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# From discrete microscopic models to macroscopic models and applications to traffic flow.

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Joint work with W. Salazar

**Conference on "New Trends in Optimal Control"**

23-27 June 2014, Tours

# Plan

- 1 Motivations
- 2 Homogenization of traffic flows models

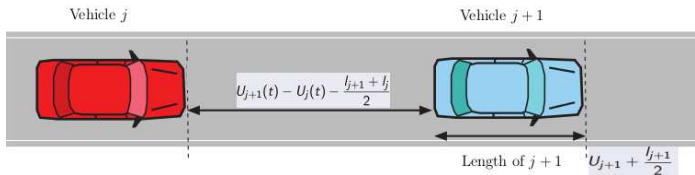
# Plan

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# Bando model

- Second order discrete model with  $n_0 \in \mathbb{N}$  types of vehicles :

$$\ddot{U}_j(t) = a_j \left( V_j \left( U_{j+1}(t) - U_j(t) - \frac{l_{j+1} + l_j}{2} \right) - \dot{U}_j(t) \right). \quad (1)$$



- $U_j$ : position of the vehicle  $j$ .
- $a_j$ : sensibility of the driver  $j$  ( $a_{j+n_0} = a_j$ ).
- $V_j$ : Optimal velocity function (OVF) of the driver  $j$  ( $V_{j+n_0} = V_j$ ).

# Passing from micro to macro

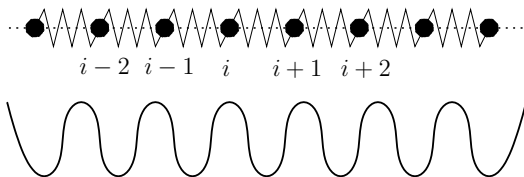
- Goal : Describe the traffic in term of density of vehicles, i.e. passing from the microscopic model to a macroscopic one.
- LWR macroscopic model:

$$\rho_t + (\rho v(\rho))_x = 0$$

where  $v$  is the average speed of vehicles.

# Existing results for the Frenkel-Kontorova model

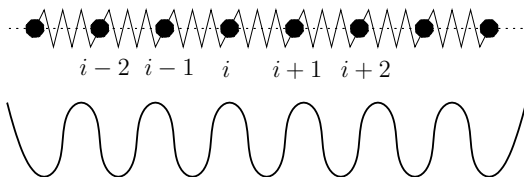
## Existing results for the Frenkel-Kontorova model



$$m \frac{d^2 U_j}{dt^2} + \gamma \frac{dU_j}{dt} = (U_{j+1} - U_j) + (U_{j-1} - U_j) + \sin(2\pi U_j) + f$$



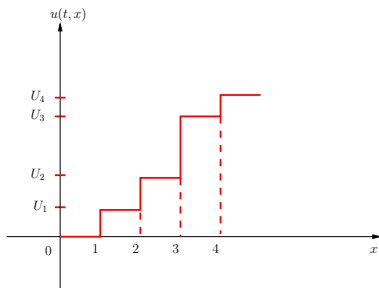
## Existing results for the Frenkel-Kontorova model



$$m \frac{d^2 U_j}{dt^2} + \gamma \frac{dU_j}{dt} = F_j(U_{j-m}, \dots, U_{j+m})$$

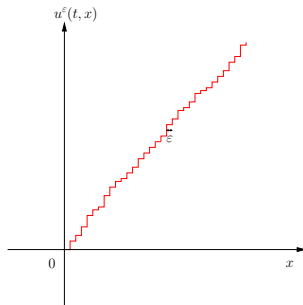
# Homogenization

$$u(t, x) = U_{[x]}(t)$$



# Rescaling

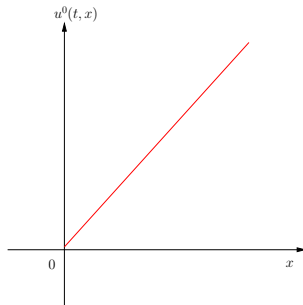
$$u^\varepsilon(t, x) = \varepsilon u\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right)$$



$$u^\varepsilon \rightarrow ?$$

# Rescaling

$$u^\varepsilon(t, x) = \varepsilon u\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right)$$



$$u^\varepsilon \rightarrow u^0$$

# Homogenization results

For  $\varepsilon > 0$ , we define

$$\begin{cases} u^\varepsilon(t, x) = \varepsilon u\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right), \\ u^\varepsilon(0, x) = u_0(x) \end{cases}$$

## Theorem (F., Imbert, Monneau)

*Under certain assumptions on  $u_0$ , we have  $u^\varepsilon \rightarrow u^0$  with*

$$\begin{cases} u_t^0 = \bar{F}(u_x^0), \\ u^0(0, x) = u_0(x) \end{cases}$$

- The density of particles  $\rho = \frac{1}{u_x^0}$  satisfies formally a conservation law

$$\rho_t = (\bar{H}(\rho))_x \quad \text{with} \quad \bar{H}(\rho) = -\rho \bar{F}(1/\rho)$$

# Ingredients of the proof

- Main idea: inject the system of particles into a (system of) PDE.
- If  $m$  is small, the system of PDE satisfies a comparison principle
- Order of the particles (link between the  $F_j$ ).
- Notion of hull functions

# Plan

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# Change of variables

$$\ddot{U}_j(t) = a_j \left( V_j (U_{j+1}(t) - U_j(t)) - \dot{U}_j(t) \right).$$

- We set

$$\Xi_j(t) = U_j(t) + \frac{1}{\alpha} \dot{U}_j(t) \quad \text{with} \quad \alpha = \frac{1}{2} \min_{j \in \{1, \dots, n_0\}} (a_j).$$

Then

$$\begin{cases} \dot{U}_j(t) = \alpha (\Xi_j(t) - U_j(t)) \\ \dot{\Xi}_j(t) = (a_j - \alpha)(U_j(t) - \Xi_j(t)) + \frac{a_j}{\alpha} V_j (U_{j+1}(t) - U_j(t)). \end{cases}$$



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- This system is monotone if  $a_j \geq 4\|V'\|_\infty$

# Injection in a system of PDE

For  $j \in \mathbb{Z}$ , pour tout  $(t, x) \in (0, +\infty) \times \mathbb{R}$ ,

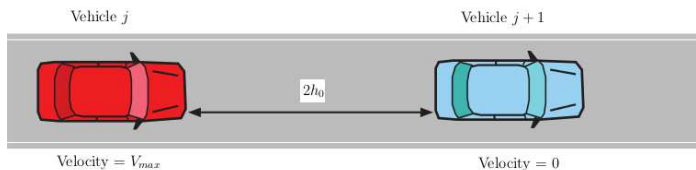
$$u_j(t, x) = U_{j+n_0[x]}(t) \quad \text{and} \quad \xi_j(t, x) = \Xi_{j+n_0[x]}(t).$$

Then

$$\left\{ \begin{array}{l} \frac{\partial u_j}{\partial t} = \alpha(\xi_j(t, x) - u_j(t, x)) \\ \frac{\partial \xi_j}{\partial t} = (a_j - \alpha)(u_j - \xi_j) + \frac{a_j}{\alpha} V_j(u_{j+1} - u_j) \\ u_{j+n_0}(t, x) = u_j(t, x + 1) \\ \xi_{j+n_0}(t, x) = \xi_j(t, x + 1). \end{array} \right. \quad (2)$$

# Ordering the vehicles

- We assume that  $V_j(h) = 0$  for  $h \leq 2h_0$  and we consider the worst case



- Using the solution of this ode as barrier, we get  $U_{j+1} \geq U_j + h_0$  for a good choice of  $h_0$
- This implies that  $\Xi_{j+1} \geq \Xi_j$

# Notion of hull function

# Hull function $h$

$$\begin{cases} \frac{\partial u_j}{\partial t} = \alpha(\xi_j(t, x) - u_j(t, x)) \\ \frac{\partial \xi_j}{\partial t} = (a_j - \alpha)(u_j - \xi_j) + \frac{a_j}{\alpha} V_j (u_{j+1} - u_j) \end{cases}$$

- We search particular solutions of the form

$$u_j(t, x) = h_j(\lambda t + px), \quad \xi_j(t, x) = g_j(\lambda t + px)$$

$(h_j, g_j)$  = hull functions

$\lambda$  = mean velocity

$\frac{n_0}{p}$  = mean density

$$\frac{U_{i+n_0 l} - U_i}{l} \rightarrow p \quad \text{as } l \rightarrow +\infty$$

# Equation of the hull functions

$$\begin{cases} \lambda = \alpha(g_j(z) - h_j(z)) \\ \lambda = (a_j - \alpha)(h_j(z) - g_j(z)) + \frac{a_j}{\alpha} V_j (h_{j+1}(z) - h_j(z)) \end{cases}$$

# Existence of hull functions

## Theorem (F., Salazar)

*For every  $p \in (0, +\infty)$ , there exists a unique  $\lambda := \overline{F}(p)$  for which there exists hull functions  $(h_j, g_j)$ . Moreover,  $(h_j, g_j)$  can be constructed such that*

$$h_j(z) = z + h_j(0), \quad g_j(z) = z + g_j(0).$$

# Properties of $\lambda$

## Theorem (F., Salazar)

- *Monotonicity* :  $\lambda$  is non-decreasing
- *Upper boundary* :  $\lim_{p \rightarrow +\infty} \lambda(p) = \min_{j \in \{1, \dots, n_0\}} \|V_j\|_\infty$
- *zero velocity* : If  $p \leq 2h_0n_0$ , then  $\lambda = 0$

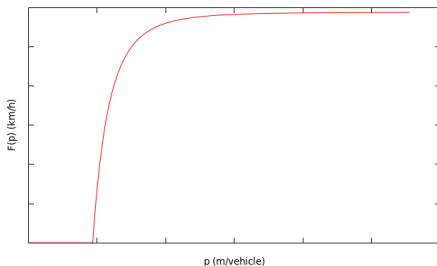


Figure : Schematic representation of the effective Hamiltonian.



# Construction of the hull functions

# Existence of hull functions

## Theorem (F., Salazar)

For every  $p > 0$ , there exists some  $(\bar{u}_j^\infty, \bar{\xi}_j^\infty)$  such that there exists a unique  $\lambda =: \bar{F}(q)$  such that

$$\left( (u_j^\infty(t, x))_j, (\xi_j^\infty(t, x))_j \right) = \left( (px + \lambda t + \bar{u}_j^\infty)_j, (px + \lambda t + \bar{\xi}_j^\infty)_j \right),$$

is a solution of (2). Moreover, if we define  $(h_j, g_j)$  such that

$$u_j^\infty(t, x) = h_j(\lambda t + qx), \quad \text{and} \quad \xi_j^\infty(t, x) = g_j(\lambda t + qx)$$

then  $(h_j, g_j)$  is a hull function and satisfies

$$h_j(z) = z + h_j(0) \quad \text{and} \quad g_j(z) = z + g_j(0).$$

# Ideas to construct $h$

① **(Initial data)**  $u(0, x) = \xi(0, x) = px$

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$$|u_j(t, 0) - \lambda t| \leq C \quad \text{and} \quad |\xi_j(t, 0) - \lambda t| \leq C$$

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⑤ **(Translation at infinity)** By considering

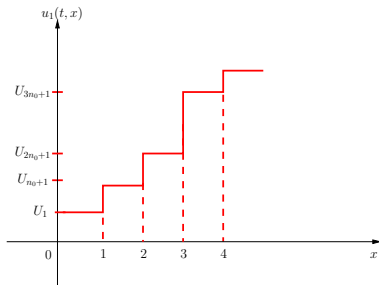
$(u_j(t + n, y) - \lambda n, \xi_j(t + n, y) - \lambda n)$  as  $n \rightarrow +\infty$ , one can construct global in time solution  $(u_j^\infty, \xi_j^\infty)$

# Homogenization results



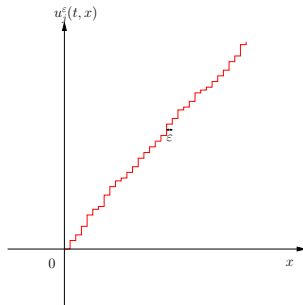
# Homogenization

$$u_j(t, x) = U_{j+n_0 \lfloor x \rfloor}(t)$$



# Rescaling

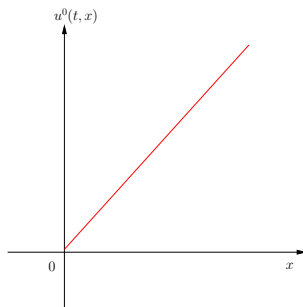
$$u_j^\varepsilon(t, x) = \varepsilon u_j \left( \frac{t}{\varepsilon}, \frac{x}{\varepsilon} \right)$$



$$u_j^\varepsilon \rightarrow ?$$

# Rescaling

$$u_j^\varepsilon(t, x) = \varepsilon u_j \left( \frac{t}{\varepsilon}, \frac{x}{\varepsilon} \right)$$



$$u_j^\varepsilon \rightarrow u^0$$

# Homogenization results

For  $\varepsilon > 0$ , we define

$$\begin{cases} u_j^\varepsilon(t, x) = \varepsilon u_j \left( \frac{t}{\varepsilon}, \frac{x}{\varepsilon} \right), & \xi_j^\varepsilon(t, x) = \varepsilon \xi_j \left( \frac{t}{\varepsilon}, \frac{x}{\varepsilon} \right) \\ u_j^\varepsilon(0, x) = u_0 \left( x + \frac{j\varepsilon}{n_0} \right), & \xi_j^\varepsilon(0, x) = u_0 \left( x + \frac{j\varepsilon}{n_0} \right) \end{cases}$$

## Theorem (F., Imbert, Monneau)

*Under certain assumptions on  $u_0$ , we have  $u_j^\varepsilon \rightarrow u^0$  and  $\xi_j^\varepsilon \rightarrow u^0$  with*

$$\begin{cases} u_t^0 = \bar{F}(u_x^0), \\ u^0(0, x) = u_0(x) \end{cases}$$

- Ansatz for the proof :

$$u_j^\varepsilon(t, x) \simeq \varepsilon h_j \left( \frac{u^0(t, x)}{\varepsilon} \right), \quad \xi_j^\varepsilon(t, x) \simeq \varepsilon g_j \left( \frac{u^0(t, x)}{\varepsilon} \right)$$

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- Ansatz for the proof :

$$u_j^\varepsilon(t, x) \simeq u^0(t, x) + \varepsilon h_j(0), \quad \xi_j^\varepsilon(t, x) \simeq u^0(t, x) + \varepsilon g_j(0)$$

# Formal idea of the proof

- $v_j^\varepsilon(t, x) := u^0(t, x) + \varepsilon h_j(0), \quad w_j^\varepsilon(t, x) := u^0(t, x) + \varepsilon g_j(0)$

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- $v_j^\varepsilon(t, x) := u^0(t, x) + \varepsilon h_j(0)$ ,  $w_j^\varepsilon(t, x) := u^0(t, x) + \varepsilon g_j(0)$
- We then have very formally

$$(v_j^\varepsilon)_t(t, x) = u_t^0 = \bar{F}(u_x^0) = \alpha(g_j(0) - h_j(0)) = \frac{\alpha}{\varepsilon}(v_j^\varepsilon - w_j^\varepsilon)$$

$$\begin{aligned} (w_j^\varepsilon)_t(t, x) &= u_t^0 = \bar{F}(u_x^0) \\ &= (a_j - \alpha)(h_j(0) - g_j(0)) + \frac{a_j}{\alpha} V_j(h_{j+1}(0) - h_j(0)) \\ &= (a_j - \alpha) \frac{v_j^\varepsilon - w_j^\varepsilon}{\varepsilon} + \frac{a_j}{\alpha} V_j \left( \frac{v_{j+1}^\varepsilon - v_j^\varepsilon}{\varepsilon} \right) \end{aligned}$$

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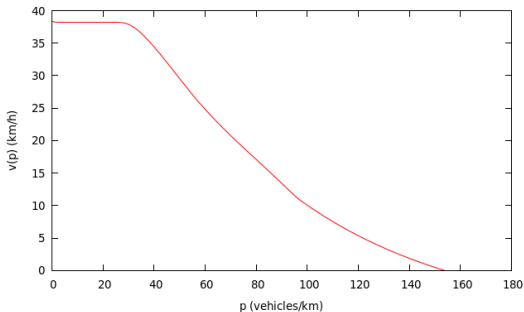
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- $(v_j^\varepsilon, w_j^\varepsilon)$  and  $(u_j^\varepsilon, \xi_j^\varepsilon)$  satisfy (almost) the same equation, so  $u_j^\varepsilon \simeq v_j^\varepsilon$  and  $\xi_j^\varepsilon \simeq w_j^\varepsilon$ .



# An example of computation of $\overline{F}$ (W. Salazar)



# Conclusions and Perspectives

- Conclusions :
  - Homogenization results for discrete traffic flow models
  - This allows to add microscopic phenomena in the modeling (red light, car crashes,....)
- Perspectives :
  - Study of car crashes, red light
  - Homogenization on networks
  - Numerical computation of  $\overline{F}$
  - Homogenization in random media