Discrete mean field games: the short-stage limit

Juan Pablo Maldonado López

Equipe Combinatoire et Optimisation, Université Pierre et Marie Curie, Paris

Tours, June 26, 2014
Outline

1. Introduction
2. The N-player game
3. Mean field equilibrium
4. The short-stage model
Mean field games

Mean field games models aim to understand the behavior of a large number of identical players, where each tries to optimize its position in space and time, but whose preferences are determined by the choices of the other players.
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- Two important features: dynamics and anticipation (backward-forward structure.)
- Most of the studied models are in continuous time: the "backward" part corresponding to a Hamilton-Jacobi PDE and the "forward" part corresponding to a Fokker-Planck PDE.
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- Our model builds on results from Adlakha, Johari and Weintraub (2012).
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The N-player game

Notation

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- The transition function:
  \[ Q : \mathcal{X} \times A \times \mathcal{P}(\mathcal{X}) \to \mathcal{P}(\mathcal{X}) \]
The game with finitely many players

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- The states of the players at time $t = 0$ are chosen i.i.d using the distribution $m_0$.
- We reserve capital letters for random variables and lower case letters for their realizations.
Development of the game

- At stage $t = 0, 1 \ldots T - 1$, player $i$ observes his own state $x_{t,N}^i$ and the average position on the state space of all the players, $m_{t,N}$. 
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- Each player $i$ receives the payoff $\ell(x^i_{t,N}, a^i_{t,N}, m_{t,N})$. 
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- Players choose simultaneously and independently their actions $a_{t,N}^i$.
- Each player $i$ receives the payoff $\ell(x_{t,N}^i, a_{t,N}^i, m_{t,N})$.
- The new state $X_{t+1,N}^i$ is chosen randomly using the transition function $Q(x_{t,N}^i, a_{t,N}^i, m_{t,N})$. 
The random average distribution is $M_{t+1,N} := \frac{1}{N} \sum_{j \neq i} \delta_{X_{t+1,N}^j}$. 
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At the beginning of stage $t + 1$, the realization of $X_{t+1, N}^i$ and $M_{t+1, N}$, denoted $x_{t+1, N}^i$ and $m_{t+1, N}$ respectively, are observed, and the situation is repeated.
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- At stage \( t = T \) a final payoff \( g(x_{T,N}^i, m_{T,N}) \) is allocated.
A behavioral strategy for player $i$ is a vector $\pi^i = (\pi^i_t)_{t=0}^{T-1}$ where $\pi^i_t : \mathcal{H}_t \rightarrow \mathcal{P}(A)$ and $\mathcal{H}_t = (\mathcal{X} \times A \times \mathcal{P}(\mathcal{X}))^t$ is the set of all possible histories up to date $t$. Denote by $\mathcal{S}$ the set of behavioral strategies.
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Strategies

- A **behavioral strategy** for player $i$ is a vector $\pi^i = (\pi^i_t)_{t=0}^{T-1}$ where $\pi^i_t : \mathcal{H}_t \to \mathcal{P}(A)$ and $\mathcal{H}_t = (\mathcal{X} \times A \times \mathcal{P}(\mathcal{X}))^t$ is the set of all possible histories up to date $t$. Denote by $\mathcal{S}$ the set of behavioral strategies.

- A **Markovian strategy** for player $i$ is a vector $\sigma^i = (\sigma^i_t)_{t=0}^{T-1}$ such that $\sigma^i_t : \mathcal{X} \to \mathcal{P}(A)$.

- A **stationary strategy** is a function $\sigma : \mathcal{X} \to \mathcal{P}(A)$. 

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- A **strategy profile** is a vector $\pi = (\pi^i)_{i \in I}$, where $\pi^i$ is a behavioral strategy of player $i$. 
The payoff of player $i$, when using the strategy $\pi^i$ and when his adversaries use the strategy profile $\pi^{-i} \in S^{N-1}$ is

$$J^i_N(x, m_0, \pi^i, \pi^{-i}) := \mathbb{E}_\pi^Q \left[ \sum_{t \in T} \ell(x^i_t, N, a^i_t, N, m_t, N) + g(x^i_T, N, m_T, N) \right].$$
Nash equilibrium

**Definition**

An $\epsilon$-Nash equilibrium where $\epsilon > 0$, is a strategy profile $(\pi^i)_{i \in I}$ such that, for all player $i$ and all behavioral strategy $\tau$,

$$J^i_N(x, m_0, \tau, \pi^{-i}) \leq J^i_N(x, m_0, \pi^i, \pi^{-i}) + \epsilon.$$
How to play well in the $N$ player game?

- The trick: Hide the complexity of the problem in a parameter $\mathbf{m} \in \mathcal{P}(\mathcal{X})$. 
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- Compute an optimal strategy for a single player with $\mathbf{m}$ fixed.
- Everyone’s strategy creates a vector of state distributions $\mathbf{m}' = (m'_0, m'_1, \ldots, m'_T)$.
- If $\mathbf{m} = \mathbf{m}'$, we are happy.
Dynamic programming

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\[ V(s, x, m) = \max_{a \in A} \left\{ \ell(x, a, m_s) + \mathbb{E}^Q V(s + 1, x_{s+1}, m) \right\} \]

with terminal condition \( V(T, x, m) = g(x, m_T) \).
The forward component

Now consider a Markovian strategy $\sigma \in \Sigma$ and $m_0$ fixed and let $m_0^\sigma := m_0$. We define, for $t \geq 0$:
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$$m_{t+1}^\sigma(x) := \sum_{y \in \mathcal{X}} Q(y, \sigma_t(y), m_t^\sigma)(x) \cdot m_t^\sigma(y).$$
Definition: MFE

Let $m_0$ given. A **mean field equilibrium** is a pair $(\sigma, m) = \left((\sigma_t)_{t=0}^{T-1}, (m_t)_{t=1}^T\right)$ such that:

1. $\sigma$ is the optimal strategy in the one player game $\Gamma_m$, computed using the dynamic programming equation.
2. $m$ is the trajectory followed by $m_0$ according to the mass equation for the strategy $\sigma$. 
Existence

Proposition

(M., 2013) There exists a mean field equilibrium for the finite horizon game in the following cases:

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- If there exists a unique maximizer for the right hand side of dynamic programming equation for \( m \) and for each \((s, x)\).
- The transitions are independent of the state distribution, i.e.

\[
Q(x, a, m)(y) \equiv Q(x, a)(y), \quad \forall (x, y, a, m).
\]
Approximation

**Proposition**

(M., 2013) Let x be a fixed initial state, $\sigma, m$ a mean field equilibrium and $(a_{t,N}^i)_{t \in T}$ an arbitrary sequence of actions of player i. Consider the following two trajectories:

1. The trajectory of player i defined by $X_{t+1,N}^i \sim Q(x_{t,N}^i, a_{t,N}^i, m_t, \sigma, N)$.
2. The trajectory defined by $X_{t+1}^i \sim Q(x_{t}^i, a_{t,N}^i, m_t)$.

The following estimate holds:

$$\max_{i=1,\ldots,N} \mathbb{E} \left( \max_{s \leq T} \| X_s^i - X_{s,N}^i \|_\infty \right) \leq \frac{L_Q T |x'| \exp(T(\| Q \|_\infty + L_Q + 1))}{\sqrt{N}}$$
Some remarks

- In continuous time mean field games, the complementary approach of studying the limit behaviour of equilibria of $N$ player games as $N \to +\infty$ has been developed by Bardi (2012) for the linear-quadratic case and by Lasry and Lions (2007) and Feleqi (2013) for games with several populations of players and ergodic payoffs.
Some remarks (cont.)

- This construction is "robust" with respect to the number of players: players can "play well" even if they do not know the exact number of players.
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- In general, the set of Nash equilibria of $N$ player might contain equilibria that depend on $N$ as in the driving game.
Driving game

- $N$ players have to choose whether to drive on the left or on the right in a two-way road.
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- Some equilibria: everyone on the left, everyone on the right, if $N$ is even, everyone on the left; $N$ odd, everyone on the right.
- The $N$ player game has $2^N$ equilibria, while the game with infinitely many players has only two.
The discounted $N$-player game

The $\lambda-$ discounted $N$ player game is the game with payoff:

$$J^\lambda_N(x, m_0, \pi^i, \pi^{-i}) := \mathbb{E}^Q_\pi \left[ \sum_{t=1}^\infty (1 - \lambda)^{t-1} \ell(x^i_t, N, a^i_t, N, m_t, N) \mid x^i_0 = x \right].$$

for $\lambda \in (0, 1]$. 
Discounted payoff: One player

For the discounted case, one can define a mean field equilibrium as follows: The value of the one-player game $\Gamma^\lambda_m, (m \in \mathcal{P}(\mathcal{X}) \text{ is fixed})$ with payoff:
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$$V^\lambda(x, m) := \sup_{\sigma} \mathbb{E}^Q \left[ \sum_{t=1}^{\infty} (1 - \lambda)^{t-1} \ell(x_t, \sigma(x_t), m) \right]$$

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For the discounted case, one can define a mean field equilibrium as follows: The value of the one-player game $\Gamma_\lambda^m$, ($m \in \mathcal{P}(\mathcal{X})$ is fixed) with payoff:

$$V_\lambda(x, m) := \sup_{\sigma} \mathbb{E}^Q \left[ \sum_{t=1}^{\infty} (1 - \lambda)^{t-1} \ell(x_t, \sigma(x_t), m) \right]$$

satisfies:

$$V_\lambda(x, m) = \max_{a \in A} \left\{ \ell(x, a, m) + (1 - \lambda) \sum_{y \in \mathcal{X}} V_\lambda(y, m) Q(x, a, m)(y) \right\}.$$
The mean field equilibrium in this case is a fixed point of the maps
\( \Psi_\lambda : \mathcal{P}(\mathcal{X}) \mapsto \Sigma_s \) and \( \Phi_\lambda : \Sigma_s \mapsto \mathcal{P}(\mathcal{X}) : \\
\begin{align*}
m &\mapsto \text{Optimal stationary strategies in } \Gamma^\lambda_m, \\
\sigma \in \Sigma_s &\mapsto \text{Inv. dist. of the MC with transition } Q(\cdot, \sigma(\cdot), m)
\end{align*}
Existence

Proposition

$\Phi_\lambda \circ \Psi_\lambda$ has a fixed point, i.e. there exists a stationary mean field equilibrium in the following cases:

- If for every stationary strategy $\sigma$ and all $m \in P(X)$, the Markov chain with transition law $Q(\cdot, \cdot, \sigma(x), m)$ has a unique stationary distribution.
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Q(x, a, m)(y) =: Q(x, a)(y), \quad \forall (x, y, a, m).
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The short-stage equilibrium

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We adapt an idea recently introduced by Neyman (2013) and Cardaliaguet et al. (2013) to our model. The aim (informally) is to construct an approximation by Friedman/Fleming discretization of a game in continuous time.

Consider a function $\mu : X \times X \times A \times \mathcal{P}(X) \to \mathbb{R}$ bounded and such that, for all $(x, a, m)$:

$$
\mu(x, y, a, m) \geq 0, \; x \neq y, \quad \mu(x, x, a, m) = -\sum_{y \neq x} \mu(x, y, a, m).
$$
Short-stage, one-player game

Let $\rho > 0$. The payoff in continuous time we want to approximate is:

$$\int_{0}^{\infty} e^{-\rho t} \ell(x_t, \sigma(x_t), m) dt$$

where $(x_t)_{t \geq 0}$ is a continuous time Markov chain whose transition semigroup has generator $\mu(\cdot, \sigma(\cdot), m)$. 
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\[
\rho V_{\rho,\delta}(x, m) = \max_{a \in A} \left\{ \ell(x, a, m) + e^{-\rho \delta} \sum_{y \in X} \mu(x, y, a, m) V_{\rho,\delta}(y, m) \right\}
\]
The short-stage limit

Proposition

The equation

$$\rho f(x, m) = \max_{a \in A} \left\{ \ell(x, a, m) + \sum_{y \in \mathcal{X}} \mu(x, y, a, m) f(y, m) \right\}$$

has a unique fixed point, denoted $V_\rho$. Moreover, $V_{\rho \delta} \to V_\rho$ uniformly as $\delta \to 0$. 
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Proof.

Use the stationary strategy \( \sigma^\rho \) given by this equation in the game with stage \( \delta \).

\[\Box\]
The limit mass equation

Now let $\sigma_\delta$ be a fixed stationary strategy for the game with stage $\delta$ and let $m' \in \mathcal{P}(\mathcal{X})$. Let $L[\sigma_\delta, m'] \in \mathbb{R}^{\mathcal{X} \times \mathcal{X}}$ be defined by

$$L[\sigma_\delta, m']_{x,y} = \mu(x, y, \sigma_\delta(x), m').$$
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For $\sigma, m'$ given, the associated invariant distribution $m$ must solve

$$\delta L[\sigma, m'] \cdot m = 0 \iff L[\sigma, m'] \cdot m = 0.$$

Combining the two limit equations we obtain the following:
The short-stage model

The limit system

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Definition

A limit stationary mean field equilibrium is a pair \((\sigma, m) \in \Sigma_s \times \mathcal{P}(\mathcal{X})\) such that \(\sigma\) is the stationary strategy derived from

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and \(m\) solves:

\[
L[\sigma^\rho, m] \cdot m = 0.
\]
Approximation

Theorem

(M.2013). For every $\epsilon > 0$ there exists $\delta_0 > 0$ and $N_0 \in \mathbb{N}$ such that, for all $\delta < \delta_0$ and $N > N_0$, the strategy provided by the limit stationary mean field equilibrium is a $2\epsilon$-Nash equilibrium of the discounted mean field game with discount factor $\lambda = 1 - e^{-\rho\delta}$. 
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- Now choose $K_0$ such that

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(1 - \rho \delta_0)^{K_0} \| \ell \|_\infty < \epsilon/2.
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- Now choose $K_0$ such that
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  (1 - \rho \delta_0)^{K_0} \| \ell \|_\infty < \epsilon/2.
  \]
- Finally, take $N_0$ as in the bound derived for the error in the game with $K_0$ stages.
Example: Online hotel booking

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v) Frequent interaction is desirable in this example!
Some remarks

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- Mean field games provide an extremely simple strategy that does not need to keep track of the other players.
- However, the mean field equilibrium might not be unique! Unless the players agree somehow on which equilibrium to play, it is hard to predict anything.
A possible way out (Repeated driving game)

Example

Consider the repeated version of the driving game with N players with the following adaptation mechanism:

i) Each player chooses left or right with probability $\frac{1}{2}$ on the first stage.
A possible way out (Repeated driving game)

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Consider the repeated version of the driving game with $N$ players with the following adaptation mechanism:

i) Each player chooses left or right with probability $\frac{1}{2}$ on the first stage.

ii) On the second stage, observing the realizations of the first stage, each player looks at everyone’s choice (and recalls its own) and imitates the choice of the majority.
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Consider the repeated version of the driving game with $N$ players with the following adaptation mechanism:

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ii) On the second stage, observing the realizations of the first stage, each player looks at everyone’s choice (and recalls its own) and imitates the choice of the majority.

iii) Thus, from stage three, the players follow a mean field equilibrium.
Possible extensions

- How do players find the mean field equilibrium? Incorporate learning mechanisms.
Possible extensions

- How do players find the mean field equilibrium? Incorporate learning mechanisms.
- Validation of this model in applications: Several economic applications in the paper of Adlakha, Johari, Weintraub (2012), applications for dynamic auctions by Iyer, Johari, Sundararajan (2011).
Thank you!