HAL
open science

Propagation of singularities for semiconcave solutions of Hamilton-Jacobi equations

Marco Mazzola, Piermarco Cannarsa, Carlo Sinestrari

To cite this version:

Marco Mazzola, Piermarco Cannarsa, Carlo Sinestrari. Propagation of singularities for semiconcave solutions of Hamilton-Jacobi equations. NETCO 2014 - New Trends in Optimal Control, Jun 2014, Tours, France. hal-01024745

HAL Id: hal-01024745

https://inria.hal.science/hal-01024745v1

Submitted on 18 Jul 2014

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Propagation of Singularities for Semiconcave Solutions of Hamilton-Jacobi Equations

Marco Mazzola

Université Pierre et Marie Curie (Paris 6)

P. Cannarsa and C. Sinestrari

NetCo 2014, Tours, 23-27 june 2014

Outline of the talk

Local propagation of singularities

Minimizing generalized characteristics

Global propagation of singularities

Semiconcave functions

Let $Q \subset \boldsymbol{R}^N$ be open and $u : Q \to \boldsymbol{R}$ be a semiconcave function with semiconcavity constant C, i.e.

$$\lambda u(X) + (1-\lambda)u(Y) - u(\lambda X + (1-\lambda)Y) \leq C\lambda(1-\lambda)\frac{|X-Y|^2}{2}$$

for every $\lambda \in [0,1]$ and $X, Y \in Q$ such that $[X, Y] \subset Q$.

Denote by $\Sigma(u)$ the set of points where u is not differentiable.

We are interested in the structure of the set $\Sigma(u)$.

Generalized gradients

The set of limiting gradients and the superdifferential of u at $X \in Q$ are defined respectively by

$$D^* u(X) = \left\{ P \in \boldsymbol{R}^N \,:\, Q \setminus \Sigma(u) \ni X_i \to X,\ Du(X_i) \to P \right\}$$

and

$$D^+ u(X) = \left\{ P \in \boldsymbol{R}^N \,:\, \limsup_{Y \to X} \frac{u(Y) - u(X) - \langle P, Y - X \rangle}{|Y - X|} \leq 0 \right\}.$$

If u is semiconcave with constant C, then

- $$D^+ u(X) = \text{co}\ D^* u(X)\,;$$

- $$\langle P - Q, X - Y \rangle \leq C|X - Y|^2$$

  for every $P \in D^+ u(X)$, $Q \in D^+ u(Y)$ and $X, Y \in Q$ such that $[X, Y] \subset Q$.

Local propagation for semiconcave functions

Theorem (Albano - Cannarsa (1999))

Let u be semiconcave and $X_0 \in \Sigma(u)$. If

$$\partial D^+ u(X_0) \setminus D^* u(X_0) \neq \emptyset\,,$$

then there exist $T > 0$ and a nonconstant Lipschitz continuous arc $\xi : [0, T] \to Q$ such that $\xi(0) = X_0$ and $\xi(t) \in \Sigma(u)$ for all $t \in [0, T]$.

Local propagation for semiconcave functions

Theorem (Albano - Cannarsa (1999))

Let u be semiconcave and $X_0 \in \Sigma(u)$. If

$$\partial D^+ u(X_0) \setminus D^* u(X_0) \neq \emptyset\,,$$

then there exist $T > 0$ and a nonconstant Lipschitz continuous arc $\xi : [0, T] \to Q$ such that $\xi(0) = X_0$ and $\xi(t) \in \Sigma(u)$ for all $t \in [0, T]$.

When u is the solution of a Hamilton-Jacobi equation, the condition above turns out to be necessary and sufficient for the propagation of the singularity at X_0.

Local propagation for solutions of HJ equations

$$F(Du(X)) = 0 \qquad \text{a.e. } X \in Q$$

Let $F \in C^1(\boldsymbol{R}^N)$ satisfy

- F is convex;
- the sublevel sets of u are strictly convex.

Theorem (Albano - Cannarsa (2002), Cannarsa - Yu (2009))

Let u be a semiconcave solution of

$$F(Du(X)) = 0 \qquad a.e.\ X \in Q$$

and $X_0 \in \Sigma(u)$. If

$$0 \notin DF(D^+u(X_0)),$$

then there exist $T > 0$ and a nonconstant Lipschitz continuous arc $\xi : [0, T] \to Q$ such that $\xi(0) = X_0$ and $\xi(t) \in \Sigma(u)$ for all $t \in [0, T]$.

Evolutionary HJ equation

Let $N = 1 + n$.
Consider

$$F(P) = \tau + H(p)$$

with $P = (\tau, p) \in \boldsymbol{R} \times \boldsymbol{R}^n$ and $Q = (0, +\infty) \times \Omega$, $\Omega \subset \boldsymbol{R}^n$.
This case correspond to the HJ equation

$$\begin{cases} u_t(t,x) + H(\nabla u(t,x)) = 0 & \text{a.e. } (t,x) \in Q \\ u(t,x) = \varphi(t,x) & \text{for } (t,x) \in \partial Q\,. \end{cases}$$

Suppose that $\varphi : \bar{Q} \to \boldsymbol{R}$ is continuous and its restriction to $\{0\} \times \Omega$ is Lipschitz continuous.

Hopf representation formula

$$u(t,x) = \min_{\substack{(s,y) \in \partial Q \\ s < t}} \left[(t-s)\, H^* \left(\frac{x-y}{t-s} \right) + \varphi(s,y) \right].$$

Evolutionary HJ equation

Question: Is it possible to propagate singularities globally in time? i.e., given $X_0 = (t_0, x_0) \in \Sigma(u)$, does a Lipschitz continuous arc $\gamma : [0, +\infty) \to \boldsymbol{R}^n$ exist such that $\gamma(0) = x_0$ and

$$(t_0 + s, \gamma(s)) \in \Sigma(u) \qquad \forall\, s \in [0, +\infty)\,?$$

Evolutionary HJ equation

Question: Is it possible to propagate singularities globally in time? i.e., given $X_0 = (t_0, x_0) \in \Sigma(u)$, does a Lipschitz continuous arc $\gamma : [0, +\infty) \to \boldsymbol{R}^n$ exist such that $\gamma(0) = x_0$ and

$$(t_0 + s, \gamma(s)) \in \Sigma(u) \qquad \forall\, s \in [0, +\infty)\,?$$

Yes, in the case $n = 1$ (Dafermos, 1977).

Outline of the talk

Local propagation of singularities

Minimizing generalized characteristics

Global propagation of singularities

Characteristics

Consider the Hamilton-Jacobi equation

$$(HJ) \qquad F(Du(X)) = 0 \qquad X \in Q\,,$$

with F of class C^1 and convex.

If $u \in C^1(Q)$ is a classical solution of (HJ), a curve $\xi : [0, T] \to Q$ is a characteristic if it satisfies

$$\xi'(t) = DF(Du(\xi(t))) \qquad \text{a.e. } t \in [0, T]\,.$$

Generalized characteristics

Let u be a semiconcave solution of (HJ).

A Lipschitz continuous curve $\xi : [0, T] \to Q$ is a generalized characteristic if it satisfies

$$\xi'(t) \in \text{co } DF(D^+u(\xi(t))) \qquad \text{a.e. } t \in [0, T].$$

The proof of (Dafermos, 1977) crucially depends on the dimension 1.

Minimizing generalized characteristics

Notice that $F(P) = 0$ for every limiting gradient P.
Consequently, $X \in \Sigma(u)$ if and only if

$$\min_{P \in D^+u(X)} F(P) < 0\,.$$

Sufficient conditions are provided in the literature for the existence of generalized characteristics that are "energy minimizing" (Cannarsa - Yu, 2009), (Stromberg, 2013).

Minimizing generalized characteristics

Theorem (Cannarsa - Yu (2009))

If for any $X_0 \in Q$ there exists a unique generalized characteristic starting from X_0, then any generalized characteristic $\xi : [0, T_0) \to Q$ admits right derivative $\dot{\xi}^+(s)$ for all $s \in [0, T_0)$, this is right-continuous and is given by

$$\dot{\xi}^+(s) = DF(P(s)),$$

where $P(s) \in D^+u(\xi(s))$ is such that

$$F(P(s)) \leq F(P) \qquad \forall\, P \in D^+u(\xi(s)).$$

Evolutionary HJ equation

In the case of the evolutionary HJ equation

$$F(P) = \tau + \frac{1}{2} Ap \cdot p$$

with $P = (\tau, p) \in \boldsymbol{R} \times \boldsymbol{R}^n$, $Q = (0, +\infty) \times \Omega$, $\Omega \subset \boldsymbol{R}^n$ and A positive definite, the uniqueness of generalized characteristics, given the initial data, is a consequence of Gronwall's Lemma.

Following (Cannarsa - Yu, 2009), it is possible to propagate a singularity X_0 locally along the minimizing generalized characterstic

$$\dot{\xi}^+(s) = DF(\tau(s), p(s)) = \begin{pmatrix} 1 \\ Ap(s) \end{pmatrix}, \qquad \xi(0) = X_0 \,.$$

Outline of the talk

Local propagation of singularities

Minimizing generalized characteristics

Global propagation of singularities

Global propagation of singularities

Question: Is it possible to propagate singularities globally in time along generalized characteristics?

So far, besides (Dafermos, 1977), an affirmative answer has been given only in a few particular cases.

The eikonal equation

$$\begin{cases} |Du(X)|^2 = 1 & \text{a.e. } X \in Q \\ u(X) = 0 & \text{on } \partial Q \end{cases}$$

The unique nonnegative viscosity solution is $u = d_{\mathbf{R}^N \setminus Q}$.

Theorem (Albano - Cannarsa - Khai Nguyen - Sinestrari (2013))

Let $X_0 \in Q$. There exists a unique Lipschitz continuous solution of

$$\xi'(t) \in D^+u(\xi(t)) \qquad a.e.\ t \in [0, +\infty), \qquad \xi(0) = X_0\,.$$

Moreover, if $X_0 \in \Sigma(u)$ then $\xi(t) \in \Sigma(u)$ for all $t \in [0, +\infty)$.

The eikonal equation

- Application: Q is homotopically equivalent to $\Sigma(u)$.
- The result holds true on riemannian manifolds.

The eikonal equation

- Application: Q is homotopically equivalent to $\Sigma(u)$.
- The result holds true on riemannian manifolds.

The proof strongly relies on the semiconcavity of u^2 with constant 2. This yields

$$\langle u(X)P - u(Y)Q, X - Y\rangle \leq |X - Y|^2$$

for every $P \in D^+u(X)$, $Q \in D^+u(Y)$ and $X, Y \in \boldsymbol{R}^n$.

$$u_t(t,x) + \frac{|\nabla u(t,x)|^2}{2} = 0$$

Fixed $t > 0$, the function $u(t,\cdot)$ is semiconcave with constant $\frac{1}{t}$. This implies

$$\langle p - q, x - y \rangle \leq \frac{|x-y|^2}{t}$$

for every $p \in \nabla^+ u(t,x)$, $q \in \nabla^+ u(t,y)$ and $x, y \in \boldsymbol{R}^n$.
We need an estimate on the monotonicity of D^+u jointly in time and space.

$$u_t(t,x) + \frac{|\nabla u(t,x)|^2}{2} = 0$$

Fixed $t > 0$, the function $u(t,\cdot)$ is semiconcave with constant $\frac{1}{t}$. This implies

$$\langle p - q, x - y \rangle \leq \frac{|x-y|^2}{t}$$

for every $p \in \nabla^+ u(t,x)$, $q \in \nabla^+ u(t,y)$ and $x, y \in \boldsymbol{R}^n$.
We need an estimate on the monotonicity of $D^+ u$ jointly in time and space.

Using the Hopf formula, we find

$$\left\langle \begin{pmatrix} \tau - \sigma \\ p - q \end{pmatrix}, \begin{pmatrix} t - s \\ x - y \end{pmatrix} \right\rangle \leq \frac{|x-y|^2}{t}$$

$$-\frac{(t-s)^2}{t}\sigma - \frac{t-s}{t}[\langle p, x-y\rangle + u(t,x) - u(s,y)]$$

$\forall\, (\tau, p) \in D^+ u(t,x)$, $(\sigma, q) \in D^+ u(s,y)$ and $t, s > 0$, $x, y \in \boldsymbol{R}^n$.

$$u_t + H(\nabla u) = 0 \text{ a.e. } (t,x) \in (0,+\infty)\times\Omega =: Q, \quad H(p) = \frac{1}{2}Ap\cdot p$$

For any $(t_0, x_0) \in Q$ there exist $T_0 > 0$ and a Lipschitz continuous arc $\xi : [0, T_0) \to Q$ such that $\xi(0) = (t_0, x_0)$, the right derivative $\dot{\xi}^+(s)$ does exist for all $s \in [0, T_0)$, it is right-continuous and satisfies

$$\dot{\xi}^+(s) = DF(\tau(s), p(s)) = \begin{pmatrix} 1 \\ \nabla H(p(s)) \end{pmatrix},$$

where $(\tau(s), p(s)) \in D^+u(\xi(s))$ is such that

$$F(\tau(s), p(s)) \leq F(\tau, p) \qquad \forall\, (\tau, p) \in D^+u(\xi(s)).$$

Using the monotonicty above, it is possible to obtain (formally)

$$\frac{d}{ds}F(\tau(s), p(s)) \leq -\frac{2}{s}F(\tau(s), p(s)).$$

Local propagation of singularities

$$u_t + H(\nabla u) = 0 \text{ a.e. } (t,x) \in (0,+\infty)\times\Omega =: Q, \quad H(p) = \frac{1}{2}Ap\cdot p$$

Theorem

Let $(t_0, x_0) \in Q$ and $\bar{t} < t_0$ be such that

$$u(t,x) = \min_{y\in\Omega}\left[(t-\bar{t})H^*\left(\frac{x-y}{t-\bar{t}}\right) + u(\bar{t},y)\right]$$

There exist $T_1 > 0$ and a Lipschitz continuous arc $\gamma : [0, T_1) \to \Omega$ starting from x_0 and such that

$$\min_{(\tau,p)\in D^+u(t_0+s,\gamma(s))} F(\tau,p) \le \left(\frac{t_0-\bar{t}}{t_0+s-\bar{t}}\right)^2 \min_{(\tau_0,p_0)\in D^+u(t_0,x_0)} F(\tau_0,p_0)$$

for every $s \in [0, T_1)$.

Example

The estimate above is somehow mild:

Let $n = 1$, $A = Id$, $\Omega = \boldsymbol{R}^N$ and $\varphi(0, x) = \frac{(|x|-1)^2}{2\varepsilon}$.

The Hopf formula yields

$$u(t, x) = \frac{1}{2} \frac{(|x| - 1)^2}{t + \varepsilon} .$$

We obtain

$$\arg\min_{(\tau,p)\in D^+u(t_0+s,\gamma(s))} F(\tau, p) = \left\{ \left(-\frac{1}{2(t_0 + s + \varepsilon)^2}, 0 \right) \right\}$$

$$\frac{\min_{(\tau,p)\in D^+u(t_0+s,\gamma(s))} F(\tau, p)}{\min_{(\tau_0,p_0)\in D^+u(t_0,0)} F(\tau_0, p_0)} = \left(\frac{t_0 + \varepsilon}{t_0 + s + \varepsilon} \right)^2$$

Global propagation of singularities

A maximality argument yields the global propagation:

Theorem

Let (t_0, x_0) be a singular point of u. Then there exist $T \in (0, +\infty]$ and a Lipschitz continuous arc $\gamma : [0, T) \to \boldsymbol{R}^n$ starting from x_0, satisfying

$$(t_0 + s, \gamma(s)) \in \Sigma(u) \qquad \forall\, s \in [0, T)$$

and such that $\lim_{s \to T} \gamma(s) \in \partial\Omega$ whenever $T < +\infty$.

Future directions

- Generalize to semiconcave functions on riemannian manifolds: up to now, we can prove a global propagation on manifolds with nonnegative curvature;
- Generalize to the case of more complex structures of the function H.

Future directions

- Generalize to semiconcave functions on riemannian manifolds: up to now, we can prove a global propagation on manifolds with nonnegative curvature;
- Generalize to the case of more complex structures of the function H.

...the end.
Thank you!