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# Propagation of Singularities for Semiconcave Solutions of Hamilton-Jacobi Equations

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# Outline of the talk

Local propagation of singularities

Minimizing generalized characteristics

Global propagation of singularities

## Semiconcave functions

Let  $Q \subset \mathbf{R}^N$  be open and  $u : Q \rightarrow \mathbf{R}$  be a **semiconcave** function with semiconcavity constant  $C$ , i.e.

$$\lambda u(X) + (1 - \lambda)u(Y) - u(\lambda X + (1 - \lambda)Y) \leq C\lambda(1 - \lambda) \frac{|X - Y|^2}{2}$$

for every  $\lambda \in [0, 1]$  and  $X, Y \in Q$  such that  $[X, Y] \subset Q$ .

Denote by  $\Sigma(u)$  the set of points where  $u$  is not differentiable.

We are interested in the structure of the set  $\Sigma(u)$ .

## Generalized gradients

The set of **limiting gradients** and the **superdifferential** of  $u$  at  $X \in Q$  are defined respectively by

$$D^*u(X) = \left\{ P \in \mathbb{R}^N : Q \setminus \Sigma(u) \ni X_i \rightarrow X, Du(X_i) \rightarrow P \right\}$$

and

$$D^+u(X) = \left\{ P \in \mathbb{R}^N : \limsup_{Y \rightarrow X} \frac{u(Y) - u(X) - \langle P, Y - X \rangle}{|Y - X|} \leq 0 \right\}.$$

If  $u$  is semiconcave with constant  $C$ , then

- 

$$D^+u(X) = \text{co } D^*u(X);$$

- 

$$\langle P - Q, X - Y \rangle \leq C|X - Y|^2$$

for every  $P \in D^+u(X)$ ,  $Q \in D^+u(Y)$  and  $X, Y \in Q$  such that  $[X, Y] \subset Q$ .

# Local propagation for semiconcave functions

Theorem (Albano - Cannarsa (1999))

Let  $u$  be semiconcave and  $X_0 \in \Sigma(u)$ . If

$$\partial D^+ u(X_0) \setminus D^* u(X_0) \neq \emptyset,$$

then there exist  $T > 0$  and a nonconstant Lipschitz continuous arc  $\xi : [0, T] \rightarrow Q$  such that  $\xi(0) = X_0$  and  $\xi(t) \in \Sigma(u)$  for all  $t \in [0, T]$ .

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When  $u$  is the **solution of a Hamilton-Jacobi equation**, the condition above turns out to be necessary and sufficient for the propagation of the singularity at  $X_0$ .

## Local propagation for solutions of HJ equations

$$F(Du(X)) = 0 \quad \text{a.e. } X \in Q$$

Let  $F \in C^1(\mathbb{R}^N)$  satisfy

- $F$  is **convex**;
- the sublevel sets of  $u$  are **strictly convex**.

Theorem (Albano - Cannarsa (2002), Cannarsa - Yu (2009))

Let  $u$  be a semiconcave solution of

$$F(Du(X)) = 0 \quad \text{a.e. } X \in Q$$

and  $X_0 \in \Sigma(u)$ . If

$$0 \notin DF(D^+u(X_0)),$$

then there exist  $T > 0$  and a nonconstant Lipschitz continuous arc  $\xi : [0, T] \rightarrow Q$  such that  $\xi(0) = X_0$  and  $\xi(t) \in \Sigma(u)$  for all  $t \in [0, T]$ .



## Evolutionary HJ equation

Let  $N = 1 + n$ .

Consider

$$F(P) = \tau + H(p)$$

with  $P = (\tau, p) \in \mathbf{R} \times \mathbf{R}^n$  and  $Q = (0, +\infty) \times \Omega$ ,  $\Omega \subset \mathbf{R}^n$ .

This case correspond to the HJ equation

$$\begin{cases} u_t(t, x) + H(\nabla u(t, x)) = 0 & \text{a.e. } (t, x) \in Q \\ u(t, x) = \varphi(t, x) & \text{for } (t, x) \in \partial Q. \end{cases}$$

Suppose that  $\varphi : \bar{Q} \rightarrow \mathbf{R}$  is continuous and its restriction to  $\{0\} \times \Omega$  is Lipschitz continuous.

### Hopf representation formula

$$u(t, x) = \min_{\substack{(s, y) \in \partial Q \\ s < t}} \left[ (t - s) H^* \left( \frac{x - y}{t - s} \right) + \varphi(s, y) \right].$$

## Evolutionary HJ equation

**Question:** Is it possible to propagate singularities globally in time?  
i.e., given  $X_0 = (t_0, x_0) \in \Sigma(u)$ , does a Lipschitz continuous arc  
 $\gamma : [0, +\infty) \rightarrow \mathbf{R}^n$  exist such that  $\gamma(0) = x_0$  and

$$(t_0 + s, \gamma(s)) \in \Sigma(u) \quad \forall s \in [0, +\infty)?$$

## Evolutionary HJ equation

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$$(t_0 + s, \gamma(s)) \in \Sigma(u) \quad \forall s \in [0, +\infty)?$$

Yes, in the case  $n = 1$  (Dafermos, 1977).

# Outline of the talk

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# Characteristics

Consider the Hamilton-Jacobi equation

$$(HJ) \quad F(Du(X)) = 0 \quad X \in Q,$$

with  $F$  of class  $C^1$  and convex.

If  $u \in C^1(Q)$  is a classical solution of (HJ), a curve  $\xi : [0, T] \rightarrow Q$  is a **characteristic** if it satisfies

$$\xi'(t) = DF(Du(\xi(t))) \quad \text{a.e. } t \in [0, T].$$

# Generalized characteristics

Let  $u$  be a semiconcave solution of (HJ).

A Lipschitz continuous curve  $\xi : [0, T] \rightarrow Q$  is a **generalized characteristic** if it satisfies

$$\xi'(t) \in \text{co } DF(D^+ u(\xi(t))) \quad \text{a.e. } t \in [0, T].$$

The proof of (Dafermos, 1977) crucially depends on the dimension 1.

# Minimizing generalized characteristics

Notice that  $F(P) = 0$  for every limiting gradient  $P$ .  
Consequently,  $X \in \Sigma(u)$  if and only if

$$\min_{P \in D^+u(X)} F(P) < 0.$$

Sufficient conditions are provided in the literature for the existence of generalized characteristics that are "energy minimizing" (Cannarsa - Yu, 2009), (Stromberg, 2013).

# Minimizing generalized characteristics

## Theorem (Cannarsa - Yu (2009))

If for any  $X_0 \in Q$  there exists a *unique generalized characteristic* starting from  $X_0$ , then any generalized characteristic  $\xi : [0, T_0) \rightarrow Q$  admits right derivative  $\dot{\xi}^+(s)$  for all  $s \in [0, T_0)$ , this is right-continuous and is given by

$$\dot{\xi}^+(s) = DF(P(s)),$$

where  $P(s) \in D^+u(\xi(s))$  is such that

$$F(P(s)) \leq F(P) \quad \forall P \in D^+u(\xi(s)).$$



## Evolutionary HJ equation

In the case of the evolutionary HJ equation

$$F(P) = \tau + \frac{1}{2}Ap \cdot p$$

with  $P = (\tau, p) \in \mathbf{R} \times \mathbf{R}^n$ ,  $Q = (0, +\infty) \times \Omega$ ,  $\Omega \subset \mathbf{R}^n$  and  $A$  positive definite, the uniqueness of generalized characteristics, given the initial data, is a consequence of Gronwall's Lemma.

Following (Cannarsa - Yu, 2009), it is possible to propagate a singularity  $X_0$  locally along the minimizing generalized characteristic

$$\dot{\xi}^+(s) = DF(\tau(s), p(s)) = \begin{pmatrix} 1 \\ Ap(s) \end{pmatrix}, \quad \xi(0) = X_0.$$

# Outline of the talk

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# Global propagation of singularities

**Question:** Is it possible to propagate singularities globally in time along generalized characteristics?

So far, besides ([Dafermos, 1977](#)), an affirmative answer has been given only in a few particular cases.

# The eikonal equation

$$\begin{cases} |Du(X)|^2 = 1 & \text{a.e. } X \in Q \\ u(X) = 0 & \text{on } \partial Q \end{cases}$$

The unique nonnegative viscosity solution is  $u = d_{R^N \setminus Q}$ .

Theorem (Albano - Cannarsa - Khai Nguyen - Sinestrari (2013))

Let  $X_0 \in Q$ . There exists a unique Lipschitz continuous solution of

$$\xi'(t) \in D^+ u(\xi(t)) \quad \text{a.e. } t \in [0, +\infty), \quad \xi(0) = X_0.$$

Moreover, if  $X_0 \in \Sigma(u)$  then  $\xi(t) \in \Sigma(u)$  for all  $t \in [0, +\infty)$ .

# The eikonal equation

- **Application:**  $Q$  is homotopically equivalent to  $\Sigma(u)$ .
- The result holds true on riemannian manifolds.

# The eikonal equation

- **Application:**  $Q$  is homotopically equivalent to  $\Sigma(u)$ .
- The result holds true on riemannian manifolds.

The proof strongly relies on the semiconcavity of  $u^2$  with constant 2. This yields

$$\langle u(X)P - u(Y)Q, X - Y \rangle \leq |X - Y|^2$$

for every  $P \in D^+u(X)$ ,  $Q \in D^+u(Y)$  and  $X, Y \in \mathbf{R}^n$ .

$$u_t(t, x) + \frac{|\nabla u(t, x)|^2}{2} = 0$$

Fixed  $t > 0$ , the function  $u(t, \cdot)$  is semiconcave with constant  $\frac{1}{t}$ .  
This implies

$$\langle p - q, x - y \rangle \leq \frac{|x - y|^2}{t}$$

for every  $p \in \nabla^+ u(t, x)$ ,  $q \in \nabla^+ u(t, y)$  and  $x, y \in \mathbf{R}^n$ .

We need an estimate on the monotonicity of  $D^+ u$  jointly in time and space.

$$u_t(t, x) + \frac{|\nabla u(t, x)|^2}{2} = 0$$

Fixed  $t > 0$ , the function  $u(t, \cdot)$  is semiconcave with constant  $\frac{1}{t}$ .  
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We need an estimate on the monotonicity of  $D^+ u$  jointly in time and space.

Using the Hopf formula, we find

$$\left\langle \begin{pmatrix} \tau - \sigma \\ p - q \end{pmatrix}, \begin{pmatrix} t - s \\ x - y \end{pmatrix} \right\rangle \leq \frac{|x - y|^2}{t} - \frac{(t - s)^2}{t} \sigma - \frac{t - s}{t} [\langle p, x - y \rangle + u(t, x) - u(s, y)]$$

$\forall (\tau, p) \in D^+ u(t, x)$ ,  $(\sigma, q) \in D^+ u(s, y)$  and  $t, s \geq 0$ ,  $x, y \in \mathbf{R}^n$ .



$$u_t + H(\nabla u) = 0 \quad \text{a.e. } (t, x) \in (0, +\infty) \times \Omega =: Q, \quad H(p) = \frac{1}{2} A p \cdot p$$

For any  $(t_0, x_0) \in Q$  there exist  $T_0 > 0$  and a Lipschitz continuous arc  $\xi : [0, T_0) \rightarrow Q$  such that  $\xi(0) = (t_0, x_0)$ , the right derivative  $\dot{\xi}^+(s)$  does exist for all  $s \in [0, T_0)$ , it is right-continuous and satisfies

$$\dot{\xi}^+(s) = DF(\tau(s), p(s)) = \begin{pmatrix} 1 \\ \nabla H(p(s)) \end{pmatrix},$$

where  $(\tau(s), p(s)) \in D^+ u(\xi(s))$  is such that

$$F(\tau(s), p(s)) \leq F(\tau, p) \quad \forall (\tau, p) \in D^+ u(\xi(s)).$$

Using the monotonicity above, it is possible to obtain (formally)

$$\frac{d}{ds} F(\tau(s), p(s)) \leq -\frac{2}{s} F(\tau(s), p(s)).$$

## Local propagation of singularities

$$u_t + H(\nabla u) = 0 \text{ a.e. } (t, x) \in (0, +\infty) \times \Omega =: Q, \quad H(p) = \frac{1}{2} A p \cdot p$$

### Theorem

Let  $(t_0, x_0) \in Q$  and  $\bar{t} < t_0$  be such that

$$u(t, x) = \min_{y \in \Omega} \left[ (t - \bar{t}) H^* \left( \frac{x - y}{t - \bar{t}} \right) + u(\bar{t}, y) \right]$$

There exist  $T_1 > 0$  and a Lipschitz continuous arc  $\gamma : [0, T_1) \rightarrow \Omega$  starting from  $x_0$  and such that

$$\min_{(\tau, p) \in D^+ u(t_0 + s, \gamma(s))} F(\tau, p) \leq \left( \frac{t_0 - \bar{t}}{t_0 + s - \bar{t}} \right)^2 \min_{(\tau_0, p_0) \in D^+ u(t_0, x_0)} F(\tau_0, p_0)$$

for every  $s \in [0, T_1)$ .

## Example

The estimate above is somehow mild:

Let  $n = 1$ ,  $A = Id$ ,  $\Omega = \mathbf{R}^N$  and  $\varphi(0, x) = \frac{(|x|-1)^2}{2\varepsilon}$ .

The Hopf formula yields

$$u(t, x) = \frac{1}{2} \frac{(|x| - 1)^2}{t + \varepsilon}.$$

We obtain

$$\arg \min_{(\tau, p) \in D^+ u(t_0 + s, \gamma(s))} F(\tau, p) = \left\{ \left( -\frac{1}{2(t_0 + s + \varepsilon)^2}, 0 \right) \right\}$$

$$\frac{\min_{(\tau, p) \in D^+ u(t_0 + s, \gamma(s))} F(\tau, p)}{\min_{(\tau_0, p_0) \in D^+ u(t_0, 0)} F(\tau_0, p_0)} = \left( \frac{t_0 + \varepsilon}{t_0 + s + \varepsilon} \right)^2$$

# Global propagation of singularities

A maximality argument yields the global propagation:

## Theorem

*Let  $(t_0, x_0)$  be a singular point of  $u$ . Then there exist  $T \in (0, +\infty]$  and a Lipschitz continuous arc  $\gamma : [0, T) \rightarrow \mathbf{R}^n$  starting from  $x_0$ , satisfying*

$$(t_0 + s, \gamma(s)) \in \Sigma(u) \quad \forall s \in [0, T)$$

*and such that  $\lim_{s \rightarrow T} \gamma(s) \in \partial\Omega$  whenever  $T < +\infty$ .*

## Future directions

- Generalize to semiconcave functions on riemannian manifolds: up to now, we can prove a global propagation on manifolds with nonnegative curvature;
- Generalize to the case of more complex structures of the function  $H$ .

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- Generalize to semiconcave functions on riemannian manifolds: up to now, we can prove a global propagation on manifolds with nonnegative curvature;
- Generalize to the case of more complex structures of the function  $H$ .

*...the end.*  
*Thank you!*