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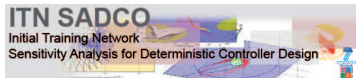
# Reconstruction of Independent sub-domains in a Hamilton-Jacobi Equation and its Use for Parallel Computation

Adriano Festa

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25th June 2014

**NetCo 2014, Tours, France.**



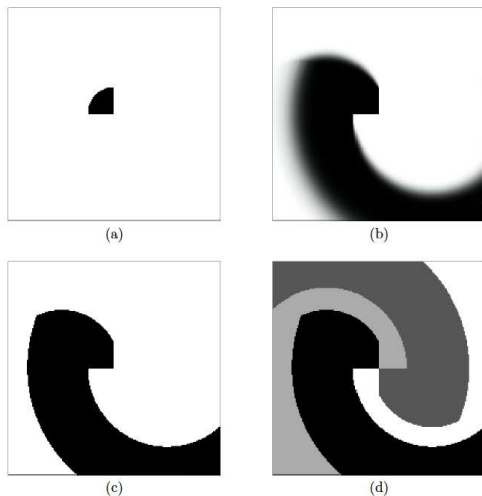
# Introduction

# Patchy Decomposition

- Cacace, Cristiani, Falcone and Picarelli, *A patchy dynamic programming scheme for a class of Hamilton-Jacobi-Bellman equation*, SIAM J. Sc. Comp. (2012)

- ▶ *Reconstruction of some “Sub-Domains of Invariance” through the resolution of the problem on a coarse grid passing by the synthesis of the controls*
- ▶ **Goal:** solve the problem in parallel on a fine grid
- ▶ **Good point:** Some cases of interest where this idea works well
- ▶ **Open questions:**
  - ▶ Convergence, error introduced,
  - ▶ Extension of this idea to a wider class of problems

## Patchy Decomposition: example



**Figure:** Some steps of the *Patchy Algorithm* (thanks to the authors)

# Decomposition for Differential Games

(Deduced from F. Vinter, preprint 2014)

Let us consider, for an  $H$  not necessarily convex

$$\begin{cases} \lambda v(x) + H(x, Dv(x)) = 0 & x \in \Omega \\ v(x) = g(x) & x \in \Gamma \end{cases}$$

Considered a **decomposition of the boundary**  $\Gamma := \bigcup_{i \in \mathcal{I}} \Gamma_i$ , with  $\mathcal{I} := \{1, \dots, m\} \subset \mathbb{N}$ , we call  $v_i : \bar{\Omega} \rightarrow \mathbb{R}$  a **Lipschitz continuous viscosity solution** of the problem

$$\begin{cases} \lambda v_i(x) + H(x, Dv_i(x)) = 0 & x \in \Omega \\ v_i(x) = g_i(x) & x \in \Gamma \end{cases}$$

where the functions  $g_i : \Gamma \rightarrow \mathbb{R}$  is a regular function such that

$$\begin{aligned} g_i(x) &= g(x), \text{ if } x \in \Gamma_i, \\ g_i(x) &> g(x), \text{ otherwise.} \end{aligned}$$

Define

$$I(x) := \{i \in \{1, \dots, m\} \mid v_i(x) = \min_j v_j(x)\},$$
$$\Sigma := \{x \in \mathbb{R}^N \mid \text{card}(I(x)) > 1\}.$$

### Theorem

Assume the following condition satisfied: for arbitrary  $x \in \Sigma$ , **any convex combination**  $\{\alpha_i \mid i \in I(x)\}$  and any collection of vectors  $\{p_i \in \partial^L v_i(x) \mid i \in I(x)\}$  we have

$$\lambda \bar{v} + H \left( x, \sum_i \alpha_i p_i \right) \leq 0. \quad (\text{E})$$

Then, for all  $x \in \mathbb{R}^N \setminus \mathcal{T}$ ,

$$v(x) = \bar{v}(x) := \min_i \{v_1(x), \dots, v_m(x)\}.$$

## Example (I):

The equation considered is, with  $\Omega := (-1, 1) \times (-1, 1)$ ,

$$\begin{cases} \max_{a \in B(0,1)} \{a \cdot Dv(x)\} = 1 & x \in \Omega \\ v(x) = 0 & x_1 \in \partial\Omega; \end{cases}$$

(evidently the solution is the **distance function** from the boundary), we associate, considered to the easiest division  $\partial\Omega = \Gamma := \cup_j \Gamma_j = \cup [(\pm 1, \pm 1), (\pm 1, \pm 1)]$ , to the  $\Gamma_1 := [(-1, -1), (-1, 1)]$  the function  $g_1 : \partial\Omega \rightarrow \mathbb{R}$  defined as

$$\begin{cases} g_1(x) := 0 & x \in \Gamma_1 \\ g_1(x) := \gamma(1 + x_2) & x \in \Gamma \setminus \Gamma_1; \end{cases}$$

with  $\gamma > 0$ .

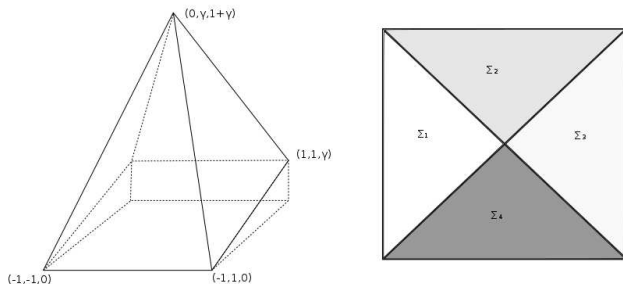


## Example (II):

The **unique viscosity solution** of such a problem is

$$v_1(x) = (1 + \gamma) - \max(|x_1 - \gamma|, |x_2|)$$

and finally, the original value function  $v(x) = 1 - \max(|x_1|, |x_2|)$  is recovered as  $v(x) = \min_{i=1, \dots, 4} v_i(x)$ .



**Figure:** the auxiliary solution  $v_1$  and the “regions of coincidence”.

# Independent subdomains reconstruction

# Differential Games Problem

Let the dynamics be given by

$$\begin{cases} \dot{y}(t) = f(y(t), a(t), b(t)), & a.e. \\ y(0) = x, \end{cases}$$

$x \in \Omega \subseteq \mathbb{R}^n$  open,  $a, b \in \mathcal{A}, \mathcal{B} = \{\mathbb{R}^+ \rightarrow A, \text{ measurable}\}$ ,  $A, B$  compact sets. A solution is a **trajectory**  $y_x(t, a(t), b(t))$ .

The goal is to find the **sup – inf optimum** over  $\mathcal{A}, \mathcal{B}$  of

$$J_x(a, b) := \int_0^{\tau_x(a, b)} l(y_x(s, a(s), b(s)), a(s), b(s)) e^{-\lambda s} ds \\ + e^{-\lambda \tau_x(a, b)} g(y_x(\tau_x(a, b))), \quad \lambda \geq 0,$$

where  $\tau_x(a, b) := \min \{t \in [0, +\infty) \mid y_x(t, a(t), b(t)) \notin \Omega\}$ .

the value function of this problem is

$$v(x) := \sup_{\phi \in \Phi} \inf_{a \in \mathcal{A}} J_x(a, \phi(a)),$$

$\Phi := \{\phi : \mathcal{A} \rightarrow \mathcal{B} : t > 0, a(s) = \tilde{a}(s) \text{ for all } s \leq t$   
**implies**  $\phi[a](s) = \phi[\tilde{a}](s) \text{ for all } s \leq t\}$ .

we will assume the *Isaacs' conditions* verified.

### Theorem

The value function of the problem is a **viscosity solution** of the HJ equation associated with

$$H(x, p) := \min_{b \in \mathcal{B}} \max_{a \in \mathcal{A}} \{-f(x, a, b) \cdot p - l(x, a, b)\}.$$

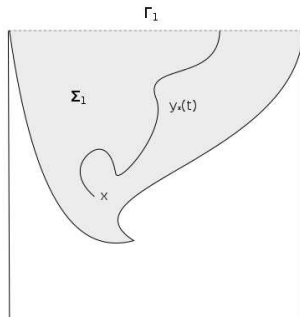
# Independent Sub-Domains

## Definition

A closed subset  $\Sigma \subseteq \bar{\Omega}$  is an **independent sub-domain** of the problem (11) if, given a point  $x \in \Sigma$  and an **optimal control**  $(\bar{a}(t), \bar{\phi}(\bar{a}(t)))$

(i.e.  $J_x(\bar{a}, \bar{\phi}(\bar{a})) \leq J_x(a, \bar{\phi}(a))$  for every choice of  $a \in \mathcal{A}$ , and  $J_x(\bar{a}, \bar{\phi}(\bar{a})) \geq J_x(\bar{a}, \phi(\bar{a}))$  for every choice of  $\phi \in \Phi$ ),

the trajectory  $y_x(\bar{a}(t), \bar{\phi}(\bar{a}(t))) \in \Sigma$  for  $t \in [0, \tau_x(\bar{a}, \bar{\phi}(\bar{a}))]$ .



# Independent Domains Decomposition

## Proposition

Given a collection of  $n - 1$  dimensional subsets  $\{\Gamma_i\}_{i \in \mathcal{I}}$  such that  $\Gamma = \cup_{i=1}^m \Gamma_i$ , the sets defined as

$$\Sigma_i := \{x \in \bar{\Omega} \mid v_i(x) = v(x)\}, \quad i = 1, \dots, m$$

where  $v_i, v$  are defined accordingly to Theorem (1), are *independent sub-domains* of the original problem.

## Proof.

By contradiction using the DPP. □

# Numeric framework, SemiLagrangian scheme

Consider a **structured grid of simplices**  $S_j$ , such that  $\bar{\Omega} \in \cup_j S_j$ , denoting  $x_m$ ,  $m = 1, \dots, N$ , the nodes of the triangulation,

$$\Delta x := \max_j \text{diam}(S_j)$$

$G$  is *internal nodes*,  $\partial G$  *boundary points*,  $\Psi$  *ghost nodes*.

Mapping the values at the nodes onto  $V = (V_1, \dots, V_N)$ , it is possible to obtain the following scheme in **fixed point form**

$$V = T(V),$$

$$[T(V)]_i = \begin{cases} \max_{b \in B} \min_{a \in A} \left\{ \frac{1}{1+\lambda h} \mathbb{I}[V](x_i - hf(x_i, a, b)) - hl(x_i, a, b) \right\} & x_i \in G, \\ g(x_i) & x_i \in \partial G, \\ +\infty & x_i \in \Psi. \end{cases}$$

# An estimate needed

## Theorem

*Let  $v$  and  $V$  be the solutions of the continuous and the discrete problem. Under some regularity hypotheses*

$$\|v - V\|_{\infty} \leq C(\Delta x)^q,$$

*where  $C$  is a positive constant independent from  $\Delta x$ ,  $q \in \mathbb{R}^+$  depending on the regularity of the problem.*



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**Examples in the SL case:** differential games with a Lipschitz solution

$$\|v - V\|_{\infty} \leq Ch^{\frac{1}{2}} \left( 1 + \left( \frac{\Delta x}{h} \right)^2 \right),$$

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$$\|v - V\|_{\infty} \leq Ch^{\frac{1}{2}} \left( 1 + \left( \frac{\Delta x}{h} \right)^2 \right),$$

Optimal Control problem with  $\lambda > 0$ :

$$\|v - V\|_{\infty} \leq 2(M_v + M_{v_h})h^{\frac{1}{2}} + \left( \frac{L_f}{\lambda(\lambda - L_f)} \frac{\Delta x}{h} \right).$$

# A necessary condition

## Proposition

Assumed  $x \in \Omega$  such that  $v_i(x) = v(x)$  for a certain  $i \in \mathbb{I}$ , an  $\epsilon \in [0, \Delta x)$  and a direction  $d \in B(0, 1)$ , such that, for a  $x_j$  node of the grid  $G$ ,  $x_j = x + \epsilon d$  and  $v(x_j) < v_i(x_j)$ ; then

$$|V_i(x_j) - V(x_j)| \leq 2(C(\Delta x)^q + M\Delta x)$$

$C$  as in the previous statement and  $M := \max\{L_{v_i}, i \in \mathbb{I}\}$  where  $L_{v_i}$  is the Lipschitz constant of the function  $v_i$ .

# Independent Subdomains Reconstruction

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## INDEPENDENT SUB-D. RECONSTRUCTION ALGORITHM (IRA).

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- Given a collection of indices vectors such that  $\text{union}(x_i, i = 1, \dots, m) = y$
  - 1) (Resolution of auxiliary problems)  
for  $i = 1 \dots m$  solve iteratively the problem  
     $V_i = T_i(V_i)$  with  $\partial G := \{x_j | j \in y_i\}$   
end
  - 2) (Reconstruction of the value function)  
obtain  $V$  as  $V = \min_{i=1 \dots m} \{V_i\}$
  - 3) (Reconstruction of the Sub-Domains)  
for  $i=1 \dots m$   
    initialize  $w_i = \emptyset$   
    for  $j=1 \dots N$   
        if  $\|V_i(j) - V(j)\| \leq \varepsilon(\Delta x)$   
        then add  $j$  to vector  $w_i$   
    end  
    the index set of the  $i$ -subset is  $w_i$   
end
-

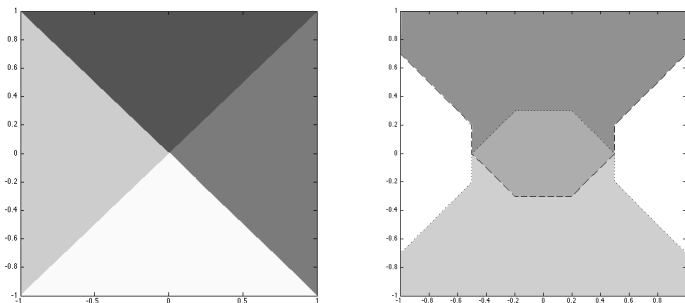
# Some Observations

(IRA) builds an approximation of the **independent sub-domains**.  
Moreover we have:

- ▶ the approximation **exceeds the desired** set, i.e.  $\Sigma_i \subseteq \bar{\Sigma}_i$ .
- ▶ The same relation holds for the **projection of the approximation** on a finer grid, i.e. for  $\Delta x_1 \geq \Delta x_2$ ,

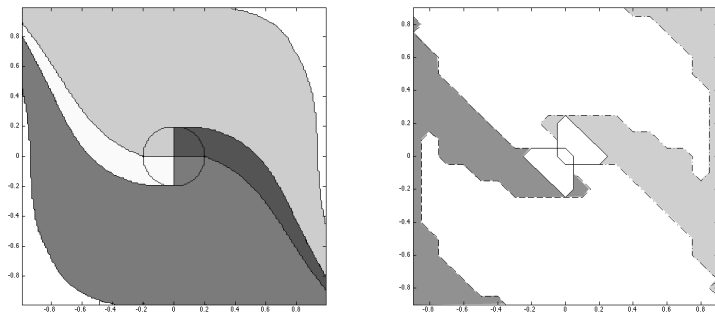
$$Proj_{\Delta x_2} \bar{\Sigma}_i^{\Delta x_1} \supseteq \Sigma_i$$

# Example of Reconstruction (I)



**Figure:** Distance function: Exact decomposition and two (of the four) approximated independent subsets found with a course grind of  $15^2$  points.

## Example of Reconstruction (II)



**Figure:** Van Der Pol: Exact decomposition and two (of the four) approximated independent subsets with a course grind of  $15^2$  points.

# Use for Parallel Computing



Consider:

- ▶ a **coarse grid**  $K$  of discretization step  $\Delta x_K$
- ▶ a **fine grid**  $G$  of step  $\Delta x_G$

on the domain  $\Omega$ .

The two triangulations are chosen such that  $\Delta x_G \ll \Delta x_K$ .

# Independent Sub-Domain Parallelization

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## INDEPENDENT-SETS ALGORITHM (ISA).

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- Given a collection of indices vectors such that  $\text{union}(z_i, i = 1, \dots, M) = y$
  - 1) **(Reconstruction of the approximated independent sub-domains)**  
Using IRA get a collection  $w_i, i = 1, \dots, M$  of indexes sets.
  - 2) **(Projection on the fine grid)**  
Extend indexes sets  $w_i$  on the fine grid  $G_j$  getting  $W_j$
  - 3) **(Resolution on the fine grid)**  
for  $i = 1, \dots, M$  solve iteratively the problem
$$V_i = T(V_i)$$
end
  - 4) **(Assembly of the final Solution)**  
for  $j = 1, \dots, N_2$ 
$$\bar{V}(x_j) = \min\{V_i \text{ s.t. } j \in w_i\}$$
end
-

# Convergence Result:

## Proposition

Called  $\bar{V}$  the **exact discrete solution** of the (ISA) algorithm (i.e. all  $V_i = T_i(V_i)$  are verified exactly) and  $v$  the exact solution of the **original continuous problem**, then, for a  $C > 0$ , independent from  $\Delta x$ , the following estimate holds

$$\|\bar{V} - v\|_{\infty} \leq C(\Delta x_G)^q.$$

# Convergence Result:

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$$\|\bar{V} - v\|_{\infty} \leq C(\Delta x_G)^q.$$

## Proof.

We know that the inclusion property holds. Then for  $x_j \in G$ , it is assured the existence of  $i \in \{1, \dots, M\}$  s.t.  $v(x_j) = v_i(x_j)$ , and  $|\bar{V}(x_j) - V_i(x_j)| = 0$ . Then

$$|\bar{V}(x_j) - v(x_j)| \leq |\bar{V}(x_j) - V_i(x_j)| + |V_i(x_j) - v(x_j)| \leq \|V_i - v_i\|_{\infty} \leq C(\Delta x_G)^q$$

for the arbitrariness of the choice of  $x_j$  we have the thesis. □

# A convergence result for a domain partition:

## Corollary

Let us consider *a partition of the grid  $K$*  in subsets  $B_i$ ,  $i = 1, \dots, M$  such that for every  $i = 1, \dots, M$

- $Z_i \subseteq B_i$
- $\forall z_j \in B_i, |V_i(j) - V(j)| \leq 2(C(\Delta x_K)^q + M\Delta x_K)$
- each  $B_i$  is connected (for each  $z_j \in B_i$ , there exists a direction  $d$  s.t. for  $r \in [0, \Delta x_K]$ ,  $z_j + dr = z_j \in B_i$  with  $j \neq \bar{j}$ )

then the following estimate for  $\hat{V} := \min\{V_i(j) | x_j \in B_i\}$ , where the  $V_i$  is calculated on the projection on  $G$  of the set  $B_i$ :

$$\|\hat{V} - v\|_\infty \leq C\Delta x_G + 2(C(\Delta x_K)^q + M\Delta x_K)$$

# Tests: Distance function

**Eikonal equation** on the set  $\Omega := (-1, 1)^2$  with the **boundary value** fixed to zero on  $\Gamma := \partial\Omega$ .

$\lambda = 1$  (non linear monotone scaling of the solution);

the correct viscosity solution is

$$v(x) + \max_{a \in B(0,1)} \{a \cdot Dv(x)\} = 1, \quad v(x) = 1 - \frac{\min\{e^{|x_1|}, e^{|x_2|}\}}{e}.$$

N. of variables	$\Delta x$	Time elapsed	$\max_i  \bar{\Sigma}_i / \Omega $	$\max_i  \Sigma_i / \Omega $
$5^2$	0.4	$1 \cdot 10^{-3}$	50%	25%
$7^2$	0.28	$2 \cdot 10^{-3}$	43%	
$10^2$	0.2	$4 \cdot 10^{-3}$	38%	
$15^2$	0.133	$2 \cdot 10^{-2}$	35%	
$20^2$	0.1	$5 \cdot 10^{-2}$	33%	
$40^2$	0.05	3	29%	
$50^2$	0.04	11	28.3%	

# Tests: Distance function

**Table:** Time necessary for the resolution on the original domain and the reduced one with various discretization steps ( $15^2$  (IRA))

N. of variables	$\Delta x$	Time on $\Omega$	Time on $\max_j  \bar{\Sigma}_j $
$25^2$	0.08	0.13	0.035
$50^2$	0.04	7.02	0.68
$75^2$	0.026	57.5	4.8
$100^2$	0.02	$1.5 \cdot 10^3$	16.6
$200^2$	0.01	—	$3 \cdot 10^3$

**Table:** Approximation error Error in norm  $\Delta_\infty$  (and  $\Delta_1$ )

	$50^2$	$100^2$	$200^2$
<b>original</b>	$1.2 \cdot 10^{-2} (1.1 \cdot 10^{-2})$	$6.5 \cdot 10^{-3} (3.6 \cdot 10^{-3})$	$2.5 \cdot 10^{-3} (1.6 \cdot 10^{-3})$
<b>2-subs.</b>	$1.2 \cdot 10^{-2} (7.2 \cdot 10^{-3})$	$6.5 \cdot 10^{-3} (3.7 \cdot 10^{-3})$	$2.5 \cdot 10^{-3} (1.4 \cdot 10^{-3})$
<b>4-subs.</b>	$9 \cdot 10^{-3} (7.2 \cdot 10^{-3})$	$4.6 \cdot 10^{-3} (3.6 \cdot 10^{-3})$	$1.4 \cdot 10^{-3} (1.3 \cdot 10^{-3})$
<b>8-subs.</b>	$9 \cdot 10^{-3} (7.2 \cdot 10^{-3})$	$4.6 \cdot 10^{-3} (3.6 \cdot 10^{-3})$	$1.4 \cdot 10^{-3} (1.3 \cdot 10^{-3})$

## Tests: Van Der Pool Oscillator

We consider  $\Gamma := \partial B(0, \rho)$  (in this case  $\rho = 0.2$ ) and  $\Omega := (-1, 1)^2 \setminus \bar{B}(0, \rho)$ . The **nonlinear system** will be:

$$f(x, a) = \begin{pmatrix} x_2 \\ (1 - x_1^2)x_2 - x_1 + a \end{pmatrix}.$$

The others parameters of the system are:

$$A = [-1, 1], \quad \lambda = 1, \quad l(x, y, a) = (x_1^2 + x_2^2)^{\frac{1}{2}}, \quad g(x) \equiv 0.$$

**Table:** Van Der Pol: Comparison of the accuracy of the decomposition with various discretization steps

N. of variables	$\Delta x$	Time elapsed	$\max_i  \bar{\Sigma}_i / \Omega $	$\max_i  \Sigma_i / \Omega $
$5^2$	0.4	$1.4 \cdot 10^{-3}$	62%	42.2%
$10^2$	0.2	0.011	55%	
$20^2$	0.1	0.103	47%	
$30^2$	0.06	1.47	45%	
$40^2$	0.05	5.6	44.6%	
$50^2$	0.04	16.3	44.1%	



# Tests: Van Der Pool Oscillator

**Table:** Van Der Pol: Approximation error Error in norm  $\Delta_\infty$  (and  $\Delta_1$ )

	$50^2$	$100^2$	$200^2$
<b>original</b>	0.09(0.07)	0.03(0.01)	$0.01(6 \cdot 10^{-3})$
<b>2-subsets</b>	0.09(0.07)	0.03(0.01)	$0.01(6 \cdot 10^{-3})$
<b>4-subsets</b>	0.09(0.07)	0.03(0.01)	$0.01(6 \cdot 10^{-3})$
<b>8-subsets</b>	0.09(0.07)	0.03(0.01)	$0.01(6 \cdot 10^{-3})$

## Tests: A Pursuit-Evasion game

We will consider a **Pursuit evasion game**, where two agents have the opposite goal to reduce/postpone the time of capture.

The dynamics considered are the following:

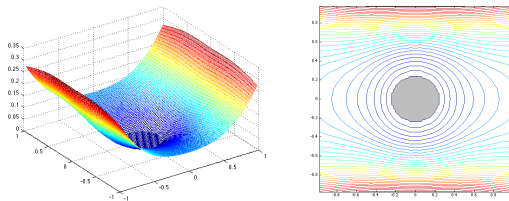
$$f(x, a, b) := \begin{pmatrix} f_1(x)(a_1 - b_1) \\ f_2(x)(a_2 - b_2) \end{pmatrix}$$

where the functions  $f_1, f_2$  are  $f_1(x) := x_2 + 1$  and  $f_2(x) := 1$ .  
The running cost  $l(x, a, b) := x_1 + 0.1$

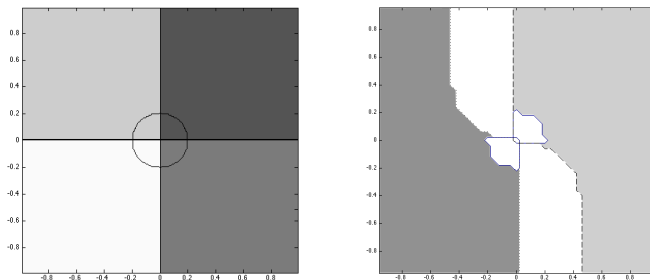
**Controls** are  $A = B(0, 1)$  for the pursuer and  $B = B(0, 1/2)$  for the evader.

**Capture** happens trajectory is driven to touch  $B(0, 0.2)$ , then  
 $\Gamma := \partial B(0, 0.2)$

# Tests: A Pursuit-Evasion game



**Figure:** Approximated value function



**Figure:** Exact decomposition and two (of the four) approximated independent subsets found with a course grind of  $40^2$  points.

# Tests: A Pursuit-Evasion game

**Table:** Comparison of the accuracy of the decomposition with various discretization steps

N. of variables	$\Delta x$	Time elapsed	$\max_i  \bar{\Sigma}_i / \Omega $	$\max_i  \Sigma_i / \Omega $
$5^2$	0.4	$10^{-3}$	60%	
$10^2$	0.2	0.008	46%	
$30^2$	0.06	1.38	38%	25%
$50^2$	0.04	15.9	36.1%	

The HJ equation associated verifies the **decomposability condition (E)**; because considering the norm (it is possible because  $|f_i(x)| > 0$  for  $i = 1, 2$ )

$$\|p\|_* := \max_{a \in B(0,1)} \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} a^T \cdot p$$

the Hamiltonian associated is equivalent to

$$H(x, p) := \|p\|_* - \frac{\|p\|_*^2}{2} - (x_1^2 + 0.1)$$

evidently convex everywhere with respect to  $p$ .




# Independent Domain Parallelization: Summary

- ▶ The gain is variable, but can be **considerable high** (no communication between the sub-domains)
- ▶ There is a direct dependence of the technique on the **initial decomposition** of the boundary
- ▶ Convergence for  $\Delta_G \rightarrow 0^+$  is guaranteed and the error in the approximation **is bounded** by an estimate
- ▶ It is proved also **convergence for a partition decomposition** (as the *Patchy Dec.*) but it is guaranteed a convergence only for  $\Delta_K \rightarrow 0^+$ , and relative estimation of the error.



# Conclusions and Perspectives

- ▶ We have shown a **constructive manner to obtain a decomposition of the domain** of a HJ eq. verifying  $(E)$ , in **independent subdomains**, subsets which have the properties of being computed, independently from each others.
- ▶ A critical point is **the estimation of the function  $\varepsilon(\cdot)$** .
- ▶ The critical occurrence about the **balance of the dimension of the subsets** can be solved with a recursive refinement of the division of  $\Gamma$ .
- ▶ A more serious limit appears in **presence of flat regions**: in that case it is, for the moment, impossible to obtain a satisfactory reduction of the dimension of the decomposed domains without solving the problem on a sufficiently fine grid.

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Thank you.