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Stability of solutions to Hamilton-Jacobi equations on closed domains arising in optimal control

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Consider a Hamiltonian $H : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ that is convex in the last variable and the Hamilton-Jacobi equation

$$-v_t + H(t, x, -v_x) = 0, \quad v(T, \cdot) = \varphi(\cdot). \quad (1)$$

Let $H^*(t, x, \cdot) : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be the Fenchel conjugate of $H(t, x, \cdot)$ and consider the Calculus of Variations problem

$$v(t_0, x_0) = \inf \{\varphi(x(T)) + \int_{t_0}^{T} H^*(t, x(t), \dot{x}(t)) dt : x \in W^{1,1}([t_0, T), x(t_0) = x_0\}.$$
It is well known that under some assumptions on $H$, $v$ is the unique viscosity solution of (1). However, $H^*$ may have infinite values. **Question**: can we associate to $H$ mappings $f : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}^n$ and $\ell : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}$ inheriting Lipschitz type regularity properties of $H$ and such that

$$H(t, x, p) = \max_{u \in U} (\langle p, f(t, x, u) \rangle - \ell(t, x, u)),$$

where $U$ is a compact subset of a finite dimensional space.
Faithful Representation of Hamiltonian

It is well known that under some assumptions on $H$, $\nu$ is the unique viscosity solution of (1). However, $H^*$ may have infinite values. **Question**: can we associate to $H$ mappings $f : [0, T] \times \mathbb{R}^n \times U \rightarrow \mathbb{R}^n$ and $\ell : [0, T] \times \mathbb{R}^n \times U \rightarrow \mathbb{R}$ inheriting Lipschitz type regularity properties of $H$ and such that

\[
H(t, x, p) = \max_{u \in U} (\langle p, f(t, x, u) \rangle - \ell(t, x, u)), \quad (2)
\]

where $U$ is a compact subset of a finite dimensional space.

The answer is indeed positive.
That is $H$ is equal to the Hamiltonian of a Bolza optimal control problem.
Let $K$ be a closed nonempty subset of $\mathbb{R}^n$.

$$V(t_0, x_0) = \inf \left\{ \varphi(x(T)) + \int_{t_0}^{T} \ell(t, x(t), u(t)) dt \mid (x, u) \in S_{[t_0, T]}(x_0) \right\}$$

Here, $S_{[t_0, T]}(x_0)$ denotes the set of all trajectory-control pairs of the control system under state constraint

$$\begin{cases}
\dot{x}(s) = f(s, x(s), u(s)), & u(s) \in U \text{ a.e. in } [t_0, T] \\
x(t_0) = x_0 \\
x(s) \in K, & \forall s \in [t_0, T]
\end{cases}$$

Under appropriate assumptions, $V$ is the unique solution to the Hamilton-Jacobi equation on the set $[0, T] \times K$. 
Assumptions

(H1) $H(t, x, \cdot)$ is convex for any $(t, x) \in [0, T] \times \mathbb{R}^n$.

(H2) For any $R > 0$ there exists an integrable function $c_R \in L^1([0, T])$ such that for all $x, y \in B_R$ and $p \in \mathbb{R}^n$

$$|H(t, x, p) - H(t, y, p)| \leq c_R(t)(1 + |p|)|x - y|.$$

(H3) There exists an integrable function $c \in L^1([0, T])$ such that

$$|H(t, x, p) - H(t, x, q)| \leq c(t)(1 + |x|)|p - q|$$

for all $(t, x) \in [0, T] \times \mathbb{R}^n$ and $p, q \in \mathbb{R}^n$. 
(H4) $H^*(t, x, \cdot)$ is bounded on its domain for all $(t, x) \in [0, T] \times \mathbb{R}^n$.

(H5) For every $R > 0$ there exists $M_R > 0$ such that for all $(t, x) \in [0, T] \times B_R$ and $v \in \text{dom}(H^*(t, x, \cdot))$ we have

$$H^*(t, x, v) = \max_{p \in B(0, M_R)} (\langle v, p \rangle - H(t, x, p)).$$

(H6) For every $R > 0$ there exists an absolutely continuous $a_R \in W^{1,1}(0, T)$ such that for all $x \in B_R$, $p \in \mathbb{R}^n$ and $t, s \in [0, T]$

$$|H(t, x, p) - H(s, x, p)| \leq (1 + |p|)|a_R(t) - a_R(s)|.$$
If (H1)-(H6) hold true, then \( \exists f : [0, T] \times \mathbb{R}^n \times B \rightarrow \mathbb{R}^n \), such that for \( \ell : [0, T] \times \mathbb{R}^n \times B \rightarrow \mathbb{R} \cup \{+\infty\} \) defined by

\[
\ell(t, x, u) = H^*(t, x, f(t, x, u))
\]

we have

(A1) \( H(t, x, p) = \max_{u \in B} (\langle p, f(t, x, u) \rangle - \ell(t, x, u)) \), \( \forall (t, x, p) \).

(A2) For any \( R > 0 \) and for all \( t \in [0, T], x, y \in B_R, u, v \in B \)

\[
\begin{cases}
|f(t, x, u) - f(t, y, u)| \leq 10nc_R(t)|x - y| \\
|f(t, x, u) - f(t, x, v)| \leq 5n(1 + R)c(t)|u - v|.
\end{cases}
\]

(A3) \( |f(t, x, u)| \leq c(t)(1 + |x|) \) for all \( (t, x, u) \in [0, T] \times \mathbb{R}^n \times B \).
Regularity of dynamics and the cost function

Theorem

(A4) \( \ell \) takes finite values and for any \( R > 0, \ t \in [0, T], \ x, y \in B_R, \ u, v \in B \)

\[
\begin{align*}
|\ell(t, x, u) - \ell(t, y, u)| &\leq (1 + 11nM_R)c_R(t)|x - y|, \\
|\ell(t, x, u) - \ell(t, x, v)| &\leq 5nM_R(1 + R)c(t)|u - v|.
\end{align*}
\]

and

\[
|\ell(t, x, u) - \ell(s, x, u)| \leq (1 + 11nM_R)|a_R(t) - a_R(s)|.
\]

We associate to \( f, \ell, \varphi \) the Bolza optimal control problem.
If assumptions (H1)-(H6) hold true for continuous $H_i : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \ i \geq 1$, then there exist $f_i : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}^n$, $\ell_i : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}$ satisfying (A1)-(A4), which are standard in control theory.

**Theorem**

If $H_i$ converge uniformly on compacts to some $H : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and satisfy $(H1) - (H6)$ with the same $M_R, c_R(\cdot), c(\cdot), a_R(\cdot)$, then the representations $f_i, \ell_i$ can be chosen to converge to some $f, \ell$ and satisfying (A1)-(A4) with the same $M_R, c_R(\cdot), c(\cdot), a_R(\cdot)$.
State constraints

We assume that the closed sets $K$ and $K_i$ are defined by the multiple inequality constraints

$$K = \bigcap_{j=1}^{m} \{ x : g^j(x) \leq 0 \}$$

$$K_i = \bigcap_{j=1}^{m} \{ x : g_i^j(x) \leq 0 \},$$

where $g_i^j : \mathbb{R}^n \to \mathbb{R}$ and $g^j : \mathbb{R}^n \to \mathbb{R}$, $j = 1, \ldots, m$, $i = 1, 2, \ldots$ are continuously differentiable functions satisfying (G1)

i) For any $R > 0$ there exists $A_R > 0$ such that $|\nabla g_i^j(x)| \leq A_R$, for any $x \in RB$, $\nabla g_i^j$ is $A_R$-Lipschitz on $RB$.

ii) $\nabla g_i^j \to \nabla g^j$ uniformly on compacts and $g_i^j(0) \to g^j(0)$, when $i \to \infty$, for any $j = 1, \ldots, m$. 
For any \( x \in \mathbb{R}^n \) denote by \( I(x) \) the set of active indices at \( x \) for \( g(\cdot) = (g^1(\cdot), \ldots, g^m(\cdot)) \), i.e.

\[
I(x) = \{ j : g^j(x) = 0 \}.
\]

\textbf{(IPC)} For any \( R > 0 \) there exists \( \rho_R > 0 \) such that for every \( x \in K \cap RB \) with \( I(x) \neq \emptyset \) and every \( s \in [0, T] \)

\[
\inf_{v \in \text{dom}(H^*(t,x,\cdot))} \max_{j \in I(x)} \langle \nabla g^j(x), v \rangle \leq -\rho_R.
\]
Viscosity solutions of Hamilton-Jacobi equation

**Definition**

A continuous function \( V : [0, T] \times K \to \mathbb{R} \) is called a viscosity solution of Hamilton-Jacobi equation on the closed set \([0, T] \times K\) if \( V(T, \cdot) = \phi(\cdot) \) and

i) for all \((t, x) \in [0, T] \times \text{Int}K\) and all \((p_t, p_x) \in \partial_- V(t, x)\)

\[-p_t + H(t, x, -p_x) \geq 0\]

ii) for all \((t, x) \in [0, T] \times \text{Int}K\) and all \((p_t, p_x) \in \partial_+ V(t, x)\)

\[-p_t + H(t, x, -p_x) \leq 0,\]

where \(\partial_{\pm} V(t, x)\) are the Frechet super/subdifferential of \( V \) at \((t, x)\).
Bilateral solution of Hamilton-Jacobi equation

**Definition**

$V : [0, T] \times K \to \mathbb{R}$ is called a bilateral solution of Hamilton-Jacobi equation on the closed set $[0, T] \times K$ if $V(T, \cdot) = \varphi(\cdot)$ and

i) for all $(t, x) \in [0, T] \times \text{Int}K$ and all $(p_t, p_x) \in \partial_- V(t, x)$

$$-p_t + H(t, x, -p_x) = 0$$

ii) for all $(t, x) \in [0, T] \times \partial K$ and all $(p_t, p_x) \in \partial_- V(t, x)$

$$-p_t + H(t, x, -p_x) \geq 0.$$
Uniqueness of solutions of Hamilton-Jacobi equation

**Theorem**

If assumptions (H1)-(H6) and (IPC) hold true. Then the associated value function of the Bolza optimal control problem is the unique bilateral solution of the Hamilton-Jacobi equation on $[0, T] \times K$ in the class of continuous functions.

**Theorem**

If assumptions (H1)-(H6) and (IPC) hold true. Then the associated value function of the Bolza optimal control problem is the unique viscosity solution of the Hamilton-Jacobi equation on $[0, T] \times K$ in the class of continuous functions.
Stability of Solutions to Hamilton-Jacobi equation

Theorem

Suppose that $H_i$ satisfy (H1)-(H6) with the same $a_R(\cdot), c_R(\cdot), c(\cdot), M_R$ and converge uniformly on compacts to a Hamiltonian $H$. Assume that (IPC) holds true. If $\varphi_i$ converge to $\varphi$ uniformly on compacts, then the unique solutions $V_i$ to Hamilton-Jacobi equation with $H_i, \varphi_i, K_i$ converge uniformly on compacts contained in the interior of $K$ to the unique solution of Hamilton-Jacobi equation with $H, \varphi, K$. 
Suppose that $H_i$ satisfy (H1)-(H6) with the same $a_R(\cdot), c_R(\cdot), c(\cdot), M_R$ and converge uniformly on compacts to a Hamiltonian $H$. Assume that (IPC) holds true. Then

$$\lim_{i \to \infty} \text{graph} V_i = \text{graph} V$$

and

$$\lim_{i \to \infty} \text{epi} V_i = \text{epi} V,$$

where the limit is taken in the Kuratowski sense.
Thank you for your attention.