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Stability of solutions to Hamilton-Jacobi equations on closed domains arising in optimal control

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Hamilton-Jacobi equation

Consider a Hamiltonian $H : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ that is convex in the last variable and the Hamilton-Jacobi equation

$$-v_t + H(t, x, -v_x) = 0, \quad v(T, \cdot) = \varphi(\cdot). \quad (1)$$

Let $H^*(t, x, \cdot) : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ be the Fenchel conjugate of $H(t, x, \cdot)$ and consider the Calculus of Variations problem

$$v(t_0, x_0) = \inf \left\{ \varphi(x(T)) + \int_{t_0}^T H^*(t, x(t), \dot{x}(t)) dt : \right.$$

$$\left. x \in W^{1,1}([t_0, T], x(t_0) = x_0) \right\}.$$

Faithful Representation of Hamiltonian

It is well known that under some assumptions on H , v is the unique viscosity solution of (1). However, H^* may have infinite values.

Question: can we associate to H mappings $f : [0, T] \times \mathbb{R}^n \times U \rightarrow \mathbb{R}^n$ and $\ell : [0, T] \times \mathbb{R}^n \times U \rightarrow \mathbb{R}$ inheriting Lipschitz type regularity properties of H and such that

$$H(t, x, p) = \max_{u \in U} (\langle p, f(t, x, u) \rangle - \ell(t, x, u)), \quad (2)$$

where U is a compact subset of a finite dimensional space.

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$$H(t, x, p) = \max_{u \in U} (\langle p, f(t, x, u) \rangle - \ell(t, x, u)), \quad (2)$$

where U is a compact subset of a finite dimensional space.

The answer is indeed positive.

That is H is equal to the Hamiltonian of a Bolza optimal control problem.

Value function of state constrained Bolza problem

Let K be a closed nonempty subset of \mathbb{R}^n .

$$V(t_0, x_0) = \inf \left\{ \varphi(x(T)) + \int_{t_0}^T \ell(t, x(t), u(t)) dt \mid (x, u) \in S_{[t_0, T]}(x_0) \right\}$$

Here, $S_{[t_0, T]}(x_0)$ denotes the set of all trajectory-control pairs of the control system under state constraint

$$\begin{cases} \dot{x}(s) = f(s, x(s), u(s)), & u(s) \in U \text{ a.e. in } [t_0, T] \\ x(t_0) = x_0 \\ x(s) \in K, & \forall s \in [t_0, T] \end{cases}$$

Under appropriate assumptions, V is the unique solution to the Hamilton-Jacobi equation on the set $[0, T] \times K$.

Assumptions

(H1) $H(t, x, \cdot)$ is convex for any $(t, x) \in [0, T] \times \mathbb{R}^n$.

(H2) For any $R > 0$ there exists an integrable function $c_R \in L^1([0, T])$ such that for all $x, y \in B_R$ and $p \in \mathbb{R}^n$

$$|H(t, x, p) - H(t, y, p)| \leq c_R(t)(1 + |p|)|x - y|.$$

(H3) There exists an integrable function $c \in L^1([0, T])$ such that

$$|H(t, x, p) - H(t, x, q)| \leq c(t)(1 + |x|)|p - q|$$

for all $(t, x) \in [0, T] \times \mathbb{R}^n$ and $p, q \in \mathbb{R}^n$.

(H4) $H^*(t, x, \cdot)$ is bounded on its domain for all $(t, x) \in [0, T] \times \mathbb{R}^n$.

(H5) For every $R > 0$ there exists $M_R > 0$ such that for all $(t, x) \in [0, T] \times B_R$ and $v \in \text{dom}(H^*(t, x, \cdot))$ we have

$$H^*(t, x, v) = \max_{p \in B(0, M_R)} (\langle v, p \rangle - H(t, x, p)).$$

(H6) For every $R > 0$ there exists an absolutely continuous $a_R \in W^{1,1}(0, T)$ such that for all $x \in B_R$, $p \in \mathbb{R}^n$ and $t, s \in [0, T]$

$$|H(t, x, p) - H(s, x, p)| \leq (1 + |p|)|a_R(t) - a_R(s)|.$$

Regularity of dynamics and the cost function

Theorem

If (H1)-(H6) hold true, then $\exists f : [0, T] \times \mathbb{R}^n \times B \rightarrow \mathbb{R}^n$, such that for $\ell : [0, T] \times \mathbb{R}^n \times B \rightarrow \mathbb{R} \cup \{+\infty\}$ defined by

$\ell(t, x, u) = H^*(t, x, f(t, x, u))$ we have

(A1) $H(t, x, p) = \max_{u \in B} (\langle p, f(t, x, u) \rangle - \ell(t, x, u))$, $\forall (t, x, p)$.

(A2) For any $R > 0$ and for all $t \in [0, T]$, $x, y \in B_R$, $u, v \in B$

$$\begin{cases} |f(t, x, u) - f(t, y, u)| \leq 10nc_R(t)|x - y| \\ |f(t, x, u) - f(t, x, v)| \leq 5n(1 + R)c(t)|u - v|. \end{cases}$$

(A3) $|f(t, x, u)| \leq c(t)(1 + |x|)$ for all $(t, x, u) \in [0, T] \times \mathbb{R}^n \times B$.

Regularity of dynamics and the cost function

Theorem

(A4) ℓ takes finite values and for any $R > 0$, $t \in [0, T]$, $x, y \in B_R$, $u, v \in B$

$$\begin{cases} |\ell(t, x, u) - \ell(t, y, u)| \leq (1 + 11nM_R)c_R(t)|x - y|, \\ |\ell(t, x, u) - \ell(t, x, v)| \leq 5nM_R(1 + R)c(t)|u - v|. \end{cases}$$

and

$$|\ell(t, x, u) - \ell(s, x, u)| \leq (1 + 11nM_R)|a_R(t) - a_R(s)|.$$

We associate to f, ℓ, φ the Bolza optimal control problem.

Stability of representations

If assumptions (H1)-(H6) hold true for continuous $H_i : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$, $i \geq 1$, then there exist $f_i : [0, T] \times \mathbb{R}^n \times U \rightarrow \mathbb{R}^n$, $\ell_i : [0, T] \times \mathbb{R}^n \times U \rightarrow \mathbb{R}$ satisfying (A1)-(A4), which are standard in control theory.

Theorem

If H_i converge uniformly on compacts to some $H : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ and satisfy (H1) – (H6) with the same $M_R, c_R(\cdot), c(\cdot), a_R(\cdot)$, then the representations f_i, ℓ_i can be chosen to converge to some f, ℓ and satisfying (A1)-(A4) with the same $M_R, c_R(\cdot), c(\cdot), a_R(\cdot)$.

State constraints

We assume that the closed sets K and K_i are defined by the multiple inequality constraints

$$K \doteq \bigcap_{j=1}^m \{x : g^j(x) \leq 0\}$$

$$K_i \doteq \bigcap_{j=1}^m \{x : g_i^j(x) \leq 0\},$$

where $g_i^j : \mathbb{R}^n \rightarrow \mathbb{R}$ and $g^j : \mathbb{R}^n \rightarrow \mathbb{R}$, $j = 1, \dots, m$, $i = 1, 2, \dots$ are continuously differentiable functions satisfying

(G1)

i) For any $R > 0$ there exists $A_R > 0$ such that $|\nabla g_i^j(x)| \leq A_R$, for any $x \in RB$, ∇g_i^j is A_R -Lipschitz on RB .

ii) $\nabla g_i^j \rightarrow \nabla g^j$ uniformly on compacts and $g_i^j(0) \rightarrow g^j(0)$, when $i \rightarrow \infty$, for any $j = 1, \dots, m$.

Inward pointing condition

For any $x \in \mathbb{R}^n$ denote by $I(x)$ the set of active indices at x for $g(\cdot) = (g^1(\cdot), \dots, g^m(\cdot))$, i.e.

$$I(x) = \{j : g^j(x) = 0\}.$$

(IPC) For any $R > 0$ there exists $\rho_R > 0$ such that for every $x \in K \cap RB$ with $I(x) \neq \emptyset$ and every $s \in [0, T]$

$$\inf_{v \in \text{dom}(H^*(t,x,\cdot))} \max_{j \in I(x)} \langle \nabla g^j(x), v \rangle \leq -\rho_R.$$

Viscosity solutions of Hamilton-Jacobi equation

Definition

A continuous function $V : [0, T] \times K \rightarrow \mathbb{R}$ is called a viscosity solution of Hamilton-Jacobi equation on the closed set $[0, T] \times K$ if $V(T, \cdot) = \varphi(\cdot)$ and

i) for all $(t, x) \in [0, T] \times \text{Int}K$ and all $(p_t, p_x) \in \partial_- V(t, x)$

$$-p_t + H(t, x, -p_x) \geq 0$$

ii) for all $(t, x) \in [0, T] \times \text{Int}K$ and all $(p_t, p_x) \in \partial_+ V(t, x)$

$$-p_t + H(t, x, -p_x) \leq 0,$$

where $\partial_{\pm} V(t, x)$ are the Frechet super/subdifferential of V at (t, x) .

Bilateral solution of Hamilton-Jacobi equation

Definition

$V : [0, T] \times K \rightarrow \mathbb{R}$ is called a bilateral solution of Hamilton-Jacobi equation on the closed set $[0, T] \times K$ if $V(T, \cdot) = \varphi(\cdot)$ and

i) for all $(t, x) \in [0, T] \times \text{Int}K$ and all $(p_t, p_x) \in \partial_- V(t, x)$

$$-p_t + H(t, x, -p_x) = 0$$

ii) for all $(t, x) \in [0, T] \times \partial K$ and all $(p_t, p_x) \in \partial_- V(t, x)$

$$-p_t + H(t, x, -p_x) \geq 0.$$

Uniqueness of solutions of Hamilton-Jacobi equation

Theorem

If assumptions (H1)-(H6) and (IPC) hold true. Then the associated value function of the Bolza optimal control problem is the unique bilateral solution of the Hamilton-Jacobi equation on $[0, T] \times K$ in the class of continuous functions.

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If assumptions (H1)-(H6) and (IPC) hold true. Then the associated value function of the Bolza optimal control problem is the unique viscosity solution of the Hamilton-Jacobi equation on $[0, T] \times K$ in the class of continuous functions.

Stability of Solutions to Hamilton-Jacobi equation

Theorem

Suppose that H_i satisfy (H1)-(H6) with the same $a_R(\cdot)$, $c_R(\cdot)$, $c(\cdot)$, M_R and converge uniformly on compacts to a Hamiltonian H . Assume that (IPC) holds true. If φ_i converge to φ uniformly on compacts, then the unique solutions V_i to Hamilton-Jacobi equation with H_i, φ_i, K_i converge uniformly on **compacts contained in the interior of K** to the unique solution of Hamilton-Jacobi equation with H, φ, K .

Stability of Value functions: Corollary

Corollary

Suppose that H_i satisfy (H1)-(H6) with the same $a_R(\cdot)$, $c_R(\cdot)$, $c(\cdot)$, M_R and converge uniformly on compacts to a Hamiltonian H . Assume that (IPC) holds true. Then

$$\text{Lim}_{i \rightarrow \infty} \text{graph} V_i = \text{graph} V$$

and

$$\text{Lim}_{i \rightarrow \infty} \text{epi} V_i = \text{epi} V,$$

where the limit is taken in the Kuratowski sense.

Thank you for your attention.