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# Necessary Conditions for Implicit and DAE Control Systems

Maria Do Rosário de Pinho

► **To cite this version:**

Maria Do Rosário de Pinho. Necessary Conditions for Implicit and DAE Control Systems. NETCO 2014 - New Trends in Optimal Control, Jun 2014, Tours, France. hal-01024782

**HAL Id: hal-01024782**

**<https://inria.hal.science/hal-01024782>**

Submitted on 16 Jul 2014

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# Necessary Conditions for Implicit and DAE Control Systems

Maria do Rosário de Pinho

University of Porto, Faculdade de Engenharia

**ITN SADCO**  
Initial Training Network  
Sensitivity Analysis for Deterministic Controller Design

**FCT**  
Fundação para a Ciência e a Tecnologia  
MINISTÉRIO DA CIÊNCIA E TECNOLOGIA

  
**COMPETE**  
PROGRAMA OPERACIONAL FACTORES DE COMPETITIVIDADE



  
**FEDER**

# Motivation

- Connection between Implicit Systems and DAE
- Recent Work on DAE (MG presentation in Bayreuth, e.g.)
- Questions on NC for Implicit Systems
  - Answer: see paper [1]
  - Really? Theorem 6.1? Theorem 7.1? How??????

This is, then, a clarification on applicability of results in [1] and... we introduce a **hybrid version** of Theorem 6.1

[1] *Optimal control problems with mixed constraints*, Clarke & dP, SIAM J. Control Optim., 48 (2010)

# Outline

- Implicit System Optimal Control Problem
- Necessary Conditions and the Hybrid Version
- Special Cases: 1, 2 and 3
- Alternative NCO for cases 2 and 3
- Implicit Problems with Additional Constraints
- How to deal with equality state constraints

Flavor  
of  
Future  
work

# Implicit System OC Problem

$$\begin{aligned} &\text{Minimize} && l(x(a), x(b)) \\ &\text{subject to} && \\ & && 0 = f(t, x(t), \dot{x}(t), u(t)) \text{ a.e.} \\ & && u(t) \in U \text{ a.e.} \\ & && (x(a), x(b)) \in E \end{aligned}$$

where

$$l : \mathbf{R}^n \times \mathbf{R}^n \rightarrow \mathbf{R}$$

$$f : [a, b] \times \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^k \rightarrow \mathbf{R}^N$$

$U \subset \mathbf{R}^k$  is compact

$E \subset \mathbf{R}^n \times \mathbf{R}^n$  closed set

Essential here is the following **Reformulation**

$$\begin{cases} \dot{x}(t) &= v, \\ 0 &= f(t, x, v, u). \end{cases} \quad (1)$$

We say that an admissible process for  $(P)$  is  $(x^*, u^*)$  is a *local minimum* when it minimizes the cost  $l(x(a), x(b))$  over all the admissible processes  $(x, v, u)$  such that

$$|x(t) - x^*(t)| \leq \varepsilon$$

for some  $\varepsilon > 0$ .

**Introducing:**

$$\mathbf{S}(t) := \{(x, v, u) : (f(t, x, v, u), u) \in \Phi \times U\}, \quad \Phi = \{0\}$$

$$S_\varepsilon^*(t) := \{(x, v, u) \in S(t) : |x(t) - x^*(t)| \leq \varepsilon\},$$

$$S(t, u) := \{(x, v) : f(t, x, v, u) \in \Phi\}, \quad u \in U$$

$$S_\varepsilon^*(t, u) := \{(x, v) \in S(t, u) : |x(t) - x^*(t)| \leq \varepsilon\}$$

For  
hybrid  
case



# Assumptions

The function  $l$  is locally Lipschitz,  $E$  closed,  $U$  compact.

The function  $(t, x, v, u) \rightarrow f(t, x, v, u)$  is  $\mathcal{L} \times \mathcal{B}$  measurable.

There exists a constant  $k_f$  such that, for a.e.  $t \in [a, b]$ , for every  $u \in U$ ,  $(x_i, v_i, u_i)$  in a neigh. of  $S_\varepsilon^*(t)$ , we have

$$|f(t, x_1, v_1, u_1) - f(t, x_2, v_2, u_2)| \leq k_f[|x_1 - x_2| + |v_1 - v_2| + |u_1 - u_2|].$$

There exists constant  $M$  such that, for a.e.  $t \in [a, b]$ ,  $(x_i, v_i, u_i) \in S_\varepsilon^*(t)$ :

$$\text{(BS)} \quad (\alpha, \beta, \gamma) \in N_{S(t)}^P(x, v, u) \implies |\alpha| \leq M|(\beta, \gamma)|.$$

# NCO for Implicit Problems (Clarke&dP 2010)

There exist  $p \in W^{1,1}([a, b]; \mathbf{R}^n)$  and a scalar  $\lambda_0 \geq 0$  such that:

$$\|p\|_{\infty} + \lambda_0 > 0,$$

$$(p(a), -p(b)) \in N_E^L(x^*(a), x^*(b)) + \lambda_0 \partial^L l(x^*(a), x^*(b)),$$

For almost every  $t \in [a, b]$

$$(-\dot{p}(t), 0, 0) \in \partial_{x,v,u}^C \langle p(t), \dot{x}^*(t) \rangle - N_{S(t)}^C(x^*(t), \dot{x}^*(t), u^*(t))$$

For all  $u \in U$ ,  $(x^*(t), v)$  such that  $f(t, x^*(t), v, u) = 0$ ,

(W)

$$\langle p(t), v \rangle \leq \langle p(t), \dot{x}^*(t) \rangle.$$



# Hybrid Version of Theorem 6.1 in [1]

proved by “putting together”  
Theorem 3.2 with Theorem 6.1

# Assumptions: what changes

For almost every  $t \in [a, b]$ , for every  $(x_i, v_i, u_i)$  with  $|x_i - x^*(t)| \leq \varepsilon$ , we have

$$|f(t, x_1, v_1, u_1) - f(t, x_2, v_2, u_2)| \leq k_\psi [|x_1 - x_2| + |v_1 - v_2| + |u_1 - u_2|].$$

For almost every  $t \in [a, b]$ , for every  $u \in U$ ,  $(x_i, v_i)$  in a neigh. of  $S_\varepsilon^*(t, u)$ , we have

$$|f(t, x_1, v_1, u) - f(t, x_2, v_2, u)| \leq k_f [|x_1 - x_2| + |v_1 - v_2|].$$

There exists constant  $M$  such that, for a.e.  $t \in [a, b]$ ,  $(x_i, v_i, u_i) \in S_\varepsilon^*(t)$ :

$$(\alpha, \beta, \gamma) \in N_{S(t)}^P(x, v, u) \implies |\alpha| \leq M |(\beta, \gamma)|.$$

There exists constant  $M$  such that, for a.e.  $t \in [a, b]$ ,  $u \in U$ :

$$(x, v) \in S_\varepsilon^*(t, u), \quad (\alpha, \beta) \in N_{S(t, u)}^P \implies |\alpha| \leq M |\beta|.$$

(L)

(BS)

# NCO: Hybrid Theorem 6.1

There exist  $p \in W^{1,1}([a, b]; \mathbf{R}^n)$  and a scalar  $\lambda_0 \geq 0$  such that:

$$\|p\|_\infty + \lambda_0 > 0,$$

$$(p(a), -p(b)) \in N_E^L(x^*(a), x^*(b)) + \lambda_0 \partial^L l(x^*(a), x^*(b)),$$

For almost every  $t \in [a, b]$

~~$$(-\dot{p}(t), 0, 0) \in \partial_{x,v,u}^C \langle p(t), \dot{x}^*(t) \rangle - N_{S(t)}^C(x^*(t), \dot{x}^*(t), u^*(t))$$~~

For all  $u \in U$ ,  $(x^*(t), v)$  such that  $f(t, x^*(t), v, u) = 0$ ,

$$\langle p(t), v \rangle \leq \langle p(t), \dot{x}^*(t) \rangle.$$

$$(-\dot{p}(t), 0) \in \partial_{x,v}^C \langle p(t), \dot{x}^*(t) \rangle - N_{S(t, u^*(t))}^C(x^*(t), \dot{x}^*(t))$$

We have two difficulties:

Bounded Slope (BS) assumption and Euler inclusion in NCO

Both defined in terms of normal cones...but, as far as (BS),

For all  $\lambda \in \mathbf{R}^N$ , we have

$$(\alpha, \beta) \in \partial_{x,v}^L \langle \lambda, f(t, x, v, u) \rangle \implies |\lambda| \leq K_S |\beta|.$$

↓

(BS) with  $M = K_S k_f$

***And in the Smooth Case...***

# Smooth Case (hybrid)

Suppose  $(x, v, u) \rightarrow f(t, x, u, v)$  is  $C^1$ . Then our assumption

$$\text{For all } \lambda \in \mathbf{R}^N, (\alpha, \beta) \in \partial_{x,v}^L \langle \lambda, f(t, x, v, u) \rangle \implies |\lambda| \leq K_S |\beta|.$$



$$\text{For all } \lambda \in \mathbf{R}^N, \nabla_v \langle \lambda, f(t, x, v, u) \rangle = 0 \implies \lambda = 0.$$

Meaning that the  $(N)$  rows of

$$J_v f(t, x, v, u) \in \mathcal{M}_{N \times n}$$

are linearly independent, i.e., the matrix  $J_v f(t, x, v, u)$  has **full row rank**.

So  $(n \geq N)$ .

# Smooth Case (full row rank)

$$(-\dot{p}(t), 0) \in \partial_{x,v}^C \langle p(t), \dot{x}^*(t) \rangle - N_{S(t,u^*(t))}^C(x^*(t), \dot{x}^*(t))$$

↓

$$(-\dot{p}(t), 0) = \langle p(t), \dot{x}^*(t) \rangle - \nabla_{x,v} \langle \lambda(t), f(t, x^*(t), \dot{x}^*(t), u^*(t)) \rangle$$

And, by (W), we get  $\mu(t) \in N_U^C(u^*(t))$ :

$$-\mu(t) = \nabla_u \langle \lambda(t), f(t, x^*(t), \dot{x}^*(t), u^*(t)) \rangle.$$

Summarizing we have  $\lambda(t) \in \mathbf{R}^N$  and  $\mu(t) \in N_U^C(u^*(t))$ :

$$(NC) \quad \begin{cases} \dot{p}(t) &= \nabla_x \langle \lambda(t), f(t, x^*(t), \dot{x}^*(t), u^*(t)) \rangle \\ p(t) &= \nabla_v \langle \lambda(t), f(t, x^*(t), \dot{x}^*(t), u^*(t)) \rangle \\ -\mu(t) &= \nabla_u \langle \lambda(t), f(t, x^*(t), \dot{x}^*(t), u^*(t)) \rangle \end{cases}$$

$$\text{and } |\lambda(t)| \leq K_S |p(t)|.$$

# Special Case

Let  $f(t, x, v, u) = Ev - \tilde{f}(t, x, u)$ ,  $E \in \mathcal{M}_{N \times n}$ ,  $\text{rank}(E) = r$ .

**Case 1:**  $E$  constant and with full row rank i.e,  $r = N$ . Then  $N \leq n$ .

We have  $EE^T \in \mathcal{M}_{N \times N}$  nonsingular and from (NC) we get

$$\lambda(t) = \left(EE^T\right)^{-1} Ep(t)$$

and then

$$(2) \begin{cases} -\dot{p}(t) &= \nabla_x \langle (EE^T)^{-1} Ep(t), \tilde{f}(t, x^*(t), u^*(t)) \rangle \\ \mu(t) &= \nabla_u \langle (EE^T)^{-1} Ep(t), \tilde{f}(t, x^*(t), u^*(t)) \rangle \end{cases}$$

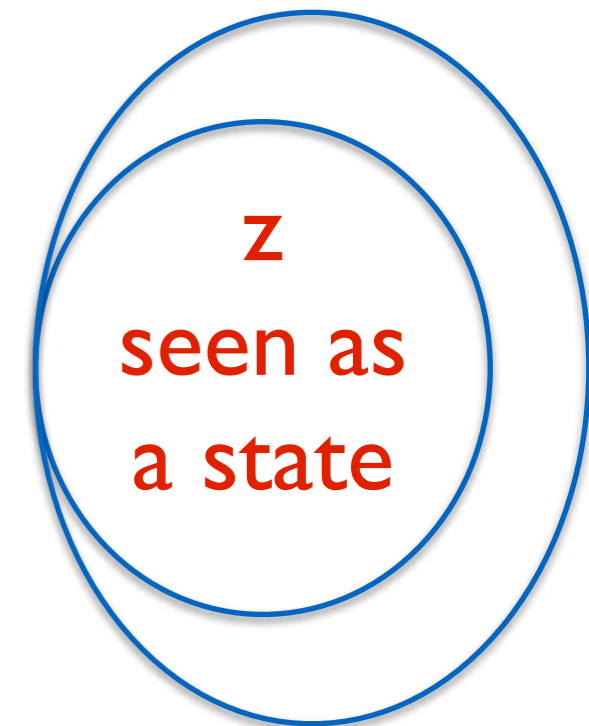
# Special Case

If  $E = [I \ 0]$ , then with  $x = (y, z) \in \mathbf{R}^N \times \mathbf{R}^{n-N}$  and  $f(t, x, \dot{x}, u) = 0$  becomes

$$\dot{y} - \tilde{f}(t, y, z, u) = 0.$$

Here  $z$  is a **state**. Our problem is now written as

$$(P) \quad \left\{ \begin{array}{l} \text{Minimize} \quad l(y(a), z(a), y(b), z(b)) \\ \text{subject to} \\ \dot{y}(t) - \tilde{f}(t, y(t), z(t), u(t)) = 0 \text{ a.e.} \\ u(t) \in U \text{ a.e.} \\ (y(a), z(a), y(b), z(b)) \in E \end{array} \right.$$



Let us write all the details of the NCO.

Set

$$p = (p_1, p_2) \in \mathbf{R}^N \times \mathbf{R}^{n-N}.$$



# NCO for Special Case

$$\|p\|_\infty + \lambda_0 > 0,$$

$$-\dot{p}(t) = \nabla_x \langle p_1(t), \tilde{f}(t, y^*(t), z^*(t), u^*(t)) \rangle$$

$$\mu(t) = \nabla_u \langle p_1(t), \tilde{f}(t, y^*(t), z^*(t), u^*(t)) \rangle$$

$$\max_{u \in U} \langle p_1(t), \tilde{f}(t, y^*(t), z^*(t), u) \rangle = \langle p_1(t), \tilde{f}(t, y^*(t), z^*(t), u^*(t)) \rangle.$$

$$(p_1(a), p_2(a), -p_1(b), -p_2(b)) \in N_E^L(y^*(a), z^*(a), y^*(b), z^*(b)) + \lambda_0 \partial^L l(y^*(a), z^*(a), y^*(b), z^*(b)),$$

Recall we are assuming  $x = (y, z)$  and

$$f(t, x, v, u) = Ev - \tilde{f}(t, x, u), \quad E \in \mathcal{M}_{N \times n}, \quad n \geq N, \quad \text{rank}(E) = N.$$

# Now...Case 2 and 3

Application of Theorem is not possible because

$$\text{rank}(J_v f(t, x, v, u)) < N$$

E



How to compensate for lack of rank of E?

# What if the matrix $E$ is not of full row rank?

**Case 2:**  $n \leq N$  and  $\text{rank}(E) = n$ , i.e.,  $E$  of full column rank.

For simplicity:  $E = \begin{bmatrix} E_c \\ 0 \end{bmatrix}$  and  $\tilde{f} = \begin{bmatrix} \tilde{f}_1 \\ \tilde{f}_2 \end{bmatrix}$ .

$$Ev - \tilde{f}(t, x, u) \longrightarrow \begin{cases} \dot{x}(t) = E_c^{-1} \tilde{f}_1(t, x, u) \\ 0 = \tilde{f}_2(t, x, u) \end{cases}$$

Use  
NCO  
for  
Mixed  
Constraints

**Case 3:**  $\text{rank}(E) = r < \min\{n, N\}$ ,  $r > 0$ .

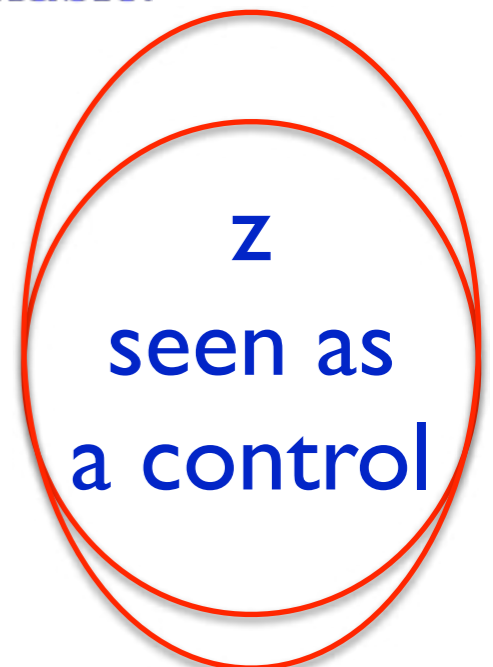
For simplicity:  $E = \begin{bmatrix} E_r & 0 \\ 0 & 0 \end{bmatrix}$ ,  $x = (y, z) \in \mathbf{R}^r \times \mathbf{R}^{n-r}$  and  $\tilde{f} = \begin{bmatrix} \tilde{f}_d \\ \tilde{f}_a \end{bmatrix}$ .

$$Ev - \tilde{f}(t, x, u) \implies \begin{cases} \dot{y}(t) = E_c^{-1} \tilde{f}_d(t, y, z, u) \\ 0 = \tilde{f}_a(t, y, z, u) \end{cases}$$

$$Ev - \tilde{f}(t, x, u) \implies \begin{cases} \dot{y}(t) = E_c^{-1} \tilde{f}_d(t, y, z, u) \\ 0 = \tilde{f}_a(t, y, z, u) \end{cases}$$

Can we treat  $z$  as a control? If yes, take  $w = (z, u)$  as the control variable.

$$(P_3) \quad \left\{ \begin{array}{l} \text{Minimize} \quad l(y(a), y(b)) \\ \text{subject to} \\ \dot{y}(t) = E_c^{-1} \tilde{f}_d(t, y(t), z(t), u(t)) \text{ a.e.} \\ 0 = \tilde{f}_a(t, y(t), z(t), u(t)) \text{ a.e.} \\ u(t) \in U \text{ a.e.} \\ (y(a), y(b)) \in E \end{array} \right.$$



If  $\forall \lambda \in \mathbf{R}^{N-r}, \nabla_z \langle \lambda, \tilde{f}_a(t, y, z, u) \rangle = 0 \implies \lambda = 0$ , we have an **index one** DAE.

For nonsmooth  $\tilde{f}$ , see hybrid NCO for mixed constrained problems (Th. 7.4 in [1]).

But...

- what if we can treat  $z$  as a control but

$$\forall \lambda \in \mathbf{R}^{N-r}, \nabla_z \langle \lambda, \tilde{f}_a(t, y, z, u) \rangle = 0 \not\Rightarrow \lambda = 0?$$

Does

$$\forall \lambda \in \mathbf{R}^{N-r}, \nabla_{z,u} \langle \lambda, \tilde{f}_a(t, y, z, u) \rangle \in -N_U^C(u) \implies \lambda = 0$$

hold?

If it does and Lipschitz conditions with respect to  $u$  also hold, apply Theorem 7.1 in [1] (NCO for mixed constraints, not hybrid).

We assume  $(x,v,u) \rightarrow f(t,x,v,u)$  to be loc. Lipschitz

- what if  $z$  cannot be treated as a control?

(1) Go back to

$$\begin{cases} \dot{x}(t) &= v, \\ 0 &= f(t, x, v, u). \end{cases}$$

(2) If  $(x, v, u) \rightarrow f(t, x, v, v)$  is locally Lipschitz and, for all  $\lambda \in \mathbf{R}^N$ , we have

$$(\alpha, \beta, \gamma) \in \partial_{x,v,u}^L \langle \lambda, f(t, x, v, u) \rangle \implies |\lambda| \leq M|(\beta, \gamma)|,$$

then **THEOREM 6.1** applies.

If  $(x, v, u) \rightarrow f(t, x, v, v)$  is strict differentiable at  $(x^*(t), \dot{x}^*(t), u^*(t))$ , then

$$\begin{cases} \dot{p}(t) &= \nabla_x \langle \lambda(t), f(t, x^*(t), \dot{x}^*(t), u^*(t)) \rangle \\ p(t) &= \nabla_v \langle \lambda(t), f(t, x^*(t), \dot{x}^*(t), u^*(t)) \rangle \\ -\mu(t) &= \nabla_u \langle \lambda(t), f(t, x^*(t), \dot{x}^*(t), u^*(t)) \rangle \end{cases}$$

# Summarizing the Smooth Special Cases

$$f(t, x, v, u) = Ev - \tilde{f}(t, x, v, u), \quad f \text{ smooth}, \quad E \in \mathcal{M}_{N \times n}.$$

- Case 1.  $N \leq n$  and  $\text{rank } J_v \tilde{f}(t, x, v, u) = N$ .

Apply Hybrid version of Theorem 6.1 presented here.

- Case 2.  $N \geq n$  and  $\text{rank } J_v \tilde{f}(t, x, v, u) = n$ .

Mixed constraint Apply Theorem 7.1 in [1].

- Case 3.  $\text{rank } J_v \tilde{f}(t, x, v, u) < \min\{n, N\}$ : Semi-explicit DAE.

★ Case 3.a). Control is now  $w = (z, u)$ . **z seen as a control**

Mixed constraint Apply 7.1 or hybrid version of it.

★ Case 3.b). Control is  $u$  only. **z seen as a state**

Theorem 6.1 if... Use initial formulation and compensate lack of rank of  $E$ .

*Approach applies also to Nonsmooth Problems*

# Implicit Systems with Additional Constraints

Based on latest developments (Boccia, Bettiol, & Vinter in SICOP 2013 and A. Boccia 2014) Theorem 7.1 and Theorem 6.1 can now be adapted to cover additional state and mixed constraints.

This means we can now treat the problem

Minimize  $l(x(a), x(b))$

subject to

$$0 = f(t, x(t), \dot{x}(t), u(t)) \text{ a.e.}$$

$$0 \geq g(t, x(t), u(t)) \text{ a.e.}$$

$$0 = b(t, x(t), u(t)) \text{ a.e.}$$

$$0 \geq h(t, x(t)) \text{ for all } t$$

$$u(t) \in U \text{ a.e.}$$

$$(x(a), x(b)) \in E$$

← Mixed  
← Mixed  
← State



# Extra Equality State Constraints

Minimize  $l(x(a), x(b))$

subject to

$$0 = f(t, x(t), \dot{x}(t), u(t)) \text{ a.e.}$$

$$0 \geq g(t, x(t), u(t)) \text{ a.e.}$$

$$0 = b(t, x(t), u(t)) \text{ a.e.}$$

$$0 \geq h_1(t, x(t)) \text{ for all } t$$

$$0 = h_2(t, x(t)) \text{ for all } t$$

$$u(t) \in U \text{ a.e.}$$

$$(x(a), x(b)) \in E$$



# How to deal with Equality State Constraints

$$0 = h_2(t, x(t)) \text{ for all } t$$

If we can differentiate  $h$ :

$$h_t + b_x \dot{x}(t) = 0 \quad \xrightarrow{\dot{x}(t)=v} \quad h_t + h_x v = 0$$

Mixed constraint.

Will NCO hold?

Any admissible  $x$  is such that  $x(t) \in h_2^{-1}(0)$ .

We can write  $x(t) \in X(t)$ , where  $X(t) = \{x : (h_1(t, x), h_2(t, x)) \in \mathbf{R}_- \times \{0\}\}$ .

Will NCO hold?

Or we can plug  $x(t) \in h_2^{-1}(0)$  in the definition of local minimum.

No multiplier!!!

Or .... some more ideas?

Thank you  
for  
the attention

Special Thanks to the  
Organizers  
of NETCO 14



# Invitation to meet in Cascais September 8-10, 2014

See information in

<http://paginas.fe.up.pt/~mrpinho/NHOC2014/>