

Stability of solutions to Hamilton-Jacobi equations on closed domains arising in optimal control

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Consider a Hamiltonian $H : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ that is convex in the last variable and the Hamilton-Jacobi equation

$$-v_t + H(t, x, -v_x) = 0, \quad v(T, \cdot) = \varphi(\cdot).$$
(1)

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Let $H^*(t, x, \cdot) : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be the Fenchel conjugate of $H(t, x, \cdot)$ and consider the Calculus of Variations problem

$$v(t_0, x_0) = \inf \{ \varphi(x(T)) + \int_{t_0}^T H^*(t, x(t), \dot{x}(t)) dt :$$

$$x \in W^{1,1}([t_0, T), x(t_0) = x_0)$$

It is well known that under some assumptions on H, v is the unique viscosity solution of (1). However, H^* may have infinite values. Question: can we associate to H mappings $f: [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}^n$ and $\ell: [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}$ inheriting Lipschitz type regularity properties of H and such that

$$H(t,x,p) = \max_{u \in U} (\langle p, f(t,x,u) \rangle - \ell(t,x,u)), \qquad (2)$$

where U is a compact subset of a finite dimensional space.

It is well known that under some assumptions on H, v is the unique viscosity solution of (1). However, H^* may have infinite values. Question: can we associate to H mappings $f: [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}^n$ and $\ell: [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}$ inheriting Lipschitz type regularity properties of H and such that

$$H(t,x,p) = \max_{u \in U} (\langle p, f(t,x,u) \rangle - \ell(t,x,u)), \qquad (2)$$

where U is a compact subset of a finite dimensional space.

The answer is indeed positive.

That is *H* is equal to the Hamiltonian of a Bolza optimal control problem.

Let K be a closed nonempty subset of \mathbb{R}^n .

$$V(t_0, x_0) = \inf \left\{ \varphi(x(T)) + \int_{t_0}^T \ell(t, x(t), u(t)) dt \right| (x, u) \in S_{[t_0, T]}(x_0) \right\}$$

Here, $S_{[t_0,T]}(x_0)$ denotes the set of all trajectory-control pairs of the control system under state constraint

$$\begin{cases} \dot{x}(s) = f(s, x(s), u(s)), \ u(s) \in U \ a.e. \ in \ [t_0, T] \\ x(t_0) = x_0 \\ x(s) \in K, \ \forall s \in [t_0, T] \end{cases}$$

Under appropriate assumptions, V is the unique solution to the Hamilton-Jacobi equation on the set $[0, T] \times K$.

- **(H1)** $H(t,x,\cdot)$ is convex for any $(t,x) \in [0,T] \times \mathbb{R}^n$.
- **(H2)** For any R > 0 there exists an integrable function $c_R \in L^1([0, T])$ such that for all $x, y \in B_R$ and $p \in \mathbb{R}^n$

$$|H(t,x,p)-H(t,y,p)|\leq c_R(t)(1+|p|)|x-y|.$$

(H3) There exists an integrable function $c \in L^1([0, T])$ such that

$$|H(t,x,p)-H(t,x,q)|\leq c(t)(1+|x|)|p-q|$$

for all $(t, x) \in [0, T] \times \mathbb{R}^n$ and $p, q \in \mathbb{R}^n$.

(H4) $H^*(t, x, \cdot)$ is bounded on its domain for all $(t, x) \in [0, T] \times \mathbb{R}^n$.

(H5) For every R > 0 there exists $M_R > 0$ such that for all $(t, x) \in [0, T] \times B_R$ and $v \in dom(H^*(t, x, \cdot))$ we have

$$H^*(t, x, v) = \max_{p \in B(0, M_R)} (\langle v, p \rangle - H(t, x, p)).$$

(H6) For every R > 0 there exists an absolutely continuous $a_R \in W^{1,1}(0, T)$ such that for all $x \in B_R$, $p \in \mathbb{R}^n$ and $t, s \in [0, T]$

$$|H(t,x,p) - H(s,x,p)| \le (1+|p|)|a_R(t) - a_R(s)|.$$

Theorem

If (H1)-(H6) hold true, then $\exists f : [0, T] \times \mathbb{R}^n \times B \to \mathbb{R}^n$, such that for $\ell : [0, T] \times \mathbb{R}^n \times B \to \mathbb{R} \cup \{+\infty\}$ defined by $\ell(t, x, u) = H^*(t, x, f(t, x, u))$ we have (A1) $H(t, x, p) = \max_{u \in B} (\langle p, f(t, x, u) \rangle - \ell(t, x, u)), \forall (t, x, p).$ (A2) For any R > 0 and for all $t \in [0, T]$, $x, y \in B_R$, $u, v \in B$

$$||f(t,x,u) - f(t,y,u)| \le 10nc_R(t)|x-y|$$

 $||f(t,x,u) - f(t,x,v)| \le 5n(1+R)c(t)|u-v|.$

(A3) $|f(t,x,u)| \le c(t)(1+|x|)$ for all $(t,x,u) \in [0,T] \times \mathbb{R}^n \times B$.

Regularity of dynamics and the cost function

Theorem

(A4) ℓ takes finite values and for any R > 0, $t \in [0, T]$, $x, y \in B_R$, $u, v \in B$

$$\begin{cases} |\ell(t,x,u) - \ell(t,y,u)| \leq (1+11nM_R)c_R(t)|x-y|, \ |\ell(t,x,u) - \ell(t,x,v)| \leq 5nM_R(1+R)c(t)|u-v|. \end{cases}$$

and

$$|\ell(t, x, u) - \ell(s, x, u)| \le (1 + 11 n M_R) |a_R(t) - a_R(s)|.$$

We associate to f, ℓ, φ the Bolza optimal control problem.

If assumptions (H1)-(H6) hold true for continuous $H_i : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, i \ge 1$, then there exist $f_i : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}^n, \ell_i : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}$ satisfying (A1)-(A4), which are standard in control theory.

Theorem

If H_i converge uniformly on compacts to some $H : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and satisfy (H1) - (H6) with the same $M_R, c_R(\cdot), c(\cdot), a_R(\cdot)$, then the representations f_i, ℓ_i can be chosen to converge to some f, ℓ and satisfying (A1)-(A4) with the same $M_R, c_R(\cdot), c(\cdot), a_R(\cdot).$

State constraints

We assume that the closed sets K and K_i are defined by the multiple inequality constraints

$$egin{aligned} &\mathcal{K}\doteq igcap_{j=1}^m\{x:g^j(x)\leq 0\}\ &\mathcal{K}_i\doteq igcap_{j=1}^m\{x:g^j_i(x)\leq 0\}. \end{aligned}$$

where $g_i^j : \mathbb{R}^n \to \mathbb{R}$ and $g^j : \mathbb{R}^n \to \mathbb{R}$, j = 1, ..., m, i = 1, 2, ... are continuously differentiable functions satisfying (G1) i) For any R > 0 there exists $A_R > 0$ such that $|\nabla g_i^j(x)| \le A_R$, for any $x \in RB$, ∇g_i^j is A_R -Lipschitz on RB. ii) $\nabla g_i^j \to \nabla g^j$ uniformly on compacts and $g_i^j(0) \to g^j(0)$, when $i \to \infty$, for any j = 1, ..., m. For any $x \in \mathbb{R}^n$ denote by I(x) the set of active indices at x for $g(\cdot) = (g^1(\cdot), ..., g^m(\cdot))$, i.e.

$$I(x) = \{j : g^j(x) = 0\}.$$

(IPC) For any R > 0 there exists $\rho_R > 0$ such that for every $x \in K \cap RB$ with $I(x) \neq \emptyset$ and every $s \in [0, T]$

$$\inf_{\substack{ v \in dom(H^*(t,x,\cdot)) \ j \in I(x)}} \max \langle
abla g^j(x), v
angle \leq -
ho_R.$$

Viscosity solutions of Hamilton-Jacobi equation

Definition

A continuous function $V : [0, T] \times K \to \mathbb{R}$ is called a viscosity solution of Hamilton-Jacobi equation on the closed set $[0, T] \times K$ if $V(T, \cdot) = \varphi(\cdot)$ and i) for all $(t, x) \in [0, T] \times IntK$ and all $(p_t, p_x) \in \partial_- V(t, x)$

$$-p_t + H(t, x, -p_x) \geq 0$$

ii) for all $(t,x) \in [0,T] imes$ IntK and all $(p_t,p_x) \in \partial_+ V(t,x)$

$$-p_t+H(t,x,-p_x)\leq 0,$$

where $\partial_{\pm}V(t,x)$ are the Frechet super/subdifferential of V at (t,x).

Definition

 $V : [0, T] \times K \to \mathbb{R}$ is called a bilateral solution of Hamilton-Jacobi equation on the closed set $[0, T] \times K$ if $V(T, \cdot) = \varphi(\cdot)$ and i) for all $(t, x) \in [0, T] \times IntK$ and all $(p_t, p_x) \in \partial_- V(t, x)$

$$-p_t + H(t, x, -p_x) = 0$$

ii) for all $(t,x) \in [0,T] imes \partial K$ and all $(p_t,p_x) \in \partial_- V(t,x)$

$$-p_t + H(t, x, -p_x) \geq 0.$$

Uniqueness of solutions of Hamilton-Jacobi equation

Theorem

If assumptions (H1)-(H6) and (IPC) hold true. Then the associated value function of the Bolza optimal control problem is the unique bilaterial solution of the Hamilton-Jacobi equation on $[0, T] \times K$ in the class of continuous functions.

Theorem

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Stability of Solutions to Hamilton-Jacobi equation

Theorem

Suppose that H_i satisfy (H1)-(H6) with the same $a_R(\cdot), c_R(\cdot), c(\cdot), M_R$ and converge uniformly on compacts to a Hamiltonian H. Assume that (IPC) holds true. If φ_i converge to φ uniformly on compacts, then the unique solutions V_i to Hamilton-Jacobi equation with H_i, φ_i, K_i converge uniformly on compacts contained in the interior of K to the unique solution of Hamilton-Jacobi equation with H, φ, K .

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Stability of Value functions: Corollary

Corollary

Suppose that H_i satisfy (H1)-(H6) with the same $a_R(\cdot), c_R(\cdot), c(\cdot), M_R$ and converge uniformly on compacts to a Hamiltonian H. Assume that (IPC) holds true. Then

 $Lim_{i \to \infty}graphV_i = graphV$

and

$$Lim_{i\to\infty}epiV_i=epiV,$$

where the limit is taken in the Kuratowski sense.

Thank you for your attention.