

Rescaling

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$$u_j^\varepsilon \rightarrow u^0$$

Homogenization results

For $\varepsilon > 0$, we define

$$\begin{cases} u_j^\varepsilon(t, x) = \varepsilon u_j \left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon} \right), & \xi_j^\varepsilon(t, x) = \varepsilon \xi_j \left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon} \right) \\ u_j^\varepsilon(0, x) = u_0 \left(x + \frac{j\varepsilon}{n_0} \right), & \xi_j^\varepsilon(0, x) = u_0 \left(x + \frac{j\varepsilon}{n_0} \right) \end{cases}$$

Theorem (F., Imbert, Monneau)

Under certain assumptions on u_0 , we have $u_j^\varepsilon \rightarrow u^0$ and $\xi_j^\varepsilon \rightarrow u^0$ with

$$\begin{cases} u_t^0 = \bar{F}(u_x^0), \\ u^0(0, x) = u_0(x) \end{cases}$$

- Ansatz for the proof :

$$u_j^\varepsilon(t, x) \simeq \varepsilon h_j \left(\frac{u^0(t, x)}{\varepsilon} \right), \quad \xi_j^\varepsilon(t, x) \simeq \varepsilon g_j \left(\frac{u^0(t, x)}{\varepsilon} \right)$$

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$$u_j^\varepsilon(t, x) \simeq u^0(t, x) + \varepsilon h_j(0), \quad \xi_j^\varepsilon(t, x) \simeq u^0(t, x) + \varepsilon g_j(0)$$

Formal idea of the proof

- $v_j^\varepsilon(t, x) := u^0(t, x) + \varepsilon h_j(0), \quad w_j^\varepsilon(t, x) := u^0(t, x) + \varepsilon g_j(0)$

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- $v_j^\varepsilon(t, x) := u^0(t, x) + \varepsilon h_j(0)$, $w_j^\varepsilon(t, x) := u^0(t, x) + \varepsilon g_j(0)$
- We then have very formally

$$(v_j^\varepsilon)_t(t, x) = u_t^0 = \bar{F}(u_x^0) = \alpha(g_j(0) - h_j(0)) = \frac{\alpha}{\varepsilon}(v_j^\varepsilon - w_j^\varepsilon)$$

$$\begin{aligned} (w_j^\varepsilon)_t(t, x) &= u_t^0 = \bar{F}(u_x^0) \\ &= (a_j - \alpha)(h_j(0) - g_j(0)) + \frac{a_j}{\alpha} V_j(h_{j+1}(0) - h_j(0)) \\ &= (a_j - \alpha) \frac{v_j^\varepsilon - w_j^\varepsilon}{\varepsilon} + \frac{a_j}{\alpha} V_j \left(\frac{v_{j+1}^\varepsilon - v_j^\varepsilon}{\varepsilon} \right) \end{aligned}$$

Formal idea of the proof

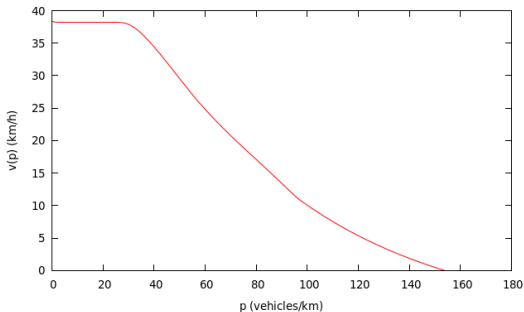
- $v_j^\varepsilon(t, x) := u^0(t, x) + \varepsilon h_j(0)$, $w_j^\varepsilon(t, x) := u^0(t, x) + \varepsilon g_j(0)$
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- $(v_j^\varepsilon, w_j^\varepsilon)$ and $(u_j^\varepsilon, \xi_j^\varepsilon)$ satisfy (almost) the same equation, so $u_j^\varepsilon \simeq v_j^\varepsilon$ and $\xi_j^\varepsilon \simeq w_j^\varepsilon$.

An example of computation of \overline{F} (W. Salazar)



Conclusions and Perspectives

- Conclusions :
 - Homogenization results for discrete traffic flow models
 - This allows to add microscopic phenomena in the modeling (red light, car crashes,....)
- Perspectives :
 - Study of car crashes, red light
 - Homogenization on networks
 - Numerical computation of \overline{F}
 - Homogenization in random media