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The Minimum Principle for Delayed Optimal Control Problems with State Constraints

Helmut Maurer¹, Laurenz Göllmann², Richard Vinter³

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New Trends in Optimal Control, Tours, June 23–27, 2014

Challenges for Optimal Control Problems **with Delays**

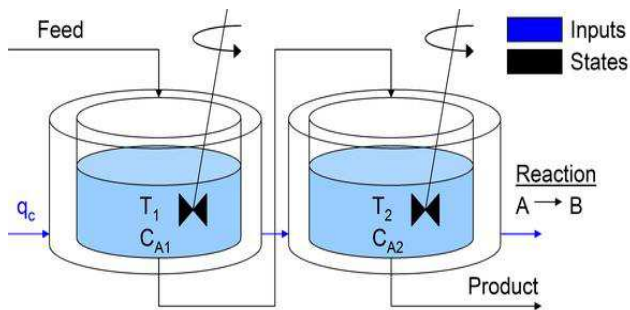
Theory and Numerics for **non-delayed** optimal control problems with control and state constraints are in a mature state-of-art:

- 1 Necessary and sufficient conditions,
- 2 Stability and sensitivity analysis,
- 3 Numerical methods: Boundary value methods, Discretization and NLP, Semismooth Newton methods,
- 4 Real-time control techniques for perturbed extremals.

CHALLENGE: Establish similar theoretical and numerical methods for **delayed (retarded)** optimal control problems.

- 1 **Example:** Two-stage Continuous Stirred Tank Reactor (CSTR)
- 2 Formulation of Optimal Control Problems with Delays in State and Control Variables
- 3 Minimum Principle
- 4 Numerical Treatment: Discretize and Optimize
- 5 **Example:** Optimal Control of the Innate Immune Response

Two-Stage Continuous Stirred Tank Reactor (CSTR)



Time delays are caused by transport between the two tanks.

Delayed Optimal Control Problem with State Constraints

State $x(t) \in \mathbb{R}^n$, Control $u(t) \in \mathbb{R}^m$, Delays $d_x, d_u \geq 0$.

Dynamics and Boundary Conditions

$$\dot{x}(t) = f(x(t), x(t - d_x), u(t), u(t - d_u)), \quad \text{a.e. } t \in [0, t_f],$$

$$x(t) = x_0(t), \quad t \in [-d_x, 0],$$

$$u(t) = u_0(t), \quad t \in [-d_u, 0],$$

$$\psi(x(t_f)) = 0 \in \mathbb{R}^q, \quad (0 \leq q \leq n).$$

Control and State Constraints

$$u(t) \in U \subset \mathbb{R}^m, \quad S(x(t)) \leq 0, \quad t \in [0, t_f] \quad (S : \mathbb{R}^n \rightarrow \mathbb{R}^k).$$

Minimize

$$J(u, x) = g(x(t_f)) + \int_0^{t_f} f_0(x(t), x(t - d_x), u(t), u(t - d_u)) dt$$

Literature on optimal control with time-delays

Time delays in state variables and pure control constraints:

Kharatishvili (1961), Oguztörelı (1966), Banks (1968), Halanay (1968), Soliman, Ray (1970, chemical engineering), Warga (1972, [abstract theory, optimization in Banach spaces](#)), Guinn (1976, [transform delayed problems to standard problems](#)), Colonius, Hinrichsen (1978), Clarke, Wolenski (1991), Dadebo, Luus (1992), Mordukhovich, Wang (2003–).

Time delays in state variables and pure state constraints:

Angell, Kirsch (1990).

State and control delays and mixed control–state constraints:

single delay : Göllmann, Kern, Maurer (OCAM 2009)
multiple delays : Göllmann, Maurer (JIMO 2014)

Optimal control problems with state constraints

Use the [transformation method of Guinn \(1976\)](#) and transform an optimal control problem **with delays** and state constraints to a standard **non-delayed** optimal control problem with state constraints. Then apply the [necessary conditions for non-delayed problems](#):

- Jacobson, Lele, Speyer (1975): KKT conditions in Banach spaces.
- Maurer (1979) : Regularity of multipliers for state constraints.
- Hartl, Sethi, Thomsen (SIAM Review 1995): Survey on Maximum Principles.
- Vinter (2000): (Nonsmooth) Optimal Control

Use methodology for [mixed control-state constraints](#) in

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The smooth case: all functions are differentiable

Hamiltonian (Pontryagin) Function

$$H(x, y, \lambda, u, v) := \lambda_0 f_0(t, x, y, u, v) + \lambda f(t, x, y, u, v)$$

- y variable with $y(t) = x(t - d_x)$
- v variable with $v(t) = u(t - d_u)$
- $\lambda \in \mathbb{R}^n$, $\lambda_0 \in \mathbb{R}$ adjoint (costate) variable

Let $(u, x) \in \mathcal{L}^\infty([0, t_f], \mathbb{R}^m) \times \mathcal{W}^{1,\infty}([0, t_f], \mathbb{R}^n)$ be a locally optimal pair of functions. Then there exist

- an adjoint function $\lambda \in \mathcal{BV}([0, t_f], \mathbb{R}^n)$ and $\lambda_0 \geq 0$,
- a multiplier $\rho \in \mathbb{R}^q$ (associated with terminal conditions),
- a multiplier function (measure) $\mu \in \mathcal{BV}([0, t_f], \mathbb{R}^k)$,

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such that the following conditions are satisfied for a.e. $t \in [0, t_f]$:

Minimum Principle

(i) **Advanced adjoint equation and transversality condition:**

$$\lambda(t) = \int_t^{t_f} (H_x(s) + \chi_{[0, t_f - d_x]}(s) H_y(s + d_x)) ds + \int_t^{t_f} S_x(x(s)) d\mu(s) \\ + (\lambda_0 g + \rho\psi)_x(x(t_f)), \quad (\text{if } S(x(t_f)) < 0)$$

where $H_x(t)$ and $H_y(t + d_x)$ denote evaluations along the optimal trajectory and $\chi_{[0, t_f - d_x]}$ is the **characteristic function**.

(ii) **Minimum Condition:**

$$H(t) + \chi_{[0, t_f - d_u]}(t) H(t + d_u) \\ = \min_{w \in U} [H(x(t), y(t), \lambda(t), w, v(t)) \\ + \chi_{[0, t_f - d_u]}(t) H(t + d_u) H(x(t + d_u), y(t), \lambda(t + d_u), u(t + d_u), w)]$$

(iii) **Multiplier condition and complementarity condition:**

$$d\mu(t) \geq 0, \quad \int_0^{t_f} S(x(t)) d\mu(t) = 0$$

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(iii) **Multiplier condition and complementarity condition:**

$$d\mu(t) \geq 0, \quad \int_0^{t_f} S(x(t)) d\mu(t) = 0$$

Regularity conditions for $d\mu(t) = \eta(t)dt$ if $d_u = 0$

Boundary arc : $S(x(t)) = 0$ for $t_1 \leq t \leq t_2$.

Assumption : $u(t) \in \text{int}(U)$ for $t_1 < t < t_2$.

Under certain **regularity conditions** we have $d\mu(t) = \eta(t) dt$ with a multiplier $\eta(t)$ for all $t_1 < t < t_2$.

Adjoint equation and jump conditions

$$\dot{\lambda}(t) = -H_x(t) - \chi_{[0, t_f - d_x]}(t) H_y(t + d_x) - \eta(t) S_x(x(t))$$

$$\lambda(t_k+) = \lambda(t_k-) - \nu_k S_x(x(t_k)), \quad \nu_k \geq 0$$

at each contact or junction time t_k , $\nu_k = \mu(t_k+) - \mu(t_k-)$

Minimum condition on the boundary

$$H_u(t) = 0.$$

This condition allows to compute the multiplier $\eta = \eta(x, \lambda)$.

Numerical Treatment: Discretize and Optimize

- 1 Choose a **suitable stepsize** $h = t_f/N$, $N \in \mathbf{N}$, adapted to the delays d_x, d_y (commensurability). Use the implicit or explicit Euler integration scheme for the dynamic equations and define the **associated NLP** having

$$u(t_i) \approx u_i \in \mathbb{R}^m, \quad x(t_i) \approx x_i \in \mathbb{R}^n, \quad i = 0, \dots, N$$

as decision variables

- 2 Apply **NLP-Solvers** (eg. IPOPT, LOQO embedded in AMPL source code) to solve the NLP
- 3 The associated **Lagrange multipliers** ($\hat{\lambda}_i$) for the discretized dynamic equations and state constraint ($\hat{\mu}_i$) give a **consistent approximation of the costate and the multiplier function**

$$\hat{\lambda}(t_i) \approx \hat{\lambda}_i, \quad \hat{\eta}(t_i) \approx \frac{1}{h} \hat{\eta}_i \quad (i = 0, \dots, N), \quad \hat{v} \approx \hat{v}_N.$$

Göllmann, Kern, Maurer (2009) for two single state/control delays,
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Model of the immune response

Dynamic model of the immune response:

Asachenko A, Marchuk G, Mohler R, Zuev S,
Disease Dynamics, Birkhäuser, Boston, 1994.

Optimal control:

Stengel RF, Ghigliazza R, Kulkarni N, Laplace O,
Optimal control of innate immune response,
Optimal Control Applications and Methods **23**, 91–104 (2002),

Simultaneous optimization of all control variables and delays:

L. Göllmann, H. Maurer: Optimal control problems with multiple
time-delays, JIMO 2014.

Here we consider a more realistic **modified optimal control problem**.

Innate Immune Response: state and control variables

State variables:

- $x_1(t)$: concentration of **pathogen**
(=concentration of associated **antigen**)
- $x_2(t)$: concentration of **plasma cells**,
which are carriers and producers of antibodies
- $x_3(t)$: concentration of **antibodies**, which kill the pathogen
(=concentration of **immunoglobulins**)
- $x_4(t)$: relative characteristic of a **damaged organ**
(**0 = healthy, 1 = dead**)

Control variables:

- $u_1(t)$: pathogen killer
- $u_2(t)$: plasma cell enhancer
- $u_3(t)$: antibody enhancer
- $u_4(t)$: organ healing factor

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Generic dynamical model of the immune response

$$\dot{x}_1(t) = (1 - x_3(t))x_1(t) - u_1(t - d_u)x_1(t),$$

$$\dot{x}_2(t) = 3A(x_4(t))x_1(t - d_x)x_3(t - d_x) - (x_2(t) - 2) + u_2(t)x_2(t),$$

$$\dot{x}_3(t) = x_2(t) - (1.5 + 0.5x_1(t))x_3(t) + u_3(t)x_3(t),$$

$$\dot{x}_4(t) = x_1(t) - x_4(t) - u_4(t)x_4(t).$$

Immune deficiency function triggered by target organ damage

$$A(x_4) = \left\{ \begin{array}{ll} \cos(\pi x_4), & 0 \leq x_4 \leq 0.5 \\ 0 & 0.5 \leq x_4 \end{array} \right\}.$$

For $0.5 \leq x_4(t)$ the production of plasma cells stops.

State delay $d_x \geq 0$ in variables x_1 and x_3 .

Control delay $d_u \geq 0$ in variable u_1 .

Initial conditions ($d = 0$): $x_2(0) = 2$, $x_3(0) = 4/3$, $x_4(0) = 0$

Case 1 : $x_1(0) = 1.5$, decay, requires no therapy (control)

Case 2 : $x_1(0) = 2.0$, slower decay, requires no therapy

Case 3 : $x_1(0) = 3.0$, diverges without control (lethal case)

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Optimal control model: cost functional

L^2 -functional quadratic in control: Stengel et al.

$$\begin{aligned} \text{Minimize } J_2(x, u) &= x_1(t_f)^2 + x_4(t_f)^2 \\ &+ \int_0^{t_f} (x_1^2 + x_4^2 + u_1^2 + u_2^2 + u_3^2 + u_4^2) dt \end{aligned}$$

Control constraints: $0 \leq u_i(t) \leq u_{\max} = 1, i = 1, \dots, 4$

L^1 -functional linear in control

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State Constraint

As result of organ damage triggered **Immune deficiency function**

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with ceasing plasma cells production for $0.5 \leq x_4(t)$, we impose

State Constraint

$x_4(t) \leq \alpha$ with threshold $\alpha < 0.5$ i.e.

$$S(x_4(t)) := x_4(t) - \alpha \leq 0$$

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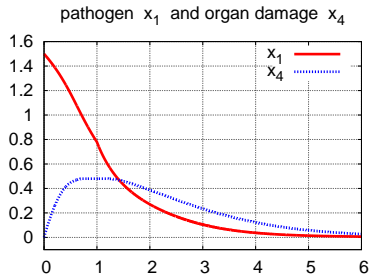
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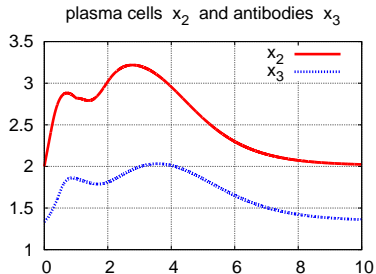
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Results for L^2 -functional, $d_x = d_u = 1.0$, $\alpha = 4.8$

Optimal state variables:



pathogen concentration
organ damage

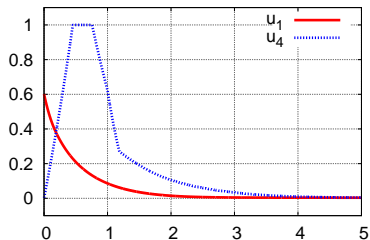


plasma cell concentration
immunoglobulins

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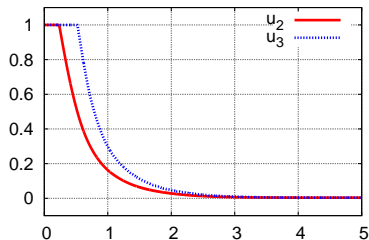
Optimal control variables:

pathogen killer u_1 and organ healing factor u_4



pathogen killer
organ healing factor

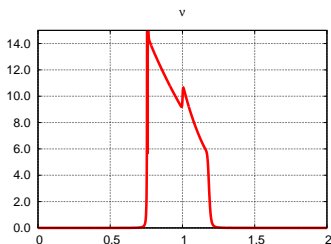
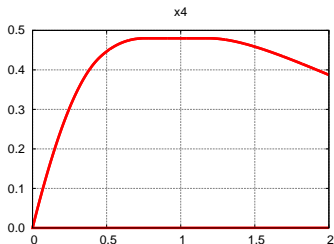
plasma cell enhancer u_2 , antibody enhancer u_3



plasma cell enhancer
antibody enhancer

Results for L^2 -functional, $d_x = d_u = 1.0$, $\alpha = 4.8$

Zoom: Optimal state x_4 : Boundary arc observed



organ healing factor

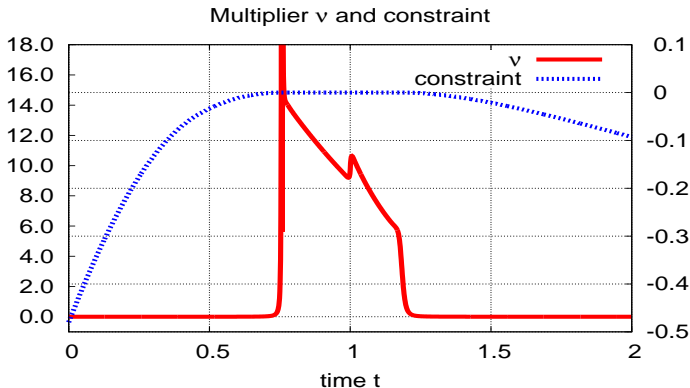
multiplier

First order necessary conditions, in particular $H_{u_4} = 0$, yield the multiplier on boundary arc $x_4(t) = \alpha$:

$$\eta(x, y, v_1, \lambda) = -2\alpha + 3\pi\lambda_2 \sin(\pi\alpha)y_1y_2 + \lambda_4 \frac{x_1}{\alpha} - \frac{2}{\alpha^2}((1-x_3)x_1 - v_1x_1)$$

Results for L^2 -functional, $d_x = d_u = 1.0$, $\alpha = 4.8$

Zoom into boundary arc $x_4(t) = 4.8$:

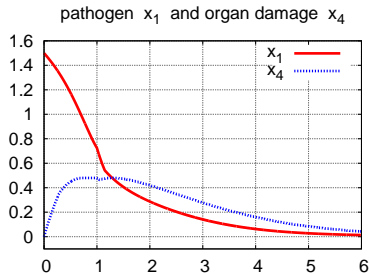


right axis: constraint $S(x_4) - \alpha \leq 0$.

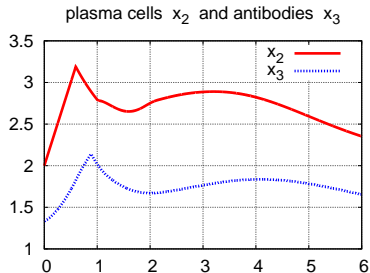
left axis: corresponding multiplier $\eta(t)$.

Results for L^1 -functional, $d_x = d_u = 1.0$, $\alpha = 4.8$

Optimal state variables:



pathogen concentration
organ damage

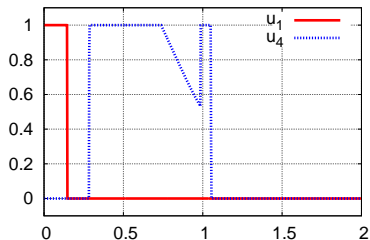


plasma cell concentration
immunoglobulins

Results for L^1 -functional, $d_x = d_u = 1.0$, $\alpha = 4.8$

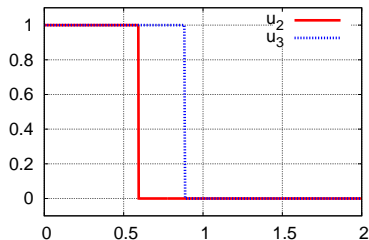
Optimal control variables:

pathogen killer u_1 and organ healing factor u_4



pathogen killer
organ healing factor

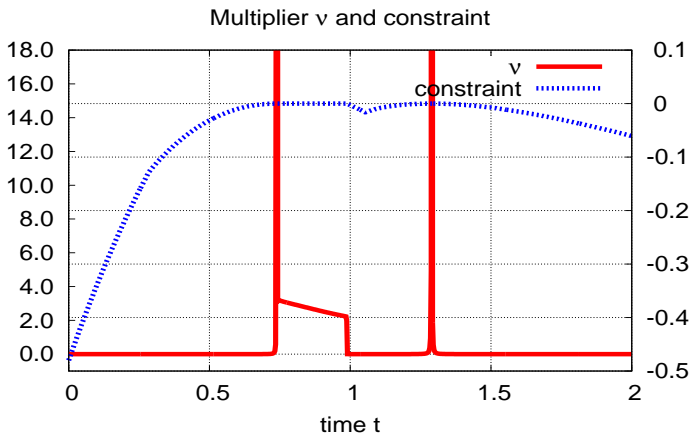
plasma cell enhancer u_2 , antibody enhancer u_3



plasma cell enhancer
antibody enhancer

Results for L^1 -functional, $d_x = d_u = 1.0$, $\alpha = 4.8$

Zoom into boundary arc $x_4(t) = 4.8$:

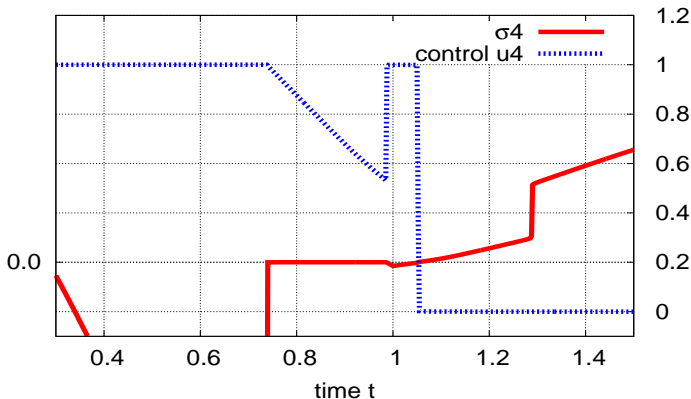


overlay: constraint $S(x_4) - \alpha \leq 0$ (right axis)
and corresponding multiplier $\eta(t)$ (left axis)

Results for L^1 -functional, $d_x = d_u = 1.0$, $\alpha = 4.8$

Zoom into boundary arc $x_4(t) = 4.8$:

switching function σ_4 and healing factor u_4



right axis: control u_4 .

left axis: corresponding switching function $\sigma(t) = \frac{\partial H}{\partial u_4}$.

Further applications and future work

- 1 Optimal investment and dividend decisions of a firm
- 2 Expediting the transition from non-renewable to renewable energy (Maurer, Semmler)
- 3 Biomedical applications: optimal protocols in cancer treatment and immunology, optimal tapering of steroids
- 4 Verifiable sufficient conditions
- 5 Time and state dependent delays



Thank you for your attention !