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# Discrete mean field games: the short-stage limit

Juan Pablo Maldonado López

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Tours, June 26, 2014



# Outline

- 1 Introduction
- 2 The N-player game
- 3 Mean field equilibrium
- 4 The short-stage model

# Mean field games

Mean field games models aim to understand the behavior of a large number of identical players, where each tries to optimize its position in space and time, but whose preferences are determined by the choices of the other players.

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- Introduced by Huang, Caines and Malhamé (2003, 2006) and by Lasry and Lions (2006,2007).
- Two important features: dynamics and anticipation (backward-forward structure.)
- Most of the studied models are in continuous time: the "backward" part corresponding to a Hamilton-Jacobi PDE and the "forward" part corresponding to a Fokker-Planck PDE.



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- Gomes, Mohr and Souza (2010) introduced a model for discrete time mean field games, but they consider a continuum of players instead and study asymptotic behaviour of the game as the time horizon tends to infinity. Our asymptotic results concern the number of players.
- Our model builds on results from Adlakha, Johari and Weintraub (2012).

# Notation

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- The states of the players at time  $t = 0$  are chosen i.i.d using the distribution  $m_0$ .
- We reserve capital letters for random variables and lower case letters for their realizations.

# Development of the game

- At stage  $t = 0, 1 \dots T - 1$ , player  $i$  observes his own state  $x_{t,N}^i$  and the average position on the state space of all the players,  $m_{t,N}$ .

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- Players choose simultaneously and independently their actions  $a_{t,N}^i$ .
- Each player  $i$  receives the payoff  $\ell(x_{t,N}^i, a_{t,N}^i, m_{t,N})$ .
- The new state  $X_{t+1,N}^i$  is chosen randomly using the transition function  $Q(x_{t,N}^i, a_{t,N}^i, m_{t,N})$ .

- The random average distribution is  $M_{t+1,N} := \frac{1}{N} \sum_{j \neq i} \delta_{X_{t+1,N}^j}$ .



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- At the beginning of stage  $t + 1$ , the realization of  $X_{t+1,N}^i$  and  $M_{t+1,N}$ , denoted  $x_{t+1,N}^i$  and  $m_{t+1,N}$  respectively, are observed, and the situation is repeated.

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- At stage  $t = T$  a final payoff  $g(x_{T,N}^i, m_{T,N})$  is allocated.

# Strategies

- A **behavioral strategy** for player  $i$  is a vector  $\pi^i = (\pi_t^i)_{t=0}^{T-1}$  where  $\pi_t^i : \mathcal{H}_t \rightarrow \mathcal{P}(A)$  and  $\mathcal{H}_t = (\mathcal{X} \times A \times \mathcal{P}(\mathcal{X}))^t$  is the set of all possible histories up to date  $t$ . Denote by  $\mathcal{S}$  the set of behavioral strategies.

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- A **Markovian strategy** for player  $i$  is a vector  $\sigma^i = (\sigma_t^i)_{t=0}^{T-1}$  such that  $\sigma_t^i : \mathcal{X} \rightarrow \mathcal{P}(A)$ .
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- A **stationary strategy** is a function  $\sigma : \mathcal{X} \rightarrow \mathcal{P}(A)$ .
- A **strategy profile** is a vector  $\pi = (\pi^i)_{i \in I}$ , where  $\pi^i$  is a behavioral strategy of player  $i$ .

# Payoff

The payoff of player  $i$ , when using the strategy  $\pi^i$  and when his adversaries use the strategy profile  $\pi^{-i} \in \mathcal{S}^{N-1}$  is

$$J_N^i(x, m_0, \pi^i, \pi^{-i}) := \mathbb{E}_\pi^Q \left[ \sum_{t \in \mathcal{T}} \ell(x_{t,N}^i, a_{t,N}^i, m_{t,N}) + g(x_{T,N}^i, m_{T,N}) \right].$$

# Nash equilibrium

## Definition

An  $\epsilon$ -**Nash equilibrium** where  $\epsilon > 0$ , is a strategy profile  $(\pi^i)_{i \in I}$  such that, for all player  $i$  and all behavioral strategy  $\tau$ ,

$$J_N^i(x, m_0, \tau, \pi^{-i}) \leq J_N^i(x, m_0, \pi^i, \pi^{-i}) + \epsilon.$$

# How to play well in the N player game?

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- Compute an optimal strategy for a single player with  $\mathbf{m}$  fixed.
- Everyone's strategy creates a vector of state distributions  $\mathbf{m}' = (m_0, m'_1, \dots, m'_T)$
- If  $\mathbf{m} = \mathbf{m}'$ , we are happy.

# Dynamic programming

From the familiar arguments we obtain the following dynamic programming equation:

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$$V(s, x, \mathbf{m}) = \max_{a \in A} \left\{ \ell(x, a, m_s) + \mathbb{E}^Q V(s+1, x_{s+1}, \mathbf{m}) \right\}$$

with terminal condition  $V(T, x, \mathbf{m}) = g(x, m_T)$ .

# The forward component

Now consider a Markovian strategy  $\sigma \in \Sigma$  and  $m_0$  fixed and let  $m_0^\sigma := m_0$ . We define, for  $t \geq 0$  :

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$$m_{t+1}^\sigma(x) := \sum_{y \in \mathcal{X}} Q(y, \sigma_t(y), m_t^\sigma)(x) \cdot m_t^\sigma(y).$$



# Definition: MFE

## Definition

Let  $m_0$  given. A **mean field equilibrium** is a pair

$(\sigma, \mathbf{m}) = \left( (\sigma_t)_{t=0}^{T-1}, (m_t)_{t=1}^T \right)$  such that:

- ①  $\sigma$  is the optimal strategy in the one player game  $\Gamma_{\mathbf{m}}$ , computed using the dynamic programming equation.
- ②  $\mathbf{m}$  is the trajectory followed by  $m_0$  according to the mass equation for the strategy  $\sigma$ .

# Existence

## Proposition

(M.,2013) *There exists a mean field equilibrium for the finite horizon game in the following cases:*

- *If there exists a unique maximizer for the right hand side of dynamic programming equation for  $\mathbf{m}$  and for each  $(s, x)$ .*

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- *The transitions are independent of the state distribution, i.e.*

$$Q(x, a, m)(y) =: Q(x, a)(y), \quad \forall(x, y, a, m).$$

# Approximation

## Proposition

(M., 2013) Let  $x$  be a fixed initial state,  $(\sigma, \mathbf{m})$  a mean field equilibrium and  $(a_{t,N}^i)_{t \in \mathcal{T}}$  an arbitrary sequence of actions of player  $i$ . Consider the following two trajectories:

- 1 The trajectory of player  $i$  defined by  $X_{t+1,N}^i \sim Q(x_{t,N}^i, a_{t,N}^i, m_{t,\sigma,N})$ .
- 2 The trajectory defined by  $X_{t+1}^i \sim Q(x_t^i, a_{t,N}^i, m_t)$ .

The following estimate holds:

$$\max_{i=1,\dots,N} \mathbb{E} \left( \max_{s \leq T} \|X_s^i - X_{s,N}^i\|_\infty \right) \leq \frac{L_Q T |\mathcal{X}| \exp(T(\|Q\|_\infty + L_Q + 1))}{\sqrt{N}}$$

# Some remarks

- In continuous time mean field games, the complementary approach of studying the limit behaviour of equilibria of  $N$  player games as  $N \rightarrow +\infty$  has been developed by Bardi (2012) for the linear-quadratic case and by Lasry and Lions (2007) and Feleqi (2013) for games with several populations of players and ergodic payoffs.

## Some remarks(cont.)

- This construction is "robust" with respect to the number of players: players can "play well" even if they do not know the exact number of players.

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# Driving game

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- Some equilibria: everyone on the left, everyone on the right, if  $N$  is even, everyone on the left;  $N$  odd, everyone on the right.
- The  $N$  player game has  $2^{\lfloor N/2 \rfloor}$  equilibria, while the game with infinitely many players has only two.

# The discounted $N$ -player game

The  $\lambda$ -discounted  $N$  player game is the game with payoff:

$$J_N^{\lambda,i}(x, m_0, \pi^i, \pi^{-i}) := \mathbb{E}_{\pi}^Q \left[ \sum_{t=1}^{\infty} (1 - \lambda)^{t-1} \ell(x_{t,N}^i, a_{t,N}^i, m_{t,N}) \mid x_0^i = x \right].$$

for  $\lambda \in (0, 1]$ .

## Discounted payoff: One player

For the discounted case, one can define a mean field equilibrium as follows:  
The value of the one-player game  $\Gamma_m^\lambda$ , ( $m \in \mathcal{P}(\mathcal{X})$  is fixed) with payoff:

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$$V_\lambda(x, m) := \sup_{\sigma} \mathbb{E}^Q \left[ \sum_{t=1}^{\infty} (1 - \lambda)^{t-1} \ell(x_t, \sigma(x_t), m) \right]$$

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satisfies:

$$V_\lambda(x, m) = \max_{a \in A} \left\{ \ell(x, a, m) + (1 - \lambda) \sum_{y \in \mathcal{X}} V_\lambda(y, m) Q(x, a, m)(y) \right\}.$$

# Stationary mean field equilibrium

The mean field equilibrium in this case is a fixed point of the maps  $\Psi_\lambda : \mathcal{P}(\mathcal{X}) \mapsto \Sigma_s$  and  $\Phi_\lambda : \Sigma_s \rightsquigarrow \mathcal{P}(\mathcal{X})$ :

$$\begin{aligned}
 m &\rightsquigarrow \text{Optimal stationary strategies in } \Gamma_m^\lambda, \\
 \sigma \in \Sigma_s &\rightsquigarrow \text{Inv. dist. of the MC with transition } Q(\cdot, \sigma(\cdot), m)
 \end{aligned}$$



# Existence

## Proposition

$\Phi_\lambda \circ \Psi_\lambda$  has a fixed point, i.e. there exists a stationary mean field equilibrium in the following cases:

- If for every stationary strategy  $\sigma$  and all  $m \in \mathcal{P}(\mathcal{X})$ , the Markov chain with transition law  $Q(\cdot, \cdot, \sigma(x), m)$  has a unique stationary distribution.

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# The short-stage equilibrium

We adapt an idea recently introduced by Neyman(2013) and Cardaliaguet et al.(2013) to our model. The aim (informally) is to construct an approximation by Friedman/Fleming discretization of a game in continuous time.

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We adapt an idea recently introduced by Neyman(2013) and Cardaliaguet et al.(2013) to our model. The aim (informally) is to construct an approximation by Friedman/Fleming discretization of a game in continuous time.

Consider a function  $\mu : \mathcal{X} \times \mathcal{X} \times A \times \mathcal{P}(\mathcal{X}) \rightarrow \mathbb{R}$  bounded and such that, for all  $(x, a, m)$ :

$$\mu(x, y, a, m) \geq 0, \quad x \neq y, \quad \mu(x, x, a, m) = - \sum_{y \neq x} \mu(x, y, a, m).$$

# Short-stage, one-player game

Let  $\rho > 0$ . The payoff in continuous time we want to approximate is:

$$\int_0^{\infty} e^{-\rho t} \ell(x_t, \sigma(x_t), m) dt$$

where  $(x_t)_{t \geq 0}$  is a continuous time Markov chain whose transition semigroup has generator  $\mu(\cdot, \sigma(\cdot), m)$ .

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$$\rho V_{\rho, \delta}(x, m) = \max_{a \in A} \left\{ \ell(x, a, m) + e^{-\rho\delta} \sum_{y \in \mathcal{X}} \mu(x, y, a, m) V_{\rho, \delta}(y, m) \right\}$$



# The short-stage limit

## Proposition

*The equation*

$$\rho f(x, m) = \max_{a \in A} \left\{ \ell(x, a, m) + \sum_{y \in \mathcal{X}} \mu(x, y, a, m) f(y, m) \right\}$$

*has a unique fixed point, denoted  $V_\rho$ . Moreover,  $V_{\rho\delta}^\delta \rightarrow V_\rho$  uniformly as  $\delta \rightarrow 0$ .*

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## Proof.

Use the stationary strategy  $\sigma^\rho$  given by this equation in the game with stage  $\delta$ . □

# The limit mass equation

Now let  $\sigma_\delta$  be a fixed stationary strategy for the game with stage  $\delta$  and let  $m' \in \mathcal{P}(\mathcal{X})$ . Let  $L[\sigma_\delta, m'] \in \mathbb{R}^{\mathcal{X} \times \mathcal{X}}$  be defined by

$$L[\sigma_\delta, m']_{x,y} = \mu(x, y, \sigma_\delta(x), m').$$

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For  $\sigma, m'$  given, the associated invariant distribution  $m$  must solve

$$\delta L[\sigma, m'] \cdot m = 0 \iff L[\sigma, m'] \cdot m = 0$$

# The limit system

Combining the two limit equations we obtain the following:

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## Definition

A **limit stationary mean field equilibrium** is a pair  $(\sigma, m) \in \Sigma_s \times \mathcal{P}(\mathcal{X})$  such that  $\sigma$  is the stationary strategy derived from

$$\rho V^\rho(x, m) = \max_{a \in A} \left\{ \ell(x, a, m) + \sum_{y \in \mathcal{X}} \mu(x, y, a, m) V^\rho(y, m) \right\}$$

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and  $m$  solves:

$$L[\sigma^\rho, m] \cdot m = 0.$$



# Approximation

## Theorem

(M.2013). For every  $\epsilon > 0$  there exists  $\delta_0 > 0$  and  $N_0 \in \mathbb{N}$  such that, for all  $\delta < \delta_0$  and  $N > N_0$ , the strategy provided by the limit stationary mean field equilibrium is a  $2\epsilon$ -Nash equilibrium of the discounted mean field game with discount factor  $\lambda = 1 - e^{-\rho\delta}$ .

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## Proof.

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- Now choose  $K_0$  such that

$$(1 - \rho\delta_0)^{K_0} \|\ell\|_\infty < \epsilon/2.$$

## Proof.

Collect everything:

- Choose  $\delta_0$  first, from the uniform convergence of the value functions.
- Now choose  $K_0$  such that

$$(1 - \rho\delta_0)^{K_0} \|\ell\|_\infty < \epsilon/2.$$

- Finally, take  $N_0$  as in the bound derived for the error in the game with  $K_0$  stages.

□

# Example: Online hotel booking

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- v) Frequent interaction is desirable in this example!*

# Some remarks

- Mean field games provide an extremely simple strategy that does not need to keep track of the other players.

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- Mean field games provide an extremely simple strategy that does not need to keep track of the other players.
- However, the mean field equilibrium might not be unique! Unless the players agree somehow on which equilibrium to play, it is hard to predict anything.

# A possible way out (Repeated driving game)

## Example

*Consider the repeated version of the driving game with  $N$  players with the following adaptation mechanism:*

- i) *Each player chooses left or right with probability  $\frac{1}{2}$  on the first stage.*

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- ii) On the second stage, observing the realizations of the first stage, each player looks at everyone's choice (and recalls its own) and imitates the choice of the majority.*
- iii) Thus, from stage three, the players follow a mean field equilibrium.*

# Possible extensions

- How do players find the mean field equilibrium? Incorporate learning mechanisms.



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- How do players find the mean field equilibrium? Incorporate learning mechanisms.
- Validation of this model in applications: Several economic applications in the paper of Adlakha, Johari, Weintraub(2012), applications for dynamic auctions by Iyer, Johari, Sundararajan (2011).

Thank you!